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Conditions Equivalent to Unit Representations of Ordered Relational Structures

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This paper studies several concepts about subgroups of automorphisms of linearly ordered relational structures. In particular, it focuses on conditions that are equivalent to the translations (automorphisms with no fixed points plus the identity) forming a homogeneous, Archimedean ordered group under the asymptotic order. For the automorphisms of an ordered relational structure, these properties are equivalent to the structure having a numerical representation whose scale type lies between, but does not exclude, the ratio and interval types (Alper, 1987; Luce, 1987; Narens, 1981a, 1981b). One result of this paper (Theorem 5) is that for the automorphisms of an ordered relational structure the following necessary conditions for such a representation are also sufficient: the asymptotic order induced on the automorphisms is connected, the structure is homogeneous, the translations are Archimedean, and the dilations (i.e., automorphisms with fixed points) are Archimedean relative to all automorphisms. The last property is equivalent to there being at most one proper, nontrivial convex subgroup. Contrary to hope, these results do not seem to lead to a simpler proof than Alper's for the Dedekind complete case. The paper concludes with an examination of the structure of parallel automorphisms.

Key Words: Alper–Narens theorem; Archimedeanness; dilations; homogeneity; infinitesimals; ordered relational structures; translations.

1. INTRODUCTION

Many measurement theorists believe that understanding the group of automorphisms of an ordered relational structure is essential to understanding both the existence of numerical representations as well as the scale type in the sense of Stevens (1946, 1951). To be more specific, suppose $\mathscr{X} = \langle X, \succsim, S_1, ..., S_k \rangle$ is an

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ordered relational structure in the sense that X is a set, called the domain of $\mathscr{X}$, $\succsim$ is a simple ordering[1] on X, and the S_i, $i=1, ..., k$, are relations of finite order on X. An automorphism is any mapping α of X onto X that maintains the structure in the sense that for $a, b \in X$,

$$a \succsim b \Leftrightarrow \alpha(a) \succsim \alpha(b),$$

and if the order of S_i is $n(i)$, then for all $a_1, a_2, ..., a_{n(i)} \in X$ and all $i=1, 2, ..., k$,

$$(a_1, a_2, ..., a_{n(i)}) \in S_i \Leftrightarrow (\alpha(a_1), \alpha(a_2), ..., \alpha(a_{n(i)})) \in S_i.$$

Two relational structures are of the same type if they have the same number k of defining relations and the ith relation of each, $i=1, ..., k$, is of the same finite order $n(i)$. It is easy to verify that if ϕ is an isomorphic representation of $\mathscr{X}$ onto a real structure $\mathscr{R}=\{R, \geqslant, R_1, ..., R_k\}$, $R \subseteq \mathbb{R}$, then $\mathscr{R}$ is of the same type as $\mathscr{X}$ and for any automorphism α of $\mathscr{X}$, $\phi\alpha$ is another equally good isomorphic representation onto $\mathscr{R}$. Moreover, if ψ is another isomorphism onto $\mathscr{R}$, then $\phi^{-1}\psi=\alpha$ is an automorphism of $\mathscr{X}$. Of course, as is well known, the automorphisms form a group under function composition with identity ι.

These automorphisms form a subgroup of the order automorphisms of the linearly ordered set $\langle X, \succsim \rangle$, and for the first four theorems of this paper it is sufficient to suppose that we are working with an arbitrary nontrivial subgroup $\mathscr{A}$ of the order automorphisms of $\langle X, \succsim \rangle$. In Theorem 5, $\mathscr{A}$ will be taken to be the full automorphism group of an ordered relational structure.

Narens (1981a, 1981b) has shown that it is useful to partition $\mathscr{A}$ into the following subsets.

DEFINITION 1. Let $\mathscr{A}$ be a subgroup of the order-automorphism group of a linearly ordered structure $\mathscr{X}=\langle X, \succsim \rangle$.

(i) An automorphism $\tau \in \mathscr{A}$ is said to be a *translation* if either it is the identity map ι or it has no fixed points, i.e., for every $a \in X$, $\tau(a) \neq a$. The set of translations is denoted $\mathscr{T}$.[2]

(ii) An automorphism $\delta \in \mathscr{A}$ is said to be a *proper dilation* if it is not a translation. Thus, it has at least one fixed point. The set of proper dilations is denoted $\mathscr{D}^*$. The set of dilations is $\mathscr{D}=\mathscr{D}^* \cup \{\iota\}$, where ι denotes the identity map. For $a \in X$, the set $\mathscr{D}_a^*$ consists of all proper dilations for which a is a fixed point, and $\mathscr{D}_a=\mathscr{D}_a^* \cup \{\iota\}$.

The terms translation and dilation make good sense when the structure has a real representation in which the automorphisms form a subgroup of the affine group, as is the case in Theorem 5. In more general contexts, however, these terms can be misleading.

[1] Empirical attributes of course yield a weak ordering, so we simply model the equivalence classes as elements.

[2] Strictly speaking, the set of translaltions should be written $\mathscr{T}_{\mathscr{A}}$, but because $\mathscr{A}$ is fixed throughout the discussion I supress the subscript.

We will repeatedly invoke the easily verified fact that $\mathscr{T}$ and $\mathscr{D}$ are both closed under inverses. Note, however, that $\mathscr{T}$ need not be closed under function composition, so it cannot be assumed to be a group. It is easily seen that each $\mathscr{D}_a$ forms a group under function composition.

Luce (1992) extended these concepts to structures having singular points, i.e., points that are fixed under all automorphisms or, put another way, $\mathscr{A} = \mathscr{D}_a$ for some $a \in X$. The distinction then becomes whether or not an automorphism has a fixed point other than the singular ones. Singular points are excluded in this paper.

Narens (1981a, 1981b) introduced three important concepts which are similar to well-known concepts in permutation group theory.

Definition 2. Suppose that $\mathscr{X}$ is a linearly ordered structure with $\mathscr{A}$ a subgroup of the order automorphisms, that $\mathscr{G} \subseteq \mathscr{A}$, and that M and N are finite integers $\geqslant 1$.

(i) The set $\mathscr{G}$ is said to be *M-point homogeneous* if and only if for any $a_1 \prec a_2 \prec \cdots \prec a_M$ and $b_1 \prec b_2 \prec \cdots \prec b_M$, $a_i, b_i \in X$, $i = 1, ..., M$, there exists a $\gamma \in \mathscr{G}$ such that $\gamma(a_i) = b_i$, $i = 1, ..., M$. The structure is called M-point homogeneous when $\mathscr{A}$ is. The term *homogeneous* is used to mean that the structure is at least 1-point homogeneous.[3]

(ii) The set $\mathscr{G}$ is said to be *N-point unique* if and only if whenever two automorphisms agree at N or more distinct points, then they agree everywhere. The set $\mathscr{G}$ is said to be *finitely unique* if for some finite integer N it is N-point unique.

(iii) The asymptotic order $\succsim'$ induced on $\mathscr{A}$ is defined by the following: for $\alpha, \beta \in \mathscr{A}$, $\alpha \succsim' \beta$ if there exists some $a \in X$ such that, for all $b \succ a$, $\alpha(b) \succcurlyeq \beta(b)$.

It is easy to verify that $\succsim'$ is reflexive and transitive, but it need not be connected. To assume that it is, as we do in various theorems below, says among other things that two automorphisms do not cross back and forth infinitely many times. In connection with this point, keep in mind that $\alpha \succsim' \iota$ is equivalent to $\iota \succsim' \alpha^{-1}$.

The following important result[4] is due to the combined efforts of Alper (1985, 1987) and Narens (1981a, 1981b). If $\mathscr{X}$ has a real representation *onto* the ordered real numbers $\mathbb{R}$ and its automorphism group is M-point homogeneous and N-point unique, then $\succsim'$ is connected, $M \leqslant N \leqslant 2$, and the set of translations $\mathscr{T}$ forms a 1-point homogeneous Archimedean ordered group (see Definition 5 below). The homogeneity was used to map $\mathscr{X}$ onto $\mathscr{T}$ which, in turn, was mapped via Hölder's (1901) theorem onto the additive real numbers in such a way that a translation takes the difference form $x \to x + s$. The dilations, if any, were shown to be of the form $x \to rx + s$, where $r \neq 1$ lies in a multiplicative subgroup of positive real numbers and s is any real number. For the details of this construction, see the proof of Theorem 5.

We also know from Cohen and Narens (1979) and Luce and Narens (1985) that much the same conclusion holds for $\mathscr{X} = \langle X, \succcurlyeq, \circ \rangle$, where $\circ$ is a positive[5] binary

[3] In the literature on ordered permutation groups, the term homogeneous is sometimes taken to be equivalent to 1-point homogeneity and sometimes equivalent to M-point homogeneous for all M.

[4] Alper received the 1990 Young Investigator Award from the Society of Mathematical Psychology for this research.

[5] For each $a, b \in X$, $a \circ b \succ a$ and $a \circ b \succ b$.

operation, under the same homogeneity and uniqueness assumptions but without requiring the structure to form a continuum.

The concern here is to try to understand more fully what gives rise to the various aspects of the major properties that $\mathscr{T}$ be homogeneous, a group, and Archimedean. In particular, I wish to see the degree to which the Alper–Narens restriction to the continuum can be replaced by, presumably weaker, Archimedean assumptions. This work continues, and in a number of ways improves, the work begun in Luce (1986, 1987) and summarized by Luce, Krantz, Suppes, and Tversky (1990, Chap. 20). Perhaps the most important result here is Theorem 5, which shows that $\mathscr{T}$ being a homogeneous, Archimedean ordered group is equivalent to the following: $\succsim'$ is connected, $\mathscr{A}$ is homogeneous, $\mathscr{T}$ is Archimedean, and $\mathscr{D}$ is Archimedean in $\mathscr{A}$. This result is arrived at by studying a number of interrelated concepts. Many, but not all, of the results reported in Theorems 3 and 4 are needed for Theorem 5.

In considering the various concepts that are introduced it is useful to keep in mind the real representation of Alper and Narens in which the automorphism group is a subgroup of the affine group $x \to rx + s$, where r is from a multiplicative subgroup of the positive real numbers and s is any real number.

2. ELEMENTARY UNIQUENESS AND HOMOGENEITY PROPERTIES

2.1. Normality and Uniformity

The first result concerns, primarily, properties of $\mathscr{T}$. Only some of it is new.

DEFINITION 3. Suppose $\mathscr{G} \subseteq \mathscr{A}$.

(i) The set $\mathscr{G}$ is said to be *normal* (in $\mathscr{A}$) if for every $\alpha \in \mathscr{A}$ and $\gamma \in \mathscr{G}$, then $\alpha\gamma\alpha^{-1} \in \mathscr{G}$.

(ii) The set $\mathscr{G}$ is said to be *uniform*[6] if for all $\alpha, \beta \in \mathscr{G}$ either $\alpha = \beta$ or for all $x \in X$, $\alpha(x) \succ \beta(x)$, or for all $x \in X$, $\alpha(x) \prec \beta(x)$.

Note that (i) is a standard algebraic concept and (ii) is important in the measurement literature. For example, in the affine group, the subgroup of translations $\tau(x) = x + s$ is normal because if $\alpha(x) = rx + t$, then $\alpha^{-1}(x) = (x - t)/r$, so

$$\alpha\tau\alpha^{-1}(x) = r([(x-t)/r] + s) + t = x + rs,$$

which is a translation. It is also obviously uniform.

In some of the formulations below it is convenient to use the following notation: Suppose $\mathscr{G}$ and $\mathscr{H}$ are subsets of automorphisms, then $\mathscr{G}\mathscr{H} = \{\gamma\eta \colon \gamma \in \mathscr{G} \text{ and } \eta \in \mathscr{H}\}$.

THEOREM 1. *Suppose that $\mathscr{X} = \langle X, \succsim \rangle$ is a linearly ordered structure, $\mathscr{A}$ is a subgroup of the order automorphisms, $\mathscr{T}$ is the set of translations of $\mathscr{A}$, $\mathscr{D}^*$ is the proper dilations of $\mathscr{A}$, and $\mathscr{D}_a$ is the dilations of $\mathscr{A}$ with the point a fixed. Then*:

[6] This choice of term agrees reasonably closely with Levine's (1971) usage and deviates from Luce (1986), where I spoke of the pairs of automorphism as being uncrossed.

(i) *$\mathcal{T}$ is a normal subset of $\mathcal{A}$.*

(ii) *If $\mathcal{G}$ is a uniform subset and $\iota \in \mathcal{G}$, then $\mathcal{G} \subseteq \mathcal{T}$.*

(iii) *The following statements are equivalent:*

(a) *$\mathcal{T}$ is* 1*-point unique.*

(b) *$\mathcal{T}$ is a group.*

(c) $\mathcal{T}\mathcal{T} \subseteq \mathcal{T}$.

(d) $\mathcal{T}\mathcal{D}^* \subseteq \mathcal{D}^*$.

(e) *If a decomposition $\alpha = \tau\delta_a$ exists, where $\alpha \in \mathcal{A}$, $\tau \in \mathcal{T}$, and $\delta_a \in \mathcal{D}_a$, then it is unique; i.e., if also $\alpha = \tau'\delta'_a$, $\tau' \in \mathcal{T}$, and $\delta'_a \in \mathcal{D}_a$, then $\tau' = \tau$ and $\delta'_a = \delta_a$.*

(iv) *If $\mathcal{T}$ is uniform, then $\mathcal{T}$ is a group.*

(v) *$\mathcal{T}$ is homogeneous if and only if $\mathcal{A}$ is homogeneous and for all $a \in X$, $\mathcal{D}^* \subseteq \mathcal{T}\mathcal{D}^*_a$.*

(vi) *If $\mathcal{A}$ is finitely unique and $\mathcal{T}$ is commutative, then $\mathcal{T}$ is* 1*-point unique.*

(vii) *If $\mathcal{A}$ is finitely unique and $\mathcal{T}$ is a homogeneous group, then $\mathcal{A}$ is* 2*-point unique.*

Proof.

(i) Suppose $\tau \in \mathcal{T}$, $\alpha \in \mathcal{A}$, and $a \in X$, then by definition of a translation $\alpha\tau\alpha^{-1}(a) \neq \alpha\alpha^{-1}(a) = a$. Thus $\alpha\tau\alpha^{-1} \in \mathcal{T}$, proving that $\mathcal{T}$ is normal.

(ii) Suppose $\gamma \in \mathcal{G} \cap \mathcal{D}$, then for some $a \in X$, $\gamma(a) = a = \iota(a)$. Because $\iota \in \mathcal{G}$ and $\mathcal{G}$ is uniform, $\gamma = \iota$.

(iii) The equivalence of (a), (b), and (c) is established as Theorem 2.1 of Luce (1986) (see also Theorem[7] 20.4 of Luce *et al.*, 1990, p. 118).

(a) ⇒ (d) By Theorem 20.4 of Luce *et al.* (1990) we know that for $\tau \in \mathcal{T}$ and $\delta \in \mathcal{D}^*$, τ^{-1} and δ must intersect; i.e., for some $a \in X$, $\tau^{-1}(a) = \delta(a)$, whence $\tau\delta(a) = a$, i.e., $\tau\delta \in \mathcal{D}^*$.

(d) ⇒ (e) Suppose $\alpha = \tau\delta_a = \tau'\delta'_a$, where $\tau, \tau' \in \mathcal{T}$ and $\delta_a, \delta'_a \in \mathcal{D}^*_a$ with $\delta_a \neq \delta'_a$. Because $\delta_a(\delta'_a)^{-1} \in \mathcal{D}^*_a$, by (d) $\tau' = \tau\delta_a(\delta'_a)^{-1} \in \mathcal{D}^*$, contrary to the assumption that $\tau' \in \mathcal{T}$.

(e) ⇒ (c) Suppose there exist $\tau, \sigma \in \mathcal{T}$ and for some $a \in X$, $\sigma\tau = \delta_a \in \mathcal{D}_a$. Then, $\tau = \sigma^{-1}\delta_a = \tau\iota$. By (e), $\tau = \sigma^{-1}$, so $\sigma\tau = \iota$, proving (c) holds.

(iv) Suppose that for some $a \in X$ and $\sigma, \tau \in \mathcal{T}$, $\sigma(a) = \tau(a)$. Then, by uniformity, $\sigma = \tau$, proving $\mathcal{T}$ is 1-point unique and thus a group by part (iii).

(v) Suppose $\mathcal{T}$ is homogeneous, then, of course, so is $\mathcal{A}$. Suppose $a \in X$ and $\alpha \in \mathcal{D}^*$. If $\alpha \in \mathcal{D}^*_a$ we are done. Otherwise, by the homogeneity of $\mathcal{T}$ there exists some $\tau \in \mathcal{T}$ such that $\tau\alpha(a) = a$. Set $\beta = \tau\alpha$, so $\alpha = \tau^{-1}\beta$, where $\beta \in \mathcal{D}^*_a$, proving $\mathcal{D}^* \subseteq \mathcal{T}\mathcal{D}^*_a$. Conversely, suppose $x, y \in X$. By the homogeneity of $\mathcal{A}$, for some $\alpha \in \mathcal{A}$, $\alpha(x) = y$. If $\alpha \in \mathcal{T}$ we are done. Otherwise, choose $\beta \in \mathcal{D}^*_y$ and $\tau \in \mathcal{T}$ such that $\alpha = \tau\beta$. By

[7] In referring to results from this book, I prefix the theorem or definition number by the chapter number.

part (i), there exists $\sigma \in \mathscr{T}$ such that $\alpha = \tau\beta = \beta\sigma$, so $y = \alpha(x) = \beta\sigma(x)$, whence $\sigma(x) = \beta^{-1}(y) = y$, proving the homogeneity of $\mathscr{T}$.

(vi)[8] Suppose there exist $\sigma, \tau \in \mathscr{T}$, $\sigma \neq \iota$, $\tau \neq \iota$, and $a \in X$ such that $\sigma(a) = \tau(a)$. So, for any n, the commutativity of $\mathscr{T}$ yields $\tau^n(a) = \tau^{n-1}\sigma(a) = \sigma\tau^{n-1}(a)$. So τ agrees with σ at each $\tau^n(a)$. These points are all distinct since τ is a translation, so finite uniqueness implies $\sigma = \tau$, proving that $\mathscr{T}$ is 1-point unique.

(vii) Suppose $\delta \in \mathscr{D}$ is such that $\delta(a) = a$ and $\delta(b) = b$, $a \neq b$. By the homogeneity of $\mathscr{T}$, there exists $\tau \in \mathscr{T}$ such that $\tau(a) = b$. By part (i), there exists $\eta = \delta\tau\delta^{-1} \in \mathscr{T}$. So, $\eta(a) = \delta\tau\delta^{-1}(a) = \delta\tau(a) = \delta(b) = b = \tau(a)$. Because $\mathscr{T}$ is a group and so, by part (iii), is 1-point unique, $\eta = \tau$. Thus, $\delta\tau(b) = \tau\delta(b) = \tau(b)$, so $\tau^2(a) = \tau(b)$ is another fixed point of δ. Continuing inductively, $\tau^n(a)$ is a fixed point of δ for each n. Note that they must all be distinct because $\tau^n(a) \succ \tau^{n-1}(a)$. Thus, by finite uniqueness $\delta = \iota$, proving $\mathscr{A}$ is 2-point unique. ∎

Observe that Theorem 1(i) immediately implies that for $\tau \in \mathscr{T}$ and $\delta_a \in \mathscr{D}_a$ there exists $\sigma \in \mathscr{T}$ such that $\tau\delta_a = \delta_a\sigma$.

2.2. *Parallel Automorphisms*

Earlier the subgroup $\mathscr{A}$ of automorphisms was divided into the set $\mathscr{T}$ and the subgroups $\mathscr{D}_a$, $a \in X$. We now introduce and study a classification into what I call parallel automorphisms. It has $\mathscr{T}$ in common with the former classification, but as we shall see, the other sets of parallel automorphisms cut across the set of groups $\mathscr{D}_a$. The term parallel is suggested by the example of the affine transformations: if $\beta(x) = rx + s$ and $\tau(x) = x + t$, then the graph of $\alpha(x) = \tau\beta(x) = rx + s + t$ is clearly parallel to that of β. But, just as with dilations and translations, the term can be misleading in sufficiently general contexts.

Definition 4. For $\alpha, \beta \in \mathscr{A}$, α and β are *parallel*, denoted $\alpha \approx \beta$, if and only if there exists some $\tau \in \mathscr{T}$ such that $\alpha = \tau\beta$.

Put another way, the set of elements parallel to β is the right coset $\mathscr{T}\beta$, and several aspects of the following Theorem are simply special cases of well-known properties of cosets.

Theorem 2. *Suppose that the conditions of Theorem* 1 *hold. Then*:

(i) We have $\alpha \approx \beta$ *if and only if either* $\alpha = \beta$ *or* α *and* β *have no point in common.*

(ii) *Suppose* $\mathscr{T}$ *is a group. Then:*

(a) $\approx$ *is an equivalence relation and* $\mathscr{T}$ *is an equivalence class of* $\approx$.

(b) *If* $\mathscr{G}$ *is both a subgroup of* $\mathscr{A}$ *and an equivalence class of* $\approx$, *then* $\mathscr{G} = \mathscr{T}$.

(c) *If* $\delta_a, \eta_a \in \mathscr{D}_a$, $\delta \approx \delta_a$, *and* $\eta \approx \eta_a$, *then* $\delta\eta \approx \delta_a\eta_a$.

(d) *If, for each* $a \in X$, $\mathscr{D}_a^*$ *is commutative and* $\mathscr{D}^* = \mathscr{T}\mathscr{D}_a^*$, *then the equivalence classes of* $\approx$ *are normal, i.e., for all* $\alpha, \beta \in \mathscr{A}$, $\alpha\beta \approx \beta\alpha$.

(iii) *If* $\mathscr{T}$ *is uniform, then each equivalence class of* $\approx$ *is uniform.*

[8] This proof is a simple adaptation of the Corollary to Theorem 20.4 of Luce *et al.* (1990, p. 119).

Proof.

(i) Suppose $\alpha \approx \beta$ and $\alpha(a) = \beta(a)$, then for some $\tau \in \mathscr{T}$, $\beta(a) = \alpha(a) = \tau\beta(a)$, so $\tau = \iota$, and $\alpha = \beta$. The converse is trivial.

(ii)(a) The reflexivity and symmetry of $\approx$ are obvious. To show transitivity, suppose $\alpha \approx \beta$ and $\beta \approx \gamma$, i.e., for some $\tau, \sigma \in \mathscr{T}$, $\alpha = \tau\beta$ and $\beta = \sigma\gamma$. Thus, $\alpha = \tau\sigma\gamma$. Because $\mathscr{T}$ is a group, $\tau\sigma \in \mathscr{T}$ so $\alpha \approx \gamma$. To show that $\mathscr{T}$ is itself an equivalence class of $\approx$, suppose $\tau \in \mathscr{T}$ and $\sigma \approx \tau$, then by definition $\sigma = \eta\tau$ for some $\eta \in \mathscr{T}$. Because $\mathscr{T}$ is a group, this means $\sigma \in \mathscr{T}$. Clearly any two elements of $\mathscr{T}$ are equivalent.

(b) Suppose $\mathscr{G}$ is a group that is also an equivalence class of $\approx$. If $\gamma \in \mathscr{G}$, then $\gamma \approx \iota$, i.e., for some $\tau \in \mathscr{T}$, $\gamma = \tau\iota = \tau$, so $\mathscr{G} \subseteq \mathscr{T}$. Thus, by part (a), $\mathscr{G} = \mathscr{T}$.

(c) By hypothesis there exist $\sigma, \tau \in \mathscr{T}$ such that $\delta = \sigma\delta_a$ and $\eta = \tau\eta_a$. Thus, $\delta\eta = \sigma\delta_a\tau\eta_a$. By Theorem 1(i), there exists $\varsigma \in \mathscr{T}$ such that $\varsigma\delta_a = \delta_a\tau$, whence the conclusion with $\tau' = \sigma\varsigma \in \mathscr{T}$ because $\mathscr{T}$ is a group.

(d) Using the hypothesis,

$$\mathscr{A} = \mathscr{T} \cup \mathscr{D}^* = \mathscr{T} \cup \mathscr{T}\mathscr{D}_a^* = \mathscr{T}(\{i\} \cup \mathscr{D}_a^*) = \mathscr{T}\mathscr{D}_a.$$

Thus, we may write $\alpha = \sigma\delta_a$ and $\beta = \tau\eta_a$, where $\sigma, \tau \in \mathscr{T}$ and $\delta_a, \eta_a \in \mathscr{D}_a$. Then, for some $\gamma \in \mathscr{T}$,

$$\begin{aligned} \alpha\beta\alpha^{-1}\beta^{-1} &= \delta_a\eta_a\gamma\eta_a^{-1}\delta_a^{-1} && \text{[repeated use of Theorem 1(i)]} \\ &= \delta_a\eta_a\gamma(\delta_a\eta_a)^{-1} && \text{[property of inverses]} \\ &\in \mathscr{T} && \text{[Theorem 1(i)].} \end{aligned}$$

So, $\alpha\beta\alpha^{-1}\beta^{-1} \in \mathscr{T}$ which is equivalent to $\alpha\beta \approx \beta\alpha$.

(iii) By Theorem 1(iv) we know that $\mathscr{T}$ is a group, so by part (ii)(a) $\approx$ is an equivalence relation. Suppose $\alpha \approx \beta$, i.e., for some $\tau \in \mathscr{T}$, $\alpha = \tau\beta$. With no loss of generality, suppose for some x, $\tau\beta(x) = \alpha(x) \succ \beta(x)$. By the uniformity of $\mathscr{T}$, we know that for any $y \in X$, $\tau(y) \succ y$. Thus, for all y, $\alpha(y) = \tau\beta(y) \succ \beta(y)$, proving that each equivalence class is uniform. ∎

3. CONVEX SUBSETS, ARCHIMEDEANNESS, AND INFINITESIMALS

This section begins with two well-known concepts, convexity and Archimedeanness, some of whose properties are examined in Theorem 3. It then turns to the important set of infinitesimals, and Theorem 4 examines relations among all three. Most of these results are used in proving Theorem 5.

Definition 5. Suppose $\mathscr{X}$ is a linearly ordered structure, $\mathscr{A}$ is a subgroup of the order automorphisms, and $\succsim'$ is the induced ordering on $\mathscr{A}$. Suppose $\mathscr{G} \subseteq \mathscr{A}$.

(i) The set $\mathscr{G}$ is said to be *convex with respect to* $\succsim'$ if for any $\alpha \in \mathscr{A}$ and $\gamma, \eta \in \mathscr{G}$ for which $\gamma \succsim' \alpha \succsim' \eta$, then $\alpha \in \mathscr{G}$.

(ii) The set $\mathscr{G}$ is said to be *Archimedean relative to* $\mathscr{H}$, where $\mathscr{G} \subseteq \mathscr{H} \subseteq \mathscr{A}$, provided that $\mathscr{G} \neq \{\iota\}$ and that for each $\eta \in \mathscr{H}$ and $\gamma \in \mathscr{G}$ with $\gamma \succ' \iota$ there is some integer n such that $\gamma^n \succsim' \eta$. When $\mathscr{G}$ is Archimedean relative to $\mathscr{G}$, we simply say that $\mathscr{G}$ is Archimedean.

(iii) Suppose $\iota \in \mathscr{G}$, then define $\overline{\mathscr{G}} = (\mathscr{A} \backslash \mathscr{G}) \cup \{\iota\}$.

Note that when $\mathscr{G}$ is closed under inverses, convexity is equivalent to the condition that for any $\alpha \in \mathscr{A}$, $\alpha \succ' \iota$, and $\gamma \in \mathscr{G}$ for which $\gamma \succ' \alpha$ then $\alpha \in \mathscr{G}$.

Note further that $\overline{\overline{\mathscr{G}}} = \mathscr{G}$, that $\mathscr{G} \cap \overline{\mathscr{G}} = \{\iota\}$, and that $\mathscr{T} = \overline{\mathscr{D}}$.

It is easy to see that the subgroup of translations of the affine group of real automorphisms is, in addition to being uniform and therefore a group, convex and Archimedean. Moreover, the affine dilations are Archimedean relative to the entire group of affine automorphisms.

THEOREM 3. *Suppose that the conditions of Theorem* 1 *hold, the asymptotic order* $\succsim'$ *on* $\mathscr{A}$ *is connected, and* $\mathscr{G}$, $\mathscr{H} \subseteq \mathscr{A}$ *are closed under inverses. Then*:

(i) *The following three statements are equivalent:*

(a) The set $\mathscr{G}$ *is convex.*

(b) *For any* $\alpha \in \mathscr{G}$ *and* $\beta \in \overline{\mathscr{G}}$ *with* $\beta \succ' \iota$, *then* $\beta \succ' \alpha$.

(c) *If* $\alpha, \beta \in \overline{\mathscr{G}}$ *and* $\alpha \succ' \beta$, *then either* $\alpha\beta^{-1} \in \mathscr{G}$ *or for all* $\gamma \in \mathscr{G}$, $\alpha \succ' \gamma\beta$.

(ii) *If* $\mathscr{G}$ *and* $\mathscr{H}$ *are convex, then either* $\mathscr{G} \subseteq \mathscr{H}$ *or* $\mathscr{H} \subseteq \mathscr{G}$.

(iii) *If* $\mathscr{G}$ *is convex, then* $\overline{\mathscr{G}}$ *is Archimedean relative to* $\mathscr{A}$ *if and only if* $\overline{\mathscr{G}}$ *is Archimedean.*

(iv) *Suppose* $\mathscr{H}$ *is Archimedean and* $\mathscr{G}$ *is a convex subgroup of* $\mathscr{A}$.

(a) *Then either* $\mathscr{H} \subseteq \mathscr{G}$ *or* $\mathscr{G} \subseteq \overline{\mathscr{H}}$.

(b) *If in addition* $\mathscr{H}$ *is Archimedean in* $\mathscr{A}$, *then either* $\mathscr{G} = \mathscr{A}$ *or* $\mathscr{G} \subseteq \overline{\mathscr{H}}$.[9,10]

Proof.

(i) (a) ⇒ (b) If $\alpha \succsim' \beta \succ \iota$, then the convexity of $\mathscr{G}$ implies $\beta \in \mathscr{G}$, contrary to choice. So $\beta \succ' \alpha$.

(b) ⇒ (c) Suppose $\alpha, \beta \in \overline{\mathscr{G}}$ with $\alpha \succ' \beta$. Then $\alpha\beta^{-1} \succ' \iota$ and either $\alpha\beta^{-1} \in \mathscr{G}$ or, by (b), for any $\gamma \in \mathscr{G}$, $\alpha\beta^{-1} \succ' \gamma$, whence $\alpha \succ' \gamma\beta$.

(c) ⇒ (a) Suppose $\beta, \gamma \in \mathscr{G}$ and for some $\alpha \in \mathscr{A}$, $\beta \succsim' \alpha \succsim' \gamma$. With no loss of generality, $\alpha \succ' \iota$ because if not then we work with $\gamma^{-1} \succsim' \alpha^{-1} \succsim' \beta^{-1}$. Suppose $\alpha \in \overline{\mathscr{G}}$, then by (c) either $\alpha\iota^{-1} = \alpha \in \mathscr{G}$ or $\alpha \succ' \beta$, contrary to the assumption that $\beta \succsim' \alpha$. So $\mathscr{G}$ is convex.

(ii) This is a well-known property of convex subsets of linearly ordered sets that are closed under inverses.

[9] This assertion is clearly related to parts (iii) and (iv) of Theorem 2.2 of Luce (1987); see Theorem 3(iii) below.

[10] Theorem 2.3(ii) of Luce (1986) showed that if $\mathscr{D}$ is Archimedean relative to $\mathscr{A}$ and $\mathscr{G}$ is a convex subgroup of $\mathscr{A}$, then either $\mathscr{G} \subseteq \mathscr{T}$ or $\mathscr{G} = \mathscr{A}$. This follows immediately by setting $\mathscr{H} = \mathscr{D}$.

(iii) It suffices to show that if $\overline{\mathscr{G}}$ is Archimedean, then it is Archimedean relative to $\mathscr{A}$. Suppose $\beta \in \overline{\mathscr{G}}$, with $\beta \succ' \iota$. Suppose, first, $\alpha \in \overline{\mathscr{G}}$, then by the fact that $\overline{\mathscr{G}}$ is Archimedean, we know that for some integer n $\beta^n \succsim' \alpha$. So, it suffices to assume that $\alpha \in \mathscr{G}$. By the convexity of $\mathscr{G}$, if $\alpha \succsim' \beta \succ' \iota \succ' \alpha^{-1}$, then $\beta \in \mathscr{G}$, contrary to choice. So, $\beta \succ' \alpha$, establishing that $\overline{\mathscr{G}}$ is Archimedean relative to $\mathscr{A}$.

(iv)(a) If $\mathscr{G} \subseteq \overline{\mathscr{H}}$ we are done. Therefore, suppose $\gamma \in \mathscr{H} \cap \mathscr{G}$, and without loss of generality, we can assume $\gamma \succ' \iota$. By the Archimedeanness of $\mathscr{H}$, for any $\alpha \in \mathscr{H}$ with $\alpha \succ' \iota$, there exists an n such that $\gamma^n \succ' \alpha$. Because $\mathscr{G}$ is a convex group, $\alpha \in \mathscr{G}$ so $\mathscr{H} \subseteq \mathscr{G}$.

(b) Next, suppose $\mathscr{H}$ is Archimedean in $\mathscr{A}$. If $\mathscr{G} \subseteq \overline{\mathscr{H}}$ does not hold, then by part (iv)(a) we know $\mathscr{H} \subseteq G$. Choose $\alpha \in \mathscr{A}$, $\alpha \succ' \iota$. So, there exist some $\beta \in \mathscr{H} \subseteq G$ and integer n such that $\beta^n \succsim' \alpha \succ' \iota$. By assumption, $\mathscr{G}$ is convex, so $\alpha \in \mathscr{G}$. Thus, $\mathscr{G} = \mathscr{A}$. ▮

COROLLARY 3.1.. *Under the conditions of Theorem* 3, *suppose that $\mathscr{H}$ and $\overline{\mathscr{H}}$ are each closed under inverses and are Archimedean, that $\mathscr{G}$ and $\mathscr{H}$ are convex, and that $\mathscr{G}$ is a nontrivial group. Then either $\mathscr{G} = \mathscr{H}$ or $\mathscr{G} = \mathscr{A}$.*

Proof. With $\mathscr{H}$ and $\overline{\mathscr{H}}$ each playing the role of $\mathscr{H}$ in two applications of part (iv), we have both ($\mathscr{G} \subseteq \mathscr{H}$ or $\overline{\mathscr{H}} \subseteq \mathscr{G}$) and ($\mathscr{H} \subseteq \mathscr{G}$ or $\mathscr{G} \subseteq \overline{\mathscr{H}}$). So there are four possibilities: (i) $\mathscr{G} = \mathscr{H}$, (ii) $\overline{\mathscr{H}} \subseteq \mathscr{G}$ and $\mathscr{H} \subseteq \mathscr{G}$ implying $\mathscr{G} = \mathscr{A}$, (iii) $\mathscr{G} \subseteq \mathscr{H}$ and $\mathscr{G} \subseteq \overline{\mathscr{H}}$, which is clearly impossible for a nontrivial $\mathscr{G}$, and (iv) $\mathscr{G} = \overline{\mathscr{H}}$. To rule out (iv), note that by Theorem 3(ii) either $\overline{\mathscr{H}} = \mathscr{G} \subseteq H$ or $\mathscr{H} \subseteq G = \overline{\mathscr{H}}$, both of which are impossible. ▮

We next define a convex subgroup introduced by Narens (1981b) and used by Alper (1985) and Luce (1986). The present results improve the earlier ones.

DEFINITION 6. Define

$$\mathscr{I} = \{\sigma \colon \sigma \in \mathscr{A} \text{ and there exists } \alpha \in \mathscr{A} \text{ such that for every } n,\ \alpha \succ' \sigma^n, \sigma^{-n}\}.$$

Elements of $\mathscr{I}$ are called *infinitesimals.*

Note that if $\mathscr{A} \neq \{\iota\}$, $\iota \in \mathscr{I}$ and $\mathscr{I}$ is closed under inverses. Note also that if for some $\alpha \neq \iota$ and all n, $\alpha \succ' \sigma^n$, then the assertion is true for all $\alpha \in \overline{\mathscr{I}}$. The reason is that if for some $\beta \in \overline{\mathscr{I}}$, $\beta \neq \alpha$ and some m, $\sigma^m \succ' \beta$, then for all n, $\alpha \succ' \sigma^{mn} \succ' \beta^n$, so $\beta \in \mathscr{I}$, contrary to choice.

Returning once again to the affine group as an example, it is easy to verify there that $\mathscr{T} = \mathscr{I}$.

In the following result, we distinguish carefully properties that do and do not explicitly depend upon the defining property of $\mathscr{T}$, namely, no fixed points except for ι.

THEOREM 4. *Suppose that the conditions of Theorem* 1 *hold and that the asymptotic ordering $\succsim'$ of $\mathscr{A}$ is connected. Then:*

(i) *$\mathcal{I}$ is a normal, convex subgroup.*

(ii) *Either $\mathcal{I} = \mathcal{A}$ or $\mathcal{I}$ is the maximal, proper, convex subgroup.*[11]

(iii) *Suppose $\mathcal{I} \neq \{\iota\}$. Then the following are equivalent:*

(a) *$\mathcal{I}$ is Archimedean.*

(b) *$\mathcal{I}$ has no proper, nontrivial, convex subgroup.*

(c) *If for $\alpha \in \mathcal{A}$ and some $\sigma \in \mathcal{I}$,* for all integers n, $\alpha \succ' \sigma^n, \sigma^{-n}$, *then this is true for all $\sigma \in \mathcal{I}$.*

(iv) *Suppose $\mathcal{G}$ is nontrivial and closed under inverses. Then it follows that:*

(a) *If $\overline{\mathcal{G}}$ is Archimedean, then either $\mathcal{I} \subseteq \mathcal{G}$* or *$\overline{\mathcal{G}} \subseteq \mathcal{I}$.*

(b) *$\overline{\mathcal{G}}$ is Archimedean relative to $\mathcal{A}$ if and only if $\mathcal{I} \subseteq \mathcal{G}$.*

(c) *If $\mathcal{I} \subseteq \mathcal{G}$ and $\mathcal{G}$ is Archimedean, then either $\mathcal{I} = \{\iota\}$ or $\mathcal{I} = \mathcal{G}$.*

(v) *Suppose that $\mathcal{A}$ is homogeneous. Then it follows that*

(a) *either $\mathcal{T} = \mathcal{A}$ or $\mathcal{I} \neq \{\iota\}$; and*

(b) *if $\mathcal{T} \subseteq \mathcal{I}$, then $\mathcal{I}$ is homogeneous.*

Proof. (i) Suppose $\sigma \in \mathcal{I}$, $\sigma \neq \iota$. With no loss of generality because $\iota, \sigma^{-1} \in \mathcal{I}$, we suppose $\sigma \succ' \iota$. Now, suppose $\eta \in \mathcal{A}$, $\eta \succ' \iota$, and $\sigma \succ' \eta \succ' \eta^{-1}$. By the definition of $\mathcal{I}$, for some $\alpha \in \mathcal{A}$ and every n, $\alpha \succ' \sigma^n \succ' \eta^n \succ' \eta^{-n}$. Thus $\eta \in \mathcal{I}$, establishing that $\mathcal{I}$ is convex.

Because function composition is associative, we need only establish the closure of $\mathcal{I}$ to prove it is a group. To that end, suppose $\sigma, \eta \in \mathcal{I}$ and with no loss of generality $\sigma \succsim' \eta$. Again, there exists $\alpha \in \mathcal{A}$ such that for all n, $\alpha \succ' \sigma^n \succsim' \eta^n$. Thus, for all n, $\alpha \succ' \sigma^{2n} \succsim' (\sigma\eta)^n$, so $\sigma\eta \in \mathcal{I}$.

To show that $\mathcal{I}$ is normal, suppose $\alpha \in \mathcal{A}$ and $\sigma \in \mathcal{I}$. By definition of $\mathcal{I}$, for some $\beta \in \mathcal{A}$ and all n, $\beta \succ' \sigma^n, \sigma^{-n}$. Since $(\alpha\sigma\alpha^{-1})^n = \alpha\sigma^n\alpha^{-1}$, we have for all n,

$$\alpha\beta\alpha^{-1} \succ' \alpha\sigma^n\alpha^{-1} = (\alpha\sigma\alpha^{-1})^n,$$

and similarly for $\alpha\sigma^{-n}\alpha^{-1}$, thereby proving $\alpha\sigma\alpha^{-1} \in \mathcal{I}$.

(ii) Suppose $\mathcal{I} \neq \mathcal{A}$ and $\mathcal{G}$ is a convex subgroup with $\mathcal{I} \subset \mathcal{G} \subset \mathcal{A}$. Let $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{G} \backslash \mathcal{I}$. Because $\beta \notin \mathcal{I}$, there exists some m such that $\beta^m \succ' \alpha$, so by the convexity of $\mathcal{G}$ we have $\alpha \in \mathcal{G}$, proving $\mathcal{G} = A$.

(iii)(a) $\Rightarrow$ (b) Suppose $\mathcal{H}$ is a nontrivial convex subgroup of $\mathcal{I}$. Let $\sigma \in \mathcal{H}$, $\sigma \succ \iota$, and $\eta \in \mathcal{I}$. With no loss of generality, assume $\eta \succ' \iota$. Because $\mathcal{I}$ is Archimedean, there is an integer n such that $\sigma^n \succsim' \eta$. The convexity of $\mathcal{H}$ implies $\eta \in \mathcal{H}$ so $\mathcal{I} = H$.

(b) $\Rightarrow$ (c) Let $\mathcal{H}_\alpha = \{\sigma$: for each integer n, $\alpha \succ' \sigma^n, \sigma^{-n}\}$. Clearly, $\mathcal{H}_\alpha$ is a convex subset of $\mathcal{I}$. To show $\mathcal{H}_\alpha$ is closed and therefore a subgroup, suppose $\sigma, \eta \in \mathcal{H}_\alpha$, $\sigma \succsim' \eta \succ' \iota$. Because $\sigma^2 \in \mathcal{H}_\alpha$ and $\sigma^2 \succsim' \sigma\eta$, the convexity of $\mathcal{H}_\alpha$ implies closure. So, by hypothesis $\mathcal{H}_\alpha = \mathcal{I}$.

(c) $\Rightarrow$ (a) Suppose $\mathcal{I}$ is not Archimedean, i.e., for some $\sigma, \eta \in \mathcal{I}$, $\eta \succ' \iota$, and for each integer n, $\eta \succsim' \sigma^n, \sigma^{-n}$. But (c) implies $\eta \succ' \eta^n$, which is impossible. So $\mathcal{I}$ is Archimedean.

[11] I thank a referee for pointing this out.

(iv)(a) This follows immediately from part (i) and Theorem 3(iv).

(b) Assume that $\overline{\mathscr{G}}$ is Archimedean relative to $\mathscr{A}$. Suppose $\sigma \in \mathscr{I}$, $\sigma \succ' \iota$. If $\sigma \in \mathscr{G}$ we are done. So, suppose $\sigma \in \overline{\mathscr{G}}$. By the definition of $\mathscr{I}$ there exists some α such that for each integer n, $\alpha \succ' \sigma^n$. However, by the hypothesis that $\overline{\mathscr{G}}$ is Archimedean relative to $\mathscr{A}$, we know that for some m, $\sigma^m \succsim' \alpha$, which is a contradiction. So $\mathscr{I} \subseteq \mathscr{G}$. Conversely, suppose that $\overline{\mathscr{G}}$ is not Archimedean in $\mathscr{A}$, i.e., for some $\alpha \in \mathscr{A}$, all $\beta \in \overline{\mathscr{G}}$, $\beta \succ' \iota$, and every integer n, $\alpha \succ' \beta^n$. Thus, $\beta \in \mathscr{I} \subseteq \mathscr{G}$, contrary to the choice of $\beta \in \overline{\mathscr{G}}$. So, $\overline{\mathscr{G}}$ must be Archimedean in $\mathscr{A}$.

(c) Suppose $\sigma \in \mathscr{I}$, $\sigma \succ' \iota$, and $\beta \in \mathscr{G}$, $\beta \succ' \iota$. Because $\mathscr{G}$ is Archimedean, for some integer n, $\sigma^n \succsim' \beta \succ' \iota$. By part (i), $\mathscr{I}$ is convex, so $\beta \in \mathscr{I} \subseteq G$, thus proving $\mathscr{G} = \mathscr{I}$.

(v)(a) Suppose $\mathscr{I} = \{\iota\}$, then $\mathscr{A}$ is Archimedean, so, by Hölder's theorem, it is an Archimedean ordered group which means it is commutative. Choose $\alpha \in \mathscr{A}$. Suppose for some $a \in X$, $\alpha(a) = a$. For any $b \in X$, the homogeneity of $\mathscr{A}$ implies there exists $\beta \in \mathscr{A}$ such that $\beta(a) = b$. Thus $\alpha(b) = \alpha\beta(a) = \beta\alpha(a) = \beta(a) = b$. So $\alpha = \iota$, which proves $\mathscr{A} = T$.

(b) If $\mathscr{A} = \mathscr{T}$ we are done. So, by part (a), $\mathscr{I} \neq \{\iota\}$. Because by part (ii) $\mathscr{I}$ is the maximal proper convex subgroup, we know from Fuchs (1963, p. 50) that for any $\alpha, \beta \in \mathscr{A}$, $\alpha\beta\alpha^{-1}\beta^{-1} \in \mathscr{I}$. To show the homogeneity of $\mathscr{I}$, suppose $x, y \in X$. Following Alper (1987) (see proof of Lemma 20.12 of Luce *et al.*, 1990, p. 135), the homogeneity of $\mathscr{A}$ implies there is a β such that $\beta(x) = y$. If $\beta \in \mathscr{T}$ we are done since, by assumption, $\mathscr{T} \subseteq \mathscr{I}$. If not, let a be a fixed point of β, and by homogeneity of $\mathscr{A}$ choose α so that $\alpha(y) = a$. Then

$$\alpha^{-1}\beta^{-1}\alpha\beta(x) = \alpha^{-1}\beta^{-1}\alpha(y) = \alpha^{-1}\beta^{-1}(a) = \alpha^{-1}(a) = y,$$

proving that $\mathscr{I}$ is homogeneous. ▮

COROLLARY 4.1. *Suppose the conditions of Theorem* 4 *hold, that* $\mathscr{G}$ *is closed under inverses and is Archimedean*, that $\overline{\mathscr{G}}$ *is Archimedean relative to* $\mathscr{A}$, *and that* $\mathscr{I} \neq \{\iota\}$. *Then it follows that*

(i) $\mathscr{I} = \mathscr{G}$, *and*

(ii) *either* $\mathscr{I} = \mathscr{A}$ *or* $\mathscr{I}$ *is the unique nontrivial, proper convex subgroup.*

Proof. (i) By part (iv)(b) we see that $\mathscr{I} \subseteq \mathscr{G}$. Then, by part (iv)(c) and the assumption that $\mathscr{I} \neq \{\iota\}$, we have $\mathscr{I} = \mathscr{G}$.

(ii) By what we have just shown, $\mathscr{I}$ is Archimedean and, by part (iii) of the Theorem, it has no proper, nontrivial, convex subgroup. By part (ii) of the Theorem, either $\mathscr{I} = \mathscr{A}$ or $\mathscr{I}$ is the maximal proper, convex subgroup. ▮

COROLLARY 4.2. *Suppose the conditions of Theorem* 4 *hold. Then it follows that:*

(i) *If* $\mathscr{T}$ *is a convex*, 1-*point homogeneous group, then for each* $a \in X$, $\mathscr{D}_a$ *is Archimedean if and only if* $\mathscr{D}$ *is Archimedean relative to* $\mathscr{A}$.

(ii) *If* $\mathscr{D}$ *is Archimedean relative to* $\mathscr{A}$, *then the equivalence classes of* $\approx$ *are normal.*

Proof. (i) Obviously, if $\mathscr{D}$ is Archimedean so is $\mathscr{D}_a$. Conversely, suppose $\alpha \in \mathscr{A}$ and $\delta \in \mathscr{D}_a \subseteq \mathscr{D}$, $\delta \succ \iota$. If $\alpha \in \mathscr{T}$, then by Theorem 3(i) $\delta \succ' \alpha$. So we suppose $\alpha \in \mathscr{D}_b$ for some b. By the assumption that $\mathscr{D}_a$ is Archimedean, it suffices to consider $b \neq a$. For some c, $\delta(b) = c$. Either $c = b$, in which case $\eta_b = \delta \in \mathscr{D}_b$, or $c \neq b$, in which case by the 1-point homogeneity of $\mathscr{T}$, there exists $\tau \in \mathscr{T}$ such that $\tau(c) = b$, so $\eta_b(b) = \tau\delta(b) = \tau(c) = b$, proving that $\eta_b \in \mathscr{D}_b$. Because $\mathscr{D}_b$ is Archimedean, for some n, $\eta_b^n \succ' \alpha$. By repeated use of Theorem 2(ii)(c) and the assumption that $\mathscr{T}$ is a group, there exists $\sigma \in \mathscr{T}$ such that $\sigma\delta^n = \eta_b^n \succ' \alpha$. Because $\sigma\delta^n \in \mathscr{D}$, Theorem 3(i) says there are two possibilities: (a) $\sigma\delta^n\alpha^{-1} = \eta \in \mathscr{T}$ or (b) for all $\tau \in \mathscr{T}$, $\sigma\delta^n \succ' \tau\alpha$. If (a), $\eta^{-1}\sigma\delta^n = \alpha$. By Theorem 3(i) $\delta \succ' \eta^{-1}\sigma$, so $\delta^{n+1} \succ' \eta^{-1}\sigma\delta^n \sim' \alpha$, proving the Archimedean condition. If (b), then select $\tau = \sigma$ and apply σ^{-1}, yielding $\delta^n \succ' \alpha$, which again is the Archimedean condition.

(ii) By part (iv)(b), we know $\mathscr{I} \subseteq T$. By part (ii), either $\mathscr{I} = A = T$, in which case the result is trivial or $\mathscr{I}$ is a maximal proper convex subgroup. So by Fuchs (1963, p. 50) $\alpha\beta\alpha^{-1}\beta^{-1} \in \mathscr{I} \subseteq T$, whence the conclusion. ∎

By Corollary 4.2(i), whenever $\mathscr{T}$ is a convex 1-point homogeneous group and $\mathscr{D}$ is Archimedean relative to $\mathscr{A}$, then each $\mathscr{D}_a$ is an Archimedean-ordered group, so it is commutative.

Theorem 4(iv)(a), with $\mathscr{G} = \mathscr{D}$, says that if $\mathscr{T}$ is Archimedean, either $\mathscr{I} \subseteq \mathscr{D}$ or $\mathscr{T} \subseteq \mathscr{I}$. At one point I thought the former case could be excluded without assuming $\mathscr{D}$ to be Archimedean relative to $\mathscr{A}$, but Theodore Alper (personal communication, February 24, 1995) has provided the following example to show that $\mathscr{I} \subseteq \mathscr{D}$ cannot be excluded in general. Let $\mathbb{I}$ denote the integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{N}_0(\mathbb{I})$ the set of sequences

$$\vec{a} = (a_0, a_1, a_2, \ldots),$$

where $a_i \in \mathbb{I}$ and only finitely many of them are non-zero. Define

$$\vec{a} < \vec{b} \Leftrightarrow \text{for the smallest index } j \text{ for which } \vec{a} \text{ and } \vec{b} \text{ differ, } a_j < b_j.$$

Let $\mathbb{X} = \mathbb{I} \times \mathbb{N}_0(\mathbb{I})$ be ordered lexicographically. It is not difficult to show that $\mathbb{X}$ is countable, order dense, and with neither a maximum nor a minimum element. Thus, it is order isomorphic to $\mathbb{Q}$.

For $n \in \mathbb{I}$, define $\vec{\delta}_n$ to be $\vec{0}$ when $n < 0$ and the vector of all 0s except in the nth position, which is 1, when $n \geqslant 0$. Define the mappings from $\mathbb{X}$ to $\mathbb{X}$:

$$\gamma_n(i, \vec{a}) = (i, \vec{a} + \vec{\delta}_{i-n}) \qquad \text{and} \qquad \beta_m(i, \vec{a}) = (i + m, \vec{a}).$$

One can verify the following. They are order automorphisms. Each γ_n fixes any point for which $i < n$ and none for $i \geqslant n$, β_0 is the identity, and each β_m, $m \neq 0$, is a translation.

Let $\mathscr{G}$ denote the subgroup of order automorphisms of $\mathbb{X}$ that are generated by these automorphisms. Because, as is easily verified,

$$\beta_m \beta_n = \beta_{m+n}, \qquad \gamma_m \gamma_n = \gamma_n \gamma_m, \qquad \beta_m \gamma_n = \gamma_{m+n} \beta_m,$$

one can show that each element of $\mathscr{G}$ is the product of some β followed by finitely many γs. These are translations when and only when $m \neq 0$, and dilations otherwise.

The translations are Archimedean because any automorphism is less than β_m for some sufficiently large m, the asymptotic order is connected given its definition, and $\mathscr{G}$ is homogeneous because if $\vec{a}$ is non-zero at $m_1 < m_2 < \cdots < m_p$ with values $k_1, k_2, ..., k_p$, then $\beta_m \gamma^{k_1}_{-m_1} \gamma^{k_2}_{-m_2} \cdots \gamma^{k_p}_{-m_p}$ maps $(0, \vec{0})$ to $(m, \vec{a})$. Observe that the dilations are infinitesimal, so they are not Archimedean in $\mathscr{G}$.

4. CONDITIONS EQUIVALENT TO UNIT STRUCTURE REPRESENTATIONS

The following definition can be found in Luce (1986) (see also Luce *et al*., 1990, p. 123), except there I formulated it multiplicatively whereas here I formulate it additively.

Definition 7. A real relational structure $\mathscr{R} = \langle R, \geqslant, R_1, ..., R_n \rangle$ is said to be a (additive) *homogeneous unit structure* if and only if $R \subseteq \mathbb{R}$ and there is some $T \subseteq R$ such that

1. T is a group under addition,
2. T maps R under addition onto R,
3. the set of translations of $\mathscr{R}$ is T.

A well-known example of such a structure is $\langle \mathbb{R}, \geqslant, + \rangle$ and $T = \mathbb{R}$. More complex examples are constructed in Luce and Narens (1985).

Theorem 5. *Suppose that $\mathscr{A}$ is the automorphism group of a linearly ordered relational structure. Then the following are equivalent:*

(i) *$\succsim'$ on $\mathscr{A}$ is connected, $\mathscr{A}$ is homogeneous, $\mathscr{T}$ is Archimedean, and $\mathscr{D}$ is Archimedean relative to $\mathscr{A}$.*

(ii) *$\succsim'$ on $\mathscr{A}$ is connected, $\mathscr{A}$ is homogeneous, $\mathscr{I}$ is Archimedean, and $\mathscr{T} \subseteq \mathscr{I}$.*

(iii) *$\mathscr{T}$ is a homogeneous Archimedean ordered group under $\succsim'$ and composition of functions.*

(iv) *There is an additive homogeneous unit structure representation of $\mathscr{X}$ whose automorphisms form a subgroup of the affine group on the unit structure.*

In these cases, either $\mathscr{A}$ is 1-*homogeneous and either* 1- *or* 2-*point unique or $\mathscr{A}$ is* 2-*point homogeneous and* 2-*point unique.*

Proof. (i) ⇒ (iii) By Theorem 4(v)(a) either $\mathscr{T} = \mathscr{A}$, in which case we are done, or $\mathscr{I} \neq \{\iota\}$. By Corollary 4.1 $\mathscr{T} = \mathscr{I}$. So, by Theorem 4(i), $\mathscr{T}$ is a group, and by Theorem 4(v)(b), $\mathscr{T}$ is homogeneous. Thus $\mathscr{T}$ is a homogeneous Archimedean ordered group.

(iii) $\Rightarrow$ (iv)[12] We first show that $\mathcal{T}$ is uniform. Suppose $x, y \in X$, $\tau \in \mathcal{T}$, and $\tau(x) \succ x$. By the homogeneity of $\mathcal{T}$, there is $\sigma \in \mathcal{T}$ such that $\sigma(x) = y$. Using the commutativity of $\mathcal{T}$, which follows from the fact that it is an Archimedean ordered group, $\tau(y) = \tau\sigma(x) = \sigma\tau(x) \succ \sigma(x) = y$, whence $\mathcal{T}$ is uniform.

Fix $x_0 \in X$. For each $x \in X$, by homogeneity there exists $\tau_x \in \mathcal{T}$ such that $\tau_x(x_0) = x$, and by uniformity $\tau_x \succsim' \tau_y$ if and only if $x \succcurlyeq y$. This defines a map of X onto $\mathcal{T}$, and by Hölder's theorem there is a map from $\mathcal{T}$ onto an additive subgroup $\langle T, \geqslant, + \rangle$ of $\mathbb{R}$. The composite mapping is from X to $\langle T, \geqslant, + \rangle$. The defining relations S_i of $\mathcal{X}$ relations may therefore be mapped onto real relations R_i on T. Let $\mathcal{R} = \langle T, \geqslant, R_1, ..., R_n \}$. This is a homogeneous unit structure because $\langle T, \geqslant, + \rangle$ being a group implies that properties 1 and 2 hold, and 3 holds because the translations of $\mathcal{X}$ map onto T since if $\tau \in \mathcal{T}$, then for some x, $\tau(x_0) = x$ and so $\tau = \tau_x$.

We now show that the automorphism group of $\mathcal{R}$ is a subgroup of the affine group on T.

Let τ be a translation of $\mathcal{R}$, so for any $r \in T$ and some $t \in T$, $\tau(r) = r + t$. Let α be any automorphism of $\mathcal{R}$. By Theorem 1(i) $\alpha\tau\alpha^{-1} \in T$, so from some $s(t) \in T$, $\alpha\tau\alpha^{-1}(r) = r + s(t)$. Set $p = \alpha(r)$,

$$\begin{aligned} \alpha(r) + s(t) &= p + s(t) \\ &= \alpha\tau\alpha^{-1}(p) \\ &= \alpha\tau(r) \\ &= \alpha(r + t). \end{aligned}$$

Note that since T is a subgroup of the additive real numbers, $0 \in T$, so $s(t) = \alpha(t) - \alpha(0)$. Setting $h(r) = \alpha(r) - \alpha(0)$, the above functional equation becomes

$$h(r + t) = h(r) + h(t). \tag{1}$$

Moreover, since α is order preserving, h is strictly increasing.

To solve this, we must consider two cases. First, suppose that for some $u, v \in T$, $u < v$, there is no $w \in T$ with $u < w < v$. We show that each $r \in T$ is of the form $r = u + nt$, where $t = v - u$. Because $\langle T, \geqslant, + \rangle$ is a group, there is some $p \in T$ such that $r = u + p$. By the Archimedeanness of T, there exists some integer n such that

$$u + (n - 1)\, t < r \leqslant u + nt,$$

so

$$u < r - (n - 1)\, t \leqslant u + t = v,$$

which is only possible, by hypothesis, if $r = u + nt$. With no loss of generality, we may assume $u = 0$ and $t = 1$, so Eq. (1) holds in integers. Let $h(1) = k$, then by induction $h(m) = mk$, so $\alpha(m) = a(0) + mk$, which is a subgroup of the affine group.

[12] The following proof mimics closely that of Alpers (1987) except that it deals explicitly with the discrete subcase rather than excluding it.

In the other case, T is order dense in the real numbers. Because h is strictly increasing, Eq. (1) can be extended to a real interval. It is well known in that case that its solution is $h(r) = cr$, $c > 0$, so $\alpha(r) = \alpha(0) + cr$, again a subgroup of the affine group.

(ii) ⇒ (iii) By Theorem 4(i) and (v)(b), $\mathscr{I}$ is a homogeneous, Archimedean ordered group. Following the same mapping technique as above but replacing $\mathscr{T}$ by $\mathscr{I}$, one represents each $\sigma \in \mathscr{I}$ as the real mapping $r \to s + r$ and those $\alpha \in \mathscr{A} \backslash \mathscr{I}$ by $r \to s + cr$. To show $\mathscr{T} = \mathscr{I}$, suppose we have $\tau \in \mathscr{T} \backslash \mathscr{I}$, which must be represented $r \to s + r$ if a fixed point is to be avoided. So for any map α with representation $r \to s + cr$, $c > 1$, we see that $s + cr - (ns + r) = -(n-1)\, s + (c-1)\, r$ becomes positive for sufficiently large r, so $\alpha \succsim' \tau^n$, proving $\tau \in \mathscr{I}$.

(iv) ⇒ (i)(ii)(iii) Trivial.

Clearly, by (iv), $1 \leqslant M \leqslant N \leqslant 2$, where M is the degree of homogeneity and N that of uniqueness. ∎

5. COMMENTS

First, Luce (1987) (see also Luce *et al.*, 1990, Theorem 20.7, p. 124) showed that the properties (ii) and (iv) of Theorem 5 are also equivalent to the following three conditions holding: The translations $\mathscr{T}$ of the ordered relational structure $\mathscr{X}$ form a group, $\mathscr{X}$ can be imbedded as the first component of a conjoint structure $\mathscr{C}$ that is Archimedean and solvable, and $\mathscr{X}$ distributes in $\mathscr{C}$, where *distributes* means the following. Each defining relation S_i has the property that if $(a_1, ..., a_{n(i)}) \in S_i$ and if for some p, q in the second component $(b_i, p) \sim (a_i, q)$, $i = 1, ..., n(i)$, then $(b_1, ..., b_{n(i)}) \in S_i$.

Second, Cameron (1989) constructed examples of ordered, countable structures of M-point homogeneity and N-point uniqueness for every M and N with $1 \leqslant M < N$, and Macpherson (1996)[13] did so for $M = N$. Save for, perhaps, the $M \geqslant 1$, $N = 2$ cases, Theorem 5 implies that either the translations or the dilations of these examples must be non-Archimedean.

Third, if $\mathscr{A}$ is N-point unique and $\mathscr{T}$ is homogeneous and a group, then Theorem 1(vii) shows directly that $\mathscr{A}$ is 2-point unique. Thus, Cameron's and Macpherson's results mean that in the countable case with $N > 2$, $\mathscr{T}$ either is not homogeneous or not a group.

Fourth, the proof that $\mathscr{A}$ is 2-point unique and that the automorphism group of $\mathscr{R}$ is a subgroup of the affine group was proved for the order dense case in Luce (1987) (see Corollary 2 of Theorem 20.7, Luce *et al.*, 1990, p. 124).

Fifth, can Alper's (1987) main theorem be reduced to property (i) or (ii) of Theorem 5? This does not seem implausible because often it has been possible to deduce the Dedekind completeness of specific structures from Archimedean assumptions. So, I was hoping that it might be possible to do so more directly than Alper's proof, which proceeded as follows. He used N-point uniqueness to show that the chain of distinct convex subgroups is finite. This is fairly subtle. Then by repeatedly using the theorem of Fuchs (1963, p. 50) cited in the proof of Theorem 4(v)(b) he

[13] I thank a referee for bringing this reference to my attention.

showed that the smallest one, which as we have shown turned out to be $\mathscr{T} = \mathscr{I}$, is homogeneous. I have been unable to find a more direct proof. One failed strategy was to try to show directly that the structure is 2-point unique, in which case the results of Narens (1981b) and Alper (1985) for the 2-point unique, Dedekind complete structures complete the proof.

6. THE FACTOR GROUP $\mathscr{A}/\mathscr{T}$

For the case where the translations are a group, the relation $\approx$ of parallel is an equivalence relation and $\mathscr{T}$ is an equivalence class (Theorem 2(ii)). On the resulting factor group $\mathbb{P} = \mathscr{A}/\mathscr{T}$ we may define an ordering and operation.

Definition 8. Suppose that $\mathscr{T}$ is a group, $\mathbb{P} = \mathscr{A}/\mathscr{T}$, and $\gtrsim'$ is connected. For $\mathscr{G}, \mathscr{H} \in \mathbb{P}$, define $\geqq'$ on $\mathbb{P}$ by $\mathscr{G} \geqq' \mathscr{H}$ if and only if either $\mathscr{G} = \mathscr{H}$ or $\mathscr{G} \neq \mathscr{H}$ and for all $\gamma \in \mathscr{G}$, $\eta \in \mathscr{H}$, $\gamma \gtrsim' \eta$. Define $\oplus$ on $\mathbb{P}$ by $\mathscr{G} \oplus \mathscr{H}$ equals the equivalence class of $\approx$ containing $\gamma\eta$.

Theorem 6. *Suppose that the conditions of Theorem 1 hold, that $\gtrsim'$ is connected, and that $\mathscr{T}$ is a convex group. Then:*

(i) *Both $\geqq'$ and $\oplus$ are well defined.*

(ii) *The structure $\langle \mathbb{P}, \geqq', \oplus \rangle$ is an ordered group with identity $\mathscr{T}$.*

(iii) *If in addition $\mathscr{D}$ is Archimedean, then $\langle \mathbb{P}, \geqq', \oplus \rangle$ is an Archimedean ordered group.*

Proof. (i) and (ii) are well-known facts from convex group theory.

(iii) Suppose $\mathscr{G}, \mathscr{H} \in \mathbb{P}$, $\mathscr{G} >' \mathscr{T}$, $\gamma \in \mathscr{G}$, and $\eta \in \mathscr{H}$. By the hypothesis that $\mathscr{D}$ is Archimedean, for some n, $\gamma^n > \eta$, which implies $\mathscr{G}^n >' \mathscr{H}$. ∎

This theorem provides an alternative proof that under the conditions of Theorem 5(i) $\mathscr{T}$ is homogeneous. For by Theorem 6(iii) and Hölder's theorem we know that $\oplus$ is commutative, so for any $\alpha, \beta \in \mathscr{A}$, $[\alpha\beta] = [\alpha] \oplus [\beta] = [\beta] \oplus [\alpha] = [\beta\alpha]$, which means that for some $\tau \in \mathscr{T}$, $\alpha\beta = \tau\beta\alpha$, i.e., $\alpha\beta\alpha^{-1}\beta^{-1} \in \mathscr{T}$. Thus, as in the proof of the homogeneity of $\mathscr{I}$ (Theorem 4(v)(b)), we obtain the homogeneity of $\mathscr{T}$.

7. OPEN PROBLEMS

One clearly open problem is to find a simpler way to reduce the proof of Alper's theorem to Archimedean conditions on $\mathscr{T}$ and $\mathscr{D}$.

A second arises from the fact that in empirical practice, one works with structures, not with translations and dilations. So a task is to find (observable) structural properties that lead to one or another of the conditions of Theorem 5(i), (ii), or (iii). Relatively little is known about what, structurally, gives rise to homogeneity.

Extensive structures are one example of structural conditions that do lead to homogeneity. A little more is known about Archimedeaness. For example, Cohen and Narens (1979) showed that what we now call PCSs (see Definition 19.3 of Luce *et al.*, 1990, p. 38) have Archimedean ordered automorphism groups. A conjecture I have been unable to prove is the following. Suppose a concatenation structure $\mathscr{X} = \langle X, \succsim, \circ \rangle$ is closed, idempotent, solvable, Archimedean in differences, and homogeneous. Then $\mathscr{X}$ has an Archimedean ordered translation group.

A possibly useful observation is found in Luce and Narens (1985) (see Theorem 19.15 of Luce *et al.*, 1990, p. 82), which showed that for each $a \in X$, such a concatenation structure induces a total concatenation structure $\mathscr{X}_a$, that the translations establish isomorphisms among these induced total concatenation structures, and that the group of dilations at a, $\mathscr{D}_a$, are the automorphisms of $\mathscr{X}_a$. This establishes that $\mathscr{T}$ is a group because isomorphism is a transitive relation and also that the groups $\mathscr{D}_a$ are each Archimedean and so commutative because each $\mathscr{X}_a$ is a total concatenation structure. I have not seen how to use this information to show that $\mathscr{T}$ is Archimedean.

Considerably more needs to be done to understand the structural conditions that give rise to homogeneity and Archimedeaness of the translations.

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