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Functional Equations in Behavioral and Social Sciences

Theories in the behavioral, social, and natural sciences are often formalized by equations involving unknown functions, i.e., by functional equations. For instance, a theorist may be reluctant to make specific assumptions regarding the form of functions involved in a mathematical model, but the qualitative formulation of the model itself may impose constraints on the initially unknown functions. Such constraints often reduce the possibilities and occasionally are so severe that they restrict the possible forms to a few. A recent general reference to functional equations is Aczél and Dhombres (1989) where references to earlier and still useful surveys are given.

In the simplest cases there is one unknown, real-valued, function φ. An example, named after the famous French mathematician A. L. Cauchy, is the Cauchy equation $\varphi(x+y) = \varphi(x)+\varphi(y)$, where x, y are in the set of real numbers $\mathbb{R}$. More restrictive domains are sometimes studied. With no further restrictions, the solutions to the Cauchy equation can be wild. But assuming that φ is monotonic, or continuous, or bounded over a finite interval, the solution reduces to $\varphi(x) = cx$ for some constant c. By taking $\psi = \exp \varphi$, we get $\psi(x+y) = \psi(x)\psi(y)$, which is the so-called 'lack of memory property' and it is easy to see from the previous case that the strictly monotonic solutions are $\psi(x) = e^{cx}$, $c \neq 0$.

A celebrated psychophysical example is the connection between empirical just noticeable differences and G. T. Fechner's hypothesis that sensation corresponds to subjective differences being equal. By one interpretation of what he meant, one is led to the family of functional equations, named Abel equations after the famous Norwegian mathematician N. H. Abel, $\eta[x+g(k,x)]-\eta(x) = h(k)$, where the variable x and the parameter k are in the set of nonnegative real numbers $\mathbb{R}_+$. For the special case of Weber's law where $g(k, x) = (k-1)x$, this equation reduces to $\eta(kx) - \eta(x) = h(k) = \eta(k)-\eta\ (1)$, i.e., with $\varphi(z) = \eta(\exp z) - \eta\ (1)$, $k' = \log k$, $x' = \log x$, we have $\varphi(k'+x') = \varphi(k')+\varphi(x')$, which if k' is treated as a variable is Cauchy's equation. The restriction of strict monotonicity yields $\eta(x) = a \ln x + b$, $a \neq 0$, b constants, as the solution. For references and a general discussion of closely related issue see Falmagne (1985, Chap. 4).

Sometimes a functional equation has multiple, qualitatively different solutions. Yet the scientist arriving at the functional equation has the strong intuition that only one of these solutions is really appropriate for the scientific problem in question. Whenever this happens, the challenge is to discover additional behavioral properties that seem to be empirically correct and that serve to isolate the desired solution. A reason for fully determining the several solutions is that the same functional equation may arise in an entirely different empirical context and, for that context, one of the previously unacceptable solutions may be appropriate. So the complete characterization is clearly of interest.

As will be seen, functional equations arise in the social sciences in at least three main ways. One occurs when one knows how to measure numerically the same attribute in two different ways, which is the case more often than not. Then the two measures are related by an unknown strictly increasing function. An empirical law linking the two underlying measurement structures manifests itself as an equation restricting that unknown function.

Functional equations also arise when some invariance condition holds. The Fechner hypothesis above is an example. Others are given later.

Economic aggregation problems are a third source of functional equations. Consistent aggregation both rules out some *ad hoc* aggregation functions and leads to families of functions that are indeed consistent. Despite the intuitive appeal of additive aggregation, that assumption is inconsistent with the most common production functions used in economics. Examples are given of production functions and of permissible aggregation rules.

1. *Independent Measures of an Attribute*

1.1 *A Physical Example: Mass Measures*

Consider mass measurement where one decides which of two objects x and y is more massive by observing which arm of an equal-arm beam balance drops (in a vacuum). This provides a mass ordering. One can construct a numerical measure m of mass, unique up to multiplication by a positive constant (ratio scale), by placing pairs of objects, denoted by $x \circ y$, on the pans. Under reasonable, empirically-testable assumptions m is additive over the operation of combining, i.e., $m(x \circ y) = m(x)+m(y)$. One can also construct a measure unique up to power transformations (log-interval scale) by varying the velocity of objects and determining which has greater momentum. Under empirically reasonable assumptions, the measure of momentum is multiplicative in powers of a mass and a velocity measure, i.e., $\alpha m'(x)^{\beta} v'(x)^{\beta}$, where $\alpha, \beta > 0$ but are otherwise unspecified. It can be shown that the two measures m and m' preserve the same mass order, and so they must be related by a strictly increasing function ξ. A natural question is: What is the form of ξ?

The following qualitative distribution property is empirically justified. Suppose (x, v) denotes the object x moving at velocity v and that $(x, v) \sim (x', v')$ means the two objects have the same momentum. Suppose also that $(y, v) \sim (y', v')$. Then, distribution asserts $(x \circ y, v) \sim (x' \circ y', v')$. This condition forces ξ to be multiplicative in the sense of satisfying $\xi(rs) = \xi(r)\,\xi(s)$, where the variables are real numbers with $r > 0$, $s > 0$. By setting $\xi = \exp\varphi(\ln)$ we see from the strictly increasing solutions of the Cauchy equation that $\xi(r) = r^{\beta}$, $\beta > 0$. Thus, one mass measure is a power function of the other one. See Luce et al. (1990, p. 125).

1.2 Preferences Among Gambles

Something quite analogous occurs in studying people's preferences among uncertain alternatives, often called gambles. A binary gamble $(x,C;\, y,\bar{C})$ means that x is the consequence if event C occurs and y otherwise when a chance 'experiment' with possible outcomes $E = C \cup \bar{C}$ is conducted. The preference order over gambles, which is assumed to be connected and transitive, is denoted by $\succcurlyeq$. We assume there is a status quo e, and a consequence x is said to be a gain if $x \succcurlyeq e$ and a loss if $e \succcurlyeq x$. A well developed theory of preferences over gambles, which generalizes subjective expected utility (Savage 1954), was worked out during the 1980s (see Quiggin 1993). It establishes that under certain conditions a utility function U over gambles and a weighting function W over chance events exists such that for gains x, y, and $g = (x,C;\, y,\bar{C})$

$$U(g) = \begin{cases} U(x)W(C) + U(y)[1 - W(C)], & \text{if } x \succcurlyeq y \\ U(x)[1 - W(\bar{C})] + U(y)W(\bar{C}), & \text{if } x \preccurlyeq y \end{cases} \tag{1}$$

$U(g) = U[(x,C;\, y,\bar{C})]$ is usually written just as $U(x,C;\, y,\bar{C})$, which is a minor abuse of notation. Because we assume $x \sim (x,C;\, x,\bar{C})$, U on gambles is also on consequences as well. This representation is called rank-dependent utility because the weight used in the average depends upon the ordering of the consequences. The model can be extended to cover losses and mixed gains and losses, but we do not go into that here (see Luce 2000).

One can also consider having or receiving pairs of valued consequences and gambles. Suppose g and h are gambles (including, as special cases, pure consequences) and let $g \oplus h$ denote having or receiving both. This binary operation $\oplus$ is called joint receipt. Assume that the preference order $\succcurlyeq$ extends to the domain of joint receipts. It is plausible that, for gains, $\oplus$ and $\succcurlyeq$ satisfy the same formal properties as does mass where no change from the status quo, e, plays the role of an identity. Thus, there is a real-valued 'value measure' V that preserves the order $\succcurlyeq$ and is additive over $\oplus$, i.e., $V(x \oplus y) = V(x) + V(y)$.

A natural question is: How do U and V relate? Luce and Fishburn (1991) considered the following linking hypothesis (or putative 'law') called segregation: For all gains x, y and events C

$$(x,C;\, e,\bar{C}) \oplus y \sim (x \oplus y, C;\, y, \bar{C}) \tag{2}$$

Note that both sides yield the same thing under the same conditions, so this is a highly rational property. If we introduce the notation $G(u, v) = U[U^{-1}(u) \oplus U^{-1}(v)]$ one can show that segregation implies that for $u = U(x)$, $v = U(y)$, $w = W(C)$ and $u \geqslant v$

$$G(uw, v) = G(u, v)w + v(1 - w) \tag{3}$$

For commutative the solution to this functional equation (Luce and Fishburn 1991) is that for some real δ

$$G(u, v) = u + v - \delta uv \tag{4}$$

From this equation one can show that U and V are related as follows. If $\delta = 0$, then $U = \alpha V$. If $\delta \neq 0$, then either $U_+ = 1 - e^{-\kappa V}$ or $U_+ = e^{\kappa V} - 1$, where $U_+(x) = |\delta| U(x)$, $x \succsim e$. Thus, if one knows how V depends on the consequences, then one knows up to a constant how U_+ does also. Conditions are given that determine the form of V over money and W over probability. [In like manner, a theory for losses is possible with constant δ' and $U_-(x) = |\delta'| U(x)$, $x \prec e$.]

From these expressions and the additivity of V over $\oplus$ one can deduce formulas for the utility U over $\oplus$ and for gambles when the consequences are a gain and a loss. Some interesting non-bilinear expressions arise (Luce 2000).

Recently, Luce and A. A. J. Marley (summarized in Luce 2000) have suggested a generalized representation of binary gambles over gains, namely, for $x \succsim y \succsim e$,

$$\varphi_0[(x,C;\, y,\bar{C})] = \varphi_1[u_1(x)w_1(C)] + \varphi_2[u_2(y)w_2(\bar{C})] \tag{5}$$

where $\varphi_1(0) = \varphi_2(0) = 0$. The ranked-additive decomposition in this form has been axiomatized by Wakker (1991, 1993) and the separability of $u_1 w_1$ can be defended axiomatically for the first component (Luce 2000). Assume, as seems reasonable,

$$x \sim (x,E;\, y,\varnothing) \tag{6}$$

$$y \sim (x,\varnothing;\, y,E) \tag{7}$$

$$x \sim (x,C;\, x,\bar{C}) \tag{8}$$

where $\varnothing$ denotes the null event and E the universal one underlying the gamble. Then noting $w_1(E) = 1$, $w_2(\varnothing) = 0$, the first indifference yields $\varphi_0(x) = \varphi_1[u_1(x)]$. If we set $y = x$, $v = u_1(x)$, $w = w_1(C)$, $w_2(\bar{C})$

$= Q[w_1(C)] = Q(w)$, then the second of the above conditions implies $x \sim (x,\varnothing; x,E)$, and so $\varphi_1(v) = \varphi_0(x) = \varphi_2[u_2(x)]$ or rewriting $u_2(x) = \varphi_2^{-1}[\varphi_1(v)] = \psi(v)$. Thus, setting $\varphi = \varphi_1$ the third yields the functional equation

$$\varphi(v) = \varphi(vw) + \varphi(\psi^{-1}[\psi(v)Q(w)]),\ v \in [0,k[,\ w \in [0,1] \tag{9}$$

where the unknown functions φ, ψ are strictly increasing, and Q is strictly decreasing. Aczél et al. (2001) have determined all solutions to this equation. From these results, Luce and Marley (see Luce 2000) showed the following utility representation: For $x \succcurlyeq y \succcurlyeq e$, there exist constants $c > 0$ and $\mu > -1/k^c$ such that

$$U(x,C; y,\bar{C}) = \frac{U(x)W(C) + U(y)[1-W(C)] + \mu U(x)U(y)W(C)}{1+\mu U(y)W(C)} \tag{10}$$

where $U = (u_1)^c$ and $W = (w_1)^c$. If, however, one supposes that segregation holds, then one can show that $\mu = 0$, which of course is just the rank-dependent model.

2. Invariance Principles

2.1 Value Function for Money

Section 1.2 left open the question of how V depends upon money. The most obvious assumption is that if x, y are sums of money, then $x \oplus y = x + y$. Thus, $V(x) + V(y) = V(x \oplus y) = V(x+y)$, which is Cauchy's equation. Because V is strictly increasing, V is thus proportional to money. Some empirical work suggests that, in general, this hypothesis may be too strong (Thaler 1985). An alternative assumption is that $\oplus$ is invariant under proportional changes in money, i.e., for any $\lambda > 0$,

$$x \oplus y \sim z \Rightarrow \lambda x \oplus \lambda y \sim \lambda z \tag{11}$$

Using the additivity of V,

$$V^{-1}[V(\lambda x) + V(\lambda y)] = \lambda z = \lambda V^{-1}[V(x) + V(y)] \tag{12}$$

For $x,y \geqslant 0$, functional equation arguments lead, through multiplicative functions, to $V(x) = \alpha x^\beta$ and $x \oplus y = (x^\beta + y^\beta)^{1/\beta}$, for some $\alpha > 0$, $\beta > 0$. This result, combined with those of Sect. 1.2, gives U to within one additional free parameter β (note that κ of U_+ and α are not separately identifiable).

2.2 Prelec's Weighting Function

Section 1.2 also left open the question of the form of the weighting function W. When the gambles are characterized in terms of probabilities $p = \Pr(C)$, much empirical work shows that in general for some p_0 in]0, 1[

$$W(p) \begin{cases} > p, & \text{if } 0 < p < p_0 \\ = p, & \text{if } p = p_0 \\ < p, & \text{if } p_0 < p < 1 \end{cases} \tag{13}$$

with W strictly increasing initially concave, and then convex. Such weighting functions are described as inverse S-shaped. Until Prelec (1998), suggestions of the mathematical form have not been derived from behavioral principles. He provided a somewhat complex behavioral invariance condition leading to $W(p) = \exp[-\gamma(-\ln p)^\alpha]$. Gonzalez and Wu (1999) showed that the Prelec form fits empirical data remarkably well. Not only is this function inverse S-shaped for $\alpha < 1$, but for $\alpha = 1$ it includes the 'rational' weighting function that arises from the conditions that gambles of the form $(x,p; 0)$ have a separable representation $U(x,p; 0) = U(x)W(p)$ and that $((x,p; 0),q; 0) \sim (x,pq; 0)$ holds.

Luce (cited in Luce 2000) gave a simpler condition, called reduction invariance, equivalent to Prelec's function which generalizes a property of the rational condition to the non-rational case: For all gains x and p, q, r in [0, 1],

$$((x,p; 0), q; 0) \sim (x,r; 0) \text{ implies, for } N = 2,3, \text{ that } ((x,p^N; 0),q^N; 0) \sim (x,r^N; 0) \tag{14}$$

From this, one first shows that the right side holds for N replaced by any real exponent $\lambda > 0$. Then, using separability, one arrives at the functional equation

$$(W^{-1}[W(p) + W(q)])^\lambda = r^\lambda = W^{-1}[W(p^\lambda) + W(q^\lambda)] \tag{15}$$

Note that with $V = \ln W(\exp)$, $x = \ln p$, $y = \ln q$ this reduces to the same functional equation as that for the value of money.

2.3 Additive Conjoint Weber Laws

Consider stimulus pairs (a, x) such as a tone of intensity a in a background noise of intensity x. Denote by $P(a,x; b,y)$ the probability that a subject perceives a in x more easily than b in y. Theoretical psychophysicists wish to understand the mathematical structure of such families of probabilities.

To do so, they explore various regularity or invariance conditions. For example, a homogeneity condition called a 'conjoint Weber law' assumes that for all real $\lambda > 0$

$$P(\lambda a, \lambda x; \lambda b, \lambda y) = P(a,x; b,y) \tag{16}$$

That is, the probability is invariant under multiplication of all four intensities by any positive number λ. A second important hypothesis is that of 'additive probability' in the following sense: There are continuous functions f strictly increasing, r strictly decreasing, and H strictly increasing in the first and strictly decreasing in the second variable such that

$$P(a,x;\, b,y) = H[f(a)+r(x), f(b)+r(y)] \quad (17)$$

In place of the last equation, Falmagne and G. Iverson (see Falmagne 1985, pp. 289–91) assumed the weaker condition

$$P(a,x;\, b,y) = H\,[q(a,\, x),\, g(b,\, y)] \quad (18)$$

of simple scalability where H is as above and g is strictly increasing in the first and strictly decreasing in the second variable, and the variable pairs $(c,\, z)$ lie in subsets closed under multiplication by any $\lambda > 0$. They showed that then P must be of one of two forms:

$$P(a,x;\, b,y) = M\left[\frac{T(a/x)x}{T(b/y)y}\right] \quad (19)$$

or

$$P(a,x;\, b,y) = Q\left(\frac{a}{x}, \frac{b}{y}\right) \quad (20)$$

where $T > 0$, M, and Q are continuous, T and M are strictly increasing, and Q is strictly increasing in the first and strictly decreasing in the second variable.

In particular, they determined the form of all representations that are both additive and satisfy a conjoint Weber law as follows. The last equation when rewritten as

$$P(a,x;\, b,y) = Q(e^{\ln a - \ln x},\, e^{\ln b - \ln y}) \quad (21)$$

is additive with

$$H(u,\, v) = Q(e^{u}, e^{v}),\, f(a) = \ln a,\, r(x) = -\ln x \quad (22)$$

The first homogeneous, simple scalable form is additive if and only if

$$M\left[\frac{T(a/x)x}{T(b/y)y}\right] = H\,[f(a)+r(x), f(b)+r(y)] \quad (23)$$

Holding $b = b_0$, $y = y_0$ constant and letting

$$K(t) = T\left(\frac{b_0}{y_0}\right) y_0 M^{-1}(H[t, f(b_0)+r(y_0)]) \quad (24)$$

we get

$$K[f(a)+r(x)] = T\left(\frac{a}{x}\right)x \quad (25)$$

Thus, for all $\lambda > 0$,

$$K[f(\lambda a)+r(\lambda x)] = \lambda K[f(a)+r(x)] \quad (26)$$

This functional equation and its method of solution is similar to but somewhat more complicated than those in Sects. 2.1 and 2.2. Moreover, it has two distinct sets of solutions:

$$f(a) = Aa^{\beta}+B, \quad r(x) = Cx^{\beta}+D \quad (27)$$

and

$$f(a) = A \ln a + B, \quad r(x) = C \ln x + D \quad (28)$$

Accordingly, for all homogeneous, additive, conjoint Weber laws, the probabilities are of one of the following three forms:

$$P(a,x;\, b,y) = Q\left(\frac{a}{x}, \frac{b}{y}\right) \quad (29)$$

$$P(a,x;\, b,y) = G\left(\frac{a^{\beta}+\gamma x^{\beta}}{b^{\beta}+\gamma y^{\beta}}\right) \quad (30)$$

$$P(a,x;\, b,y) = G\left(\frac{ax^{\gamma}}{by^{\gamma}}\right) \quad (31)$$

where Q is as above, G is continuous and strictly increasing, and $\beta > 0$, $\gamma \neq 0$ are constants.

Results of this character are useful both in understanding the implications of general invariance principles and in suggesting to experimentalists how to plot and analyse their data.

3. *Aggregation*

Consider an industry with m producers using n inputs. Denote by x_{jk} the amount of inputs k used by producer j. Let $y_j = F_j(x_{j1}, \ldots, x_{jn})$ denote the output of producer j and $z_k = G_k(x_{1k}, \ldots, x_{mk})$ denote the total consumption of input k. Consistent aggregation holds if there is a macro production function F and macro input function G such that

$$\begin{aligned} F(z_1, \ldots, z_n) &= F\,[G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn})] \\ &= G[F_1(x_{11}, \ldots, x_{1n}), \ldots, F_m(x_{m1}, \ldots, x_{mn})] \\ &= G(y_1, \ldots, y_m) \end{aligned} \quad (32)$$

This equation is called $m \times n$ generalized bisymmetry. Although a full treatment deals with general domains (see, e.g., Aczél and Maksa 1996, Aczél 1997), here it is assumed that all of the variables and functions lie in the set of non-negative real numbers $\mathbb{R}_+$. Assuming that all functions are strictly monotonic and continuous, Maksa (1999) has proved that there are continuous functions α_k, β_{jk}, γ_j, and φ, $j = 1, \ldots, m$, $k = 1, \ldots, n$ such that

$$F(z_1, \ldots, z_n) = \varphi^{-1}[\alpha_1(z_1) + \cdots + \alpha_n(z_n)] \quad (33)$$

$$F_j(x_{j1}, \ldots, x_{jn}) = \gamma_j^{-1}[\beta_{j1}(x_{j1}) + \cdots + \beta_{jn}(x_{jn})] \quad (34)$$

$$G(y_1, \ldots, y_m) = \varphi^{-1}[\gamma_1(y_1) + \cdots + \gamma_m(y_m)] \quad (35)$$

$$G_k(x_{1k}, \ldots, x_{mk}) = \alpha_k^{-1}[\beta_{1k}(x_{1k}) + \cdots + \beta_{mk}(x_{mk})] \quad (36)$$

Aggregation functions of this quasi-additive type had been obtained earlier, but under stronger regularity conditions (for a survey see Aczél 1997). Maksa's achievement was to get the same results under appreciably weaker conditions.

One way to interpret this result is that if the variables and functions are 'properly' measured, i.e., as $\varphi(F)$, $\varphi(G)$, $\gamma_j(F_j)$, $\alpha_k(G_k)$, $\alpha_k(z_k)$, $\beta_{jk}(x_{jk})$ and $\gamma_j(y_j)$, then aggregation is additive. This is, however, very different from assuming that the 'raw' inputs and outputs are related additively. Consider the following examples. Two commonly used production functions are the Cobb-Douglas (CD) function

$$F(z_1, \ldots, z_n) = a z_1^{c_1} z_2^{c_2} \ldots z_n^{c_n}, \quad a, c_k > 0 \quad (37)$$

and the Constant Elasticity of Substitution (CES) Function

$$F(z_1, \ldots, z_n) = (c_1 z_1^b + \cdots + c_n z_n^b)^{1/b}, \quad c_k > 0,\ b \neq 0 \quad (38)$$

Aczél and Maksa (1996) have shown that if any one of the aggregation functions is simple addition at the level of the 'raw' inputs or outputs, then it is impossible for any production function to be either CD or CES with $b \neq 1$. This does not say that inputs (or outputs) cannot actually be additively aggregated. Rather, it says that such additive aggregation is not consistent with common and realistic production functions. (One can prove that it is consistent only with linear production functions.)

Another question addressed is: If the F_j functions are in some sense of the same form, does that dictate that the macro F is also of that form? This is called the representativeness problem. The answer depends upon exactly what one means by 'the same form.' In a trivial sense, the answer is 'Yes' since the F and F_j's in the above quasi-additive solution have roughly the same structure. A somewhat more sophisticated answer is that it need not be so. For example, if all of the F_j are CD functions, then

$$\gamma_j(y) = \ln\left(\frac{y}{a_j}\right), \quad \beta_{jk}(x) = c_k \ln x \quad (39)$$

Or if they are all CES functions, then

$$\gamma_j(y) = y^b, \quad \beta_{jk}(x) = c_k x^b \quad (40)$$

Neither conclusion restricts φ and α_k in any way. Thus one may choose F to be CD or CES or neither.

So the general conclusion from solving the aggregation problem is that consistent and even representative aggregation is possible only for appropriately chosen functions. In general neither is possible if the aggregating functions are preselected in the 'wrong way.' The implications for the possibility of macro production models are considerable.

Related ideas and results concerning aggregation of probabilistic models of choice, with results for Luce's choice model as a special case, appear in Aczél et al. (1997).

4. Conclusions

For many static questions in the behavioral and social sciences, functional equations serve as a powerful tool in understanding theoretical constraints. Three types of constraint have been illustrated: (a) the linking of measurement structures that have a common qualitative attribute; (b) the imposition of invariance principles of various types of which three were illustrated; and (c) the question of whether one can aggregate over inputs and separately over industries and arrive at the same macro conclusion.

See also: Algorithms; Axiomatic Theories; Mathematical Psychology; Measurement, Representational Theory of; Measurement Theory: Conjoint; Ordered Relational Structures; Psychophysical Theory and Laws, History of; Utility and Subjective Probability: Contemporary Theories

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Functional Explanation: Philosophical Aspects

Functional explanation has received a great deal of philosophical attention in recent years, partly as a consequence of a surge of interest in the philosophy of biology, and partly as a consequence of attempts to make functions and functional explanation central to issues in the philosophy of psychology. In both of these areas there has been some controversy over what the correct account of functional explanation is, with the contending accounts splitting into two main camps. Both camps agree that functional explanation is dependent on an account of functions, but disagree on what the correct account is. Within evolutionary biology the favored view is that functions emerge over time as a consequence of evolutionary processes, so a certain feature of an organism will only have a function if it has a suitable history. The opposed view is that the operation of a system can be analyzed into the way in which parts of the system function to make the system behave in that manner. This version has many names, but is perhaps best known as a 'systems' or 'Cummins' account of functions and functional explanation, but it has also been called 'causal-role' functionalism. In what follows both of these views will be examined; in addition the controversy over what it is that functional explanations explain will be outlined.

1. *Functions and Systems*

The 'systems' account of functions takes as its starting point the fact of (biological) complexity and proposes to understand the complex functioning of a biological entity (or system) in terms of the working of its parts. The fact of complexity suggests that the systems are goal directed, and if this is taken for granted one can recover what is distinctive of many explanations in biology, that they invariably cite the effects of causes as explaining why the cause is there. It does this by specifying how a part of the system operates in the production of the aimed-at goal.

The best known account of how such an explanation operates is given by Cummins (1975, 1983). He distinguishes explanation of changes, which is done by 'transition' theories, from explanation of properties. Transition theories are suitable for explaining 'What caused S to acquire P?' whereas a property theory explains what it is for S to instantiate P. This is best done by constructing an analysis of S, such an analysis adverting to the properties of S's components and their organization. So the kinetic theory of gases explains what it is for a gas to have temperature via an account of the properties of the molecules contained within the gas. Cummins calls the analysis of such a system compositional analysis: the analysis of a dispositional property being called functional analysis. The general claim is that in science the ascription of a function to something is to specify a capacity, such a capacity being identified by its role in enabling a system to operate in a certain manner. Thus, in biology the capacities of an organism are explained by analyzing them into a number of systems (circulatory, digestive, etc.), such systems having a particular task to perform. How these tasks get performed is in turn explained by functional analysis, such an analysis citing the capacities of parts of these systems. Such capacities are the functions of the parts, so we get the Cummins-style definition of function:

> *X* functions as an *F* in *s* (or: the function of *X* in *s* is to *F*) relative to an analytical account *A* of *s*'s capacity to *G* just in case *X* is capable of *F*-ing in *s* and *A* appropriately and adequately accounts for *s*'s capacity to *G* by, in part, appealing to the capacity of *X* to *F* in *s* (Cummins 1975, p. 762).

This account clearly does not restrict the attribution of functions to biological organisms; one can recognize the attribution of this type of function to parts of societies within sociological and anthropological functionalism. Here the system was seen as a society or culture, and the parts were the subsocietal structures or the recurrent patterns of behavior, the function of which was to maintain the structural continuity of the whole. The fact that this account of functionality is applicable to any system has given rise to two criticisms of this approach, one seeing a problem in how to specify nonarbitrarily which behavior or capacity of the containing system is relevant for the

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