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A Simple Axiomatization of Binary Rank-Dependent Utility of Gains (Losses)

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For binary gambles composed only of gains (losses) relative to a status quo, the rank-dependent utility model with a representation that is dense in intervals is shown to be equivalent to ten elementary properties plus event commutativity and a gamble partition assumption. The proof reduces to a (difficult) functional equation that has been solved by Aczél, Maksa, and Páles (in press). © 2001 Elsevier Science (USA)

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The current literature on utility theory largely agrees that preferences among binary gambles for which both of the consequences are gains, i.e., are preferred to the status quo, e, can be represented by the following *binary rank-dependent utility* (binary RDU) representation. There exists a utility function U from the set of binary gambles, including certain consequences as a special case, into the positive real numbers $\mathbb{R}_+$, and a weighting function W from events into $[0, 1]$ such that if $\succsim$ is a preference relation among binary gambles, x, y are certain consequences $\succsim e$, and C is an uncertain or chance event with $\bar{C}$ its complement, then

$$U(x, C; y) = \begin{cases} U(x)\,W(C) + U(y)[1 - W(C)], & x \succ y \\ U(x), & x \sim y \\ U(x)[1 - W(\bar{C})] + U(y)\,W(\bar{C}), & x \prec y \end{cases} \tag{1}$$

preserves the order $\succsim$ and W preserves $\subseteq$. Actually, we are able to extend this

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representation to compound gambles where x and/or y are themselves gambles. We do this by having sufficient axioms so that the gambles can be replaced by their certainty equivalents.

An entirely parallel representation holds for the case where the consequences are seen as losses, i.e., are less preferred than the status quo.

The case of mixed consequences is considerably more problematic and mostly is not considered here (see Chechile & Butler, 2000; Chechile & Cooke, 1997; Chechile & Luce, 1999; Kahneman & Tversky, 1979; Luce, 1997, 2000; Luce & Fishburn, 1991, 1995; Tversky & Kahneman, 1992).

Given the binary cases of gains and losses, there are two obvious questions. What about binary cases where one consequence is a gain and one is a loss? This complex case is covered in Chapters 6 and 7 of Luce (2000)[1]. And what about cases where there are more than two consequences? Some authors have attempted direct axiomatizations, but others work with one or another inductive principle beginning with the binary case (Luce, 1998; Luce & Fishburn, 1995; Marley & Luce, in press). This means the binary case is of special interest.

The question to be considered here is what qualitative properties give rise to Eq. (1). Several authors have addressed this question. The first approach drew on invariance arguments leading to unit structures whose most general form under invariance by positive affine transformations is rank dependence (Luce, 1988; Luce & Narens, 1985). The second approach rested heavily on assuming that the two consequence–event pairs form an additive conjoint structure in some fashion (Luce, 1998[2]; Wakker & Tversky, 1993). And the third approach rested upon arguments involving an additive representation of the joint receipt of gambles and an assumption called segregation (Luce & Fishburn, 1991, 1995); this latter approach also leads to classes of special forms for the utility function (see Luce, 2000, pp. 152, 156–157). Here we present a still different approach that seems in many ways to be the simplest way to formulate the conditions although the proof is mathematically fairly complex. As is true of most of the axiomatizations, this one requires both the consequences and the events to be very dense and so the images of the functions U and W are dense in intervals.

1. ELEMENTARY, NECESSARY PROPERTIES OF BINARY RDU

- Let $\mathscr{E}$ denote an algebra of events that can arise in a chance experiment; in particular the null set $\varnothing \in \mathscr{E}$ and the universal $\Omega = \overline{\varnothing} \in \mathscr{E}$, where for any $C \in \mathscr{E}$, $\bar{C} = \Omega \setminus C$.
- Let $\mathscr{B}_0$ denote the set of pure gains consequences plus the status quo, which we denote e.
- A binary gamble, $(x, C; y)$, is interpreted to mean that the experiment

[1] In connection with Chapter 7, be sure to note the correction listed.

[2] There is an error in this paper. The proof given holds only for rank-independent utility where $W(C) + W(\bar{C}) = 1$. See the erratum cited.

underlying $\mathscr{E}$ is run and if event $C \in \mathscr{E}$ occurs, then the consequence to the decision maker is $x \in \mathscr{B}_0$; whereas if C fails to occur, then the consequence is $y \in \mathscr{B}_0$.

- Let $\mathscr{B}_1$ denote the union of $\mathscr{B}_0$ with all binary gambles so generated.
- Let $\mathscr{B}_1^*$ denote the union of $\mathscr{B}_0$ with all binary gambles of the form $(x, C; e)$ where $x \in \mathscr{B}_0,\ C \in \mathscr{E}$
- Let $\mathscr{B}_2$ denote the union of $\mathscr{B}_1$ with all binary gambles generated from elements of $\mathscr{E}$ whose consequences are in $\mathscr{B}_1$.
- Let $\mathscr{B}_2^*$ denote the union of $\mathscr{B}_1^*$ with all binary gambles of the form $(x, C; e)$ where $x \in \mathscr{B}_1^*,\ C \in \mathscr{E}$.
- Let $\succsim$ be a binary relation over $\mathscr{B}_2$.

The following seven elementary properties are all fairly immediate consequences of binary RDU. They are also intuitively fairly compelling.

A1. *Weak ordering.* $\succsim$ is a weak ordering, i.e., transitive and connected.

A2. *Order independence of events.* For all $x, y \in \mathscr{B}_1$ with $x, y \succ e$, and $C, D \in \mathscr{E}$,

$$(x, C; e) \succsim (x, D; e) \Leftrightarrow (y, C; e) \succsim (y, D; e). \tag{2}$$

Thus, by Eq. (2), an order $\succsim_{\mathscr{E}}$ is induced over $\mathscr{E}$ in the usual fashion.

A3. *Consequence monotonicity (on the first consequence).* For all $x, y, z \in \mathscr{B}_1$ and $C \in \mathscr{E}$ such that $C \succ_{\mathscr{E}} \varnothing$,

$$x \succsim y \Leftrightarrow (x, C; z) \succsim (y, C; z). \tag{3}$$

A4. *Monotonicity of event inclusion.* For all $x, y \in \mathscr{B}_1$ with $x \succ y$, and $C, D \in \mathscr{E}$,

$$\text{if } C \supset D, \text{ then } (x, C; y) \succ (x, D; y). \tag{4}$$

PROPOSITION 1.1. *Suppose Assumptions* A2 *and* A4 *hold. Then,* $C \supset D$ *implies* $C \succ_{\mathscr{E}} D$.

Proof. Trivial. ▌

A5. *Idempotence.* For all $x \in \mathscr{B}_1$ and $C \in \mathscr{E}$,

$$(x, C; x) \sim x. \tag{5}$$

A6. *Certainty.* For all $x, y \in \mathscr{B}_1$,

$$(x, \varnothing; y) \sim y. \tag{6}$$

A7. *Complementarity.* For all $x, y \in \mathscr{B}_1$ and $C \in \mathscr{E}$,

$$(x, C; y) \sim (y, \bar{C}; x). \tag{7}$$

PROPOSITION 1.2. *Suppose assumptions* A1, A3, *and* A7 *hold. Then consequence monotonicity holds for the second consequence, i.e., for* $x, y, z \in \mathscr{B}_1$ *and* $C \in \mathscr{E}$ *with* $\Omega \succ_{\mathscr{E}} C$,

$$x \succsim y \Leftrightarrow (z, C; x) \succsim (z, C; y). \tag{8}$$

Proof. By consequence monotonicity, Eq. (3), and complementarity, A7,

$$x \succsim y \Leftrightarrow (x, \bar{C}; z) \succsim (y, \bar{C}; z) \Leftrightarrow (z, C; x) \succsim (z, C; y). \quad \blacksquare$$

Define *standard sequences* in the usual way:

$$\begin{aligned} &(x_i, C; z) \sim (x_{i+1}, D; z), \qquad C \succ_{\mathscr{E}} D \\ &(x, C_i; z) \sim (y, C_{i+1}; z), \qquad x \succ y. \end{aligned} \tag{9}$$

The next property is necessary if RDU holds, but it is not elementary in that it cannot be rejected with finitely many observations.

A8. *Archimedeanness.* Every bounded standard sequence is finite.

DEFINITION 2.1. Any structure satisfying A1–A7 is called an *elementary rational structure*. If in addition A8 holds, it is called *Archimedean.*

2. EVENT COMMUTATIVITY AND GAINS PARTITION

Our next two necessary conditions of binary RDU are of a sufficiently different nature that it is worth placing them in a separate section.

A9. *Event commutativity.* For all $x, y \in \mathscr{B}_0$ with $x \succsim y$ and $C, D \in \mathscr{E}$,

$$((x, C; y), D; y) \sim ((x, D; y), C; y), \tag{10}$$

where the successive experiments giving rise to C and D are run independently. We say that *status-quo event commutativity* holds if and only if Eq. (10) holds with $y = e$, the status quo.

A10. *Gains partition.* There exists a 1–1 function $M: \mathscr{E} \xrightarrow{onto} \mathscr{E}$ that inverts the order $\succsim_{\mathscr{E}}$ such that for $x, x', y, y' \in \mathscr{B}_0$, with $x \succsim y$, $x' \succsim y'$, and $C, C' \in \mathscr{E}$, if

$$(x, C; e) \sim (x', C'; e) \qquad \text{and} \qquad (y, M(C); e) \sim (y', M(C'); e), \tag{11}$$

then

$$(x, C; y) \sim (x', C'; y').$$

At first blush, gains partition may seem trivial; however, it need not hold in a representation where $U(x)$ and $W(C)$ (respectively, $U(y)$ and $W[M(C)]$) play separate roles, not just as $U(x)\,W(C)$ (respectively, $U(y)\,W[M(C)]$). This type of representation has arisen, for example, in Luce's (1997, 2000) treatment of binary mixed gains and losses.

From an axiomatic perspective, A10 is not fully satisfactory because it is difficult to know how to test it because the function M simply is not specified. A consideration of how A10 is used in obtaining the RDU representation shows that its main function is in deriving the existence of a function $R: A \xrightarrow{onto} I$ such that for $x, y \in \mathscr{B}_0$ with $x \succsim y$,

$$u(x, C; y) = R(u(x)\,w(C), u(y)\,w[M(C)]),$$

where $u: \mathscr{B}_1^* \xrightarrow{onto} I$ and $w: \mathscr{E} \xrightarrow{onto} J$ are derived without the use of A10. Thus, one direction for future work aimed at obtaining a more satisfactory version of A10 might concentrate on its use in deriving the existence of the function R.

We also know that RDU requires that $M(C) = W^{-1}[1 - W(C)]$, which follows from our axioms. So future work might also consider a more direct derivation of that form. (For instance, one property of this form of M is that $M^2 = I$). The situation would be appreciably better if we had a form for $W(C)$ specified to a few parameters. No relevant theory currently exists for general events, but in the case of risk Prelec (1998; see also Luce, 2001) has justified the form $W(p) = \exp(-\gamma(-\ln p)^{\alpha})$, $\alpha > 0, \gamma > 0$. So, in that case, the form of M is known up to two parameters that can be estimated.

3. STRUCTURAL PROPERTIES

The following structural property is necessary if RDU is onto sets that are dense in intervals.

A11. *Density.* The orderings over $\mathscr{B}_2$ and $\mathscr{E}$ are both nontrivial and dense in the sense that there exist $x, y \in \mathscr{B}_2$ such that $x \succ y$; for any $x \succ y$, there exists $z \in \mathscr{B}_2$ such that $x \succ z \succ y$; there exist $C, D \in \mathscr{E}$ such that $C \succ_{\mathscr{E}} D$; for any $C \succ_{\mathscr{E}} D$, there exists $B \in \mathscr{E}$ such that $C \succ_{\mathscr{E}} B \succ_{\mathscr{E}} D$.

The next property is not necessary even for RDU that is dense in intervals.

A12. *Restricted solvability.* For all $x, x^*, x_*, y, z \in \mathscr{B}_1$ and $C, C^*, C_*, D \in \mathscr{E}$,

$$\begin{aligned}
&\text{if } (x^*, C; y) \succsim (z, D; y) \succsim (x_*, C; y),\\
&\text{then } \exists x \in \mathscr{B}_0 \text{ such that } (x, C; y) \sim (z, D; y),\\
&\text{if } (x, C^*; y) \succsim (z, D; y) \succsim (x, C_*; y),\\
&\text{then } \exists C \in \mathscr{E} \text{ such that } (x, C; y) \sim (z, D; y).
\end{aligned} \tag{12}$$

PROPOSITION 3.1. *Suppose $\langle \mathscr{B}_2, \succsim \rangle$ is an elementary rational structure that satisfies restricted solvability. Then for any gamble $(x, C; y) \in \mathscr{B}_2$, $x \succsim y$, there exists $CE(x, C; y) \in \mathscr{B}_0$ such that*

$$CE(x, C; y) \sim (x, C; y). \tag{13}$$

Proof. By complementarity, A7, certainty, A6, idempotence, A5, consequence monotonicity, Eqs. (3) and (8), and weak order, A1,

$$\begin{aligned}(x, \bar{\varnothing}; y) &\sim (y, \varnothing; x) \sim x \sim (x, C; x)\\ &\succsim (x, C; y) \succsim (y, C; y) \sim y\\ &\sim (y, \varnothing; y) \sim (y, \bar{\varnothing}; y).\end{aligned}$$

So by restricted solvability, A12, there exists $z \in \mathscr{B}_0$ such that $(z, \bar{\varnothing}; y) \sim (x, C; y)$. Then using certainty and complementarity, $z \sim (y, \varnothing; z) \sim (z, \bar{\varnothing}; y) \sim (x, C; y)$. Finally, set $CE(x, C; y) = z$. ▮

$CE(x, C; y)$ is called the *certainty equivalent* of $(x, C; y)$. Note that by A1, A3, and Proposition 1.2, the existence of certainty equivalents means that for any $(g, C; h) \in \mathscr{B}_2$, we have

$$CE(g, C; h) \sim (CE(g), C; CE(h)) \in \mathscr{B}_1.$$

We are now ready to state the major result.

4. MAJOR RESULT AND CONCLUSION

THEOREM 4.1. *Consider a structure* $\langle \mathscr{B}_2, \succsim \rangle$.

(i) *If* $\langle \mathscr{B}_2, \succsim \rangle$ *has a RDU representation, then* $\langle \mathscr{B}_2, \succsim \rangle$ *is an Archimedean elementary rational structure (*A1–A8*) that satisfies* A9, *and* A10 *with* $W[M(C)] = 1 - W(C)$.

(ii) *If in addition the representation is onto sets that are dense in intervals, then A11 also holds.*

(iii) *The necessary assumptions* A1–A10 *together with* A11 *and* A12 *are sufficient for a binary RDU representation of* $\langle \mathscr{B}_2, \succsim \rangle$ *in which* $W[M(C)] = 1 - W(C)$.

The conclusion, then, is this. Suppose a structure is Archimedean, elementary rational, A1–A8, that satisfies density, A11, and restricted solvability, A12, all of which are widely accepted and in some cases tested. Then the existence of a rank-dependent utility representation devolves to the satisfaction of event commutativity, A9, and gains partition, A10. As a step in the proof, Proposition 5.2 below shows that under the conditions of the theorem a gamble $(x, C; y)$ can be expressed as a function of the two simpler gambles $(x, C; e)$ and $(y, M(C); e) \sim (e, \overline{M(C)}; y)$. Although the function is initially unspecified, the proof establishes that it has a binary RDU representation.

We believe this to be the simplest way to axiomatize the binary rank-dependent model; however, the proof involves a functional equation whose solution is far from simple.

Note that for $M(C)=\bar{C}$, the above gives an axiomatization of the binary rank-independent utility (RIU) model for which $W(C)+W(\bar{C})=1$.

5. PROOF OF THE MAJOR RESULT

PROPOSITION 5.1. *Suppose assumptions* A1–A4, A8, *status-quo event commutativity* A9, A11, *and* A12 *hold. Then there exists a set I that is dense in an interval* $[0, a[$, $a \succ 0$, *such that* $u: \mathscr{B}_1^* \xrightarrow{onto} I$, *and a set J that is dense in* $[0, 1]$ *such that* $w: \mathscr{E} \xrightarrow{onto} J$, *and uw is an order preserving multiplicative representation of* $\mathscr{B}_1^* \times \mathscr{E}$ *with* $u(e)=0$, $w(\varnothing)=0$, *and* $w(\bar{\varnothing})=w(\Omega)=1$.

Proof. Luce (1996) noted that transitivity, monotonicity of consequences, and status-quo event commutativity are sufficient to show that the Thomsen condition holds for the trade-off structure of consequences and events. That together with the remaining conditions implies, by a standard theorem of additive conjoint representation, that there exists an order preserving multiplicative representation uw over $(\mathscr{B}_1^* \setminus [e]) \times (\mathscr{E} \setminus [\varnothing])$, where $[e]$ is the equivalence class of e under $\sim$ and $[\varnothing]$ is the equivalence class of $\varnothing$ under $\sim_{\mathscr{E}}$ (Krantz, Luce, Suppes, & Tversky, 1971, Ch. 6). We now extend this representation on $(\mathscr{B}_1^* \setminus [e]) \times (\mathscr{E} \setminus [\varnothing])$ to a multiplicative representation on the remainder of $\mathscr{B}_1^* \times \mathscr{E}$, by setting $u(e)=w(\varnothing)=0$. Then from certainty, A6, and idempotence, A5, we have for any $x \in \mathscr{B}_1^*$, $C \in \mathscr{E}$,

$$u(x, \varnothing; e)=u(e)=0=u(x)\, w(\varnothing)$$

and

$$u(e, C; e)=u(e)=0=u(e)\, w(C),$$

the required multiplicative form. Finally, we show that this extension is consistent with the multiplicative representation uw over $(\mathscr{B}_1^* \setminus [e]) \times (\mathscr{E} \setminus [\varnothing])$. Assumption A11, density, ensures that the image of u is dense in an interval $[0, a[$, where $a=\sup \{u(g) : g \in \mathscr{B}_1^*\}$, and the image of w is dense in $[0, 1]$. Because e is a minimal element relative to $\succsim$ over $\mathscr{B}_1^*$ and I is dense in $[0, a[$, it follows that we can set $u(e)=0$. And by monotonicity of event inclusion, A4, $\varnothing$ is a lower bound and $\Omega=\bar{\varnothing}$ is an upper bound under $\succsim_{\mathscr{E}}$. Thus, because the image of w is dense in $[0, 1]$, we can set $w(\varnothing)=0$ and we must have $w(\Omega)=1$. ▮

The only place where restricted solvability plays a role is in the above proof and in Proposition 3.1 establishing the existence of certainty equivalents.

The property that uw is order preserving is called *separability*.

Let

$$A=\{(X, Y) : \exists x, y \in \mathscr{B}_0, C \in \mathscr{E} \text{ such that } X=u(x)\, w(C), Y=u(y)\, w[M(C)]\}.$$

PROPOSITION 5.2. *Suppose assumptions* A1–A8, *status-quo event commutativity (*A9*), and* A10–12 *hold and that uw is a separable representation (Proposition* 5.1*). Then there exists a function* $R: A \xrightarrow{onto} I$ *that is continuous and strictly increasing in each argument and such that for* $x, y \in \mathscr{B}_0$ *with* $x \succsim y$,

$$u(x, C; y) = R(u(x)\, w(C), u(y)\, w[M(C)]), \tag{14}$$

$$R(X, 0) = X, \tag{15}$$

$$R(0, Y) = Y. \tag{16}$$

Proof. Using the existence of certainty equivalents (Proposition 3.1), we first extend u to $\mathscr{B}_1$ by setting $u(x, C; y) = u[CE(x, C; y)]$. Then for real $(X, Y) \in A$ $Z \in I$, define $R(X, Y) = Z$ if there exist $x, y \in \mathscr{B}_0, C \in \mathscr{E}$ with $X = u(x)\, w(C)$, $Y = u(y)\, w[M(C)]$, and $Z = u(x, C; y)$. Observe that assumption A10 insures that R is well defined. So Eq. (14) holds.

We need to show R is onto I. That is obvious once we establish either Eq. (15) or (16); we establish both. To that end, we have from M is 1–1, strictly decreasing, and onto $\mathscr{E}$, that $M(\varnothing) = \Omega$ and $M(\Omega) = \varnothing$, and from Proposition 5.1 that $w(\varnothing) = 0$, $w(\bar{\varnothing}) = w(E) = 1$. Choosing $X = u(x) = u(x)\, w(E), Y = 0 = u(y)\, w(\varnothing) = u(y)\, w[M(\Omega)]$, and using complementarity, A7, certainty, A6, and Eq. (13),

$$\begin{aligned} R(X, 0) &= R(u(x)\, w(\Omega), u(y)\, w[M(\Omega)]) \\ &= u[CE(x, \Omega; y)] \\ &= u[CE(y, \varnothing; x)] \\ &= u(y; \varnothing; x) \\ &= u(x) \\ &= u(x)\, w(\Omega) \\ &= X. \end{aligned}$$

Similarily, choosing $X = 0 = u(x)\, w(\varnothing)$, $Y = u(y) = u(y)\, w(\Omega) = u(y)\, w[M(\varnothing)]$, and using certainty, A6, and Eq. (13),

$$\begin{aligned} R(0, Y) &= R(u(x)\, w(\varnothing), u(y)\, w[M(\varnothing)]) \\ &= u[CE(x, \varnothing; y)] \\ &= u[CE(y)] \\ &= u(y) \\ &= u(y)\, w(\Omega) \\ &= u(y)\, w[M(\varnothing)] \\ &= Y. \end{aligned}$$

By consequence monotonicity, R is strictly increasing in each argument and because it is onto a dense interval it is continuous in each. ∎

We now introduce a function that is used in the next steps of the proof. For any $P \in [0, 1]$ for which there exists $C \in \mathscr{E}$ with $P = w(C)$, let

$$\pi(P) = w[M(C)].$$

We show that π: $[0, 1] \to [0, 1]$ is a well-defined strictly decreasing function. If $C, C' \in \mathscr{E}$ are such that $P = w(C) = w(C')$, then the fact that both w and M preserve equivalence with respect to the order $\succsim_{\mathscr{E}}$ gives that $C \sim_{\mathscr{E}} C'$ and $w[M(C)] = w[M(C')]$. Thus π is a well-defined function, and it is strictly decreasing because M inverts the order $\succsim_{\mathscr{E}}$.

Throughout the following, we use the notations:

$$P = w(C), \qquad P \in [0, 1],$$

$$Z = \frac{u(y)}{u(x)}, \qquad Z \in [0, 1] \quad \text{if} \quad x \succsim y \succ e,$$

$$\gamma(P) = \frac{\pi(P)}{P}, \qquad \gamma :]0, 1[\to]0, \infty[.$$

Note that because π is strictly decreasing in P, so is γ. Because the domain of P is dense in $[0, 1]$, π can be extended to a continuous, strictly decreasing function on $]0, 1[$. And so γ is continuous on $]0, 1[$.

Using the above notation, Eq. (14) becomes $u(x, C; y) = R[u(x)\, P, u(y)\pi(P)]$. By A2, A3, Propositions 1.1 and 1.2, and the fact that u is order preserving of CEs, R is order preserving in each of $u(x)$, $u(y)$, P, and so R can be extended continuously to intervals. From now on, we work with this extension.

The technique used in the following proposition was first used in Maksa, Marley, and Páles (2000).

Proposition 5.3. *Suppose assumptions* A1–A8, *status-quo event commutativity* (A9), *and* A10–A12 *hold. Then, for* $x, y \in \mathscr{B}_0$ *with* $x \succsim y$,

$$u(x, C; y) = \begin{cases} \dfrac{u(x)\, P}{\gamma^{-1}[Z\gamma(P)]}, & \textit{if} \quad u(x)\, P > 0 \\ u(y)\, \pi(P), & \textit{if} \quad u(x)\, P = 0. \end{cases} \tag{17}$$

Proof. For $u(x)\, P = 0$,

$$u(x, C; y) = R[u(x)\, P, u(y)\, \pi(P)] = R[0, u(y)\, \pi(P)] = u(y)\, \pi(P).$$

So assume $u(x)\, P > 0$. By Proposition 3.1, let $v = CE(x, C; y)$. Let $Q = \frac{u(x)}{u(v)} P$, where $Q \in [0, 1]$ because by consequence monotonicity and Eq. (15),

$$\begin{aligned} u(v) = u(x, C; y) \geqslant u(x, C; e) &= R(u(x)\, w(C), u(e)\, w[M(C)]) \\ &= R[u(x)\, w(C), 0] = u(x)\, w(C) = u(x)\, P. \end{aligned} \tag{18}$$

Because the image of w, J, is dense in the unit interval, we may choose a descending sequence $Q_i \in J$ converging to Q. Let $D_i \in \mathscr{E}$ be such that $w(D_i) = Q_i$ and correspondingly $w[M(D_i)] = \pi(Q_i)$. So, using Proposition 5.2

$$\begin{aligned} R[u(x)\,P, u(y)\,\pi(P)] &= u(x, C; y) \\ &= u(v) \\ &= u(v, D_i; v) \\ &= R(u(v)\,w(D_i), u(v)\,w[M(D_i)]) \\ &= R[u(v)\,Q_i, u(v)\,\pi(Q_i)]. \end{aligned}$$

By the continuity of R and taking limits

$$\begin{aligned} R[u(x)\,P, u(y)\,\pi(P)] &= \lim_{i\to\infty} R[u(v)\,Q_i, u(v)\,\pi(Q_i)] \\ &= R[u(v) \lim_{i\to\infty} Q_i, u(v) \lim_{i\to\infty} \pi(Q_i)] \\ &= R[u(v)\,Q, u(v)\,\pi(Q)] \\ &= R[u(x)\,P, u(v)\,\pi(Q)]. \end{aligned}$$

Using the monotonicity of R, we see $u(y)\,\pi(P) = u(v)\,\pi(Q)$. On eliminating $u(v)$ by Eq. (18) and using the definition of Q, we obtain

$$\gamma(Q) := \frac{\pi(Q)}{Q} = \frac{u(y)}{u(x)} \frac{\pi(P)}{P} = Z\gamma(P).$$

Therefore,

$$u(x, C; y) = u(v) = \frac{u(x)\,P}{Q} = \frac{u(x)\,P}{\gamma^{-1}[Z\gamma(P)]}. \quad \blacksquare$$

By monotonicity of consequences and idempotence it follows immediately that the structure is *intern* in the sense that for all $x \succsim y$ and $C \in \mathscr{E}$,

$$x \succsim (x, C; y) \succsim y. \tag{19}$$

Thus, from Eqs. (17) and (19) we see that for $Z, P \in \,]0, 1[$

$$P \leqslant \gamma^{-1}[Z\gamma(P)] \leqslant \frac{P}{Z}. \tag{20}$$

For the next result it is convenient to introduce two notations,

$$F(Z, P) = \gamma^{-1}[Z\gamma(P)], \qquad \text{where} \quad F: \,]0, 1[\, \times \,]0, 1[\, \longrightarrow \,]0, \infty[, \tag{21}$$

$$G(Z, P) = \frac{Z}{P} F(Z, P), \qquad \text{where} \quad Z, P \in \,]0, 1[, \tag{22}$$

where we use the continuous extension of γ. So both F and G are continuous in both variables. From Eq. (20) we have the following bounds on F and G:

$$P \leqslant F(Z, P) \leqslant \frac{P}{Z}, \qquad Z \leqslant G(Z, P) \leqslant 1. \tag{23}$$

We introduce two additional concepts about G that will be used. First, the iteration of G is defined for $i \geqslant 1$,

$$G^1(Z, P) = G(Z, P)$$
$$G^i(Z, P) = G[G^{i-1}(Z, P), P].$$

Recall that R, F, and G are all continuous.

PROPOSITION 5.4. *Under* A1–A12, *G over* $]0, 1[\times]0, 1[$ *has the following properties:*

(i) *G is strictly increasing in the first variable and strictly decreasing in the second.*

(ii) *For all $Z, P, Q \in]0, 1[$,*

$$G[G(Z, P), Q] = G[G(Z, Q), P]. \tag{24}$$

(iii) *G is Archimedean in the sense that for $Y, Z, P \in]0, 1[$, there exists a positive integer m such that $G^m(Z, P) \geqslant Y$.*

(iv) *G is solvable in the sense that for $Y, Z, P \in]0, 1[$ if there exists a non-negative integer n with*

$$G^{n+1}(Z, P) \geqslant Y > G^n(Z, P); \tag{25}$$

then there exist $Q \in]0, 1[$ with $Y = G[G^n(Z, P), Q]$.

Proof. We have $Z, P \in]0, 1[$. This allows us to restrict attention to cases where $x, y \in \mathscr{B}_0$ with $x \succ y \succ e$. In particular, we can assume that product terms such as $u(x)\,P$ have values greater than zero; i.e., the first limb of Eq. (17) holds.

(i) Let $\tilde{I} = \{r \mid \exists x, y \in \mathscr{B}_0 \text{ with } r = \frac{u(y)}{u(x)}\}$. We first show monotonicity for the first variable of G. Consider $s, x, y \in \mathscr{B}_0$ such that $s \succ x \succ y \succ e$. Then $Y = \frac{u(y)}{u(s)}$, $Z = \frac{u(y)}{u(x)} \in \tilde{I}$, and $Y < Z$. Also, consider $P \in J$, so there is $C \in \mathscr{E}$ with $w(C) = P$. Then, with the first limb of Eq. (17),

$$\begin{aligned} Y < Z &\Leftrightarrow u(s) > u(x) \\ &\Leftrightarrow u(s, C; y) > u(x, C; y) \\ &\Leftrightarrow \frac{u(y)}{u(s, C; y)} < \frac{u(y)}{u(x, C; y)} \\ &\Leftrightarrow \frac{u(y)}{u(s)\,P}\,\gamma^{-1}\left[\frac{u(y)}{u(s)}\,\gamma(P)\right] < \frac{u(y)}{u(x) P}\,\gamma^{-1}\left[\frac{u(y)}{u(x)}\,\gamma(P)\right] \\ &\Leftrightarrow \frac{Y}{P}\,F(Y, P) < \frac{Z}{P}\,F(Z, P) \\ &\Leftrightarrow G(Y, P) < G(Z, P). \end{aligned}$$

Now, we must extend this to the case where $Y, Z, P \in]0, 1[$, $Y < Z$. By density, we know there exist $Y', Z' \in \tilde{I}$ such that $Y < Y' < Z' < Z$. Thus, by what we have just shown, for $P \in J$, $G(Y', P) < G(Z', P)$. Let $\varepsilon = G(Z', P) - G(Y', P)$. Then by continuity of G and density, we can choose $Y'', Z'' \in \tilde{I}$ such that $Y < Y'' < Y' < Z' < Z'' < Z$ and $|G(Y, P) - G(Y'', P)| < \frac{\varepsilon}{4}$ and $|G(Z, P) - G(Z'', P)| < \frac{\varepsilon}{4}$. Thus, since $G(Y'', P) < G(Y', P) < G(Z', P) < G(Z'', P)$, we see that $G(Y, P) < G(Z, P)$. By the continuity of G, this extends to $P \in]0, 1[$.

Next we show monotonicity for the second variable. Fix $Z = \frac{u(y)}{u(x)}$, and consider $C, D \in \mathscr{E}$ with $P = w(C)$, $Q = w(D)$. Then

$$\begin{aligned} P > Q &\Leftrightarrow w(C) > w(D) \\ &\Leftrightarrow u(x, C; y) > u(x, D; y) \\ &\Leftrightarrow \frac{u(y)}{u(x, C; y)} < \frac{u(y)}{u(x, D; y)} \\ &\Leftrightarrow \frac{u(y)}{u(x)\, P} \gamma^{-1}\left[\frac{u(y)}{u(x)} \gamma(P)\right] < \frac{u(y)}{u(x)\, Q} \gamma^{-1}\left[\frac{u(y)}{u(x)} \gamma(Q)\right] \\ &\Leftrightarrow \frac{Z}{P} F(Z, P) < \frac{Z}{Q} F(Z, Q) \\ &\Leftrightarrow G(Z, P) < G(Z, Q). \end{aligned}$$

An argument paralleling that given for the first variable then extends this result to the case where $P, Q, Z \in]0, 1[$, $P > Q$.

(ii) Using the first limb of Eq. (17),

$$\begin{aligned} &u[((x, C; y), D; y)] \\ &\quad = u[(CE(x, C; y), D; y)] \\ &\quad = \frac{u[CE(x, C; y)]\, w(D)}{\gamma^{-1}\left(\dfrac{u(y)}{u[CE(x, C; y)]} \gamma[w(D)]\right)} \\ &\quad = \frac{u(x, C; y)\, w(D)}{\gamma^{-1}\left(\dfrac{u(y)}{u(x, C; y)} \gamma[w(D)]\right)} \\ &\quad = \frac{u(x)\, w(C)\, w(D)}{\gamma^{-1}\left(\dfrac{u(y)}{u(x)} \gamma[w(C)]\right) \gamma^{-1}\left[\dfrac{u(y)}{u(x)\, w(C)} \gamma^{-1}\left(\dfrac{u(y)}{u(x)} \gamma[w(C)]\right) \gamma[w(D)]\right]} \\ &\quad = \frac{u(x)\, PQ}{F(Z, P)\, F\left[\dfrac{Z}{P} F(Z, P), Q\right]}. \end{aligned}$$

So, assuming event commutativity, Eq. (10), we have

$$F(Z, P)\, F\left[\frac{Z}{P} F(Z, P), Q\right] = F(Z, Q)\, F\left[\frac{Z}{Q} F(Z, Q), P\right].$$

This holds for $P, Q \in J$ and $Z \in \tilde{I}$. By the continuity of F in each variable, this is extended to $P, Q, Z \in \,]0, 1[$. Multiply this equation by $\frac{Z}{PQ}$ and then rewrite it in terms of G to yield the functional equation (24).

(iii) Because G is strictly increasing in the first variable, the only way the Archimedean property can fail is for G^m to approach a limit $L < 1$. If L is a limit, then we have from Eqs. (21) and (22)

$$L = G(L, P) = \frac{L}{P} \gamma^{-1}[L\gamma(P)],$$

whence $\gamma(P) = L\gamma(P)$. So $L = 1$ since $\gamma(P) > 0$ for $P \in \,]0, 1[$.

(iv) Turning to solvability, because $G(Z, P)$ is strictly monotonically decreasing in P, Eq. (25) with the fact that $G(Z, 1) = 1$ yields

$$G[G^n(Z, P), P] = G^{n+1}(Z, P) \geqslant Y > G^n(Z, P) = G[G^n(Z, P), 1].$$

The continuity of G in the second variable yields Q such that $Y = G[G^n(Z, P), Q]$. Note that if G is Archimedean and $Y < G(Z, P)$, then the condition Eq. (25) is always met and so under solvability Q always exists. ∎

We turn now to the proof of Theorem 4.1 itself.

Equation (24) is the well known commutativity functional equation studied by Luce (1964) and Marley (1967) who adapted results from Aczél (1966). Specifically, Marley has shown that there exist functions φ: $]0, 1[\to]0, \infty[$ and $\psi :]0, 1[\to]1, \infty[$ such that

$$G(Z, P) = \varphi^{-1}[\varphi(Z)\, \psi(P)] \tag{26}$$

provided the following conditions are met: commutativity, strict monotonicity in Z, G is Archimedean, and it is solvable. Proposition 5.4 established these conditions. (Because $G(Z, P) \geqslant Z$ for all $P > 0$, we only need consider the second conditions of Definitions 9 and 10 of Marley (1967).)

Because G is strictly increasing in Z, φ is strictly increasing, and because G is strictly decreasing in P, ψ is strictly decreasing. From Eq. (23) we see

$$1 \leqslant \psi(P) \leqslant \frac{\varphi(1)}{\varphi(Z)}.$$

Putting Eqs. (21), (22), and (26) together yields

$$\begin{aligned}\frac{Z}{P}\gamma^{-1}[Z\gamma(P))] &= \frac{Z}{P}F(X, Z)\\ &= G(Z, P)\\ &= \varphi^{-1}[\varphi(Z)\,\psi(P)]. \qquad (27)\end{aligned}$$

The task thus devolves to solving this functional equation in the three unknown functions γ, φ, and ψ. Aczél, Maksa, and Páles (in press) have done so—which is not easy—yielding, for some constants $A > 0$, $k > 0$, $c < 0$,

$$\gamma(P) = \frac{(1-P^k)^{1/k}}{P},$$

$$\varphi(Z) = A\left(\frac{1-Z^k}{Z^k}\right)^c,$$

$$\psi(P) = P^{kc}.$$

Thus,

$$\pi(P) = P\gamma(P) = (1-P^k)^{1/k},$$

and

$$\varphi(Z)\,\psi(P) = \varphi\left(\frac{Z}{[(1-P^k)\,Z^k+P^k]^{1/k}}\right). \qquad (28)$$

Now, consider expressions for $u(x, C; y)$, $x \gtrsim y$. Let $P = w(C)$. We split the proof into several cases.

(i) $x \succ y \succ e$. Then $u(x) > u(y) > 0$, and so $Z = \frac{u(y)}{u(x)}$ is in $]0, 1[$. We now need to consider three cases for P, namely $P \in\,]0, 1[$, $P = 0$, and $P = 1$.

For $P \in\,]0, 1[$, using the first limb of Eq. (17) and the above solution of the functional equation, we have:

$$\begin{aligned}u(x, C; y) &= \frac{u(x)\,w(C)}{F\left(\dfrac{u(y)}{u(x)}, \gamma[w(C)]\right)}\\ &= \frac{u(x)\,P}{\dfrac{P}{Z}G(Z, P)} \qquad [\text{Def. of } G]\\ &= \frac{u(y)}{G(Z, P)}\\ &= \frac{u(y)}{\varphi^{-1}[\varphi(Z)\,\psi(P)]} \qquad [\text{Eq. } (26)]\\ &= \frac{u(y)}{\varphi^{-1}\left[\varphi\left(\dfrac{Z}{[(1-P^k)\,Z^k+P^k]^{1/k}}\right)\right]} \qquad [\text{Eq. } (28)]\\ &= \frac{u(y)[(1-P^k)\,Z^k+P^k]^{1/k}}{Z}.\end{aligned}$$

So, setting $U = u^k$ and $W = w^k$, we have

$$U(x, C; y) = U(x)\, W(C) + U(y)[1 - W(C)],$$

which is the binary rank-dependent form for the case $x \succsim y \succ e$.

For $P = 0$, i.e., $C \sim_{\mathscr{E}} \varnothing$, Eq. (16) gives

$$\begin{aligned} u(x, C; y) &= R(u(x)\, w(\varnothing), u(y)\, w[M(\varnothing)]) = R(0, u(y)\, w[M(\varnothing)]) \\ &= u(y)\, w[M(\varnothing)] = u(y)\, w(\Omega) = u(y). \end{aligned} \tag{29}$$

Taking the kth power of Eq. (29) yields $U(x, C; y) = U(y)$ which is the special case of the binary RDU form where $W(C) = P^k = 0$, $W[M(C)] = (1-P)^k = 1$.

For $P = 1$, i.e., $C \sim_{\mathscr{E}} \bar{\varnothing}$, complementarity, A7, gives

$$(x, C; y) = (x, \bar{\varnothing}; y) \sim (y, \varnothing; x),$$

and the binary RDU form follows by appying the above argument for the case $P = 0$ to the term $(y, \varnothing; x)$.

(ii) $x \sim y \succ e$, i.e., $u(x) = u(y)$. Then using idempotence, A5, we have $u(x, C; y) = u(x) = u(y)$, and so, with $U = u^k$, we have

$$U(x, C; y) = U(x)\, W(C) + U(y)[1 - W(C)].$$

(iii) $y \sim e$, i.e., $u(y) = 0$. Using Eq. (15),

$$\begin{aligned} u(x, C; y) &= R(u(x)\, w(C), u(y)\, w[M(C)]) \\ &= R[u(x)\, w(C), 0]] \\ &= u(x)\, w(C). \end{aligned}$$

Thus, taking the k power yields $U(x, C; y) = U(x)\, W(C)$.

Complementarity, A7, yields the case where $x \precsim y$. ∎

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