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A Psychophysical Theory of Intensity Proportions, Joint Presentations, and Matches

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Empirically testable assumptions relate 3 psychophysical primitives: presentations of pairs of physical intensities (e.g., pure tones of the same frequency and phase to the 2 ears or 2 successive tones to both ears); a respondent's ordering of such signal pairs by perceived intensity (e.g., loudness); and judgments about 2 pairs of stimuli being related as some proportion (numerical factor, as in magnitude production). Explicit behavioral assumptions lead to 2 families of psychophysical functions, one corresponding to unbiased joint presentations and the other to biased ones. Under an invariance assumption, the psychophysical functions in the unbiased case are approximate power functions, and those in the biased case are exact power functions. A number of testable predictions are made. The mathematics involved draws from publications in utility theory and mathematics but with a reinterpretation of the primitives.

This article offers a psychophysical theory for physical intensities that rests on three psychophysical primitives. The first concerns the joint presentation of pairs of intensities, which are called *stimuli* in contrast to their component signal intensities. A pure tone example is applying different intensities of the same frequency, in phase, to the two ears. This and two other interpretations are detailed below. The second primitive describes a respondent's ordering of stimuli according to a subjective intensity attribute. An example is the ordering of pure tone sounds of the same frequency according to loudness. When the same intensity is presented to both ears, the loudness order is identical to that of physical intensity, but in general, the loudness order is not physically determined. One does not know a priori if 50 dB to the left ear and 57 dB to the right is louder or not than 55 dB to the left and 53 dB to the right. The third primitive is the respondent's subjective determination of an intensity interval that is some prescribed proportion of another intensity interval. When the lower end of each interval is the respondent's threshold, these judgments are called *magnitude productions* (Stevens, 1975).

The assumptions of the theory relate these three attributes—joint presentations, ordering, and proportions—in terms of several testable hypotheses. An invariance postulate sharply limits the form of the psychophysical function to either a power function or something that for most of the range closely approximates a power function. Assuming that such representations hold for two modalities, the forms of cross-modality matches are predicted. Another, testable, invariance principle limits the form of a weighting function that arises in describing proportion judgments.

The results presented in this article suggest a substantial empirical program. If the data from some interpretation of the primitives support the several qualitative assumptions described here, then the article establishes that a fairly comprehensive theory of numerical representations holds for global psychophysical judgments. This theory integrates three major methods—proportions, summation over joint presentations, and matches. If only some of the assumptions are supported empirically, the failures will clarify which aspects of the theory need modification. One such modification occurred during the development of this article. Originally, Luce (2000) presented the theory only for the unbiased (commutative) cases. When a psychophysical test rejected that assumption, the noncommutative cases reported here were developed as well.

Most of the mathematics involved is not presented here because it is fairly complicated. For the unbiased case, the theory, which had been developed originally as a theory of utility, was summarized in Luce (2000). The theory for the biased case was developed by Aczél, Luce, and Ng (in press). Here, I interpret the primitives differently from the utility interpretation. The mathematical results constraining the form of the psychophysical function are found in Luce (2000) and Marchant and Luce (in press), and those constraining the possible forms for the weighting function are from Luce (2001) and Prelec (1998).

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Basic Setup[1]

The Primitives

Let $\mathbb{R}_+ = \{x|x$ is a real number, $x \geq 0\}$ denote the set of non-negative real numbers ordered by $\geq$ and endowed with both addition (+) and multiplication (·).

- The structure $\langle \mathbb{R}_+, \geq, +, \cdot \rangle$ is a numerical representation of physical signal intensities in some fixed units, for example, the intensity of pure tones of the same frequency, in which + represents intensity superposition and $r \cdot x$ represents increasing the intensity x by a factor of r. As usual, multiplication is abbreviated to rx. The additive identity 0 represents the presentation of no intensity at all. Over $\mathbb{R}_+$, $x \geq y$ denotes the physical ordering of intensities. Of course, the physical intensities that can be presented to people are, in practice and for ethical reasons, necessarily bounded from above. That fact is not modelled in the present theory. In addition, for many psychophysical interpretations, signals of sufficiently low intensities are not detectable. The entire presentation is greatly simplified by working with suprathreshold intensity measures. Let ε denote the relevant absolute threshold;[2] then when we speak of presenting intensity x, from a purely physical perspective we actually present $x' = x + \varepsilon$. These suprathreshold intensities can also be represented by $\langle \mathbb{R}_+, \geq, +, \cdot \rangle$
- Let x, u denote two suprathreshold intensities, and let (x, u) denote their *joint presentation* that is treated by the subject as a single entity.[3] It is important to distinguish the left and right variables in (x, u). Possible interpretations and properties are given below. We must assume that there may be distinct left and right thresholds, denoted ε_l and ε_r, and so the actual physical intensities are $x' = x + \varepsilon_l$, $u' = u + \varepsilon_r$. For notational clarity, let

$$T_l = \{x|x = x' - \varepsilon_l, x' \geq \varepsilon_l \geq 0\}, \quad (1)$$

$$T_r = \{u|u = u' - \varepsilon_r, u' \geq \varepsilon_r \geq 0\}. \quad (2)$$

  Observe, as is obvious, that $T_l = T_r = \mathbb{R}_+$. Sometimes the T notation is used for increased clarity, but mostly I use either $\mathbb{R}_+$ on the assumption that the reader will recall that we are dealing with intensity increments above threshold, or I substitute statements such as $x \geq 0$, $u \geq 0$. For $x \in T_l$, $u \in T_r$, it is assumed that (x, u) can always be presented. The convention I follow most often is to use x, y for intensities in the left position and u, v for those in the right position.

- Let $\succsim$ denote a psychological ordering over $T_l \times T_r = \mathbb{R}_+ \times \mathbb{R}_+$, for example, the ordering of auditory stimuli by loudness. For $x, y \in T_l$, $u, v \in T_r$, let $a = (x, u)$ and $b = (y, v)$. Then $a \succsim b$ denotes the respondent's judgment that stimulus a is perceived as exhibiting at least as much of the attribute as stimulus b, for example, "at least as loud as" in the case of stimuli constructed from pure tones or noises. Let $\sim$ denote psychological indifference in the sense that $a \sim b$ means that both $a \succsim b$ and $b \succsim a$ hold, and the strict order $\succ$ is defined by $a \succ b \Leftrightarrow a \succsim b$ and not $(a \sim b)$. We do *not* automatically assume joint presentation is *symmetric* (unbiased) in the sense that for all $x, u \in \mathbb{R}_+$

$$(x, u) \sim (u, x). \quad (3)$$

- Next is a family of binary operations $\circ_{i,p}$ on T_i, where $i = l, r, p \geq 0$. In defining the operation on T_l, assume that intensities are presented to the left position and 0 (i.e., the threshold intensity or less) to the right position. For $x, y \in T_i$, with $x > y$, let $z = x \circ_{i,p} y > y$ denote the stimulus that the respondent judges makes the "interval" $[y, z]$ stand subjectively in proportion p to the interval $[y, x]$, where p may be any positive real number.[4] So, in the case of hearing, these operations are realized by four successive presentations to the same ear, the first two defining the intensity interval $[y, x]$ and the second two defining the intensity interval $[y, z]$, where $z = x \circ_{i,p} y$, $i = l, r$, is the respondent's choice.

Three Realizations of Joint Presentation

I describe here the three different psychophysical interpretations of (x, u) that are under empirical study. Others may be possible.

1. Let (x, u) denote the physical superposition of two intensities x and u, that is, $(x, u) = x + u$. Clearly, for this interpretation symmetry, Equation 3, holds, and it is anticipated that there are no thresholds, that is, $\varepsilon_l = \varepsilon_r = 0$. Indeed, the assumptions of extensive measurement such as closure, monotonicity, commutativity, and associativity are satisfied. This leads to the subjective measure being proportional to intensity. This is not an acceptable model for dimensions such as loudness and brightness where, for well over a century, it has been explicitly recognized that subjective intensity is not proportional to physical intensity. This interpretation may be satisfactory, however, for line lengths.[5]

2. Let the signals in a joint presentation be intensities of pure tones of the same frequency and phase or of bursts of white noise with different intensities at each ear. In particular, for $x, u \in \mathbb{R}_+$, let (x, u) denote the presentation of physical intensity $x + \varepsilon_l$ to the left ear and $u + \varepsilon_r$ to the right ear. The subjective summation corresponding to loudness is automatically carried out by the brain. Here, joint-presentation symmetry, Equation 3, is not clearly correct, and auditory data collected by Steingrimsson (2002) suggest that it holds rarely, if ever, for pure tones. Note that, except for stimuli quite near 0, there is little psychological distinction between x and $x + \varepsilon_l$. For example, suppose that the threshold is 20 dB SPL and the signal is 40 dB above the threshold, then the intensities are, respectively, 10^2 and $10^4 + 10^2$, and the latter in dB

[1] The formulation of this section owes a great deal to suggestions made by Ehtibar Dzhafarov. My earlier version of it was a good deal more complex and confusing, and he has persuaded me to follow the path taken here. I am very grateful to him.

[2] The model presented here ignores the fact that these as well as other judged intensities are, in practice, only statistically defined. I do not know how to construct a probabilistic version of the present theory or, indeed, of any theory with a strong focus on structure beyond order. This fact leads to some familiar complications in testing the model. These can be handled in fairly standard ways of, in essence, superimposing a statistical model on top of the structural one. This article does not detail this aspect of applying the theory to data.

[3] In the utility interpretation, (x, y) is taken to mean that valued entities x and y are both received (or held). There (x, y) is treated as a binary operation $\oplus$, which is called *joint receipt*. In the present context, that term does not seem very suggestive, and so at A. A. J. Marley's suggestion it is called *joint presentation* and its appraisal as *subjective summation*, as being more descriptive of the psychophysical phenomenon. In Table 1, a full glossary is provided of the utility and psychophysical terminology.

[4] The use of p for proportion follows Narens (1996), who has developed a somewhat related theory of magnitude estimation (see below). His notation is somewhat different from mine. My statement $z = x \circ_p 0$ he writes as $(z, \mathbf{p}, x)$. He does not treat the case $z = x o_p y$; presumably he would write $([y, z], \mathbf{p}, [y, x])$. I will indicate as appropriate some of the similarities and differences between his predictions and mine.

[5] Such a study is underway by John C. Baird (personal communication).

is 40.04, which is not perceptually different from 40 dB. For other domains, one can sometimes think of similar interpretations. For example, in the domain of subjective heaviness, weight x can be placed in the left hand and u in the right for an overall summed heaviness corresponding to (x, u).

3. Let (x, u) denote the presentation of intensity x for some brief duration d to both ears (or eyes, as the case may be) followed immediately by the presentation of intensity u also for duration d to both ears (eyes). The subject is instructed to sum these successive presentations with respect to loudness (brightness) and to provide a matching intensity y of duration d to both ears (eyes).[6] This interpretation seems possible for all intensity domains. One does not have much reason to suppose that this interpretation will prove to be symmetric.

In illustrations below, the second interpretation of pure tone signals to the two ears is used.

After reporting the main results, the complication is explored of having, in a single modality such as loudness judgments, several distinct interpretations of joint presentation, $(\cdot, \cdot)$, that each satisfy the assumptions.

Basic Properties of $(\cdot, \cdot)$ and $\succsim$

This subsection introduces four basic assumptions underlying the entire article.

Assumption 1. Equivalence relation: The relation $\sim$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is an equivalence relation, that is, *transitive*, for all $f, g, h \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$f \sim g \text{ and } g \sim h \Rightarrow f \sim h, \tag{4}$$

symmetric,

$$f \sim g \Leftrightarrow g \sim f, \tag{5}$$

and *reflexive*,

$$f \sim f. \tag{6}$$

This assumption is clearly an idealization; see Footnote 2. Even without going to probabilistic models, one could assume that $\succsim$ forms a semiorder and so $\sim$ is not transitive, although $\succ$ is, and so $\sim$ is not an equivalence relation. I have not yet attempted this generalization, which no doubt will be complicated.

Assumption 2. Compatibility of $\succsim$ and $\geq$: For all $x, y, u, v \in \mathbb{R}_+$,

$$(x, u) \succsim (y, u) \Leftrightarrow x \geq y, \tag{7}$$

$$(x, u) \succsim (x, v) \Leftrightarrow u \geq v. \tag{8}$$

Both conditions concern a situation in which one ear receives the same intensity in both presentations. In these special cases, the subjective ordering agrees with the physical ordering. The assumption of compatibility precludes the theory applying to anyone who is totally deaf in either ear.

Assumption 3. Solvability: For every $x, u \in \mathbb{R}_+$, there exist $y = y(x, u)$ and $v = v(x, u) \in \mathbb{R}_+$ such that

$$(x, u) \sim (y, 0) \sim (0, v). \tag{9}$$

This simply means that for any joint presentation, one can find single-ear intensities that are perceived as equally loud as the pair.

By Assumptions 1 and 2, the quantities in Assumption 3 are unique. For suppose $(x, u) \sim (0, v)$ and $(x, u) \sim (0, v')$, then using symmetry and reflexivity of $\sim$ we have

$$(0, v) \sim (x, u) \sim (0, v'),$$

and so by transitivity $(0, v) \sim (0, v')$, whence by Assumption 2, $v = v'$. Therefore, the following concepts are well defined.

Definition 1. Define the operations $\oplus_l$ and $\oplus_r$ on $\mathbb{R}_+$ in terms of Equation 9 by

$$x \oplus_l u := y(x, u), \quad x \oplus_r u := v(x, u). \tag{10}$$

Because, by definition, $x \oplus_i u$, $i = l, r$, is an intensity, this definition means that the defined $\oplus_i$ are closed on $\mathbb{R}_+$, so they truly are operations.

Proposition 1. Suppose Assumptions 1–3 hold. Then, for $x, y, u, v \in \mathbb{R}_+$,

1. $\succsim$ is a *weak*[7] *order*.
2. $\succsim$ is *weakly monotonic*, that is,

$$(x, u) \succsim (y, u) \Leftrightarrow (x, v) \succsim (y, v),$$

$$(x, u) \succsim (x, v) \Leftrightarrow (y, u) \succsim (y, v).$$

3. The defined operations $\oplus_l$ and $\oplus_r$ are each strictly increasing in each argument.
4. 0 is a *right identity* of $\oplus_l$ and a *left identity* of $\oplus_r$, that is,

$$x \oplus_l 0 = x \quad 0 \oplus_r u = u. \tag{11}$$

5. Stimuli are bounded from below by

$$x \oplus_i u \geq 0 \oplus_i 0 = 0 \quad (i = l, r).$$

6. Joint presentation symmetry, Equation 3, is equivalent to $\oplus_l \equiv \oplus_r = \oplus$ being a commutative operation, that is,

$$x \oplus u = u \oplus x. \tag{12}$$

The proof, as are all others that are new, is given in the Appendix.

Definition 2. For some $x > 0$, $u > 0$ such that $(x, 0) \sim (0, u)$, *left, no,* or *right bias* is said to hold iff $x < u$, $x = u$, or $x > u$, respectively.

Assumption 4. Consistent Bias: For all $x > 0$, $u > 0$ with $(x, 0) \sim (0, u)$, a person exhibits just one of left, no, or right bias.

This assumption not only takes into account that joint presentations are not automatically assumed to be symmetric, but asserts that the asymmetry, if it exists, is consistent in the sense that, for all suprathreshold intensities, the direction of bias is the same at all intensity levels. This assumption precludes from the theory the possibility, for example, that a person is left biased for low intensity signals and changes to right bias for high intensity ones. At present, I know of no published data concerning Assumption 4.

[6] This interpretation is from Karin Zimmer (personal communication), who has been using it for experiments. Apparently, such summation is as automatic as loudness over the two ears (Karin Zimmer & W. Ellermeier, personal communication, February 16, 2001).

[7] The adjective *weak* applies whenever the indifference relation $\sim$ is an equivalence relation different from equality.

The case of no bias turns out to yield joint presentation symmetry, Equation 3, whereas when there is a bias, joint presentation is necessarily not symmetric.

Another empirical question is whether the direction of the bias is the same at all values of another attribute of the signals. For example, is the direction of the bias dependent on the frequency of pure tones? No assumption about this enters the present theory.

It should be noted that this type of bias is, as far as I know, unrelated to differences in threshold sensitivity. Nothing in the theory precludes a person from having a right ear that is more sensitive than the left in terms of thresholds but who is left biased in the sense of Definition 2.

Most of the following results for the biased cases are worked out for left bias with brief mention of how they vary for right bias.

Proposition 2. Suppose that Assumptions 1–4 hold. Then, for $x > 0$, $u > 0$,

1. *Bias criterion:*

$$\text{left bias} \Leftrightarrow (x, 0) \succ (0, x), \tag{13}$$

$$\text{no bias} \Leftrightarrow (x, 0) \sim (0, x), \tag{14}$$

$$\text{right bias} \Leftrightarrow (x, 0) \prec (0, x). \tag{15}$$

2. *Positivity:*

$$\text{left bias} \Rightarrow x \oplus_l u \geq x \text{ and } x \oplus_r u \geq x, u, \tag{16}$$

$$\text{no bias} \Rightarrow x \oplus_l u \geq x, u \text{ and } x \oplus_r u \geq x, u, \tag{17}$$

$$\text{right bias} \Rightarrow x \oplus_l u \geq x, u \text{ and } x \oplus_r u \geq u. \tag{18}$$

Let us unpack the statements of Equation 16. The assertion $x \oplus_l u \geq x$ means that if y is such that $(y, 0) \sim (x, u)$, then $y \geq x$ and, by definition, $y = x \oplus_l u$. The assertion $x \oplus_r u \geq x, u$ is shorthand for the two statements $x \oplus_r u \geq x$ and $x \oplus_r u \geq u$. Thus, writing $v = x \oplus_r u$, we have $(0, v) \sim (x, u)$, and the inequalities are $v \geq x$ and $v \geq u$.

The conditions of Equations 13–15 are easy criteria to check for bias, if any. Note that in the biased cases, one operation exhibits positivity for both independent variables, whereas the other operation exhibits positivity only for one independent variable. This asymmetry is important below.

Conditions on Proportion Judgments

So far, nothing has been said about the family of operations $\circ_{i,p}$; this is taken up now.

Structure of Proportions

A plausible representation of subjective proportions is the following:

Definition 3. A *subjective-proportion representation* is said to hold if and only if there exists a function $\psi: \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$ that is order preserving and a function $W: \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$, with $W(1) = 1$, that are both strictly increasing such that, for $x, y, p \in \mathbb{R}_+$,

$$\frac{\psi(x \circ_p y) - \psi(y)}{\psi(x) - \psi(y)} = W(p) \quad (x > y \geq 0). \tag{19}$$

The function ψ is called a *psychophysical function* and the function W a *subjective weighting function*.

Were a person literally to follow the instructions about producing $x \circ_p y$ in terms of physical intensity, then the result would be

$$\frac{x \circ_p y - y}{x - y} = p \quad (x > y \geq 0).$$

This would happen only if there were no subjective distortions, but psychophysicists are quite confident that substantial distortions exist both in the sense that subjective intensity is not the same as physical intensity and that subjective proportions are not the same as numerical proportions. So the representation formulates the same idea but in terms of two subjective distortions, one of intensities and the other of numbers. Ultimately, I will show how to derive Equation 19 from behavioral properties.

If instead of p being given in Equation 19 and $z = x \circ_p y$ be determined by the respondent, we give z and the respondent gives p, the procedure is called *magnitude estimation*.

Note that by setting $y = 0$ in Equation 19, we obtain the form

$$\psi(x \circ_p 0) = W(p)\psi(x), \tag{20}$$

which is called a *separable representation* of $x \circ_p 0$.

Below, the operations $\circ_{i,p}$, $i = l, r$, $p \geq 0$, will each be assumed to have a subjective-proportion representation, (ψ_i, W_i).

Relation to Utility

For those unfamiliar with utility theory, this subsection may be omitted.

In the utility context, the domain is valued goods, not just money, and so the domain is more general than $\mathbb{R}_+$; the function ψ is called a *utility function* and is usually denoted U; and the representation of Equation 19 is usually rewritten as follows:

$$U(x \circ_p y) = U(x)W(p) + U(y)[1 - W(p)] \quad (x \succsim y), \tag{21}$$

the domain of p being limited to [0, 1] and interpreted as a probability. For $x \prec y$, assume complementarity in the sense that $x \circ_p y \sim y \circ_{1-p} x$, and so $U(x \circ_p y) = U(y \circ_{1-p} x)$, which follows from Equation 21. Indeed, the model can be generalized to the case of uncertain chance events C, in which case p is replaced by C. The resulting version of Equation 21 is called *rank-dependent utility*. Because this term seems somewhat inappropriate in the psychophysical interpretation, Equation 19 is called the *subjective-proportion representation* and ψ is called a *psychophysical function*, rather than a *utility function*.

Linking $\oplus$ and $\circ_p$: Segregation

Up to this point, the two structures $\langle \mathbb{R}_+, \geq, \oplus_i \rangle$ and $\langle \mathbb{R}_+, \geq, \circ_{i,p} \rangle_{p \geq 0}$ over $\mathbb{R}_+$ have been linked only by $\geq$. For any substantial results, $\oplus_i$ and $\circ_{i,p}$ must be linked to one another in some additional fashion. The proposed linking property is as follows:

Definition 4. A structure $\langle \mathbb{R}_+, \geq, \oplus, \circ_p \rangle_{p \geq 0}$ is said to satisfy *binary segregation* if and only if, for all $x, u, p \geq 0$, when 0 is a left identity,

$$(x \circ_p 0) \oplus u = (x \oplus u) \circ_p u, \tag{22}$$

and when 0 is a right identity,

$$u \oplus (x \circ_p 0) = (u \oplus x) \circ_p u. \quad (23)$$

One reason for stating segregation in two forms depending on whether 0 is a left or right identity is to be able to replace, respectively, $0 \oplus u$ and $u \oplus 0$ by just u. One can view binary segregation as asserting a form of translation invariance under $\oplus$. More specifically, assuming a left identity, which, as we saw in Proposition 1.4, is true for $\oplus_r$, binary segregation asserts that if p is the proportional relation of $[0, z]$ to $[0, x]$, that is, $z = x \circ_{r,p} 0$, then the proportion remains p when comparing $[u, x \oplus_r u]$ and $[u, z \oplus_r u]$, that is, $z \oplus_r u = (x \oplus_r u) \circ_{r,p} u$. Clearly, segregation is a property to be tested empirically. To see exactly what such a test has to be in this case, we recall, using Definition 1, that Equation 22 actually means

$$(x \circ_{r,p} 0, u) \sim (0, (x \oplus_r u) \circ_{r,p} u).$$

Thus, the judged proportion $x \circ_{r,p} 0$, which is determined in the right ear, is then placed in the left ear, whereas $(x \oplus_r u) \circ_{r,p} u$ is in the right ear. Later, we will assume that each $\langle \mathbb{R}_+, \geq, \oplus_i, \circ_{i,p} \rangle_{p \geq 0}$, $i = l, r$ satisfies segregation.

The Forms for $\psi_i(x \oplus_i u)$

Generalized Additivity

Definition 5. A structure $\langle \mathbb{R}_+, \geq, \oplus \rangle$ with 0 a left identity is said to have a *generalized additive representation* (ψ, σ_l), where $\psi, \sigma_l: \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$, and only if ψ and σ_l are each strictly increasing and for all $x, u \in \mathbb{R}_+$,

$$\psi(x \oplus u) = \sigma_l(u)\psi(x) + \psi(u). \quad (24)$$

For 0 a right identity, the representation is (ψ, σ_r) with

$$\psi(x \oplus u) = \psi(x) + \sigma_r(x)\psi(u). \quad (25)$$

The name arises from the fact that one may write Equation 24 as

$$\psi(x \oplus u) = \psi(x) + \psi(u) + \tau_l(u)\psi(x), \quad \tau_l(u) = \sigma_l(u) - 1.$$

We will focus mainly on two special cases of this expression.

An Equivalence Among Conditions

Theorem 1. Suppose that $\langle \mathbb{R}_+, \geq, \oplus, \circ_p \rangle_{p \geq 0}$ is a structure for which Assumptions 1–4 hold and 0 is a left identity of $\oplus$. Suppose that $\psi, \sigma_l, W: \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$ are strictly increasing functions with $W(1) = 1$. Then any two of the following statements imply the third:

1. The pair (ψ, W) forms a subjective proportion representation (Equation 19) of the entities $x \circ_p y$, $x > y \geq 0$.
2. Segregation, (Equation 22), holds.
3. The pair (ψ, σ_l) forms a generalized additive representation of $\oplus$ (Equation 24) and ψ is a separable representation (Equation 20) of entities of the form $x \circ_p 0$.

The proofs of this result and the two corollaries below are in Aczél et al. (in press).

Note that 0 is a two-sided identity if and only if $\sigma(0) = 1$. For the case that 0 is a right identity, Condition 3 is based on Equation 25.

Unbiased Case

Luce (1991) and Luce and Fishburn (1991) prove the result of Theorem 1 for commutative $\oplus$. Specifically, their result can be stated in the following way:

Corollary 1 to Theorem 1. Suppose the conditions of the theorem hold. Then $\oplus$ is commutative if and only if, for some $\delta \geq 0$, $\sigma_l(y) = \sigma_r(y) = 1 + \delta\psi(y)$.[8]

When joint presentation is not biased, Part 6 of Proposition 1 says $\oplus_l \equiv \oplus_r$, so we continue to suppress the subscripts r, l. Thus, by Corollary 1, we see that for all $x \geq 0$, $u \geq 0$,

$$\psi(x \oplus u) = \psi(x) + \psi(u) + \delta\psi(x)\psi(u) \quad (\delta \geq 0). \quad (26)$$

For $\delta = 0$, this says ψ is additive over $\oplus$, whereas for $\delta > 0$, ψ is superadditive over $\oplus$, that is, $\psi(x \oplus u) > \psi(x) + \psi(u)$. For $\delta > 0$, Equation 26 may be rewritten as

$$1 + \delta\psi(x \oplus u) = [1 + \delta\psi(x)]\,[1 + \delta\psi(u)],$$

and so

$$\varphi(x) = \ln[1 + \delta\psi(x)] \quad (27)$$

is an additive representation of $\oplus$ in the sense that

$$\varphi(x \oplus u) = \varphi(x) + \varphi(u).$$

The form of $\psi(x \oplus u)$ given by Equation 26 is called *polynomial additive*, or, for short, p-*additive*,[9] because it has been shown in the mathematical literature that Equation 26 is the only polynomial function that both maps 0 into 0 and that can be transformed into an additive form.

Because $\oplus$ has an additive representation, it is associative in the sense that, for $x, y, z \in \mathbb{R}_+$,

$$(x \oplus y) \oplus z = x \oplus (y \oplus z), \quad (28)$$

as well as commutative. One way to test Equation 28 empirically involves first estimating $x \oplus y$ from $(x, y) \sim (0, x \oplus y)$ and then estimating $(x \oplus y) \oplus z$ from $(x \oplus y, z) \sim (0, (x \oplus y) \oplus z)$. Each estimate will introduce some error. The right side of Equation 28 is similar.

Biased Cases

The following is parallel to the above result but for the biased case.

Corollary 2 to Theorem 1. Suppose that the conditions of Theorem 1 are met and 0 is a left identity. Then $\oplus$ is bisymmetric in the sense that, for all $x, y, u, v \geq 0$,

[8] In the utility case, the constant δ may be positive or negative. It is not difficult to show that in the negative case, ψ is bounded. This, as Ehtibar Dzhafarov pointed out to me, is inconsistent with Equation 19 when $p > 1$. So in the psychophysical case, we must assume $\delta \geq 0$. Also, in the utility context, the notation $-\delta$ has been used.

[9] This "p" has nothing to do with the proportion p.

$$(x \oplus y) \oplus (u \oplus v) = (x \oplus u) \oplus (y \oplus v), \qquad (29)$$

if and only if $\sigma_l(y) \equiv \alpha > 0$, a constant.

This second solution is more complex than the first one because there are two $\oplus$ operations as well as two forms of bias. Assume left bias. Applying the result to $\oplus_i$, $i = l, r$, and taking into account Part 4 of Proposition 1, there are $\alpha_l > 0$ and $\alpha_r > 1$ such that

$$\psi_l(x \oplus_l u) = \psi_l(x) + \alpha_l \psi_l(u), \qquad (30)$$

$$\psi_r(x \oplus_r u) = \alpha_r \psi_r(x) + \psi_r(u). \qquad (31)$$

Aczél et al. (in press) call these forms *left-weighted additive*, abbreviated *lw-additive*. The reasons are that the left signal is overweighted relative to the right one. This is clear for the case of ψ_r because $\alpha_r > 1$, and it follows for ψ_l because, as we show below in Proposition 3.2, $\alpha_l = 1/\alpha_r < 1$. This subtle asymmetry concerning the domains of the constants α_l and α_r reflects the asymmetry noted in Part 2 of Proposition 2 concerning positivity.

As with associativity, testing bisymmetry has to be translated into a series of steps determining, first, the four first-level terms, $x \oplus_i y$, and so on, and then in the l case

$$((x \oplus_l y) \oplus_l (u \oplus_l v), 0) \sim (x \oplus_l y, u \oplus_l v),$$

$$((x \oplus_l u) \oplus_l (y \oplus_l v), 0) \sim (x \oplus_l u, y \oplus_l v).$$

The test is whether or not the equality $(x \oplus_l y) \oplus_l (u \oplus_l v) = (x \oplus_l u) \oplus_l (y \oplus_l v)$ holds. Each determination is a source of error.

The case of right bias results also in Equations 30 and 31 but with the constants satisfying $\alpha_l > 1$ and $\alpha_r = 1/\alpha_l < 1$. These are called *rw-additive representations* because the right term is weighted more than the left one.

It is easy to verify the tie between the bias direction and the constant:

$$(x, 0) \prec (0, x) \Leftrightarrow x \oplus_l 0 < 0 \oplus_l x$$

$$\Leftrightarrow \psi(x) < \alpha_l \psi(x) \Leftrightarrow 1 < \alpha_l.$$

It is not difficult to show that bisymmetry and Equation 24 are equivalent to these lw-additive or rw-additive representations.

Relation Between Left and Right Representations

So far, in the biased cases, we have two distinct representations that, presumably, are related to each other through the fact that each $x \oplus_i u$ is defined in terms of (x, u). Here we characterize that relation. Recall that

$$x \oplus_l u \geq y \oplus_l v \Leftrightarrow (x, u) \succeq (y, v) \Leftrightarrow x \oplus_r u \geq y \oplus_r v,$$

so both ψ_l and ψ_r are order preserving. This means that there exists a strictly increasing function f such that $\psi_r = f(\psi_l)$. So the task is to determine f. We show the following:

Proposition 3. Suppose the conditions of Corollary 2 are satisfied and that ψ_l, ψ_r are representations for the left bias case, that Equations 30 and 31 hold, and f is defined by $\psi_r = f(\psi_l)$. Then:

1. For $t \geq 0$, there exists some $c > 0$ such that

$$f(t) = ct. \qquad (32)$$

2. The constants of Equations 30 and 31 satisfy

$$\alpha_l \alpha_r = 1. \qquad (33)$$

3. The operations $\oplus_l$ and $\oplus_r$ are related as

$$x \oplus_r u = \psi_l^{-1}[\alpha_r \psi_l(x \oplus_l u)]. \qquad (34)$$

4. $W_l \equiv W_r$ if and only if $\circ_{l,p} \equiv \circ_{r,p}$.

This result forces the right and left psychophysical functions to be essentially the same, which in turn forces $\oplus_l$ and $\oplus_r$ to be closely related. The fourth result is less satisfactory. One has to argue that the distortion $W(p)$ of p is a mental process that is independent of which ear is involved, in which case it says the operations $\circ_{l,p}$ and $\circ_{r,p}$ are identical. Whether they are or are not equal is easily checked experimentally.

Relations to Narens (1996)

Narens (1996) focused only on postulated underlying properties of proportion judgments and did not include anything about joint presentations. In particular, he arrived at (in the current notation) the separable representation Equation 20, which is a special case of Equation 19. This immediately implies

$$(x \circ_p 0) \circ_q 0 = (x \circ_q 0) \circ_p 0, \quad (x \geq 0, p \geq 0, q \geq 0), \qquad (35)$$

which is called *threshold-proportion commutativity*.[10] It has been tested for loudness by Ellermeier and Faulhammer (2000) for $p > 1$ and was well supported. Steingrimsson is investigating it for $p < 1$ (personal communication).

Narens (1996) also interpreted one of Stevens's (1975) basic assumptions as saying that the probability-reduction property holds,

$$(x \circ_p 0) \circ_q 0 = x \circ_{pq} 0,$$

which, with separability, implies that $W(p)$ satisfies

$$W(p)W(q) = W(pq),$$

and so $W(p) = p^{\gamma}$, $\gamma > 0$. However, Ellermeier and Faulhammer (2000) rejected this probability-reduction property for auditory stimuli. A far more general form for W is discussed below.

Two or More Interpretations of $\oplus$ in a Modality

It is entirely possible to have two or more quite distinct interpretations of $(\cdot, \cdot)$ and so of $\oplus_i$ while holding fixed the subjective intensity judgment, such as loudness. For example, one can have both two-ear summation and the summation of successive presentations of signals. Suppose that both interpretations are found to satisfy the assumptions made here; does this mean there are different psychophysical functions? The answer is no, because the subjective-proportion representation of Definition 4 is common to both structures. Therefore, up to the choice of unit, ψ_i is fixed. The differences will be of two types. First, each operation, $\oplus_i^{(j)}$, $i = l, r$, $j = 1, 2$, has one of three representations: p-additive, lw-additive, or rw-additive, so nine combinations of the two opera-

[10] In the utility interpretation where chance events rather than probabilities are used and the role of 0, there denoted e, is interpreted as no change from the status quo, this commutativity condition is called *status-quo event commutativity*.

tions are a priori possible. Second, one would expect parameter differences even when the same type of representation occurs. So if $\oplus_i^{(1)}$ and $\oplus_i^{(2)}$ each have a lw-additive representation, one would expect distinct parameters $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$, or, if they each have a p-additive representation, distinct δ_1 and δ_2. Thus, the psychophysical scale does not vary; however, the representations of the operations certainly do vary.

Possible Mathematical Forms for ψ

As it stands, there are now two relatively free functions, ψ and W. It is desirable to narrow down, in a principled fashion, their mathematical forms to a few parameters. Concerning ψ we have some information, such as $\psi(0) = 0$ and that the form of $\psi(x \oplus y)$ is either p-additive, lw-additive, or rw-additive. The p-additive form is either actually additive or of the form $\delta\psi(x) = e^{\kappa\varphi(x)} - 1$, $\kappa > 0$, where φ is an additive representation of $\oplus$. The idea of this section is to impose a plausible invariance principle and take advantage of the additive or near-additive form to pin down ψ to a form with only two parameters. A later section does a similar thing for W.

For ψ, a principle of multiplicative invariance is invoked that relates $\oplus$ and $+$. Luce (2000) investigated in the commutative case of Corollary 1 and Marchant and Luce (in press) have done the same for the noncommutative case of Corollary 2.

Multiplicative Invariance

Consider the following property, commonly called *multiplicative invariance:* There exists a function θ: $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $r > 0$, $x \geq 0$, $u \geq 0$,

$$rx \oplus ru = \theta(r)(x \oplus u). \tag{36}$$

Observe that if 0 is a left identity, then for $x = 0$ one has

$$ru = r0 \oplus ru = \theta(r)(0 \oplus u) = \theta(r)u,$$

whence $\theta(r) = r$, in which case the property Equation 36 is often called *homogeneity* (of Order 1). The case of the right identity is similar.

Basically, this property is saying that stretching the arguments by the same amount is the same as stretching the result by that amount.

Unbiased Case. Given that φ of Equation 27 is additive over $\oplus$, Luce (2000) proved that the multiplicative invariance condition is equivalent to the assertion that there exists some $\beta > 0$ such that

$$\varphi(x) = \mu x^\beta, \ (\beta > 0, \ \mu > 0, \ x \geq 0). \tag{37}$$

Note that $\varphi(0) = 0$ is satisfied. So for $\delta > 0$, $\lambda = \kappa\mu > 0$, and $x \in \mathbb{R}_+$,

$$\begin{aligned} \delta\psi(x) &= e^{\kappa\varphi(x)} - 1 \\ &= e^{\lambda x^\beta} - 1 \quad (\beta > 0, \lambda > 0). \end{aligned} \tag{38}$$

For $\lambda x^\beta < 0.01$, this is approximately λx^β, which accords with the results reported using magnitude estimation and production methods over much of the dynamic range. A similar computation to the one above using $\beta = 0.3$ suggests that a reasonable value for λ is 10^{-4}. An example of this function in log-log coordinates is shown in Figure 1 using $\beta = 0.3$ and a threshold ε at 2 bels, that is, 20 dB SPL. The latter seems to accord with magnitude estimation data.

Biased Case

Marchant and Luce (in press) showed that for ψ having the lw-additive form ψ with parameter $\alpha > 1$, then multiplicative invariance is equivalent to the existence of parameters $\beta > 0$, $\mu > 0$ such that

$$\psi(x) = \mu x^\beta \quad (x \geq 0). \tag{39}$$

So in this case, the psychophysical function itself is a power function. By Proposition 3, we see that β is not different for ψ_l and ψ_r.

Empirical Predictions of Power Psychophysical Functions

Left and Right Operations

Recall that Equation 34 showed that if units are chosen so $\psi_r \equiv \psi_l$, then setting $\alpha = \alpha_r$:

$$x \oplus_r u = \psi_l^{-1} [\alpha\psi_l (x \oplus_l u)].$$

Thus, if ψ_l grows as a power function, Equation 39, with exponent β, then we know

$$x \oplus_r u = \gamma(x \oplus_l u), \quad (\gamma = \alpha^{1/\beta} > 0). \tag{40}$$

In terms of dB measures,

$$10 \log \gamma = 10 \log x \oplus_r u - 10 \log x \oplus_l u.$$

This can be checked empirically using regression analysis, yielding an estimate of the factor 10 log γ.

Symmetrizing the Biased Case

Suppose we are in the noncommutative, left-bias case, and suppose that ψ grows as a power function (Equation 39). Under these assumptions we show that the above factor γ leads, in a

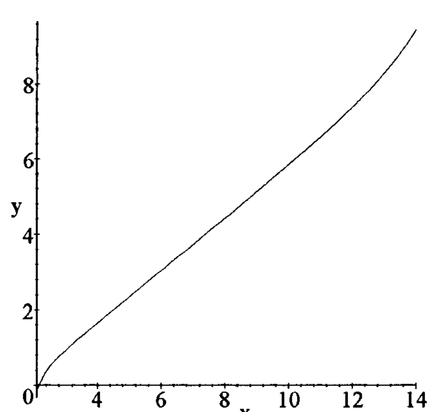

Figure 1. The function $y = \ln \psi(x) = \ln(e^{\lambda(10^x - 10^2)^\beta} - 1) + \ln \rho$. The intensity x is in bels $= dB/10$, the threshold ε is 2 bels, $\ln \rho = 8.1$, $\lambda = 0.0001$, and $\beta = 0.3$.

certain sense, to commutativity (but not associativity). Observe that for the left bias case, consider

$$\psi_r(x \oplus_r zu) = \alpha\psi_r(x) + \psi_r(zu)$$

$$= \alpha\mu x^\beta + \mu z^\beta u^\beta$$

and

$$\psi_r(u \oplus_r zx) = \alpha\psi_r(u) + \psi_r(zx)$$

$$= \alpha\mu u^\beta + \mu z^\beta x^\beta.$$

Therefore,

$$x \oplus_r zu = u \oplus_r zx \Leftrightarrow \alpha = z^\beta \Leftrightarrow z = \alpha^{1/\beta} = \gamma,$$

where the value of γ can be estimated from Equation 40. R. Steingrimsson (personal communication) is testing this prediction in the two-ear, pure-tone case. Indeed, in terms of Definition 2, which involves setting $u = 0$ above, then we have $x \oplus 0 = 0 \oplus \gamma x$. Notice that in terms of dB, the factor γ corresponds to a constant additive dB increment to the right ear independent of signal intensity above threshold or the same decrement to the left ear. In the case of right bias, the correction is a constant additive dB increment to the left ear or the same decrement to the right ear.

Define $\circledast$ by

$$x \circledast_r u := x \oplus_r \gamma u.$$

Given these formulas, it is routine to show that $\circledast$ is bisymmetric, not associative, with the representation

$$\psi_r(x \circledast_r u) = \psi_r(x \oplus \gamma u)$$

$$= \alpha\psi_r(x) + \psi_r(\gamma u)$$

$$= \alpha\mu x^\beta + \mu\gamma^\beta u^\beta$$

$$= \alpha[\psi_r(x) + \psi_r(u)].$$

This immediately yields

$$\psi_r(x \circledast_r u) = \psi_r(u \circledast_r x),$$

and so $\circledast_r$ is commutative and bisymmetric but not associative. The development for $\circledast_l$ is similar.

Cross-Modality Matching

Suppose that we have two distinct modalities, such as sound intensity and light intensity. A procedure introduced by S. S. Stevens and later summarized in Stevens (1975) was to have a respondent match, say, brightness to loudness. The empirical finding was that the matches are related by a power law. Let us consider that finding within the context of the present theory. Suppose that each modality satisfies the present theory, and distinguish them by the subscripts 1 and 2.

First, supposing both modalities are commutative, then we assume that the match of x_2 to x_1 is characterized by

$$\delta_1\psi_1(x_1) = \delta_2\psi_2(x_2), \qquad (41)$$

where δ_i is the parameter of the p-additive form. This assumes that the thresholds match. Substituting Equation 38 into Equation 41 yields

$$e^{\lambda_1 x_1^{\beta_1}} - 1 = e^{-\lambda_2 x_2^{\beta_2}} - 1,$$

and so

$$x_2 = \alpha x_1^{\beta_1/\beta_2}. \qquad (42)$$

For the noncommutative case, suppose we work with just the left or the right representations, but not both, and suppress the subscript. The same result as in the commutative case, Equation 42, follows immediately from Equation 39. For one modality commutative and the other one not commutative, the same prediction holds only approximately within a reasonable signal range.

These results warrant three comments. First, they agree with the empirical findings of Stevens and others that the matches form approximate power functions between signal intensities relative to their thresholds. Second, the predictions agree with the invariance argument advanced by Luce (1990). Third, although this power relation agrees with the psychophysical function over much of its domain in the cases where one or both of the joint presentations is commutative, it does disagree for very intense stimuli. That is to say, matching is not a fully satisfactory way to get at the psychophysical function itself in such cases.

A Possible Form for *W*

Luce (2000, 2001), modifying an idea of Prelec (1998), developed a behavioral theory for a particular form for *W*. Of course, in the utility context it was formulated only for $p \in [0, 1]$. An examination of that proof shows that this restriction comes into play only at one point when a multiplicative Cauchy equation is solved. It is easy to see what happens in that equation when $p > 1$ and the result is stated in that form:

Theorem 2. Suppose that a separable representation of the form Equation 20 holds with $\psi, W: \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$. Then, the following properties are equivalent:

1. Suppose that for $p, q \in [0, 1]$ there exists $s = s(p, q) \in [0, 1]$ or for $p, q \in]1, \infty[$ there exists $s = s(p, q) \in]1, \infty[$ such that if

$$(x \circ_p 0) \circ_q 0 = x \circ_s 0, \qquad (43)$$

then

$$(x \circ_{p^N} 0) \circ_{q^N} 0 = x \circ_{s^N} 0, \qquad (44)$$

where $N = 2, 3$.

2. For some positive constants ρ, ρ', η, η',

$$W(p) = \begin{cases} \exp[-\rho(-\ln p)^\eta] & (p \in]0, 1]) \\ \exp[\rho'(\ln p)^{\eta'}], & (p \in]1, \infty[). \end{cases} \qquad (45)$$

The result is equally true for many other pairs of *N* values in Equation 44, in particular $N = \frac{3}{2}$ and 2 are sufficient, which is more convenient empirically for $p, q > 1$ to stay within acceptable intensity bounds. One may be surprised that Equation 44 is not asserted to hold for all integers *N*, but that is unnecessary because by induction one is able to prove from Part 1 of Theorem 2 that Equation 44 actually holds for all rational numbers $N = 2^m 3^n$, where m and n run over all positive and negative integers. This set of rational numbers is dense in the positive real numbers, and because W is strictly increasing, Equation 44 can be extended to hold for N having as its domain all positive real numbers.

The Prelec (1998) form of Part 2 of Theorem 2 is quite flexible. It can be concave, convex, S shaped, or inverse S shaped over $]0, 1]$ and the same possibilities, independently, over $]1, \infty]$. It includes power functions as special cases ($\eta, \eta' = 1$). The equivalent property stated in Part 1 of Theorem 2 is called *reduction invariance*. It certainly is important to check whether or not it holds empirically.

Reduction invariance clearly holds if $s(p, q) = pq$, because $s(p^N, q^N) = p^N q^N = s(p, q)^N$. This is the multiplicative property of Narens (1996) that one gets if the numerals of magnitude estimation or production are treated as numbers. As was noted $s(p, q) = pq$ implies W is a power function. Stevens (1975) implicitly assumed this to be true, but with no real empirical justification. Indeed, once it is stated explicitly, one realizes that there is little reason to expect that it will hold, and Ellermeier and Faulhammer (2000) have established empirically that it is false for $p > 1$, $q > 1$. My model is more general in the sense that reduction invariance can hold with $s(p, q) \neq pq$. In fact, from separability, Equation 20, and Equation 43, we see that

$$W[s(p, q)] = W(p)W(q).$$

Substituting Equation 45 and taking logarithms yields

$$[-\ln s(p, q)]^{\eta} = (-\ln p)^{\eta} + (-\ln q)^{\eta} \quad (p, q \in [0, 1]), \quad (46)$$

$$[\ln s(p, q)]^{\eta'} = (\ln p)^{\eta'} + (\ln q)^{\eta'} \quad (p, q \in]1, \infty[). \quad (47)$$

So from the data of Equation 43, we can estimate $s(p, q)$, and the parameters η, η' can be estimated from, respectively, Equations 46 and 47. As an example, Figure 1 of Ellermeier and Faulhammer (p. 1507) shows for one respondent that compounding 2 and 3 as in Equation 43 corresponds approximately to a factor of 7.5, not 6. Thus, from Equation 47 we may conclude for this person $\eta' \approx 0.85$.

Qualitative Conditions for Condition 3

From an empirical perspective, perhaps the most interesting aspect of Theorem 1 is the fact that Conditions 2 and 3 together imply the subjective proportion representation, Equation 19. Condition 2, segregation, is simply a straightforward empirical assertion that can be tested independently of other aspects of the theory. Condition 3 is more complex, but for the two extreme cases discussed in the corollaries to Theorem 1 it breaks down into three testable properties. At present no analysis exists of the large family of cases aside from these two extreme ones (but see below).

The Unbiased Case

The p-additivity of ψ, Equation 26, is assured if $\langle \mathbb{R}_+, \geq, \oplus \rangle$ forms an extensive structure (see Krantz, Luce, Suppes, & Tversky, 1971, chap. 3), as it must if $\oplus$ is interpreted as the addition of physical intensities. Under other interpretations, such as presentations to the two ears, this must be justified empirically. It is well known what must be checked: Aside from monotonicity and commutativity, the most important property is associativity (Equation 28) which was discussed earlier.

Turning, then, to separability, two questions need to be addressed. First, does separability hold at all? And, second, how can one justify assuming that there is a single psychophysical function ψ that is both p-additive and separable?

The first is easily answered (see Krantz et al., 1971, chap. 6). It must be the case that the entities $x \circ_p 0$, where $x, p \in \mathbb{R}_+$, form an additive conjoint measurement structure $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succeq \rangle$. The key testable conditions are that the operation is strictly increasing (monotonic) in both variables, and the Thomsen condition of additive conjoint measurement holds. Monotonicity, which we expect to hold, is directly testable. The Thomson condition has been shown to amount to the property threshold-proportion commutativity, Equation 35, described earlier (Luce, 1996). This is also a property of Narens's (1996) axiomatization. In addition, restricted solvability, Archimedeanness, and essentialness must also be assumed to be satisfied. These definitions are standard and can be found in any treatment of additive conjoint measurement, for example, Krantz et al. (1971). There really is nothing empirical to check in these assumptions.

A stronger version of threshold-proportion commutativity is also implied by the proportion representation, Equation 19, that is called simply *proportion commutativity*;[11] that is,

$$(x \circ_p y) \circ_q y = (x \circ_q y) \circ_p y. \quad (48)$$

It is desirable to test this as well.

The answer to the second question of what property is equivalent to the simultaneous existence of a p-additive representation and a separable one with the same psychophysical function ψ is the following condition (Luce, 1996):

Definition 6. The structure $\langle \mathbb{R}_+, \geq, \oplus, \circ_p \rangle_{p \geq 0}$ is said to be *joint-presentation decomposable*[12] if and only if for each $x, p \in \mathbb{R}_+$, there exists $q = q(x, p) \in \mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$,

$$(x \oplus u) \circ_p 0 = (x \circ_p 0) \oplus (u \circ_q 0). \quad (49)$$

The special case where $q = p$ is called *simple joint-presentation decomposable*.

Note that on the left there is a single proportion judgment and on the right there are two independent ones. The formal result[13] is:

Theorem 3. Suppose the structure of Theorem 1 has a p-additive representation ψ_1 over $\oplus$ and a separable representation (ψ_2, W_2) over stimuli of the form $x \circ_p 0$. Then the following statements are equivalent.

1. There exists a constant $\nu > 0$ such that $\psi = \psi_2^{\nu}$ is p-additive and the pair (ψ, W), where $W = W_2^{\nu}$, forms a separable representation.

2. Joint-presentation decomposability (Definition 6) is satisfied.

[11] Narens (1996) simply called this property *commutativity*, but in the present theory that would be ambiguous because the term is used here in its usual algebraic sense for $\oplus$.

[12] In my work on utility, where $\oplus$ has been called *joint receipt*, this concept has been called *joint-receipt decomposable*.

[13] The result is formulated for $p, q \in [0, 1]$ in Luce (2000) and a proof is given on pp. 169–171 of that book. A careful examination of that proof shows it holds for general positive p, q as well.

This means that if commutativity and associativity have been verified, and so for all practical purposes $\langle \mathbb{R}_+, \geq, \oplus \rangle$ is an extensive structure with p-additive representations, then joint-presentation decomposability is another critical property to test empirically. It entails both estimating q for one choice of u and then showing that Equation 49 holds for a number of other choices of u. This will clearly be the most time consuming of the properties to test. Having done that, if we also show that threshold-proportion commutativity holds, then we know by Theorem 2 that the subjective intensity ψ has the proportional representation, Equation 19, over the operation $\circ_p$.

Biased Case

The remarks concerning the axiomatization of a separable representation are unchanged from the previous subsection. To axiomatize the lw-additive representation, the main thing to verify is bisymmetry (Equation 29) and, of course, to check whether the bias is to the left or the right (Krantz et al., 1971). Again, the question becomes one of finding a condition so that the same psychophysical function can be used for both. The result is closely similar to Theorem 3.

Theorem 4. Suppose the structure of Theorem 1 has a lw-additive representation ψ_1 over $\oplus$ and a separable one (ψ_2, W_2) over $x \circ_p 0$. Then the following statements are equivalent.

1. There exists a constant $\nu > 0$ such that $\psi = \psi_2^{\nu}$ is lw-additive over $\oplus$ and the pair (ψ, W), where $W = W_2^{\nu}$, forms a separable representation.

2. Simple joint-presentation decomposability (Definition 6) is satisfied, that is, for each $x, u, p \in \mathbb{R}_+$,

$$(x \oplus u) \circ_p 0 = x \circ_p 0) \oplus (u \circ_p 0). \tag{50}$$

Simple joint-presentation decomposability is a property that must be tested, but doing so is far simpler than testing joint-presentation decomposability of Theorem 3 because $q = p$ does not have to be estimated.

It is somewhat remarkable how the unbiased and biased cases are related and how, in many ways, the latter is simpler than the former.

Unit Structure Representation of $\oplus$

Observe that the two special cases given in the corollaries to Theorem 1 we have examined can be transformed into a representation of the following type, which has been studied in an axiomatic measurement context by Luce and Narens (1985). It is, in a sense, the most general class of (mostly noncommutative) binary operations that exhibit ratio scale invariance.

Definition 7. A *unit representation* (φ, F) of a strictly monotonic increasing operation $\oplus$ over $\mathbb{R}_+$ with a left identity is said to exist if and only if $\varphi: \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$ is a strictly increasing function and $F: \mathbb{R}_+ \xrightarrow{onto}]1, \infty[$ is a strictly increasing function, $F(z)/z$ is strictly decreasing, if > 0 is a constant, and

$$\varphi(x \oplus u) = \begin{cases} \varphi(u) F\left(\dfrac{\varphi(x)}{\varphi(u)}\right) & (x \geq 0, u > 0) \\ d\varphi(x) & (x \geq 0, u = 0), \end{cases} \tag{51}$$

$$\varphi(0) = 0, \tag{52}$$

$$F(0) = 1. \tag{53}$$

The property Equation 53 follows from Equations 51 and 52 with $x = 0$ and $0 \oplus u = u$.

For the symmetric case of Corollary 1 of Theorem 1, (φ, F) with $F(z) = z + 1$ is a unit representation of $\oplus$, and for the nonsymmetric case of Corollary 2 of Theorem 1, (ψ, F) with $F(z) = \alpha z + 1$ is a unit representation.

If instead of a left identity one has a right one, $x \oplus 0 = x$, then instead of Equation 51 one uses the other unit representation:

$$\varphi(x \oplus u) = \varphi(x) F^* \left(\frac{\varphi(u)}{\varphi(x)}\right) \quad (x > 0, u \geq 0).$$

The remainder of the construction is basically unchanged. As mentioned earlier, the results are formulated only for left bias in Definition 2, but what happens in the other case is noted.

Because φ and ψ both preserve the order $\succsim$, there exits a strictly increasing function $G_i: \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$ such that, for all stimuli x,

$$\psi(x) = G[\varphi(x)]. \tag{54}$$

Theorem 5. Suppose that the conditions of Theorem 1 and Equation 24 hold, and that $\oplus$ also has a unit representation (φ, F), Equation 51. Then, with $v = \varphi(y) > 0$, $z = \varphi(x)/\varphi(y) > 0$, and $A(v) = \sigma[\varphi^{-1}(v)]$, the following functional equation is satisfied:

$$A(v)G(vz) + G(v) = G[vF(z)]. \tag{55}$$

This equation has been studied by Aczél et al. (in press), and the following amounts to the corollary to their Theorem 4.

Theorem 6. Suppose that the conditions of Theorem 5 hold and that F is once and G is twice continuously differentiable. Then there are two classes of solutions to Equation 55, namely, those of Corollaries 1 and 2 to Theorem 1.

This means that either all other solutions to Equation 55 are not as smooth as assumed in this theorem or they do not have the automorphism group corresponding to ratio scales: More work is needed to understand the rest of the generalized additive forms. Among other things, I do not know how to axiomatize them in terms of $\succsim$ and $\oplus$.

Summary

Utility-Psychophysics Glossary

As was indicated earlier, much of this mathematics was first developed in the context of utility theory, where the various concepts have been given names that are appropriate in that context. However, those terms do not seem to me very appropriate in the psychophysical context and so, with some reluctance, I have introduced more meaningful terms. It seems useful, therefore, to provide in one place, Table 1, a glossary of the terms.

The Representations

Suppose that the following properties hold: those deduced in Theorems 1, 3, and 4, multiplicative invariance for $\oplus$, and reduction invariance for $\circ_p$. Then for commutative $\oplus$, there is a numer-

Table 1
A Glossary of Terms and Symbols for the Utility and Psychophysical Interpretation of the Concepts

Term	Symbol
Utility	
Status quo, no change from	e
Joint receipt	$\oplus$
Gamble	$(x, p; u)$
Utility function	U
Value function	V
RDU representation	(21)
Event commutativity	
Joint-receipt decomposition	
Psychophysics	
Absolute threshold	0
Joint presentation	(x, u), $\oplus_i$
Proportion judgment	$x \circ_{i,p} u$
Psychophysical function	ψ_i
Additive representation	φ_i
Proportion representation	(19)
Proportion commutativity	(35)
Joint-presentation decomposition	(49)

Note. Numbers in parentheses refer to numbered equations in this article. RDU = rank-dependent utility.

ical psychophysical representation of proportions and joint presentations that is specified up to six numerical constants: β, λ for the form of the psychophysical function ψ and ρ, ρ', η, η' for the form of $W(p)$. Note that the parameter δ plays no role because throughout $\delta\psi$ is a normalized function. If a second interpretation of $\oplus$ is available, one can estimate δ_1/δ_2 even though they cannot be estimated separately. The representation equations are

$$\psi(x \oplus u) = \psi(x) + \psi(u) + \delta\psi(x)\psi(u) \quad (x \geq 0, u \geq 0),$$

and

$$\psi(x \circ_p y) - \psi(y) = W(p)[\psi(x) - \psi(y)] \quad (x > y \geq 0),$$

where

$$\delta\psi(x) = \exp(\lambda x^{\beta}) - 1 \quad (x \geq 0, \beta > 0, \delta > 0, \lambda = \kappa\mu > 0).$$

Note that the p-additive and proportion forms are unchanged by multiplying everything by $\delta > 0$. The form of W is

$$W(p) = \begin{cases} \exp[-\rho(-\ln p)^{\eta}] & (p \in [0, 1]) \\ \exp[\rho'(\ln p)^{\eta'}] & (p \in]1, \infty[). \end{cases}$$

If $s = s(p, q)$ satisfies

$$(x \circ_p 0) \circ_q 0 = x \circ_s 0,$$

then

$$\begin{array}{ll} [-\ln s(p, q)]^{\eta} = (-\ln p)^{\eta} + (-\ln q)^{\eta} & (p, q \in [0, 1]) \\ [\ln s(p, q)]^{\eta'} = (\ln p)^{\eta'} + (\ln q)^{\eta'} & (p, q \in]1, \infty[) \end{array}.$$

One way to test the theory is to collect sufficient data to estimate the six parameters listed above so as to achieve some optimal fit of the representations of $\oplus$ and $\circ_p$.

For noncommutative $\oplus$, the expression for W is unchanged and that for ψ is

$$\psi(x) = \mu x^{\beta},$$

but μ may be chosen arbitrarily. There is, in addition, $\alpha > 1$ from the lw-additive form that arises in the left bias case such that

$$\psi_l(x \oplus_l u) = \psi_l(x) + \frac{1}{\alpha}\psi_l(y),$$

$$\psi_r(x \oplus_r u) = \alpha\psi_r(x) + \psi_r(y),$$

resulting again in a total of six parameters.

Testable Properties

Another way to check this theory empirically is to focus on the key properties that underlie the representations. Again, the two cases must be distinguished:

First, if $(\cdot, \cdot)$ is symmetric, then $\oplus$ is commutative,

$$x \oplus u = u \oplus x,$$

and, in addition, it is associative,

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z.$$

Segregation asserts

$$(x \circ_p 0) \oplus u = (x \oplus u) \circ_p u.$$

Theorem 3 says that the property of joint-presentation decomposition (Equation 49) is a consequence of Theorem 2, that is, there exists $q = q(x, p)$ such that for all y,

$$(x \oplus u) \circ_p 0 = (x \circ_p 0) \oplus (u \circ_q 0).$$

Although difficult to test, this property, together with proportion commutativity and segregation establishes the existence of the representation described above, but not the specific forms for ψ and W.

Alternatively and more relevant empirically, the major tests for nonsymmetric $(\cdot, \cdot)$ are the left-bias criterion,

$$x \oplus_i 0 > 0 \oplus_i x \quad (i = l, r),$$

or the right-bias criterion,

$$x \oplus_i 0 < 0 \oplus_i x \quad (i = l, r),$$

and bisymmetry,

$$(x \oplus_i y) \oplus_i (u \oplus_i v) = (x \oplus_i u) \oplus_i (y \oplus_i v) \quad (i = l, r).$$

In addition, we have assumed binary segregation in one of two forms: when 0 is a left identity,

$$(x \circ_p 0) \oplus u = (x \oplus u) \circ_p u,$$

and when 0 is a right identity,

$$u \oplus (x \circ_p 0) = (u \oplus x) \circ_p u.$$

The property of threshold-proportion commutativity, Equation 35,

$$(x \circ_{i,p} 0) \circ_{i,q} 0 = (x \circ_{i,q} 0) \circ_{i,p} 0 \quad (i = l, r),$$

follows from separability, and it goes a long way toward forcing separability. For $p > 1$, Ellermeier and Faulhammer (2000) verified that threshold-proportion commutativity holds in audition. It remains to check it for $p < 1$. It would also be desirable to test proportion commutativity (Equation 48),

$$(x \circ_{i,p} y) \circ_{i,q} y = (x \circ_{i,q} y) \circ_{i,p} y \quad (i = l, r).$$

Finally, Theorem 5 says that for there to be ψ that is both lw-additive and separable, simple joint-presentation decomposition holds, that is,

$$(x \oplus_i u) \circ_{i,p} 0 = (x \circ_{i,p} 0) \oplus_i (u \circ_{i,p} 0) \quad (i = l, r).$$

This is far easier to check than joint-presentation decomposition because $q(x, p)$ is simply p.

Independent of the bias, to verify multiplicative invariance, which underlies the form for ψ, we need to focus empirically on Equation 36,

$$rx \oplus_i ru = r(x \oplus_i u) \quad (i = l, r).$$

And to verify the Prelec form for W_i, $(i = l, r)$, Theorem 2, we need to focus empirically on reduction invariance (Equations 43 and 44): For $i = l, r$,

$$(x \circ_{i,p} 0) \circ_{i,q} 0 = x \circ_{i,s} 0,$$

implies for $N = 2, 3$ or equally well for $N = \frac{3}{2}, 2$,

$$(x \circ_{i,p^N} 0) \circ_{i,q^N} 0 = x \circ_{i,s^N} 0.$$

In this connection, it is desirable to check whether $\circ_{l,p} \equiv \circ_{r,p}$, which, if true, means there is only one W function.

As was noted early in the article, these testable properties define a substantial empirical program that will either support the current theory or indicate where changes are necessary. Such work is underway and will be published subsequently.

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(*Appendix follows*)

Appendix

Proofs

Proposition 1

Proof

1. Consider

$$(x, u) \succsim (y, v) \text{ and } (y, v) \succsim (z, w).$$

By compatibility, (Equations 7 and 8), these are equivalent to

$$x \oplus_l u \geq y \oplus_l v \text{ and } y \oplus_l v \geq z \oplus_l w,$$

whence $x \oplus_l u \geq z \oplus_l w$, which by compatibility is equivalent to $(x, u) \succsim (z, w)$. Thus, transitivity holds. Connectedness follows similarly from the fact that $\geq$ is connected.

2. Observe that by applying Assumption 2 twice

$$(x, u) \succsim (y, u) \Leftrightarrow x \geq y \Leftrightarrow (x, v) \succsim (y, v).$$

The other case is similar.

3. The monotonicity of $\oplus_i$, $i = l, r$, follows immediately from their definitions and the weak monotonicity of joint presentation.

4. By definition,

$$(x, 0) \sim (x \oplus_l 0, 0),$$

and so by compatibility, Assumption 2, $x = x \oplus_l 0$. The other case is similar.

5. From $x \geq 0$ and (7), we have $x \oplus_i u \geq 0 \oplus_i u$, and from $u \geq 0$ and (8), we have $0 \oplus_i u \geq 0 \oplus_i 0$. So $x \oplus_i u \geq 0 \oplus_i 0$. By (4), $0 \oplus_i 0 = 0$.

6. By the definition of $\oplus_l$, Assumptions 1 and 2, and joint presentation symmetry for all x, u,

$$(x, u) \sim (u, x) \Leftrightarrow (x \oplus_l u, 0) \sim (u \oplus_l x, 0) \Leftrightarrow x \oplus_l u = u \oplus_l x.$$

So $\oplus_l$ is commutative (Equation 12). A parallel proof of commutativity holds for $\oplus_r$. Using joint presentation symmetry again,

$$(x \oplus_l u, 0) \sim (x, u) \sim (0, x \oplus_r u) \sim (x \oplus_r u, 0),$$

and so, by Assumption 2, $x \oplus_l u = x \oplus_r u$, i.e., $\oplus_l \equiv \oplus_r$.

Proposition 2

Proof

1. Assume left bias. By solvability, there exists $u \in \mathbb{R}_+$ such that $(x, 0) \sim (0, u)$. By left bias, $x < u$, and so by Assumption 2,

$$x < u \Leftrightarrow (0, x) \prec (0, u) \sim (x, 0).$$

The conclusion follows by transitivity (Proposition 1.1). The other two cases are similar.

2. Using the axioms freely and the left bias property in the third line,

$$(x \oplus_l u, 0) \sim (x, u) \succ (x, 0) \Leftrightarrow x \oplus_l u > x,$$

$$(0, x \oplus_r u) \sim (x, u) \succ (0, u) \Leftrightarrow x \oplus_r u > u,$$

$$(0, x \oplus_r u) \sim (x, u) \succ (x, 0) \succ (0, x) \Leftrightarrow x \oplus_r u > x.$$

The other two cases are similar.

Proposition 3

Proof

1. From Equations 30 and 31, we have

$$\alpha_r\psi_r(x) + \psi_r(u) = f[\psi_l(x) + \alpha_l\psi_l(u)] \quad (\alpha_r > 1, \alpha_l > 0).$$

Setting u = 0,

$$\alpha_r\psi_r(x)\, f[\psi_l(x)],$$

so

$$f[\psi_l(x)] + \frac{1}{\alpha_r} f[\psi_l(u)] = f[\psi_l(x) + \alpha_l\psi_l(u)].$$

Let $r = \psi_l(x) > 0$, $s = \alpha_l\psi_l(u) > 0$, and define

$$g(s) = \frac{1}{\alpha_r} f\left(\frac{s}{\alpha_l}\right).$$

Then the functional equation becomes the Pexider equation

$$f(r) + g(s) = f(r + s) \quad (r \geq 0, s \geq 0),$$

which is known (Aczél, 1966, p. 142) to have as its only strictly monotonic increasing solutions with $f(0) = 0$,

$$f(t) = ct = g(t), \quad (c > 0).$$

2. By this result and the definition of g,

$$cs = g(s) = \frac{1}{\alpha_r} f\left(\frac{s}{\alpha_l}\right) = \frac{cs}{\alpha_r\alpha_l},$$

whence $\alpha_l\alpha_r = 1$.

3. Using Equations 30 and 31, $\psi_l = c\psi_r$, and the fact that $\alpha_l = 1/\alpha_r$,

$$\begin{aligned} \alpha_r\psi_l(x \oplus_l u) &= \alpha_r\psi_l(x) + \psi_l(u) \\ &= \mathrm{c}[\alpha_r\psi_r(x) + \psi_r(u)] \\ &= c\psi_r(x \oplus_r u) \\ &= \psi_l(x \oplus_r u). \end{aligned}$$

Taking ψ_l^{-1} yields the assertion.

4. Using the definition of subjective-proportion representation and Part 1 of Proposition 3, above,

$$\begin{aligned} \psi_r(x \circ_{l,p} y) - \psi_r(x \circ_{r,p} y) &= c\psi_l(x \circ_{l,p} y) - \psi_r(x \circ_{r,p} y) \\ &= W_l(p)\,[c\psi_l(x) - c\psi_l(y)] + c\psi_l(y) \\ &\quad - W_r(p)\,[\psi_r(x) - \psi_r(y)] - \psi_r(y) \\ &= [W_l(p) - W_r(p)]\,[\psi_r(x) - \psi_r(y)], \end{aligned}$$

in which case the result follows immediately.

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