Journal of Mathematical Psychology **45**, 167–179 (2001)
doi:10.1006/jmps.1999.1301, available online at http://www.idealibrary.com on IDEAL®

Reduction Invariance and Prelec's Weighting Functions

R. Duncan Luce

University of California—Irvine

Within the framework of separable utility theory, a condition, called reduction invariance, is shown to be equivalent to the 2-parameter family of weighting functions that Prelec (1998) derived from the condition called compound invariance. Reduction invariance, which is a variant on the reduction of compound gambles, is appreciably simpler and more easily testable than compound invariance, and a simpler proof is provided. Both conditions are generalized leading to more general weighting functions that include, as special cases, the families of functions that Prelec called exponential-power and hyperbolic logarithm and that he derived from two other invariance principles. However, of these various families, only Prelec's compound-invariance family includes, as a special case, the power function, which arises from the simplest probabilistic assumption of reduction of compound gambles.

Key Words: compound invariance; cumulative prospect theory; Prelec's weighting functions; separable utility; rank-dependent utility; reduction invariance.

Let (x, p) denote a gamble in which the consequence x occurs with probability p and nothing otherwise. A consequence can be either a pure one, such as a book or a sum of money, or a gamble, such as (y, q) where y is a pure consequence. In this paper, the only gambles under consideration are of the form (x, p) and $((y, q), p)$, where x and y range over the pure consequences and p and q can be any probabilities. Over these gambles, there is assumed to be a preference relation $\succsim$ that is a weak order (connected and transitive).

A family of such gambles is said to have a *separable representation* if there is a real-valued *utility* function U from the gambles and pure consequences,

This research was supported in part by National Science Foundation Grant SBR-9520107 to the University of California, Irvine. I thank Drazen Prelec, Peter Wakker, and a referee for a number of useful suggestions on earlier drafts.

Address correspondence and reprint requests to R. Duncan Luce, School of Social Sciences, Social Science Plaza, University of California, Irvine, CA 92697-5100. E-mail: rdluce@uci.edu.

0022-2496/01 $35.00

which may be identified with $(x, 1)$, and a strictly increasing *weighting* function[1] W: $[0, 1] \rightarrow [0, 1]$ such that for all x and p,

$$U(x, p) = U(x, 1)\, W(p) = U(x)\, W(p), \tag{1}$$

where $U(x) =: U(x, 1)$ and U preserves the order $\succsim$. When W is onto $[0, 1]$, which I assume in this paper, then it is called an *onto separable representation*. Qualitative conditions for separability are known (Luce, 1996). Of these, consequence monotonicity is very important and somewhat controversial. It simply says in this context that, for $p > 0$, $x \succsim y$ if and only if $(x, p) \succsim (y, p)$.

Of course, such a representation is unique only up to power transformations $U \rightarrow \alpha U^{\beta}$, $\alpha > 0$, $\beta > 0$. However, many theories of more general gambles impose ratio scale uniqueness on U and an absolute scale on W. We assume this degree of ratio scale uniqueness.

Although others have proposed various formulas that were chosen to fit empirical estimates of the weighting function, Prelec (1998) was the first to offer an axiomatic theory[2] for the form of the weighting function under the assumption of onto separability. He derived (his Proposition 1) from a condition called compound invariance (see Definition 1) the following form[3],

$$W(p) = \exp[-\beta(-\ln p)^{\alpha}], \qquad \alpha > 0, \beta > 0, \tag{2}$$

which we may call the *compound-invariance family* or when there is no ambiguity *Prelec's family*.

This approach has two desirable features. First, it includes as a special case $[\alpha = 1]$ the power functions,

$$W(p) = p^{\beta}, \tag{3}$$

where $\beta > 0$. As is well known, this class follows from separability and the simplest probabilistic reduction of compound gambles, namely,

$$((x, p), q) \sim (x, pq). \tag{4}$$

Although the reduction of compound gambles is not descriptive of many subjects (Keller, 1985), it does seem to me undesirable for a descriptive theory not to include the admittedly rare but possible person who does make probability calculations.

[1] Some theories, such as prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992) and rank- and sign-dependent utility (Luce, 1991; Luce & Fishburn, 1991, 1995) distinguish gains from losses, in which case two different, but similar, weighting functions are involved.

[2] He actually derived three families of forms in Propositions 1, 4, and 5 based on three different invariance principles: compound, conditional, and projection. I concentrate initially only on the first, but later show that the other two are special cases of a generalized form of reduction and compound invariance.

[3] For reasons that are not entirely clear, Prelec emphasized the special case of $\beta = 1$.

A second notable feature of Prelec's compound-invariance formula is that for $\alpha < 1$ it exhibits an inverse S-shape[4] which is true for many estimated weighting functions (Abdellaoui, 1999; Birnbaum, Coffey, Mellers, & Weiss, 1992; Gonzalez & Wu, 1999; Kahneman & Tversky, 1979; Karmarkar, 1978, 1979; Preston & Baratta, 1948; Tversky, 1967; Tversky & Kahneman, 1992; Wu & Gonzalez, 1996).

It is worth noting explicitly that of the various extant utility theories, the only property of utility that is needed to derive Eq. (2) is onto separability for the gambles (x, p). In particular, beyond assuming that Eq. (1) is unique up to multiplication by a positive constant, neither Prelec (1998) nor I assume subjective expected utility or any of its closely related generalizations such as cumulative prospect theory (Tversky & Kahneman, 1992), rank-dependent utility (Quiggin, 1993), or rank- and sign-dependent utility (e.g., Luce, 1991; Luce & Fishburn, 1991, 1995).

This paper does three major things:

- It provides a necessary and sufficient condition for Prelec's weighting functions that is more easily tested than compound invariance. In particular, there is just one antecedent condition instead of three, and the assertion holds for just two integers, 2 and 3, rather than for all integers N.
- It simplifies the proof considerably.
- Both compound and reduction invariance are generalized to include a continuum of families of functions each characterized by a strictly decreasing function. Among these are the exponential-power and hyperbolic-logarithm functions Prelec derived from other invariance principles[5]. But of these, the only family that includes the power function, Eq. (3), as a special case is Prelec's compound-invariance one.

COMPOUND INVARIANCE

Prelec's key qualitative property that gives rise to Eq. (2) derives readily from separability and the following simple property of Eq. (2): For any real number $\lambda \geqslant 1$,

$$\begin{aligned} W(p^{\lambda}) &= \exp[-\beta(-\ln p^{\lambda})^{\alpha}] \\ &= \exp[-\beta\lambda^{\alpha}(-\ln p)^{\alpha}] \\ &= (\exp[-\beta(-\ln p)^{\alpha}])^{\lambda^{\alpha}} \\ &= W(p)^{\lambda^{\alpha}}. \end{aligned} \tag{5}$$

Assuming separability and Eq. (5), Prelec observed that the following property must hold (assuming $\lambda = N$, a natural number):

[4] That is, for some p_0, $W(p_0) = p_0$; for $p \in]0, p_0[$, $W(p) > p$; for $p \in]p_0, 1[$, $W(p) < p$; and for some p_1, $W(p)$ is concave for $p \in]0, p_1[$ and convex for $p \in]p_1, 1[$.

[5] It should be noted, however, that the solutions to these other invariance principles have, in addition to the two forms mentioned, the power functions.

Definition 1. Let N be any natural number. Then *N-compound invariance* is said to hold if and only if, for consequences x, y, x', y', probabilities $p, q, r, s \in]0, 1[$, with $q < p$, $r < s$,

$$(x, p) \sim (y, q), \ (x, r) \sim (y, s), \text{ and } (x', p^N) \sim (y', q^N) \tag{6}$$

imply

$$(x', r^N) \sim (y', s^N). \tag{7}$$

When N-compound invariance holds for all natural numbers N, we say *compound invariance holds.*

His result (Proposition 1, p. 503) says, in essence, that in the presence of separability, $W: [0, 1] \xrightarrow{onto} [0, 1]$, and a suitable density of consequences, then compound invariance is equivalent to Eq. (2).

From an empirical perspective, compound invariance has the fairly serious drawback that to test it an experimenter must first estimate solutions to three indifferences. For example, if one fixes[6] x, y, x' and p and r, then one must find in Eq. (6) the solutions for q in the first indifference, s in the second, and y' in the third. The potential for error cumulating in the estimates is considerable because the estimate of y' depends on the estimate of q, and in the conclusion two estimated quantities appear on the right. So a simpler condition would be desirable, which is the reason for this paper.

REDUCTION INVARIANCE

If as in the reduction of compound gambles, Eq. (4), one is willing to work with second-order compound gambles, then the fact that Eq. (4) gives rise to a special case of Eq. (2) suggests the following simpler formulation:

Definition 2. Let N be a natural number. Then *N-reduction invariance* is said to hold if and only if, for any consequence x and probabilities $p, q, r \in]0, 1[$,

$$((x, p), q) \sim (x, r) \tag{8}$$

implies

$$((x, p^N), q^N) \sim (x, r^N). \tag{9}$$

When N-reduction invariance holds for all natural numbers N, we say *reduction invariance* holds.

[6] As Prelec has pointed out (personal communication, July 23, 1998), the proof I give in Proposition 4 of a generalization of his result allows one to fix $p = 1$ in Eq. (6). See also the discussion below of reduction invariance.

Note that the reduction of compound gambles of Eq. (4) automatically implies this condition, but not conversely.

P. Wakker (referee's report) has pointed out that one can think of reduction invariance as implied by N-compound invariance plus consequence monotonicity. Fix x, p, q, and find consequences y, y' and probability r such that

$$(x, p) \sim (y, 1), \qquad (x, r) \sim (y, q), \qquad (x, p^N) \sim (y', 1).$$

Then in Eq. (6) set $q = 1, s = q$, and note that $y' \sim (y', 1)$, so N-compound invariance and consequence monotonicity yields

$$\begin{aligned}(x, r^N) &\sim (y', q^N)\\ &\sim ((y', 1), q^N)\\ &\sim ((x, p^N), q^N).\end{aligned}$$

N-reduction invariance is clearly somewhat simpler to check empirically than is N-compound invariance: only one solution in Eq. (8), r, need be found, and then one tests for indifference in Eq. (9). (Moreover, as we shall see, this need only be done for $N = 2, 3$.) A concern is that if one fixes x, p, q and asks a subject to report the r for which Eq. (8) holds, many subjects may realize that $r = pq$ is sensible. In the absence of such an empirical study, one cannot be certain what will happen. Moreover, Keller's (1985) results suggest that perhaps a substantial fraction of subjects will not be rational in this sense. Nonetheless, we have reasons to be skeptical of procedures that ask subjects to provide probabilities and so I would be inclined to another, albeit experimentally time consuming, approach involving certainty equivalents. If g is a gamble, then its *certainty equivalent* $CE(g)$ is defined to be that sum of money such that the respondent is indifferent between g and $CE(g)$. The proposed approach involves using some form of repeated choices to home-in on the certainty equivalents of the two sides of Eqs. (8) and (9). For example, one could estimate the $CE((x, p), q)$ and the several $CE(x, r')$ for a range of values of r' centered on pq. From the interpolated psychometric function $CE(x, r')$ versus r', one then estimates the value r for which $CE((x, p), q) = CE(x, r)$. That done, Eq. (9) entails estimating $CE((x, p^N), q^N)$ and $CE(x, r^N)$ for $N = 2, 3$ and deciding if, within the noise level of the procedure, these certainty equivalents are equal. Examples of how this method has been used to study other similar properties can be found in Cho and Luce (1995) and Cho, Luce, and von Winterfeldt (1994), but see the modification suggested by Sneddon (submitted).

Originally, I stated the following proposition for reduction invariance, i.e., for any natural number N, but a referee pointed out that the proof holds just as well assuming only $N = 2, 3$, the reason being that $3^m/2^n$, where m, n are integers, form a dense subset of the positive real numbers.

Proposition 1. *Suppose that a structure of binary gambles of the form* (x, p) *and* $((y, q), p)$, *with known probabilities, is weakly ordered in preference and has a* *separable representation* (U, W) with $W: [0,1] \xrightarrow{onto} [0,1]$, *where* W *is strictly increasing in* p. *Then the following two conditions are equivalent*:

(i) N-*reduction invariance* (*Definition* 2) *holds for* $N=2, 3$.

(ii) W *is given by Eq.* (2).

Proof. (ii) $\Rightarrow$ (i). This is a trivial consequence of Eq. (5).

(i) $\Rightarrow$ (ii). Note that by separability, $r=r(p,q)$ and does not depend on x.

Our first task is to prove that N-reduction invariance holds not only for 2 and 3, but for all positive real numbers. First, by induction it clearly holds for $N=2^n, 3^n$, where n is a natural number. Next, we show it for $\frac{1}{2^n}$. Suppose $((x, p^{1/2^n}), q^{1/2^n}) \sim (x, r^{1/2^n})$; then applying N-reduction invariance using $N=2^n$ we see that $((x, p), q) \sim (x, r)$. So, by monotonicity of p and the fact that W is onto, the converse must hold as well. Applying N-reduction invariance with $N=3^m$ to $((x, p^{1/2^n}), q^{1/2^n}) \sim (x, r^{1/2^n})$ yields for $N=3^m/2^n$ that $((x, p^N), q^N) \sim (x, r^N)$. So by separability we have

$$W(p)\, W(q) = W(r) \Leftrightarrow W(p^N)\, W(q^N) = W(r^N), \qquad N = \frac{3^m}{2^n}.$$

Because $3^m/2^n$ is dense in the positive real numbers and W is strictly increasing and onto, limits exist and so it follows that for $\lambda \in]0, \infty[$,

$$W(p)\, W(q) = W(r) \Leftrightarrow W(p^\lambda)\, W(q^\lambda) = W(r^\lambda).$$

Thus,

$$(W^{-1}[W(p)\, W(q)])^\lambda = r^\lambda = W^{-1}[W(p^\lambda)\, W(q^\lambda)].$$

Setting $G(P) = -\ln W(e^{-P})$, $P = -\ln p$, and $Q = -\ln q$, this equation becomes

$$\lambda G^{-1}[G(P)+G(Q)] = G^{-1}[G(\lambda P)+G(\lambda Q)]. \tag{10}$$

This functional equation is similar[7] to, but appreciably simpler than one solved by Aczél, Luce, and Maksa (1996). Following their strategy, for fixed λ, define

$$H(Z) = G(\lambda Z); \tag{11}$$

then Eq. (10) can be rewritten as

$$G^{-1}[G(P)+G(Q)] = H^{-1}[H(P)+H(Q)]. \tag{12}$$

Now, let

$$f(V) = HG^{-1}(V) \tag{13}$$

[7] The major difference is that Q was negative and bounded in their problem whereas here it is positive and unbounded.

and set $S = G(P)$ and $T = G(Q)$. Then, Eq. (12) becomes

$$f(S+T) = f(S) + f(T), \qquad S, T \in\,]0, \infty[. \tag{14}$$

Because G is strictly increasing, so is f, and the solution to Eq. (14) is well known (Aczél, 1966, p. 34) to be of the form $f(S) = AS$, $A > 0$. So from Eqs. (11) and (13)

$$G(\lambda P) = H(P) = fG(P) = AG(P).$$

Now treat λ as a variable, then $A = A(\lambda)$. Setting $P = 1$, we see $G(\lambda) = A(\lambda)\,\beta$, where $\beta = G(1)$, whence

$$G(\lambda P) = \frac{G(\lambda)\, G(P)}{\beta}. \tag{15}$$

It is well known (Aczél, 1966, p. 39) that the only strictly increasing solution to Eq. (15) is of the form

$$G(P) = \beta P^{\alpha}, \qquad \beta > 0, \alpha > 0. \tag{16}$$

Substituting Eq. (16) back into the definition of G and solving for W yields Eq. (2). ∎

φ-REDUCTION INVARIANCE

Permutable and Transitive Families

The purpose of this section is to provide a generalization of Prelec's compound-invariance weighting family. The following concepts are found in Aczél (1966, p. 270).

DEFINITION 3. Suppose that for an interval I of real numbers, including 0, $\varphi\colon]0,1[\times I \xrightarrow{onto}]0,1[$ is a function such that $\varphi(\cdot, \lambda)$ is strictly increasing in the first variable for each $\lambda \in I$ and $\varphi(p, \cdot)$ is strictly monotonic in the second for each $p \in\,]0,1[$. The family of functions φ is said to be *permutable* if for all $p \in\,]0,1[$ and $\lambda, \mu \in I$,

$$\varphi[\varphi(p, \lambda), \mu] = \varphi[\varphi(p, \mu), \lambda], \tag{17}$$

and is said to be *transitive* if for each $p, q \in\,]0,1[$ there is some $\lambda \in I$ such that

$$\varphi(p, \lambda) = q. \tag{18}$$

The permutability condition, Eq. (17), simply says that the impact of applying two members of the family is independent of the order of application, and transitivity, Eq. (18), guarantees a certain richness to the family.

According to Aczél (1966, p. 273)[8], Eqs. (17) and (18) imply there exist a strictly decreasing function $f\colon]0,1[\xrightarrow{onto}]0,\infty[$ and a strictly monotonic function $g\colon I \xrightarrow{onto}]0,\infty[$ such that

$$\varphi(p, \lambda) = f^{-1}[f(p)\, g(\lambda)]. \tag{19}$$

The reason g is onto $]0,\infty[$ is that Eq. (18) requires that for any p, q there is a λ such that $g(\lambda) = \frac{f(p)}{f(q)}$, which can be any positive real.

Definition and Form of W

The family used by Prelec in formulating compound invariance was $\varphi(p, N) = p^N$, where N is a natural number, which satisfies Eq. (17) but not (18). As a preliminary part of the proof (similar to but more complicated than the first part of the proof of Proposition 1), he showed that the condition of Definition 1 extends to $\varphi(p, \lambda) = p^{\lambda}$, $\lambda > 0$. As is easily shown for this family both Eqs. (17) and (18) hold, $f(p) = -\ln p$, and $g(\lambda) = \lambda$.

DEFINITION 4. Suppose φ is a permutable and transitive family. *φ-reduction invariance* is said to hold if and only if, for all $p \in]0,1[$ and $\lambda \in I$,

$$((x, p), q) \sim (x, r) \Leftrightarrow ((x, \varphi(p, \lambda)), \varphi(q, \lambda)) \sim (x, \varphi(r, \lambda)). \tag{20}$$

PROPOSITION 2. *Suppose that a structure of binary gambles with known probabilities is weakly ordered in preference and has a separable representation* (U, W) with $W\colon [0,1] \xrightarrow{onto} [0,1]$, *where W is strictly increasing in p, and suppose φ is a permutable and transitive family with the representation of Eq.* (19). *Then the following two conditions are equivalent*:

(i) *φ-reduction invariance* (*Definition* 5) *holds.*

(ii) *There are positive constants α, β such that for f in Eq.* (19),

$$W(p) = \exp[-\beta f(p)^{\alpha}]. \tag{21}$$

Proof. Because the proof closely parallels that for Prelec's special case, it will be done briefly. From φ-reduction invariance, Eq. (20), and Eq. (19) we see immediately that

$$f^{-1}(fW[W^{-1}[W(p)\, W(q)]\, g(\lambda)) = W^{-1}[W(f^{-1}[f(p)\, g(\lambda)])\, W(f^{-1}[f(q)\, g(\lambda)])].$$

[8] The result there is presented in additive form on the real line. Taking exponential transformations puts it in the following multiplicative form on the positive real numbers.

Let $P = f(p)$, $Q = f(q)$, $F(X) = W[f^{-1}(X)]$, and $g(\lambda) = Z$, then

$$F^{-1}[F(P)\,F(Q)]\,Z = F^{-1}[F(PZ)\,F(QZ)],$$

or writing $G(P) = -\ln F(P)$,

$$G^{-1}[G(P) + G(Q)]\,Z = G^{-1}[G(PZ) + G(QZ)].$$

As noted in the proof of Proposition 1, the solution to this equation is $G(P) = \beta P^{\alpha}$, $\alpha, \beta > 0$. Substituting the definitions yields Eq. (21). Note that the facts $W(0) = 0$ and $W(1) = 1$ are equivalent to $f(0) = \infty$ and $f(1) = 0$. ∎

The following observation is interesting.

Proposition 3. *The only f for which W of Eq.* (21) *includes the family of power functions of Eq.* (3) *is*

$$f(p)^{\alpha} = -\frac{\gamma}{\beta} \ln p,$$

i.e., Prelec's Eq. (2).

Proof. Set Eq. (21) equal to p^{γ} and take logarithms. ∎

Thus, Prelec's condition is the only one of these generalizations that includes the person who is rational to the degree of satisfying the weakest form of reduction of compound gambles. This seems to be a strong argument for using Prelec's family—one hardly wishes to exclude from a descriptive theory that person, however rare, who is at least this rational. Moreover, a substantial fraction of the estimated weighting functions are not inverse S-shaped and seem better approximated by power functions.

Special Cases

As an example different from Prelec's let $f(p) = \frac{1-p}{p}$; then

$$\varphi(p, \lambda) = \frac{p}{p + (1-p)\,g(\lambda)}$$

and

$$W(p) = \exp\left[-\beta\left(\frac{1-p}{p}\right)^{\alpha}\right].$$

Although this family cannot include the power representation of Eq. (3) as a special case, depending on the choice of parameters it can be wholly above the main diagonal, wholly below it, or a cross it from above to below. Figure 1 gives some illustrative examples.

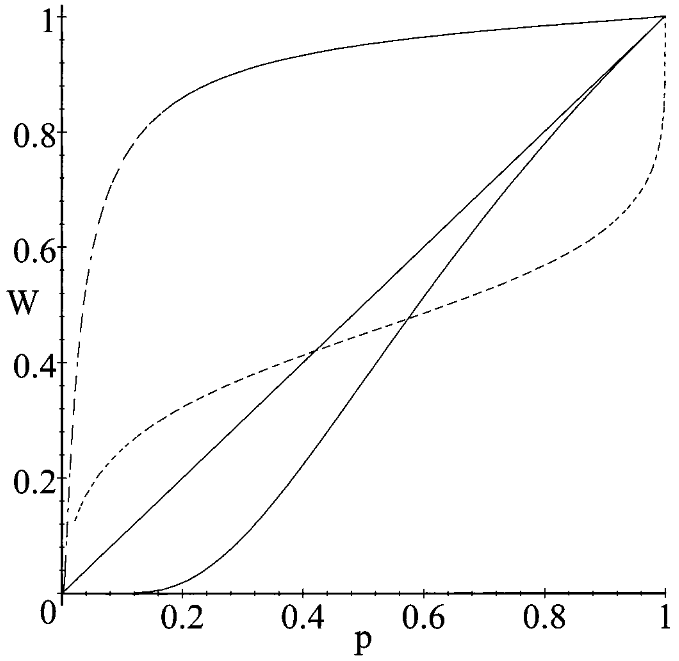

FIG. 1. Plots of $W(p)=\exp[-\beta(\frac{1-p}{p})^{\alpha}]$ for $\alpha=1$, $\beta=1$ (solid), $\alpha=.25$, $\beta=.80$ (dotted), and $\alpha=.80$, $\beta=.05$ (dashed).

From two other invariance principles, Prelec derived (Propositions 4 and 5), in addition to the power functions, two other forms[9] that he called the exponential-power function

$$W(p)=\exp\left[-\frac{\eta}{\gamma}(1-p^{\gamma})\right], \qquad \gamma\neq 0, \eta>0,$$

and the hyperbolic-logarithm function

$$W(p)=(1-\gamma\ln p)^{-\eta/\gamma}, \qquad \gamma, \eta>0.$$

Observe that these are both of the form of Eq. (21) if, respectively,

$$f(p)=\left[\frac{\eta}{\beta\gamma}(1-p^{\gamma})\right]^{1/\alpha},$$

$$f(p)=\left[\frac{\eta}{\beta\gamma}\ln(1-\gamma\ln p)\right]^{1/\alpha}.$$

As is easily verified, both functions are strictly decreasing.

[9] I have changed the symbols for the constants so as not to cause confusions when I relate these forms to Eq. (21).

φ-COMPOUND INVARIANCE

Because some theorists do not like to work with compound gambles, it is of interest to see if one can derive the form of Eq. (21) using a generalized version of compound invariance, Definition 1. Although it is fairly plausible that one should be able to do so, I include the formal proof because it includes Prelec's proof as a special case and is, I think, easier to follow than his.

DEFINITION 5. Suppose φ: $]0, 1[\times I \xrightarrow{onto}]0, 1[$ is a permutable and transitive family of functions (Definition 3). Then *φ-compound invariance* holds if and only if, for consequences x, y, x', y', probabilities p, q, r, s, with $q < p$, $r < s$, and all $\lambda \in I$,

$$(x, p) \sim (y, q),\ (x, r) \sim (y, s), \qquad \text{and} \qquad (x', \varphi(p, \lambda)) \sim (y', \varphi(q, \lambda)) \tag{22}$$

imply

$$(x', \varphi(r, \lambda)) \sim (y', \varphi(s, \lambda)). \tag{23}$$

PROPOSITION 4. *Suppose that a structure of binary gambles based on known probabilities is weakly ordered in preference and has a separable representation UW* with W: $[0, 1] \xrightarrow{onto} [0, 1]$, *where W is strictly increasing in p, and suppose φ is a permutable and transitive family with the representation of Eq.* (19). *Then the following two conditions are equivalent*:

(i) *φ-compound invariance* (*Definition* 5) *holds.*

(ii) *W satisfies Eq.* (21).

Proof. (i) ⇒ (ii). Because $W[\varphi(p, \lambda)]$ is strictly monotonic in λ and strictly increasing in p, which in turn is strictly increasing in $W(p)$, we know by Theorem 7.1 of Krantz, Luce, Suppes, and Tversky (1971) that for some function F: $]0, 1[\times I \rightarrow]0, 1[$,

$$W[\varphi(p, \lambda)] = W(f^{-1}[f(p)\, g(\lambda)]) = F[W(p), g(\lambda)].$$

Let $P = W(p)$, $Q = W(q)$, $R = W(r)$, and $S = W(s)$; then separability and the φ-compound invariance condition asserts

$$\frac{Q}{P} = \frac{S}{R} \Leftrightarrow \frac{F[Q, g(\lambda)]}{F[P, g(\lambda)]} = \frac{F[S, g(\lambda)]}{F[R, g(\lambda)]}.$$

Taking the limit as $r \rightarrow 1$ and so $R \rightarrow 1$, we see that the left side approaches $Q = PS$ and so the right equation satisfies in the limit

$$F[PS, g(\lambda)] = F[S, g(\lambda)]\, F[P, g(\lambda)].$$

This is a well-known variant of the Cauchy equation whose solution, because F is strictly increasing in the first variable, is for some function θ

$$F[P, g(\lambda)] = P^{\theta(\lambda)} \Leftrightarrow W[\varphi(p, \lambda)] = W(p)^{\theta(\lambda)}.$$

Taking logarithms of both sides of the right equation and using Eq. (19),

$$\ln W(f^{-1}[f(p)\, g(\lambda)]) = \theta(\lambda) \ln W(f^{-1}[f(p)]),$$

which with $G(Z) = -\ln W[f^{-1}(Z)]$ becomes

$$G[f(p)\, g(\lambda)] = \theta(\lambda)\, G[f(p)].$$

Setting $p = f^{-1}(1)$ yields $\theta(\lambda)\, G(1) = G(\lambda)$. So, letting $P = f(p)$, $Z = g(\lambda)$, and $\beta = G(1)$

$$G(PZ) = \frac{G(P)\, G(Z)}{\beta}.$$

This is also a variant of the Cauchy equation. Because G is strictly monotonic, the well known solution is for some $\alpha > 0$

$$G(P) = \beta P^{\alpha}.$$

Expressing this in terms of W yields Eq. (21).

(ii) $\Rightarrow$ (i) Trivial. ∎

CONCLUSIONS

Recasting compound invariance as reduction invariance, which under consequence monotonicity is a special case of the former, effects a simplicity both in testing the empirical condition and in proving the result. To do so requires using second-order compound gambles rather than just first-order ones. Replacing the power function of reduction and compound invariance by a more general permutable and transitive family φ of functions yields what were called φ-reduction and φ-compound invariance. From either of these we obtained a generalization of Prelec's Eq. (2) in which $-\ln p$ is replaced by $f(p)$, where f is strictly decreasing. Thus, there is a whole family of alternative forms, and for at least one f (and presumably an infinity of them) W has the qualitative shapes found with estimated weighting functions. The only principled way of which I am aware to select among these alternatives is to require that the family include as a special case the person who satisfies the simplest reduction of compound gambles and for whom $W(p) = p^{\gamma}$. In that case, Prelec's compound-invariance family is the unique one.

REFERENCES

Abdellaoui, M. (1999). Parameter-free elicitation of utilities and probability weighting functions. Manuscript.

Aczél, J. (1966). *Lectures on functional equations and their applications*. New York: Academic Press.

Aczél, J., Luce, R. D., & Maksa, G. (1996). Solutions to three functional equations arising from different ways of measuring utility. *Journal of Mathematical Analysis and Applications*, **204**, 451–471.

Birnbaum, M. H., Coffey, G., Mellers, B. A., & Weiss, R. (1992). Utility measurement: Configural-weight theory and the judge's point of view. *Journal of Experimental Psychology*: *Human Perception and Performance*, **18**, 331–346.

Cho, Y., & Luce, R. D. (1995). Tests of hypotheses about certainty equivalents and joint receipt of gambles. *Organizational Behavior and Human Decision Processes*, **64**, 229–248.

Cho, Y., Luce, R. D., & von Winterfeldt, D. (1994). Test of assumptions about the joint receipt of gambles in rank- and sign-dependent utility theory. *Journal of Experimental Psychology*: *Human Perception and Performance*, **20**, 931–943.

Gonzalez, R., & Wu, G. (1999). On the shape of the probability weighting function. *Cognitive Psychology*, **38**, 129–166.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, **47**, 263–291.

Karmarkar, U. S. (1978). Subjectively weighted utility: A descriptive extension of expected utility model. *Organizational Behavior and Human Performance*, **21**, 61–72.

Karmarkar, U. S. (1979). Subjectively weighted utility and the Allais paradox. *Organizational Behavior and Human Performance*, **24**, 67–72.

Keller, L. R. (1985). Testing of the "reduction of compound alternatives" principle. *OMEGA*. *The International Journal of Management Science*, **13**, 349–358.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement* (Vol. 1). San Diego: Academic Press.

Luce, R. D. (1991). Rank- and sign-dependent linear utility models for binary gambles. *Journal of Economic Theory*, **53**, 75–100.

Luce, R. D. (1996). When four distinct ways to measure utility are the same. *Journal of Mathematical Psychology*, **40**, 297–317.

Luce, R. D., & Fishburn, P. C. (1991). Rank- and sign-dependent linear utility models for finite first-order gambles. *Journal of Risk and Uncertainty*, **4**, 25–59.

Luce, R. D., & Fishburn, P. C. (1995). A note on deriving rank-dependent utility using additive joint receipts. *Journal of Risk and Uncertainty*, **11**, 5–16.

Prelec, D. (1998). The probability weighting function. *Econometrica*, **66**, 497–527.

Preston, M. G., & Baratta, P. (1948). An experimental study of the auction-value of an uncertain outcome. *American Journal of Psychology*, **61**, 183–193.

Quiggin, J. (1993). *Generalized expected utility theory: The rank-dependent model*. Boston: Kluwer Academic.

Sneddon, R. (submitted). Bias in a PEST procedure.

Tversky, A. (1967). Utility theory and additivity analysis of risky choices. *Journal of Experimental Psychology*, **75**, 27–36.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, **5**, 204–217.

Wu, G., & Gonzalez, R. (1996). Curvature of the probability weighting function. *Management Science*, **42**, 1676–1690.

Received: December 2, 1998