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Empirical Comparisons of Bilinear and Nonbilinear Utility Theories

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Several bilinear and nonbilinear utility theories are evaluated using individual data from 144 informants. The nonbilinear models are best for 67% of the informants. The nonrational property of duplex decomposition linking joint receipts and mixed gambles of gains and losses is more adequate than the rational link of general segregation for 73% of the informants. The correlations are very high and linear, even in the worst fitting case. The weighting functions are mostly inverse-S-shaped and for 84% of the informants are fit best by the class of functions proposed by D. Prelec. These fits were generally excellent, with a minimum correlation of .81, a maximum of .99, and an average of .97.

INTRODUCTION

Since Von Neumann and Morgenstern (1947) published their classic treatment of expected utility theory, many mathematical variants on it have been proposed. Characteristic of all but the most recent axiomatically based theories is that the utility of a gamble is a bilinear expression in terms of utilities of consequences and weights of events. This means that with the chance events held fixed, the utility of the consequences enter the expression linearly; and

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with the consequences held fixed, the weights of the events enter the expression linearly. These are what Miyamoto (1992) called "generic utility models."

To be specific, suppose we have a gamble in which a chance "experiment"[1] is carried out and with probability p the holder of the gamble receives a consequence, x (perhaps an amount of money, in which case the gamble is called a lottery), and with probability $1 - p$ receives a different consequence, y.[2] We denote such a gamble by $(x, p; y, 1 - p)$. Assume that the decision maker has preferences among such gambles and that this preference order $\succsim$ can be reflected numerically by a utility function, U over the lotteries, i.e.:

$$(x, p; y, 1 - p) \succsim (x', p'; y', 1 - p') \Leftrightarrow U(x, p; y, 1 - p) \geq U(x', p'; y', 1 - p').$$

The assumption of generic utility is that there are functions u_1, u_2 over consequences and w_1, w_2 over probabilities where each function w_i is strictly increasing, $w_i(0) = 0$, $w_i(1) = 1$, and

$$U(x, p; y, 1 - p) = u_1(x)w_1(p) + u_2(y)\, w_2\, (1 - p). \tag{2}$$

These models are linear in the utilities of consequences, $u_1(x)$ and $u_2(y)$, when p is held fixed. Similarly, with x and y fixed, the utility contributions of p and $1 - p$, $w_1(p)$ and $w_2(1 - p)$, each appears linearly (although they are clearly correlated). So the expression is said to be bilinear.

Until very recently, either no distinction was made between gains and losses or when the distinction was made bilinearity continued to be assumed, e.g., Kahneman and Tversky (1979), Luce (1991), Luce and Fishburn (1991), Luce and Fishburn (1995), and Tversky and Kahneman (1992). However, practical experience has suggested that there are difficulties with this form, especially in the case of mixed gains and losses. Because the case of mixed gains and losses is without question the most important for most decision making, this area needs special attention. Two separate lines of research into this area began in 1997.

Experiments of Chechile and Cooke (1997) and Chechile and Butler (2000) have demonstrated dramatically that choices between binary lotteries of a gain and a loss almost certainly cannot be described by a bilinear model of the form of Eq. (2). These experiments were based on the following observation. Reference gambles were of the form $g(r) = (z, r; -z, 1 - r)$ and comparison ones of the form $c(p) = (v_1, p; -v_2, 1 - p)$, where z, v_1, and v_2 are all positive. The experimenter fixed z and varied r, v_1, and v_2; for each such triple, the

[1] The term "experiment" is meant in the sense used by statisticians, not that of experimental science. In the laboratory a typical chance experiment is spinning a pointer over a partitioned circle or selecting a colored ball from an opaque urn of known numbers of different-colored balls.

[2] Although we state everything here in terms of known probabilities, which is relevant to the data we have, the theory in fact can be stated for abstract chance events for which probabilities need not be known. For a general survey of this research, see Luce (2000a).

respondents selected p so that $c(p) \sim g(r)$. From this and generic utility, they showed that the following regression equation had to hold:

$$\psi(r) = \lambda X - Y,$$

where, with z held fixed, $\psi(r)$ is a function only of r, X a function of v_1 and p, and Y a function of v_2 and $1 - p$. The important observation was that the slope, λ, is independent of all the variables of the experiment. Using a variety of utility and weighting functions, they fit the regression and asked if, indeed, the slope is independent of the choice of r. They concluded that this is grossly violated.

There was a serious design flaw in the first experiment, which various people recognized. It was that the choice of experimental values sometimes made it impossible for the informant to find a p for which the indifference could be achieved. Chechile and Luce (1999) described this error and reanalyzed the data after eliminating all clearly suspect responses. The conclusion that the group analysis is inconsistent with the bilinear form remained unchanged although the slope estimates changed some. The Chechile–Butler study eliminated the flaw. They also analyzed individual as well as group data and found a striking difference between the two analyses. The slopes of the crucial regression, which in any bilinear model must be positive, turned out to be negative for individuals although positive for the group data. This finding seems a bit perplexing. It seems to violate some aspects of consequence monotonicity. The current class of models all have positive slopes and so, at best, can only account for the group data, not the negative slopes of the individual subjects.

Luce (1997), building on the work of Luce and Fishburn (1991, 1995), showed how utility-bilinear or nonbilinear representations may arise naturally. The differences in the models reflect primarily potential individual differences among respondents of three major types. The first difference is pretty standard, namely whether the utility function over gains and losses, separately, is concave, convex, or linear in some moneylike measure of the consequences. This can be interpreted as only differences in the signs of the parameter values δ and δ' (see below). The other two differences are far less familiar and represent truly qualitative differences among respondents. One concerns how people aggregate the simultaneous receipt of a gain and a loss, and the other concerns how they may decompose gambles of mixed gains and losses into simpler components. All of this is described below.

The present article is based on estimated choice certainty equivalents of gambles from 144 respondents, which were collected for other reasons by Cho, Luce, and Winterfeldt (1994). We ask which of the several models, including both bilinear and nonbilinear in the case of mixed gains and losses, best fits each respondent. Bilinear and nonbilinear are the basic model types. Within these two basic forms are several variations. Essentially, these variations arise from combining the basic form of a gamble's utility, bilinear or nonbilinear, with the definition of the utility for a single consequence, i.e., exponential concave, exponential convex, or a power function. Thus, there are several possible utility forms arising from different combinations, e.g., bilinear with an exponential concave utility for positive consequences and an exponential convex utility for negative consequences. All possible combinations are examined in this study.

To the extent that the nonbilinear models are favored over the bilinear ones, the results accord with the work of Chechile and Cooke (1997) and the analysis of the average data, but not the individual analyses of individual data, of Chechile and Butler (2000). Moreover, the fits will establish the degree to which there appear to be individual differences both within a type of model (bilinear or nonbilinear) or between these types of models. A total of 20 different combinations are examined for each respondent. This number arises from four model types being coupled with five different forms for the utility function.

We first describe the main ideas involved indicating how certain families of representations arise (many specialized details are relegated to the Appendices), then we outline how we went about comparing these models, and finally we summarize what these comparisons tell us about the models, including the general form of the estimated weighting functions. In a nutshell, we obtain results consistent with the Chechile-Cooke and Chechile-Butler studies in that a substantial fraction of respondents are best fit by nonbilinear models. These models do not, however, predict the negative slopes of the latter experiment. In addition, we obtain far more refined evidence about which behavioral assumptions seem to lead to reasonably satisfactory utility representations.

UTILITY OF GAINS (LOSSES)

As noted, a good deal of classical utility theory concerns lotteries such as the binary one $(x, p; y, 1 - p)$, which, with no loss of information, can be abbreviated to $(x, p; y)$. The classical primitives were the gambles and a preference order $\succsim$ over them. Thus, if g and h are gambles, $g \succsim h$ means that g is preferred or indifferent to h. The major innovation of Luce (1991) and Luce and Fishburn (1991, 1995) was to add two additional primitives. One is a consequence, e that entails no change from the *status quo*.[3,4] This permits one to distinguish readily gains from losses. The other concept is *joint receipt* to represent the simultaneous receipt of valued things: gambles, pure consequences, and mixed gambles and consequences. Examples of joint receipts are receiving checks and bills in the mail, receiving birthday presents, most shopping, and so on. Mathematically, joint receipt is a binary operation, $\oplus$. If x and y denote two valued things, then their joint receipt, which is denoted

[3] In a slight abuse of terminology, this is abbreviated to calling e the status quo. Although it is tempting to use 0 for e, that can be misleading except when the consequences are money. Once we come to the experimental analysis itself, we will shift from e to 0 because that is what we assumed to be no change from the status quo.

[4] One must either distinguish by context whether e refers to the status quo, which is a standard algebraic notation for an identity element, or the exponental constant, which is standard in analysis or, as we do here, write the latter as exp.

$x \oplus y$, is also valued. The concept is more general in that if g and h are gambles, $g \oplus h$ denotes the receipt of both. The domain of gambles and the preference order $\gtrsim$ are extended to the closure of gambles under $\oplus$. Let $\sim$ denote indifference in the sense that $g \sim h$ is equivalent to $g \gtrsim h$ and $h \gtrsim g$.

Let f, g, and h be typical gambles or pure consequences. We assume that the status quo e serves as a weak[5] identity for $\oplus$, i.e.:

$$g \oplus e \sim g, \tag{3}$$

that $\oplus$ satisfies weak commutativity in the sense

$$g \oplus h \sim h \oplus g \tag{4}$$

and that $\oplus$ satisfies weak monotonicity in the sense

$$g \gtrsim h \Leftrightarrow f \oplus g \gtrsim f \oplus h. \tag{5}$$

What weak commutativity means depends upon the application and what interpretation is given to the order of writing the symbols. In most applications, the order is immaterial, and so weak commutativity is implicitly assumed. Weak monotonicity seems sensible so long as neither of the pairs (f, g) and (f, h) exhibits any complementarity.

Their key assumption was that over gains alone (or over losses alone) $\oplus$ and the gambles are linked by a property, which Kahneman and Tversky (1979) invoked as part of their preediting phase of evaluating gambles, but did not formalize, called *segregation*: For all probabilities p and consequences $g, h \gtrsim e$:

$$(g, p; e) \oplus h \sim (g \oplus h, p; h). \tag{6}$$

This property is totally rational in the sense that one receives exactly the same thing on the two sides: g and h with probability p and h alone with probability $1 - p$. It is simply a question of different groupings; either the joint receipt is within the gamble or external to it. Put another way, suppose we introduce a "subtraction" operation corresponding to $\oplus$, namely if $f \gtrsim g \gtrsim e$, then

$$f \ominus g \sim h \Leftrightarrow f \sim g \oplus h. \tag{7}$$

We assume that the element h exists. Then an equivalent form to Eq. (6) is for $f \gtrsim g \gtrsim e$

$$(f, p; g) \sim (f \ominus g, p; e) \oplus g. \tag{8}$$

That is, we "subtract" the smaller gain as an assured amount and the balance remains a gamble with the second consequence the status quo.

Luce and Fishburn coupled Eq. (6) with the assumption that the following

[5] The adjective "weak" means that the condition hold for $\sim$ not just $=$.

widely studied special case of generic utility holds.[6] *Binary rank-dependent utility* holds over gains if there is a utility function U over gambles and consequences[7] and weighting function W over events such that

$$U(g,p;h) = U(g)W(p) + U(h)[1 - W(p)], \; g \gtrsim h \gtrsim e. \tag{9}$$

In terms of the generic theory, $u_1 = u_2 = U$ and $w_1(p) = W(p)$ and $w_2(1 - p) = 1 - W(p)$. Note that the rank dependence disappears if, for all p, $W(p) + W(1 - p) = 1$. *(Binary) subjective expected utility* entails the stronger constraint that, for all probabilities $p, q, p + q \leq 1$, $W(p) + W(q) = W(p + q)$, from which one can conclude $W(p) = p$.

Models of this rank-dependent type were first arrived at in utility theory by Quiggin (1982), who summarized the literature of the subsequent 10 years in Quiggin (1993). Luce (2000a) also provides a summary of not only these models but those addressed in this article.

Under some structural assumptions that insure many consequences and probabilities, Luce and Fishburn (1991, 1995) show from Eqs. (3)–(6) and (9) that for all $g, h \gtrsim e$

$$U(g \oplus h) = U(g) + U(h) - \delta U(g)U(h), \tag{10}$$

where δ is a finite constant. This form is called p-*additive* because we may rewrite it as

$$1 - \delta U(g \oplus h) = [1 - \delta U(g)][1 - \delta U(h)],$$

and so

$$\kappa V(f) = sgn(-\delta) \log[1 - \delta U(f)], \tag{11}$$

where $sgn(\delta)$ is the sign of δ, is an additive representation of $\oplus$ (for further discussion, see Appendix Section "Relation of U and Y"). We call V a *value function* to distinguish it from the utility function U. The value function represents a fundamental numerical representation of a consequence's worth. This representation arises from the structual assumptions and axioms of the utility theory. The value function V is proportional to the utility function U if and only if $\delta = 0$. For $\delta > 0$ we speak of the dependence of U on V as being negative exponential and for $\delta < 0$ as exponential.

The fact that V is additive means that $\oplus$ is like $+$ in being weakly associative:

$$(f \oplus g) \oplus h \sim f \oplus (g \oplus h). \tag{12}$$

This property asserts that how triples of values goods are combined into pairs

[6] The Chechile–Cooke and Chechile–Butler studies reject the bilinear model only for mixed gains and losses, not for gains alone or losses alone.

[7] This abuse of notation is justified by the standard assumption that $(g, 1; h) \sim g$.

does not matter. The existence of this additive representation plays an important role below.

Quite different behavior is captured depending upon whether $\delta >, =, < 0$. Observe that for $\delta = 0$, it simply says that U is additive over $\oplus$, whereas for $\delta > 0$ it is subadditive and for $\delta < 0$ it is superadditive. This freedom is a major source of individual differences in the models.

A completely parallel development holds for gambles whose consequences are perceived as losses with $\succsim$ replaced by $\precsim$ and so the losses are ranked from the greatest lost to the least. This results in a distinct parameter δ'.

Thus, in terms of these two parameters, there are nine distinct possible types of people. We will look only at five of these: $\delta\delta' > 0$, $\delta\delta' < 0$, and $\delta = \delta'$ 0 (see below and Appendix section "Relation of U and V"). The mathematical details of $\delta = 0$, $\delta' \neq 0$ and of $\delta \neq 0$, $\delta' = 0$ have not been worked out.

ADDITIVE UTILITY OF MIXED GAINS AND LOSSES

It is not automatic from the above results what happens in the case of binary gambles with a gain and a loss, i.e., $(f_+, p; g_-)$ and $f_+ \oplus g_-$, where $f_+ \succsim e \succsim g_-$,. This section describes one assumption about $U(f_+ \oplus g_-)$ and the next section describes another.

Luce and Fishburn (1991), with little motive beyond convenience, assumed

$$U(f_+ \oplus g_-) = U(f_+) + U(g_-). \tag{13}$$

For technical reasons within the theory of measurement, such a model can be described as *extensive-conjoint* and abbreviated EC (see Appendix section "Extensive-Conjoint Terminology"). The term summarizes the fact that the overall joint-receipt structure is extensive for gains and losses separately and additive conjoint for the mixed case.

So given that Eq. (13) holds, the next question is how to extend that to expressions for $U(f_+, p; g_-)$. Two possibilities have been explored. One is *generalized segregation*, which extends the subtraction version of segregation, Eq. (8), to mixed consequences in a fairly natural way. It rests upon distinguishing between $(f_+, p; g_-) \succ e$ or $\prec e$, namely for $f \succsim g$,

$$(f, p; g) \sim \begin{cases} (f \ominus g, p; e) \oplus g, & \text{if } (f, p; g) \succsim e \\ (e, p; g \ominus f) \oplus f, & \text{if } (f, p; g) \prec e \end{cases} \tag{14}$$

Note that for $g \succsim e$, this reduces to segregation in the form given in Eq. (8). The psychological intuition of general segregation is that if a gamble is perceived as a gain, i.e., valued at least as much as the status quo e, then it is indifferent to the joint receipt of the smaller value, along with a gamble each of whose consequences is reduced by that value. The reduced gamble is also perceived as a gain because $f \ominus g \succsim e$. If, however, $(f, p; g)$ is seen as a loss, it is equivalent to the joint receipt of the larger value along with the reduced gamble.

Luce (1997) showed that general segregation and U additive over $\oplus$ imply a bilinear form for U of mixed gambles which is much like Eq. (9) and which is stated explicitly in Appendix section "Extensive-Conjoint (Additive) General Segregation"). All of the forms from the different models, some of which are fairly complicated, are summarized in Appendix B.

A second possible linking operation, which Luce and Fishburn (1991, 1995) studied before general segregation was proposed, is *duplex decomposition*:

$$(f_+, p; g_-) \sim (f_+, p; e) \oplus (e_+, p; g_-), \tag{15}$$

where the two gambles on the right side are to be run independently. This property was first discovered empirically by Slovic and Lichtenstein (1968), and Cho, Luce, and Winterfeldt (1994), using estimated certainty equivalents, found it was again supported in group data. Duplex decomposition describes a person who considers a gamble of one gain and one loss to be the equivalent of holding two independent gambles, each concerned with just one of the consequences. Descriptively, this means that a person evaluates a mixed gamble by partitioning it into the gains subgamble and the independent loss subgamble. Note that such a partitioning is not rational. The gamble on the left-hand side yields just one of two possible outcomes, either f_+ or g_-, whereas the pair of gambles on the right-hand side yields one of four possible pairs of outcomes, obtained by crossing $\{f_+, e\}$ with $\{e, g_-\}$.

Duplex decomposition combined with additive U again yields a bilinear form (Appendix section "Extensive-Conjoint (Additive U) Duplex Decomposition"), but one that differs significantly from Eq. (9). Specifically, in terms of the generic model one has $w_1 = W_+$ and $w_2 = W_-$. The form shown in Appendix section "Extensive-Conjoint (Additive U) Duplex Decomposition" is the same as the representation postulated by Kahneman and Tversky (1979) provided one assumes $W_- = W_+$. Later Tversky and Kahneman (1992) invoked our form. In the present article, no special assumptions are made about how the weights are related. In particular, none of the following are assumed to hold: $W_- = W_+$, $W_- (1 - p) = 1 - W_-(p)$, or $W_-(1 - p) = 1 - W_+(p)$.

These models based on additive U are labeled ECGS and ECDD for "extensive-conjoint with general segregation" and "extensive-conjoint with duplex decomposition," respectively. In all cases, they are bilinear models.

ADDITIVE VALUE FOR MIXED GAINS AND LOSSES

Associativity

As was outlined above, any utility function satisfying the p-additive form, Eq. (10), can be transformed into representation V, called a *value function*, that is additive over $\oplus$. This holds for gains and losses separately. Luce (1997) suggested that underlying this is an additive representation V that holds for all joint receipts, whether gains, losses, or mixed gains and losses. This of course means that joint receipt satisfies in addition to weak commutativity, Eq. (4), weak monotonicity, Eq. (5), and the important property of weak associativity, Eq. (12), throughout the entire structure. The additive V cases are called weak *associative* models, abbreviated "A." An empirical study of weak associativity for mixed gains and losses has been carried out by Fisher (1999). Some, but by no means all, respondents exhibited it for mixed gambles.

It turns out that having V additive throughout the structure results in a major mathematical simplicity: Knowing the forms of U for gains and losses separately determines the form of $U(f_+ \oplus g_-)$. What is notable about this is that, for $\delta\delta' \neq 0$, the resulting forms are decidedly nonadditive. That fact when coupled with either general segregation and $\delta\delta' < 0$ or duplex decomposition with $\delta\delta' \neq 0$ results in expressions for $U(f_+, p; g_-)$ that are also decidedly nonbilinear. For example, suppose duplex decomposition holds and $\delta > 0 > \delta'$, then if the gamble $(f_+, p; g_-)$ is perceived as a gain

$$\delta U(f_+, p; g_-) = \frac{\delta U(f_+)W_+(p) + |\delta'|\, U(g_-)W_-(1-p)}{1 + |\delta'|U(g_-)W_-(1-p)}.$$

We see that the numerator is a standard bilinear form, but the denominator makes the overall expression nonbilinear. Because $U(g_-) < 0$, the effect of the denominator is to increase the utility of the mixed gamble above what one would predict from the bilinear expression of the numerator. The full set of mathematical forms obtained by assuming V is additive are shown explicitly in Appendix sections "Associative (Additive V) General Segregation" and "Associative (Additive V) Duplex Decomposition".

Of course, given that the empirical literature appears to reject the bilinear models for mixed gambles, this seems a happy innovation.

The abbreviations AGS and ADD are used for the combination of the associative model with general segregation and duplex decomposition, respectively. So, we may summarize these abbreviations as in the following table:

	Additivity over $\oplus$ in mixed case	
Gamble decomposition	U (Extensive-Conjoint)	V (Associative)
General segregation	ECGS	AGS
Duplex decomposition	ECDD	ADD

Value of Money

A question not yet addressed is how the additive function $V(x)$ relates to x when x is an amount of money. One can show that if joint receipt of money exhibits invariance in multiplicative factors, i.e., if x, y are either both gains or both losses,

$$rx \oplus ry \sim r(x \oplus y),\ r > 0, \tag{16}$$

then, for positive constants α, α', β, and β'

$$V(x) = \begin{cases} \alpha x^{\beta}, & x \geq 0 \\ -\alpha'(-x)^{\beta'}, & x < 0. \end{cases} \tag{17}$$

Given the available data (see below), we concluded that it would add too many additional parameters to use the full power expression for the exponential cases, so for those we assumed V to be simply proportional to money, i.e., $\beta = \beta' = 1$. For the cases where U is proportional to V, i.e., $\delta = \delta' = 0$, we did use the power function form. This places these models at some advantage, but even so, as we will see, they turn out not to be the best fitting models for most respondents.

Because $\kappa\alpha$ and $\kappa'\alpha'$ which appear in the exponential cases, are not identifiable, we simply call the products κ and κ' for simplicity.

HOW THE FITTING WAS CARRIED OUT

So, there are four basic utility representations to be compared, ADD, AGS, ECDD, and ECGS. These four choices correspond to possible qualitative (nonparametric) individual differences in behavior. The most rational is AGS, with all of the others to some degree less rational. In addition, the specific functional forms of these models depend also on the signs of δ and δ', which again reflect a source of individual differences. As noted earlier, we opted not to pair $\delta = 0$ with $\delta' \neq 0$ and vice-versa. So, in total, there are $4 \times (4 + 1) = 20$ model types.

The only data to which we had access that were adequate for model testing—both in terms of redundancy in probabilities and consequences and in numbers of respondents, 144—were the estimated certainty equivalents of gambles obtained by Cho et al. (1994) to test directly the hypotheses of segregation for gains and losses separately and duplex decomposition for mixed alternatives. The experimental details are summarized in Appendix C. To evaluate the worth of a gamble to a respondent, they estimated the certainty equivalent (CE) of the gamble, which is defined to be the amount of money for which the person felt indifferent to the gamble when given a choice between them. The technique used, called PEST, has been described in several publications, i.e., Cho and Luce (1995); Cho et al. (1994); and Winterfeldt, Chung, Luce, and Cho (1997).

The parameters to be estimated for each model are the several unknown parameters of the utility function plus weights for each of the probabilities used. We did not assume any mathematical form for the weights, but simply estimated all of them. Below, however, we do compare for individual informants the estimated weights of the best fitting model with several mathematical forms for the weights.

Thus, the goodness-of-fit of a particular theory can be evaluated by comparing the utility of the CE of a mixed gamble to the utility found by the utility equation for the particular form, e.g., duplex decomposition and concave-convex.

TABLE 1

Fits of Models to the Data

Gain:	C	V	C	V		
Loss:	V	C	C	V	Power	Total
ADD	40.5	23	17.5	6	8.25	95.25
AGS	6	4	3	4	7.25	25.25
ECDD	2	1	1	1	4.25	9.25
ECGS	3	4	1	1	6.25	15.25
Total	51.5	32	22.5	12	26	144

Note. Each entry is the number of respondents for whom that model was the best fitting. Fractional entries mean that the fits of two or more models were not distinguishable, and so the respondent was spread equally among them.

The comparisons are done by minimizing the following sum of squares measure corrected for degrees of freedom

$$\mathrm{SS} = \frac{1}{N_{\text{gambles}} - N_{\text{weights}} - N_{\text{parameters}}} \sum_{i=1}^{N_{\text{gambles}}} (\mathbf{CE}_i - \mathrm{CE}_i)^2 \tag{18}$$

for each model.[8] Here, N_{gambles} denotes the number of CEs evaluated by an informant, N_{weights} denotes the number of weights used to compute the estimated certainty equivalents ($\mathbf{CE}_i$) from the given utility equation, and $N_{\text{parameters}}$ denotes the number of parameters used. This number is equal to two for associative theories (κ and κ') and three (κ, κ', and δ/δ') for extensive-conjoint theories.[9]

The parameter values which minimized Eq. (18) were computed using the conjugate gradient method. The basic method is detailed in Powell (1964) with useful modifications described in Brent (1973).

RESULTS OF THE MODEL FITTING

The Best Model for Each Informant

We fitted each of the 20 models, 16 exponential and 4 power functions, to each respondent's data and used the above measure to pick which one fit the best. Once done, we can distinguish whether the power functions in the proportional case of $\delta = 0$ were concave ($\beta > 1$) or convex ($\beta < 1$). The number of respondents fit best by each of the models are shown in Tables 1 and 2. Here, "C" means concave utility ($\delta > 0$ and $\delta' > 0$) and "V" means convex utility ($\delta < 0$ and $\delta' < 0$). Fractional entries arise when two models fit equally well.

[8] The program for doing this fit was written by Kiho Jeon, whom we thank.

[9] Inverting the utility equations to compute the estimated CEs cancels some parameters. In particular, the values for δ and δ' divide out in the associative theories and reduce to a ratio in the extensive-conjoint theories.

TABLE 2

Partition of the Power Function Fits According to Estimated Exponents

Gain:	C	V	C	V
Loss:	V	C	C	V
ADD	1.25	0.75	0.5	4.75
AGS	0.25	4.75	1.5	1.75
ECDD	0.25	0.75	2.5	2.75
ECGS	1.25	0.75	0.5	1.75
Total	3	7	5	11

Note. Concave corresponds to the exponent <1 and convex >1.

In Table 2, the power functions are broken up by concavity and convexity according to their estimated exponents.

Certain patterns are readily summarized.

1. The associative models do better than the extensive-conjoint ones for about 84% of the respondents.

2. The nonbilinear models (AGS with C/V or V/C and ADD with any combination of C and V) do better than the bilinear ones for about 67% of the respondents. In all likelihood, this is an underestimate because the power function models have extra degrees of freedom as compared with the exponential ones.

3. Duplex decomposition, which is a nonrational assumption, is favored in 73% of the cases over the rational general segregation.

4. The modal shape of the utility functions is concave for gains and convex for losses, but at 38% it is substantially less than a majority. This is consistent with other estimates of individual informants in the literature (see section 3.3, in Luce, 2000a).

The finding of 67% nonbilinear models seems to be in reasonable accord with the group results of Chechile and Cooke (1997) and the individual ones of Chechile and Butler (2000), who found clear rejection of the bilinear models for 9 of 12 of their respondents (75%).[10] We do not, however, explain the negative slopes of the latter study.

The Quality of the Fit

For each of the four main model types, Table 3 shows the pattern of correlations corrected for numbers of parameters. The totals in each column do not all add to 144 because in a few cases the fitting program failed to find a solution for that model. We see that, for the most part, the correlations are quite high with the distribution being best for ADD and worst for ECGS.

This finding illustrates one of the dangers of trying to select among behavioral properties on the basis of overall fits of models to data. All of the models do a

[10] The three respondents that we are classifying as not necessarily rejecting the bilinear form had estimated constant utility functions. It is not clear exactly what this means.

TABLE 3

Distribution of Correlations between Predicted and Observed CEs for Each of the Model Types

Model type correlation	ADD	AGS	ECDD	ECGS
.95–1	128	124	124	110
.90–.94	9	14	10	23
.85–.89	5	3	5	4
<.85	0	3	4	5
Total *N*	142	144	143	142

tolerable job in gross fitting, even though properly designed experiments, such as Chechile and Butler (2000), may rule out broad classes of them.

Figures 1 and 2 show scatter diagrams for the best and the worst of the best fitting models. One can see that the best fitting informant's data lies on an almost exactly straight line. The worst fitting informant's CEs are also fairly linear with two exceptions. Linear regressions of the theoretical values had an average correlation of .88.

Although most of the correlations are quite high, the predictions could of course be nonlinear. To test for that we did quadratic regressions and found that the standard error of the quadratic regressions were on average 5.6% smaller than the linear fits. However, these standard errors do not account for the extra parameter used by quadratic regression. When the standard error is computed by replacing the number of data points, n, by $n - k$, where k is the number of parameters used in fitting (two for a linear fit, three for a quadratic), the linear regressions proved superior. On average, the standard errors of the linear regression were 4.7% smaller than the quadratic standard errors.

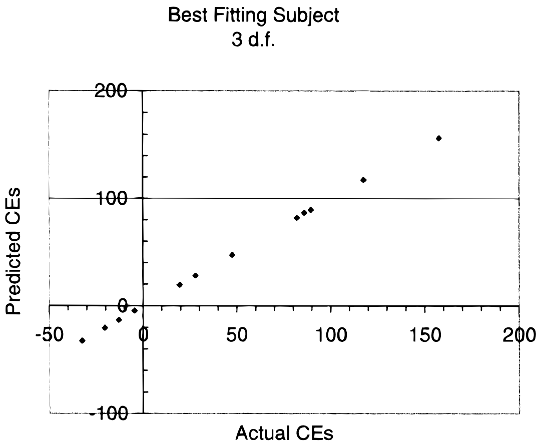

FIG. 1. Scattergram of the best fitting subject.

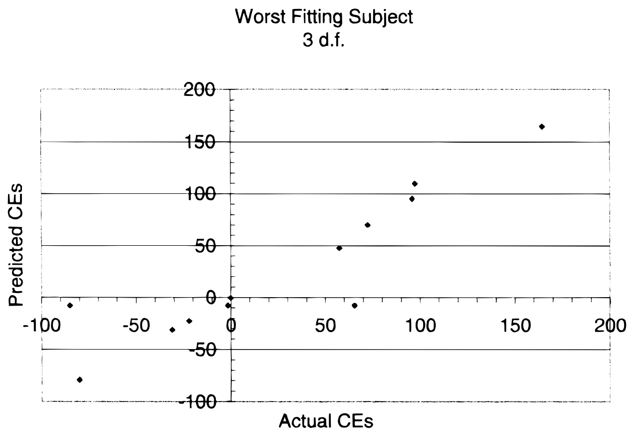

FIG. 2. Scattergram of the worst fitting subject.

The Form of the Weighting Function

We have estimated the weighting functions for each of the best fitting models for each respondent. These can be compared, at least crudely, with some of the suggestions that have appeared in the literature. We list them. Perhaps the earliest is the power function,

$$W(p) = p^{\lambda},\ \lambda > 0. \tag{19}$$

This functional form can be derived from the assumption of separability,

$$U(x, p; 0) = U(x)W(p), \tag{20}$$

with W onto [0,1], which is satisfied by the rank-dependent utility models, and the simplest probabilistic reduction of compound gambles, namely

$$((x, p; 0), q; 0) \sim (x, pq; 0). \tag{21}$$

Observe that $(x, pq; 0)$ is a binary gamble with x the consequence with probability pq and 0 the consequence with probability $1 - pq$, whereas $((x, p; 0), q; 0)$ is a compound gamble with $(x, p; 0)$ occurring with probability q and outcome 0 occurring with probability $1 - q$. As random variables, these two gambles are equivalent. However, Kahneman and Tversky (1979) tested this indifference and found, in general, that informants did not perceive them as indifferent.

Karmarkar (1978) suggested that the weighting function is of the following form which is symmetric about 1/2,

$$W(p) = \frac{p^{\lambda}}{p^{\lambda} + (1 - p)^{\lambda}},\ \lambda > 0. \tag{22}$$

This form was not derived from simpler assumptions, and it does not include the power functions as a special case although, of course, it does include the most rational of weights, $W(p) = p$, as the special case $\lambda = 1$. It seems to us undesirable to propose descriptive theories that do not include as a special case the, albeit unlikely, person who adheres to Eq. (21), which with separability implies the power function of Eq. (19). The former seems appropriate for anyone who understands probability at all.

Tversky and Kahneman (1992) fit their bilinear utility model, which is the same as our ECDD one, to data assuming U was a power function and found a nonsymmetric inverse S shape with the point p_0 where $W(p_0) = p_0$ at about .35. To accommodate that, they proposed modifying Eq. (22) as

$$W(p) = \frac{p^\lambda}{[p^\lambda + (1-p)^\lambda]^{1/\lambda}}, \lambda > 0. \tag{23}$$

Although this seemed to fit their data, it too was ad hoc and does not include the power function as a special case, but again $\lambda = 1$ yields $W(p) = p$.

Goldstein and Einhorn (1987); Gonzalez and Wu (1999); and Lattimore, Baker, and Witte (1992)[11] suggested an alternative modification of Eq. (22), namely

$$W(p) = \frac{\eta p^\lambda}{\eta p^\lambda + (1-p)^\lambda}, \lambda > 0, \eta > 0, \tag{24}$$

which is also ad hoc, fails to include the power function as a special case, and does include $W(p) = p$ ($\lambda = \eta = 1$). It can be put in a rather nice odds form

$$\Omega[W(p)] = \frac{W(p)}{1 - W(p)} = \eta\left(\frac{p}{1-p}\right)^\lambda = \eta\Omega(p)^\lambda. \tag{25}$$

Gonzalez and Wu (1999) fit this function with considerable success to data from 10 informants.

Prelec (1998) derived from a somewhat complex behavioral axiom the following form

$$W(p) = \exp(-\lambda[-\ln(-p)]^\alpha), \alpha > 0, \lambda > 0. \tag{26}$$

With appropriately chosen parameters it can exhibit inverse-S forms, S forms, power functions ($\alpha = 1$), and, of course, $W(p) = p$ ($\alpha = \lambda = 1$). Luce (2000a, 2000b) have provided a simpler derivation from Prelec's axiom and also suggested a simpler behavioral axiom, namely if for all $x > 0$ and for each p, q there exists $r = r(p, q)$ such that

$$((x, p; 0), q; 0) \sim (x, r; 0), \tag{27}$$

then for $N = 2$ and $3, \ldots$

[11] We thank a referee for pointing out the first and third of these references.

TABLE 4

A Comparison of Four Proposed Weighting Functions

Functional form	Power	Kahneman–Tversky	Karmarkar	Prelec	Total
Number of informants	32	13	11	88	144
Percentage	22	9	8	61	100

$$((x, p^N; 0), q^N; 0) \sim (x, r^N; 0). \tag{28}$$

[Note that if Eq. (21) holds, then this condition is satisfied.] Under separability, Eq. (20), this property can be shown to be equivalent to Eq. (26). Empirical testing of the underlying axiom described by Eqs. (27)–(28) needs to be carried out.

It should be noted that the predictions of Eqs. (24) and (26) are practically indistinguishable for $.01 < p < .99$. This is easily verified by putting both Eqs. (25) and (26) in log-linear form and plotting $\ln 1 - p/p$ vs $\ln(-\ln p)$; they are close to linear. We fit both models to the data and, indeed, they were practically equivalent, and so we report only the Prelec case.

The results are shown in Table 4. Prelec's weighting function has the best fit to weight values for nearly two-thirds of the respondents. Most of them exhibited inverse-S-shaped weights and none S-shaped.[12] Since Prelec's function subsumes the power function[13], its total best fit percentage is 83%. This suggests that Prelec's function is a good candidate for the proper form of the weighting function.

For each of the four weighting function forms, Table 5 shows the pattern of correlations. Similarly, Fig. 3 shows a comparison of the four functional forms. We see that, for the most part, the correlations are quite high, particularly in the case of the power function and Prelec's weighting function. Restricting our

TABLE 5

Distribution of Best Correlations between Predicted and Observed Weights for the Four Weighting Function Forms

	Weighting function form			
Correlation	Power	Kahneman–Tversky	Karmarkar	Prelec
.95–1	23	8	5	81
.90–.94	5	3	3	6
.85–.89	4	0	2	0
<.85	0	2	1	1
Total N	32	13	11	88

[12] Some care is needed in comparing studies of losses because the apparent form of the weighting function depends upon what p controls—the greater or lesser loss. In our work it was the greater loss.

[13] They were distingushed by the power function having one parameter as compared with two for the full Prelec expression.

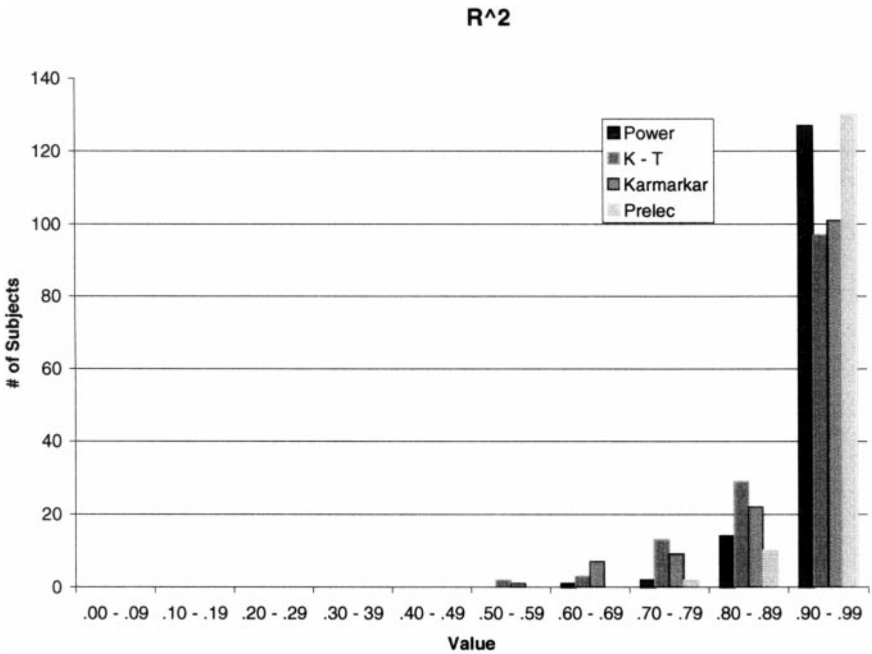

FIG. 3. Distribution of correlations for each weighting function form.

attention to just the power and Prelec function fits to the estimated weights, Fig. 4 shows the distribution of the correlations of the better fitting one. Restricting ourselves to correlations of .8 or better, we see that these models, which are both in a sense Prelec functions, account for 91% of the respondents. The other 9% of the cases are clearly not fit very well by this class of models.

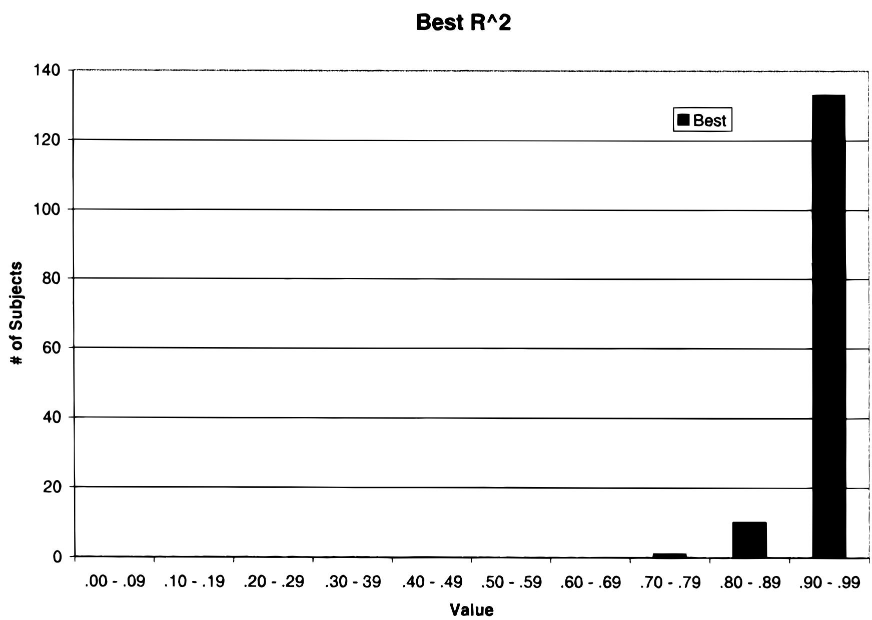

FIG. 4. Distribution of the best correlations of the weighting function forms.

GENERAL DISCUSSION AND CONCLUSIONS

Chechile and Cooke's (1997) and Chechile and Butler's (2000) experiments strongly suggested that the bilinear utility models are inadequate in the case of mixed gains and losses (9 of 12 respondents in the latter study, i.e., 75%). The present data analysis reaches the same conclusion for a substantial majority of respondents (67% of 144 respondents). We also found that the duplex decomposition condition linking mixed gambles to joint receipt was favored over the rational general segregation one.

Although these findings are suggestive, the fact is that all of the models fit these data reasonably well. Confidence in the conclusion will be increased only by exploring carefully the underlying behavioral axioms of ADD and AGS, in particular by evaluating whether joint receipt of mixed alternatives seem to be associative and by examining the descriptive adequacy of duplex decomposition and general segregation. The present data favor the nonrational duplex decomposition over the rational general segregation, but so far no attempt to compare them directly has been reported.[14]

It must be acknowledged that the data set used was an available one, not one designed explicitly to distinguish among these models. Were one to design a more ideal experiment, it would take into account that all models agree on the bilinear rank-dependent form for gains and separately for losses. Sufficient data should be collected in each domain to estimate all of the parameters involved except for δ and δ' which will only be limited to sign. Then all of the data for mixed gambles is predicted without estimating parameters except for the ratio δ/δ'. This would provide a much more stringent test of which model is best for each respondent.

In a substantial majority of cases Prelec's weighting function, Eq. (26), and equally Eq. (24), fits the weight estimates better than do either the rational power function or the other inverse-S functions that have been proposed, Eqs. (23) and (19). Two desirable qualities of the Prelec form are that it subsumes the power function as a special case and that it has a simple behavioral axiomatization. Testing the axiom, reduction invariance given by Eqs. (27) and (28), should be the next step in verifying this functional form.

If the tests of these several axioms prove favorable, one may conclude that a nonbilinear utility model with Prelec weights best describes individual decisions under risk.

APPENDIX A

Specialized Aspects of the Development

Relation of U and V

Given the expression for the value function V in terms of the utility U, Eq. (11), we can of course express U in terms of V, namely

[14] Such a study by Y.-H. Cho and Luce is nearing completion.

$$\delta = 0 \Rightarrow U = \alpha V \; (\alpha > 0) \tag{29}$$

$$\delta > 0 \Rightarrow \delta U(x) = 1 - \exp\,[-\kappa V(x)] \; (\kappa > 0) \tag{30}$$

$$\delta < 0 \Rightarrow |\delta| U(x) = \exp\,[\kappa V(x)] - 1 \; (\kappa > 0). \tag{31}$$

As was noted, the function V is additive in the sense that for either$x, y \succsim e$ or $x, y, \precsim e$,

$$x \succsim y \Leftrightarrow V(x) \geq V(y),$$
$$V(x \oplus y) = V(x) + V(y),$$
$$V(e) = 0.$$

This makes the associativity obvious.

A result in the functional equations literature establishes that for polynomials in two variables with $U(e) = 0$, the only one that is transformable into an additive representation that satisfies Eq. (10). This is the reason U meeting this condition is called *polynomial additive*, abbreviated p-*additive*.

As is evident, except for Eq. (29), V is not the same as the utility function U derived from gambles. These three cases can be called, respectively, proportional, negative exponential, and exponential. If, for sums of money, we assume that V has some simple form such as a linear or a power function, then we know the form of U up to its unit and one or two additional parameters. Although these are very tight constraints on the possible forms for the utility function, the resulting utility models for gambles exhibit considerable freedom for individual differences, which are well known to exist.

For losses, parallel expressions hold, but with different constants, namely α', δ' and κ' instead of α, δ, and κ, respectively.

Extensive-Conjoint Terminology

The term "extensive-conjoint" for the additive U structure arises for the following technical reason. The structure, which is formally much like mass measurement, satisfies the conditions of extensive measurement for gains and losses separately, and we know the behavioral properties that must be satisfied for it to have an additive representation V. And over mixed gains and losses it is an additive conjoint structure (somewhat like $\ln m = \ln V + \ln \rho$, where in the usual physical measures m is mass, V is volume, and ρ is density). Put another way, the operation $\oplus$ is associative for gains (and losses) but is conjoint (and so in general not associative) in the mixed case.

It should be noted that the assumption that $\oplus$ satisfies the axioms of additive conjoint measurement does not, by itself, force using the same utility representations as for gains and losses alone. Luce (1996) formulated a necessary and sufficient condition, which has not been studied empirically, relating the conjoint structure to the extensive one so that the same utility function applies. We do not go into that here.

APPENDIX B

Utility in the Mixed Case

We simplify the notation to $x \geq 0 \geq y$ and introduce the notation for gains

$$\mu U(x) = U_+(x) = x^\beta \; (\delta = 0,\ \beta > 0,\ \mu > 0),$$
$$\delta U(x) = U_+(x) = 1 - \exp\,(-\kappa x^\beta) \; (\delta > 0,\ \beta > 0,\ \kappa > 0),$$
$$|\delta| U(x) = U_+(x) = \exp\,(\kappa x^\beta) - 1 \; (\delta < 0,\ \beta > 0,\ \kappa > 0).$$

and for losses

$$\mu' U(x) = U_-(x) = -\,(-x)^{\beta'} \; (\delta' = 0,\ \beta' > 0,\ \mu' > 0),$$
$$\delta' U(x) = U_-(x) = 1 - \exp\,[\kappa'(-x)^{\beta'}] \; (\delta' > 0,\ \beta' > 0,\ \kappa' > 0),$$
$$|\delta'| U(x) = U_-(x) \exp\,[-\kappa(-x)^{\beta'}] - 1 \; (\delta' < 0,\ \beta' > 0,\ \kappa' > 0).$$

In the following let

$$\lambda = \begin{cases} \dfrac{\mu'}{\mu}, & \delta = \delta' = 0 \\ \left|\dfrac{\delta'}{\delta}\right| & \delta\delta' \neq 0. \end{cases}$$

We write $U_i\,(x, p; y)$ although in reality in fitting the models one uses $U_i\,[\widehat{CE}\,(x, p; y)]$, where $i = +$ for $\widehat{CE}\,(x, p; y) \geq 0$ and $i = -$ for $\widehat{CE}\,(x, p; y) < 0$.

Extensive-Conjoint (Additive) General Segregation

$$U_+(x, p; y) = U_+(x)W_+(p) + \frac{1}{\lambda}\,U_-(y)[1 - W_+(p)],$$
$$U_-(x, p; y) = \lambda U_+(x)\,[1 - W_-(1 - p)] + U_-(y)W_-(1 - p).$$

Extensive-Conjoint (Additive U) Duplex Decomposition

$$U_+(x, p; y) = U_+(x)W_+(p) + \frac{1}{\lambda}\,U_-(g)W_-(1 - p),$$
$$U_-(x, p; y) = \lambda U_+(x)W_+(p) + U_-(g)W_-(1 - p).$$

Associative (Additive V,) General Segregation

U proportional to V, i.e., $\delta = \delta' = 0$:

$$U_+(x, p; y) = U_+(x)W_+(p) + \frac{1}{\lambda} U_-(y)[1 - W_+(p)],$$

$$U_-(x, p; y) = \lambda U_+(x)[1 - W_-(1 - p)] + U_-(y) W_-(1 - p).$$

U exponential with V, gains concave, losses convex (convex \ concave in parenthesis), i.e., $\delta > 0 > \delta'$ ($\delta < 0 < \delta'$):

$$U_+ (x, p; y) = U_+(x)W_+(p) + \frac{U_-(y)}{1 + (-)\, U_-(y)} [1 - W_+(p)],$$

$$U_-(x, p; y) = \frac{U_+(x)}{1 - (+)\, U_+(x)} [1 - W_-(1 - p)] + U_-(y)W_-(1 - p).$$

U exponential with V, both gains and losses concave or both convex, i.e., $\delta\delta' > 0$:

$$U_+(x, p; y) = U_+(x)W_+(p) + U_-(y)[1 - W_+ (p)],$$

$$U_-(x, p; y) = U_+(x)[1 - W_-(1 - p)] + U_-(y)W_-(1 - p).$$

Associative (Additive V) Duplex Decomposition

U proportional to V, i.e., $\delta = \delta' = 0$:

$$U_+(x, p; y) = U_+(x)W_+(p) + \frac{1}{\lambda} U_-(y)W_-(1 - p),$$

$$U_-(x, p; y) = \lambda U_+(x)W_+(p) + U_-(y)W_-(1 - p).$$

U exponential in V, gains concave, losses convex (convex \ concave in parenthesis), i.e., $\delta > 0 > \delta'$ ($\delta < 0 < \delta'$):

$$U_+(x, p; y) = \frac{U_+(x)W_+ (p) + U_-(y)W_-(1 - p)}{1 + (-)U_-(y)W_-(1 - p)},$$

$$U_-(x, p; y) = \frac{U_+(x)W_+ (p) + U_-(y)W_-(1 - p)}{1 - (+)U_+(x)W_+(p)}.$$

U exponential in V, gains concave, losses concave (convex \ concave in parenthesis), i.e., $\delta\delta' > 0$:

$$U_+(x, p; y) = U_+(x)W_+(p) + U_-(y)W_+(1 - p) - U_+(x)W_+(p)\, U_-(y)W_-(1 - p),$$
$$U_-(x, p; y) = U_+(x)W_+(p) + U_-(y)W_-(1 - p) + U_+(x)W_+(p)U_-(y)W_-(1 - p).$$

APPENDIX C

Cho–Luce–von Winterfeldt Experiment

The stimuli used by Cho et al. (1994) experiment were the following eight forms

$$(166, p; 70), (96, p; 0),$$
$$(-166, p; -70), (-96, p; 0),$$
$$(96, p; -40), (96, p; -160),$$
$$(0, p, -40), (0, p; -160),$$

with $p = 0.2, 0.5, 0.9$. Thus, there were a total of 24 experimental gambles. In addition there were some joint receipts presented, which we did not use in our analysis, as well as filler gambles (see below) were also not used in the present analysis. The filler gambles were chosen from a set that multiplied the consequences by 1/3, 1/2, 2, and 3.

For our analysis, all losses must be written in the standard rank-dependent form, which means that the fourth row is

$$(-40, 1 - p; 0), (-160, 1 - p; 0),$$

which adds the probabilities .1, .5, and .8 for losses.

So we see that there were two gains, 96 and 166, and four losses, −40, −70, −160, and −166. For the gains there were three probabilities and for losses there were a total of six.[15]

The experiment involved finding choice certainty equivalents for these 24 gambles plus some joint receipts by an "up–down" method called PEST. The filler gambles were introduced each time the stopping criterion was reached for establishing a certainty equivalent and a filler gamble was introduced so as to maintain a constant expected separation between successive presentations of the same gamble.

Because the PEST procedure is so time consuming, each respondent was confronted with only one-half of the possible gambles. The details of how the partitioning was done are given in Cho et al. (1994). One result of this fact is that the number of weights estimated for losses varied among respondents from three to six depending upon how the partition was carried out.

[15] The computer progam we used was written so that the two 0.5 loss weights were separately estimated.

Fits of models to the data

Each entry is the number of respondents for whom that model was the best fitting. Fractional entries mean that the fits of two or more models were not distinguishable, and so the respondent was spread equally among them.

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