Two functional equations preserving functional forms

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Two functional equations are considered that are motivated by three considerations: work in utility theory and psychophysics, questions concerning when pairs of degree 1 homogeneous functions can be homomorphic and calculating their homomorphisms, and the link of the latter questions to quasilinear mean values. The first equation is

$$h(\sigma(y)x + [1 - \sigma(y)]y) = \tau(y)h(x) + [1 - \tau(y)]h(y) \quad (x \geq y \geq 0),$$

where h maps $[0, \infty[$ into a subset of $[0, \infty[$ and is strictly increasing and continuously differentiable; the functions σ and τ map $[0, \infty[$ continuously into $[0, 1]$, $\sigma(y) > 0$ for $y > 0$ but σ is not 1 on $]0, \infty[$. The solutions are fully determined. (Recently Zsolt Páles has eliminated the differentiability assumption.) The second equation is

$$h[y + f(x - y)] = h(y) + g[h(x) - h(y)] \quad (x \geq y \geq 0),$$

where h maps $[0, \infty[$ onto a subinterval of positive length of $[0, \infty[$ and is strictly increasing and twice continuously differentiable, f and g map $[0, \infty[$ onto $[0, \infty[$ and are twice differentiable, and either $f''(0) \neq 0$ or $g''(0) \neq 0$. The solutions are fully determined under these conditions. When $f''(0) = g''(0) = 0$ and h'' is not identically zero, we determine the solutions under the added assumption of analyticity. It remains an open problem to find the solutions in the latter case under the assumption of only second order differentiability. A more general open problem is to eliminate all differentiability conditions for the second equation.

functional equations in several variables and several unknown functions | homogeneous functions | homomorphisms, diffeomorphisms | order preserving | utility and psychophysics

The two functional equations to be considered here have arisen in the following way. Luce (refs. 1 and 2; two unpublished works)§ has axiomatized structures of interest in both utility theory and psychophysics that involve an ordering, $\succsim$, and two classes of (nonclosed) operations. In the utility context, one class consists of binary gambles of the form $(x, C; y)$, which is interpreted as the gamble in which a chance experiment with universal set Ω is performed and, if event C occurs, the owner of the gamble receives x, and if $\Omega \backslash C$ occurs, y is received. The other operation is $f \oplus g$ and is interpreted as having or receiving both f and g. The case dealt with in ref. 1 is where $\oplus$ is commutative, has an identity e, and the two types of operations are related by the following property called segregation:

$$(x, C; e) \oplus y \sim (x \oplus y, C; y) \quad (x \succsim y \succsim e). \qquad [1]$$

The formalism is reinterpreted in psychophysical terms in ref. 2 and in the first of the unpublished works, the details of which need not be given here, except to say that two generalizations, which were motivated by empirical results of Ragnar Steingrimsson (personal communication), were studied in succession. First, the operation $\oplus$ was no longer assumed to be commutative but did have either a right or left identity. Second, the assumption that the operation has an identity was dropped and replaced by the assumption of idempotence in the sense $x \oplus x \sim x$.

Data in both domains suggested that there is a considerable difference in the way respondents handle $(x, C; e)$ and $(x, C; y)$ with $y > e$. Thus, Luce, in the second unpublished work, was led to explore the natural generalization of Eq. **1** to right distributivity

$$(x, C; y) \oplus z \sim (x \oplus z, C; y \oplus z) \quad (x \succsim y > e, z \succsim e). \qquad [2]$$

In developing theorems describing the relations among certain basic assumptions including Eq. **2** and numerical representations of the structure, two functional equations,

$$h(\sigma(y)x + [1 - \sigma(y)]y) = \tau(y)h(x) + [1 - \tau(y)]h(y) \quad (x \geq y \geq 0) \qquad [3]$$

and

$$h[y + f(x - y)] = h(y) + g[h(x) - h(y)] \quad (x \geq y \geq 0), \qquad [4]$$

arose quite naturally. These equations do not appear to have been treated in the functional equation literature, and so we study them here.

They are also of independent mathematical interest. Indeed, define $s = e^x$, $t = e^y$, and

$$H(t) = \exp h(\ln t), \quad F(w) = \exp f(\ln w), \quad G(w) = \exp g(\ln w). \qquad [5]$$

Then Eq. **4** becomes

$$H\left[tF\left(\frac{s}{t}\right)\right] = H(t)G\left[\frac{H(s)}{H(t)}\right] \quad (s \geq t \geq 1). \qquad [6]$$

Since

$$K(s, t) = tF\left(\frac{s}{t}\right), \quad L(u, v) = vG\left(\frac{u}{v}\right) \qquad [7]$$

is the general form of degree 1 homogeneous functions (of two variables), the problem of solving Eq. **4** is equivalent to solving

$$H[K(s, t)] = L[H(s), H(t)], \quad (s \geq t \geq 1), \qquad [8]$$

that is, to *determining which pairs of degree* 1 *homogeneous functions can be homomorphic and calculating all their homomorphisms.* (The case $s < t$ is independent and can be solved similarly.)

As will be seen, the results give, under certain monotonicity and differentiability conditions, full description (containing up to five parameters) of the three functions, f, g, h, involved in Eq. **4** and, in consequence, of H, K, L in Eq. **8**.

At the end of the paper (*Note 2*), we will establish also a mathematical connection to Eq. **3**.

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THEOREM 1. *Consider the functional equation*

$$h(\sigma(y)x + [1 - \sigma(y)]y) = \tau(y)h(x) + [1 - \tau(y)]h(y) \quad (x \geq y \geq 0), \quad [9]$$

under the following conditions: The function h maps $[0, \infty[$ *into a subset of* $[0, \infty[$, *is strictly increasing and continuously differentiable; the functions* σ *and* τ *map* $[0, \infty[$ *continuously into* $[0, 1]$, $\sigma(y) > 0$ *for* $y > 0$ *but* σ *is not* 1 *on* $]0, \infty[$. *Then*

$$h(x) = cx + d \quad (c > 0, d \geq 0)$$

and $\tau = \sigma$ *on* $[0, \infty[$.

Proof: Differentiate Eq. **9** with respect to x:

$$h'(\sigma(y)x + [1 - \sigma(y)]y)\sigma(y) = \tau(y)h'(x). \quad [10]$$

If there existed an $x_0 > 0$ such that $h'(x_0) = 0$, then we would have

$$h'(\sigma(y)x_0 + [1 - \sigma(y)]y) = 0,$$

and thus, since σ is continuous and not 1, there would exist a proper interval on which $h' = 0$. That is impossible, because h is strictly increasing. Therefore $h' > 0$ on $]0, \infty[$.

Setting $x = y = t$ in Eq. **10**, we have that $h'(t)\sigma(t) = h'(t)\tau(t)$ and so $\sigma(t) = \tau(t)$ for $t > 0$ [because $h'(t) > 0$] and, by continuity, for all $t \geq 0$, as asserted. Putting that back into Eq. **10**, we have, since $\sigma(y) > 0$,

$$h'(\sigma(y)x + [1 - \sigma(y)]y) = h'(x) \quad [11]$$

for $y > 0$, and by continuity it holds for all $y \geq 0$.

Let $x_0 > 0$ and $c = h'(x_0)$. Eq. **11** and the continuity of h' and σ show that there is a proper interval $[\delta, d]$ on which $h' = c$. Let it be the largest such interval. We show that d cannot be finite (thus $d = \infty$ and the interval is open from above). Indeed, otherwise there would exist an $x > d$, as close to d as we want, for which $h'(x) \neq c$ but $\sigma(\delta)x + [1 - \sigma(\delta)]\delta \in [\delta, d]$ (because $\sigma < 1$), thus $h'(\sigma(\delta)x + [1 - \sigma(\delta)]\delta) = c$. This would contradict Eq. **11**. One proves similarly that $\delta = 0$. Therefore, $h'(x) = c$ and $h(x) = cx + d$ for all $x \in [0, \infty[$, as asserted. From $h' > 0$, $h \geq 0$, we have also $c > 0$, $d \geq 0$. □

THEOREM 2. *Suppose the functional equation*

$$h[y + f(x - y)] = h(y) + g[h(x) - h(y)] \quad (x \geq y \geq 0) \quad [12]$$

holds under the following assumptions about the functions and their domains and ranges:

(*i*) *h maps* $[0, \infty[$ *onto a subinterval of positive length of* $[0, \infty[$ *and is strictly increasing and twice continuously differentiable;*

(*ii*) *f and g map* $[0, \infty[$ *onto* $[0, \infty[$ *and are twice differentiable.*

Then one of the following holds:

(*a*) $h''(x) \equiv 0$, *in which case*

$$h(x) = cx + d, \quad f(z) = \frac{1}{c}g(cz), \quad (c > 0), \quad [13]$$

where g is an arbitrary function satisfying ii.

(*b*) *The function* h'' *is not identically 0, and* $f''(0) = g''(0) = 0$; *furthermore, either* $f'(0) = g'(0) = 0$ *or* $f'(0) = g'(0) = 1$.

(*c*) *The function* h'' *is not identically 0, and either* $f''(0) \neq 0$ *or* $g''(0) \neq 0$. *In this case, the solutions are given by*

$$h(x) = a \ln(q + re^{bx}),$$

$$f(z) = \frac{1}{b}\ln(ce^{bz} + 1 - c), \quad g(z) = a \ln(ce^{z/a} + 1 - c), \quad [14]$$

by

$$h(x) = q + ae^{bx}, \quad f(z) = \frac{1}{b}\ln(ce^{bz} + 1 - c), \quad g(z) = cz, \quad [15]$$

and by

$$h(x) = a \ln(q + px), \quad f(z) = cz, \quad g(z) = a \ln(ce^{z/a} + 1 - c) \quad [16]$$

[with the constants chosen so that the conditions i and ii are satisfied].

The functions given by Eqs. **13–16** *satisfy Eq.* **12**.

Proof: Notice that, by *ii*,

$$f(0) = g(0) = 0. \quad [17]$$

Differentiate Eq. **12** with respect to x:

$$h'[y + f(x - y)]f'(x - y) = g'[h(x) - h(y)]h'(x). \quad [18]$$

Again, if we had $h'(x_0) = 0$ for an $x_0 > 0$, then either h' or f' would be 0 on an interval, in contradiction to the strict monotonicity of h and f (the latter following from f mapping $[0, \infty[$ continuously onto $[0, \infty[$). Thus $h'(x) > 0$ for $x > 0$. Let $y = x$ in Eq. **18** to obtain (see Eq. **17**).

$$h'(x)f'(0) = h'(x)g'(0).$$

Since $h' > 0$, we have $f'(0) = g'(0)$.

Now differentiate Eq. **18** with respect to x:

$$h'[y + f(x - y)]f''(x - y) + h''[y + f(x - y)]f'(x - y)^2 = g'[h(x) - h(y)]h''(x) + g''[h(x) - h(y)]h'(x)^2.$$

Set $y = x$, to get

$$h'(x)f''(0) + h''(x)f'(0)^2 = h''(x)g'(0) + h'(x)^2g''(0),$$

that is,

$$h''(x)f'(0)[f'(0) - 1] = h'(x)[h'(x)g''(0) - f''(0)]. \quad [19]$$

We first consider the case where the left-hand side is 0. This happens in two ways:

(*a*) $h''(x) \equiv 0$. Then h is affine, as in Eq. **13** (in the rest of the cases, we will assume that h is not affine).

(*b*) The function h'' is not identically 0, in which case $f'(0)[f'(0) - 1] = 0$ and $g''(0) = f''(0) = 0$, because $h'(x) > 0$ for $x > 0$. There exists such a solution:

$$f(z) = g(z) = z, \quad h \text{ arbitrary}. \quad [20]$$

Notice, however, that for this solution, both sides of Eq. **12** are independent of y. Under analyticity assumptions, the *Proposition* at the end of this *Proof* shows that Eq. **20** is the only solution in this case (*b*).

(*c*) So we now consider Eq. **19** when the left side is not 0, in which case both $h''(x)$ is not identically 0 and $f'(0)[f'(0) - 1] \neq 0$. Recalling that $h'(x) > 0$ for $x > 0$, we have

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$$\frac{d\ \ln h'(x)}{dx} = \frac{h''(x)}{h'(x)} = \frac{h'(x)g''(0) - f''(0)}{f'(0)[f'(0) - 1]} = \alpha h'(x) + b.$$

If $\alpha = 0$ [that is, $g''(0) = 0$ and so, by supposition, $f''(0) \neq 0$, $b \neq 0$], then we have $h'(x) = \tilde{a}e^{bx}$ yielding, with $a = \tilde{a}/b$, the solution in Eq. **15**

$$h(x) = q + ae^{bx}. \qquad \textbf{[21]}$$

If $\alpha \neq 0$, then for some constant $\lambda > 0$

$$h'(x) = \lambda \exp[\alpha h(x) + bx],$$

that is,

$$\exp[-\alpha h(x) - bx]h'(x) = \lambda \neq 0.$$

With $k(x) := \exp[-\alpha h(x) - bx]$, this becomes the linear differential equation with constant coefficients

$$k'(x) + bk(x) = -\alpha\lambda.$$

The general solution of the latter is known to be either (for $b \neq 0$; writing $r = -\alpha\lambda/b \neq 0$)

$$k(x) = qe^{-bx} + r,$$

which with $a = -1/\alpha$ gives the solution in Eq. **14**

$$h(x) = a\ \ln(q + re^{bx}), \qquad \textbf{[22]}$$

for h or, if $b = 0$,

$$k(x) = q + px,$$

and then we have the solution (see Eq. **16**)

$$h(x) = (-1/\alpha)\ln(q + px) = a\ \ln(q + px) \qquad \textbf{[23]}$$

(in both cases extended by continuity to 0).

The f, g belonging to Eq. **13** satisfy Eq. **12** if

$$f(z) = \frac{1}{c}g(cz),$$

as asserted.

To determine f and g belonging to Eq. **22**, define θ, φ, and ψ by

$$\theta(y) = \frac{re^{by}}{q + re^{by}} \neq 0,$$

$$\varphi(z) = \exp bf(z), \text{ that is } f(t) = \frac{1}{b}\ln\ \varphi(t),$$

and

$$\psi(u) = \exp\left[\frac{1}{a}g(a\ \ln u)\right], \text{ that is } g(v) = a\ \ln\left(\psi\left[\exp\left(\frac{s}{a}\right)\right]\right).$$

So Eq. **12** can be written with $z = x - y$ as

$$\theta(y)\varphi(z) + 1 - \theta(y) = \psi[\theta(y)e^{bz} + 1 - \theta(y)].$$

Take the derivative with respect to z :

$$\varphi'(z)\theta(y) = \psi'[\theta(y)e^{bz} + 1 - \theta(y)]be^{bz}\theta(y).$$

We may divide by $be^{bz}\theta(y)$, since it is different from 0 :

$$\frac{\varphi'(z)}{be^{bz}} = \psi'[\theta(y)e^{bz} + 1 - \theta(y)].$$

Because the left side is independent of y and the right depends on it, there exists a constant c such that

$$\varphi'(z) = cbe^{bz}, \quad \varphi(z) = ce^{bz} + d, \quad f(z) = \frac{1}{b}\ln(d + ce^{bz})$$

and

$$\psi'(z) = c, \quad \psi(z) = cz + d', \quad g(z) = a\ \ln(d' + ce^{z/a}).$$

Because $f(0) = g(0) = 0$, we see that $d = 1 - c = d'$, thus yielding f and g in Eq. **14**.

The proof is similar but easier for the h in Eq. **23**. Here, with

$$\theta(y) := \frac{p}{q + py} \neq 0, \quad \psi(z) := \exp[(1/a)g(a\ \ln z)],$$

Eq. **12** can be written as

$$1 + \theta(y)f(z) = \psi[1 + \theta(y)z].$$

Differentiating with respect to z, we get, since $\theta(y) \neq 0$,

$$f'(z) = \psi'[1 + \theta(y)z] = c, \quad f(z) = cz,$$

because the left side of the first equation is independent of y and because $f(0) = 0$. The first equation also shows that

$$\psi'(s) = c, \quad \psi(s) = cs + d', \quad g(z) = a\ \ln(ce^{z/a} + d')$$

and, by $g(0) = 0$ (see Eq. **17**), $d' = 1 - c$. Thus we also get f, g in Eq. **16**.

We get similarly from h in Eq. **21** the f, g in Eq. **15**.

The last statement of *Theorem 2* is easy to verify. □

PROPOSITION. *If, in addition to the assumptions of the Theorem, f, g, h are analytic functions and h not affine, then the solutions in Case (b) are those given in Eq.* **20**.

Sketch of proof: The line of argument is to differentiate Eq. **12** repeatedly with respect to y and then set $x = y$. The first derivative yields $f'(0) = g'(0)$, the second that either $f'(0) = 0$ or 1, and that $f''(0) = g''(0) = 0$, and the third that $f'''(0) = g'''(0) = 0$. At this point, we proceed by induction. Assume that $f^{(i)}(0) = g^{(i)}(0) = 0$, $(i = 2, 3, \ldots, n - 1)$, then, while each differentiation with respect to y yields an increasing number of terms, after the nth differentiation, all of them except

$$h'[y + f(x - y)]f^{(n)}(x - y) - h'(y)^n g^{(n)}[h(x) - h(y)]$$

vanish when $x = y$ for the following reasons. By the induction hypothesis, all terms with $f^{(i)}(x - y)$ or $g^{(i)}[h(x) - h(y)]$ $(i = 2, 3, \ldots, n - 1)$ as factors vanish when $y = x$. On the other hand, the terms containing $h^{(n)}$ can be collected, after setting $y = x$, into one: $h^{(n)}(x)\{[1 - f'(0)]^n - 1 + f'(0)\}$, that is, 0 both if $f'(0) = 1$ (obvious) and if $f'(0) = 0$ {because 1 is cancelled in $[1 - f'(0)]^n - 1 + f'(0)$}. So, putting $y = x$ into the above difference, that too has to vanish and we get that $f^{(n)}(0) - h'(x)^{n-1}g^{(n)}(0) = 0$ (because $h' > 0$) and excluding the case of affine h this yields $f^{(n)}(0) = g^{(n)}(0) = 0$. Thus the induction is complete.

For analytic f, the case $f'(0) = 1$ is the solution given in Eq. **20**, and the case $f'(0) = 0$ means that $f \equiv g \equiv 0$, which is excluded by *ii*. □

Note 1: In view of Eqs. **14–16**, **5**, and **7**, we have for case *(c)* under conditions corresponding to *i*, *ii*, and to $f''(0) \neq 0$ or $g''(0) \neq 0$, with $m = 1/a$,

$$H(t) = (q + rt^b)^{1/m}, \qquad \textbf{[24]}$$

$$K(x, y) = [cx^b + (1 - c)y^b]^{1/b},$$

$$L(u, v) = [cu^m + (1 - c)v^m]^{1/m}; \quad \textbf{[25]}$$

$$H(t) = \rho \exp(at^b), \quad \textbf{[26]}$$

$$K(x, y) = [cx^b + (1 - c)y^b]^{1/b}, \quad L(u, v) = u^c v^{1-c}; \quad \textbf{[27]}$$

and

$$H(t) = (q + p \ln t)^{1/m}, \quad \textbf{[28]}$$

$$K(x, y) = x^c y^{1-c}, \quad L(u, v) = [cu^m + (1 - c)v^m]^{1/m} \quad \textbf{[29]}$$

as solutions of Eq. **8**.

Eq. **17**, which followed from *ii*, implies $F(1) = G(1) = 1$, thus yielding the idempotence $K(t, t) = L(t, t) = t$. The latter can be attained for all (degree 1) homogeneous K, L with $A := K(1, 1) > 0$, $B := L(1, 1) > 0$ by dividing them by A or B, respectively. We now determine when the new, not necessarily idempotent, K and L are homomorphic, and what the corresponding homomorphisms H are. In particular, in the original form, case *(a)* would give to every homogeneous K satisfying $K(t, t) = t$ (and the other conditions) as its homomorphic pair the similarly homogeneous function $K(x, y) = L(x^c, y^c)^{1/c}$ $(c \neq 0)$. When the idempotence conditions are dropped, this relation becomes $K(x, y)/A = [L(x^c, y^c)/B]^{1/c}$. That is a homomorphy if $B = A^c > 0$, the homomorphism being $H(t) = \rho t^c$ $(\rho > 0, c \neq 0)$.

For Eq. **29**, dropping the idempotence requirement leads to

$$K(x, y) = Ax^c y^{1-c}, \quad L(u, v) = [Cu^a + Dv^a]^{1/a}$$

where $C := c/B$, $D := (1 - c)/B$. The (strictly increasing) homomorphism H thus has to satisfy

$$H(Ax^c y^{1-c}) = [CH(x)^a + DH(y)^a]^{1/a}$$

or, with $M(t) := H(e^t)^a$ and $u = \ln x$, $v = \ln y$; $\varepsilon := \ln A$,

$$M(cu + (1 - c)v + \varepsilon) = CM(u) + DM(v).$$

In ref. 3, pp. 66–67, the following result has been proved (we state it in the form we will use it).

LEMMA. *The functional equation with real variables, function values, and constants*

$$M(cu + dv + \varepsilon) = CM(u) + DM(v) + E$$

$(c > 0, d > 0, C > 0, D > 0)$ *has a strictly monotonic solution if, and only if,*

$$C = c, \quad D = d$$

and then the strictly increasing solution is given by $M(v) = rv + q$, *where* $r > 0$ *and* $(c + d - 1)q = r\varepsilon - E$.

So, in our case $C = c$, $D = d = 1 - c$, $C + D = c + d = 1$, $E = 0$, so $\varepsilon = 0$, $A = 1$, and we are back to Eqs. **29** and **28**. The same is true for Eqs. **27** and **26**.

For Eq. **25**, however, dropping the idempotence conditions changes the solutions somewhat. We have then

$$H[(cx^b + dy^b)^{1/b}] = [CH(x)^m + DH(y)^m]^{1/m}$$

or, with $u = x^b$, $v = y^b$, $M(v) := H(v^{1/b})^m$,

$$M(cu + dv) = CM(u) + DM(v).$$

By the above *Lemma*, this equation has strictly increasing solutions if $c = C$, $d = D$, and then

$$K(x, y) = (Cx^b + Dy^b)^{1/b}, \quad L(u, v) = (Cu^m + Dv^m)^{1/m} \quad \textbf{[30]}$$

$(C > 0, D > 0, m \neq 0, b \neq 0)$. For H, the *Lemma* leads to $H(v^{1/b})^m = M(v) = rv + q$ if $C + D = 1$, which leads back to Eq. **24**. But, if $C + D \neq 1$ then $H(v^{1/b})^m = M(v) = rv$, giving

$$H(t) = \rho t^{b/m}. \quad \textbf{[31]}$$

Thus, under these conditions [but without supposing $K(t, t) = L(t, t) = t$], every degree 1 homogeneous function L is homomorphic to K given by

$$K(x, y) = L(x^c, y^c)^{1/c} \quad (c \neq 0)$$

with the homomorphism $(\rho > 0)$

$$H(t) = \rho t^c$$

—this contains also the solution (Eq. **30**) with the homomorphism (Eq. **31**), *whereas all pairs of homogeneous functions of two variables, under other homomorphisms, are given by Eqs.* **25***,* **27***, and* **29***, the homomorphisms being Eqs.* **24***,* **26***, or* **28***, respectively.*

(In condition *i*, we postulated h to be strictly increasing and twice continuously differentiable, so we really determined all *order-preserving* C^2 *diffeomorphisms*, a special class of homomorphisms.)

Notice that $K(x, y) = x$ and $L(u, v) = u$, which are both independent of the second variable, correspond to Eq. **20** in part (*b*) of the *Proof* of *Theorem 2.*

Note 2: The functions K and L in Eqs. **25**, **27**, and **29** turned out to satisfy idempotence anyway. They are *weighted power and geometric means*, so our result seems to be an interesting characterization of these mean values. These means are also (see, e.g., ref. 3, pp. 151–153, 287) the only homogeneous quasilinear means. A quasilinear mean is given by

$$K(x, y) = \chi^{-1}[c\chi(x) + (1 - c)\chi(y)],$$

where χ is a continuous strictly monotonic function and $c \in [0, 1]$ a constant. The quasilinear mean K is homomorphic (with an order preserving continuous homomorphism H) to another quasilinear mean

$$L(u, v) = \omega^{-1}[k\omega(u) + (1 - k)\omega(v)]$$

if, and only if, the function $h := \omega \circ H \circ \chi^{-1}$ satisfies the functional equation

$$h[cs + (1 - c)t] = kh(s) + (1 - k)h(t)].$$

By the *Lemma* above, this equation has strictly monotonic solutions if, and only if, $c = k$, and they are given by $h(x) = cx + d$. Eq. **3** is a generalization of this equation. We have solved it in *Theorem 1* as Eq. **9**, for continuously differentiable strictly increasing h (but see *Note 3* below). There we obtained the corresponding result $\sigma = \tau$ and the same affine function h.

Note 3: Zsolt Páles has succeeded (personal communication) in eliminating the differentiability assumption from *Theorem 1.*

There are two *open problems* of interest left: First, it would be desirable either to determine the complete solution in the case $f''(0) = g''(0) = 0$ (see item *b* in *Theorem 2* and the *Proposition* thereafter) under the conditions of *Theorem 2* rather than analyticity or to find a suitable condition that excludes this case. It turns out that, for the application in Luce (second unpublished work), the present result is sufficient. Second, more generally it is of interest to eliminate the differentiability conditions also from *Theorem 2.*

Notice that in *Note 1*, we determined the homomorphism H between K and L given by Eqs. **25** (or **30**), **27**, or **29** without differentiating. This is equivalent to *determining h without assuming its differentiability if, in Eq.* **12***, f, g are already known to* *be of one of the forms given in Theorem 2.* However, in the proof of that theorem, we needed the second order differentiability of h for calculating these f and g.

Bruce Ebanks (personal communication) has *determined* K, L in Eq. **8**, also *without differentiability assumption if* H *is already known to be of one of the forms Eqs.* **24***,* **26***, or* **28**. Thus, equivalently, f and g in Eq. **12** have been determined without assuming differentiability if h is already known to be of one of the forms Eqs. **21**, **22**, or **23**. But, again unfortunately, we also needed the (second order) differentiability of f and g to determine h.

So the problem of eliminating differentiability assumptions in *Theorem 2* remains open.

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3. Aczél, J. (1966) *Lectures on Functional Equations and Their Applications* (Academic, New York/London).