Mathematical Social Sciences 40 (2000) 277–295

mathematical social sciences

www.elsevier.nl/locate/econbase

Separable and additive representations of binary gambles of gains

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Received 1 April 1999; received in revised form 1 November 1999; accepted 1 November 1999

Abstract

Two approaches are taken to a new utility representation of binary gambles that is called "ratio rank-dependent utility." Both are based on known axiomatizations of a ranked-additive representation of consequence pairs (x, y) in binary gambles $(x, C; y)$ of gains with C held fixed and of a separable one of the special gambles $(x, C; e)$, where e denotes the status quo. The axiomatized version imposes the condition of status-quo event commutativity to get a functional equation that leads to the result. The other assumes, but does not axiomatize, a separable representation of the $(C; y)$ portion of the gamble. These assumptions lead to two difficult functional equations that are solved in the mathematical literature, but the former only under the assumption that the function is twice differentiable. Three behavioral conditions are shown to force this new utility representation to reduce to the standard rank-dependent utility one for gains. They are co-monotonic trade-off consistency, ranked bisymmetry, and segregation, the latter requiring the addition of an operation of joint receipt.

1. Introduction

We begin with the following primitives. The set $\mathscr{C}$ consists of pure consequences involving no uncertainty. Examples are goods purchased in reputable stores, money obtained from banks, etc. Let $e \in \mathscr{C}$ denote the consequence in which nothing happens, the status quo. Let **E** denote a chance phenomenon or experiment, in the sense used by probabilists and statisticians[1], with universal event E and an algebra $\mathscr{E}_{\mathbf{E}}$ of events. Thus,

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[1]Parzen (1960) used the phrase "chance phenomenon" whereas Feller (1950), Hayes (1963), and Rotor (1997) speak of an "experiment." This use of the term "experiment" is only loosely coupled to the uses in the experimental sciences.

0165-4896/00/$ – see front matter
PII: S0165-4896(99)00055-4

if $C \in \mathscr{E}_{\mathbf{E}}$, then $C \subseteq E$. We do not assume that a probability measure is associated with the experiment. Examples of the kinds of chance generating experiments we typically run in laboratory experiments are spinners on a partitioned pie and dice, both of which have a well known probability description, or drawing a colored ball from an opaque urn that contains some unknown mix of two colors of balls.

From these ingredients we may form binary first order gambles by assigning consequences in $\mathscr{C}$ to event partitions $\{C, \overline{C}\}$ where $C \in \mathscr{E}_{\mathbf{E}}$ and $\overline{C} = E \backslash C$. If x is assigned to C and y to $\overline{C}$, then we write the gamble as $(x, C; y, \overline{C})$ or, for simplicity of writing, just as $(x,C;y)$. Let $\succsim$ denote a weak (preference) order over consequences and gambles. Let $\mathscr{B}^+$ denote all such binary first-order gambles of gains[2], i.e., those with $x \succsim e$ and $y \succsim e$. It should be noted that the gambles cannot really be viewed as functions because we shall assume that the rank order of the consequences matters. So we will be required to make explicit certain assumptions (those below in which $\sim$ connects the two sides) that would be automatic were we dealing with functions.

In addition to first order gambles, we will also make use of second-order ones in which the experiment is run independently twice[3], and the consequences that are attached to the first running are themselves first-order gambles. A typical example is $((x, C; y), B; (u, D; v))$. So, in the first experiment if B occurs one receives the gamble $(x, C; y)$ and that is resolved by running the experiment again yielding x if C occurs and y otherwise. If, however, $\overline{B}$ occurs, then the second gamble is $(u, D; v)$ and it is resolved by running the experiment a second time. There will be no reason to go to higher order gambles.

A detailed discussion of this type of framework is provided by Luce (2000).

2. Singly separable additive utility

Wakker (1991, 1993) has provided axiomatic conditions on $\langle \mathscr{B}^+, \succsim \rangle$ such that there is a numerical, order-preserving, rank-dependent representation of the following character: For all consequences $x,y \in \mathscr{C}$ with $x \succsim y \succsim e$, and for each experiment $\mathbf{E}$ with universal set E, there are real-valued functions Θ with domain $\mathscr{B}^+$ and $\Phi_{\mathbf{E}}$ and $\Psi_{\mathbf{E}}$, both with domain $\mathscr{C} \times 2^E$, such that

$$\Theta(x, C; y) = \Phi_{\mathbf{E}}(x, C) + \Psi_{\mathbf{E}}(y, C). \tag{1}$$

In what follows, it will be simpler to dispense with the subscripts and show the dependence on $\mathbf{E}$ in terms of E as follows:

$$\Phi(x, C, E) := \Phi_{\mathbf{E}}(x, C) \quad \text{and} \quad \Psi(y, C, E) := \Psi_{\mathbf{E}}(y, C).$$

[2]Results paralleling those of this paper hold for the case $\mathscr{B}^-$ of all such binary gambles of losses, i.e., those with $x \precsim y \precsim e$. The case of gambles involving both gains and losses can be considerably more complex (Luce, 1997)

[3]We can generalize this to running two distinct, independent experiments, but for present purposed we do not need that generalization.

Because the three functions are unique up to positive affine transformations with a common unit, we may select them so that $\Phi(e, C, E) = 0$ and $\Psi(e, C, E) = 0$. By setting $y = e$, we see that $\Theta(x, C; e) = \Phi(x, C, E)$ represents $\succsim$ restricted to gambles of the form $(x, C; e)$.

Luce (1996) has also provided an axiomatization of gambles of the form $(x, C; e)$ so that they have a separable representation

$$u_1(x, C; e) = u_1(x)w_1(C), \tag{2}$$

where $u_1(e) = 0$, $w_1(\emptyset) = 0$, and $w_1(E) = 1$. (Note that the certainty property below insures that u_1 is defined over consequences as well as gambles.) Several fairly elementary axioms, which are often interpreted as capturing aspects of rationality, are crucial in doing this: For all first-order gambles $x, y, z \succsim e$ and events $C, D \subseteqq E$

Certainty: $(x, E; y) \sim x$.

Idempotence: $(x, C; x) \sim x$.

Complementarity: $(x, C; y) \sim (y, \overline{C}; x)$.

Consequence Monotonicity on First Component: For $\emptyset \subset C \subset E$, $x \succsim y \Leftrightarrow (x, C; z) \succsim (y, C; z)$.

Order Independence of Events: For $x, y \succ e$,

$$(x, C; e) \succsim (x, D; e) \Leftrightarrow (y, C; e) \succsim (y, D; e).$$

Status-Quo Event Commutativity for Any Consequence:

$$((x, C; e), D; e) \sim ((x, D; e), C; e). \tag{3}$$

Because u_1 and Φ preserve the same order, there is a strictly increasing function φ with $\varphi(0) = 0$ such that $\Phi = \varphi u_1$, so we can axiomatically justify a representation of the form: For $x \succsim y \succsim e$,

$$\Theta(x, C; y) = \varphi[u_1(x)w_1(C)] + \Psi(y, C, E).$$

Further, setting $C = E$ and $y = e$, and using both the certainty axiom and $\Psi(e, C; E) = 0$ yields

$$\Theta(x) = \varphi[u_1(x)]. \tag{4}$$

Under Luce's (1996) conditions, one can also justify the existence of certainty equivalents $CE(x, C; y) \sim (x, C; y)$, where CE is a mapping from binary gambles into sure consequences, and so we can write

$$\Theta(x, C; y) = \varphi(u_1[CE(x, C; y)]) = \varphi[u_1(x, C; y)].$$

Thus, the axiomatically justified representation is

$$\varphi[u_1(x, C; y)] = \varphi[u_1(x)w_1(C)] + \Psi(y, C, E), \tag{5}$$

where $\varphi(0) = 0$, $\Psi(e, C, E) = 0$, and φ is strictly increasing.

Because of the rank dependence constraint that $x \succsim y$, we cannot make a similar argument to show that the Ψ term has a separable representation. (Luce, 1998, claimed otherwise, but he was in error.) If one drops the rank dependence constraint, one can make the argument and use it to derive a binary rank-independent form that is closely related to SEU (Luce, 1998). However, as we shall see in Section 4 there is a way of getting separability on the second component. Here we proceed without it.

Assuming the certainty axiom and setting $C = E$ in Eq. (5), we get

$$\varphi[u_1(x)] = \varphi[u_1(x)] + \Psi(y, E; E),$$

and so $\Psi(y, E; E) = 0$. Similarly, assuming the idempotence axiom and setting $x = y = z$ in Eq. (5), we get

$$\varphi\big[u_1(z)\big] - \varphi\big[u_1(z)w_1(C)\big] = \Psi(z, C, E).$$

Substituting this back into Eq. (5) yields: for $x \succsim y \succsim e$,

$$\varphi\big[u_1(x, C; y)\big] = \varphi\big[u_1(x)w_1(C)\big] + \varphi\big[u_1(y)\big] - \varphi\big[u_1(y)w_1(C)\big], \tag{6}$$

i.e.,

$$u_1(x, C; y) = \varphi^{-1}\big(\varphi\big[u_1(x)w_1(C)\big] + \varphi\big[u_1(y)\big] - \varphi\big[u_1(y)w_1(C)\big]\big). \tag{7}$$

Since Eq. (7) gives a general representation of $u_1(x, C; y)$ in terms of φ, $u_1(x)$, $u_1(y)$, and $w_1(C)$, further conditions are required to restrict the form of φ. We know of two ways to get such added structure, and the results are very closely related.

3. Ratio rank-dependent utility

Although we used status-quo event commutativity in arriving at the representation in Eq. (7), we did not assume the more general property of *event commutativity* that for each consequence x,y, $x \succsim y \succsim e$, and events C, D

$$((x, C; y), D; y) \sim ((x, D; y), C; y). \tag{8}$$

Observe that status-quo event commutativity, Eq. (3), is the special case of event commutativity where $y = e$. The general property does not follow from the special one in general, but in Section 5.5 we show a case where it does.

So we now impose event commutativity which has received empirical support in (Chung et al., 1994).

Proposition 1. *Assume that φ satisfies Eq. (6) and that event commutativity holds. Let $k = \limsup u_1$. Then φ satisfies the functional equation for $k > X \geq Y \geq 0$ and W, $Z \in [0, 1]$,*

$$\begin{aligned}&\varphi\big(\varphi^{-1}[\varphi(XW) + \varphi(Y) - \varphi(YW)]Z\big) - \varphi(YZ)\\&\quad = \varphi\big(\varphi^{-1}[\varphi(XZ) + \varphi(Y) - \varphi(YZ)]W\big) - \varphi(YW).\end{aligned} \tag{9}$$

All proofs are found in Section 7.

Aczél and Maksa (2000) have solved this equation under the assumption that the domains and ranges are intervals and φ is twice differentiable. Their four solutions are for some constants α, β, γ, q

$$\varphi(v) = \frac{1}{\beta q} \ln(\alpha + \beta v^q) + \gamma \tag{10}$$

$$\varphi(v) = \frac{1}{\alpha q} v^q + \gamma \tag{11}$$

$$\varphi(v) = \frac{1}{\alpha} \ln(-\alpha \ln v + \beta) + \gamma \tag{12}$$

$$\varphi(v) = -\alpha \ln v + \beta \tag{13}$$

Two requirements that we have on φ are that $\varphi(0)=0$ and that φ is strictly increasing. Imposing these conditions on each of Eqs. (10)–(13) in turn:

(i) From Eq. (10) and $\varphi(0)=0$, we have $\gamma = -(1/\beta q) \ln \alpha$. Thus, $\alpha > 0$. Setting $\mu = \beta/\alpha$, Eq. (10) becomes

$$\varphi(v) = \frac{1}{\beta q} \ln(1 + \mu v^q), \quad q > 0. \tag{14}$$

Because φ is strictly increasing and $\alpha > 0$, we see from $\varphi'(v) > 0$ that $\mu\beta > 0$. Also, because φ is well defined, we must have $1 + \mu v^q > 0$. Thus, for $\mu > 0$ this imposes no restriction, but for $\mu < 0$, it means that $\mu > -1/k^q$.

(ii) From Eq. (11) and $\varphi(0)=0$, we see that $\gamma = 0$, and so the second case is

$$\varphi(v) = \frac{1}{\alpha q} v^q, \quad q > 0. \tag{15}$$

Because φ is strictly increasing $\alpha > 0$.

(iii) and (iv) Eqs. (12) and (13) are ruled out by the condition $\varphi(0)=0$ because either $\varphi(v) \equiv 0$, which violates the strict monotonicity of φ, or it fails because $\ln 0 = -\infty$.

So we need only see what Eqs. (14) and (15) imply. From the literature on additive conjoint measurement, conditions sufficient for the relevant mappings to be onto real intervals are well known, and so they are not restated here. In practice it means that the consequences must include money among them and that the events are like those underlying continuous probability.

Theorem 1. *Assume that Eq. (6) and event commutativity both hold. Suppose the image of u_1 is the real interval $[0, k[$ and w_1 is onto $[0, 1]$. Letting $U = u_1^q$ and $W = w_1^q$, then Eq. (14) implies the following order preserving representation of gambles of gains: There exists some $\mu \neq 0$, $\mu > -1/k^q$ such that for $x \succsim y \succsim e$,*

$$U(x, C; y) = \begin{cases} \dfrac{U(x)W(C) + U(y)[1 - W(C)] + \mu U(x)U(y)W(C)}{1 + \mu U(y)W(C)}, & \text{if } x \succ y \succsim e \\ U(x), & \text{if } x \sim y \succsim e \\ \dfrac{U(x)[1 - W(\overline{C})] + U(y)W(\overline{C}) + \mu U(x)U(y)W(\overline{C})}{1 + \mu U(x)W(\overline{C})} & \text{if } y \succ x \succsim e. \end{cases} \tag{16}$$

Eq. (15) yields the special case of Eq. (16) with $\mu = 0$, *which is the standard rank-dependent expected utility model.*

Observe that the unit of μ is the inverse of the unit of U.

We hope that eventually the same results will be shown under only the assumption that φ is strictly increasing, rather than that plus the strong condition that it is also second-order differentiable. However, in a personal communication, Dr. János Aczél, an expert on functional equations, conjectures that this will not be easily done. So we have elected to present the result as now known.

This representation with $\mu \neq 0$ is, as far as we know, different from anything that has previously appeared in the literature. Even the numerator is somewhat different from somewhat related forms not involving the W terms discussed in Keeney and Raiffa (1976). The case of $\mu = 0$ is the familiar rank-dependent, (subjective) utility model, RDU. We call the representation of Eq. (16) *ratio rank-dependent utility*, RRDU, where the adjective ‘‘ratio’’ refers to the fact of its being the ratio of two polynomials. (Given that mathematicians call the ratio of two polynomials a *rational* function, we were tempted to use that adjective; however, given the informal uses of ‘‘rational’’ in discussing various aspects of utility theory, a referee persuaded us not to.)

It would be nice if we could provide some intuitive interpretation of this form with $\mu \neq 0$, but we know of none beyond the axioms that give rise to it.

Although we used event commutativity, Eq. (8), in deriving the RRDU representation, it is well to verify that it is in fact satisfied, which is certainly not immediately obvious. Indeed, when we first encountered RRDU, which is certainly not a fully symmetric formula, we anticipated that event commutativity would force RDU. It doesn't.

Proposition 2. *RRDU satisfies event commutativity.*

4. Doubly separable additive utility

In this section we investigate the assumption that there are also functions u_2 and w_2 with $u_2(e) = 0$, $w_2(\emptyset) = 0$, and $w_2(E) = 1$, such that

$$\varphi[u_1(x, C; y)] = \varphi[u_1(x)w_1(C)] + \psi[u_2(y)w_2(C)]. \tag{17}$$

We do not know how to axiomatize the right hand term in a fully satisfactory fashion. We outline two approaches and cite their drawbacks.

The first issue is to axiomatize the separability of the (y, C) tradeoff. One way to do this rests on a stronger variant of event commutativity that is sufficient,[4] namely, for $x \succsim y \succsim e$ and events $C, D \in \mathscr{E}_{\mathbf{E}}$,

$$(x, C; (x, D; y)) \sim (x, C; (x, D; y)). \tag{18}$$

(This would follow from event commutativity and complementarity if the only restrictions were $x \succsim e$ and $y \succsim e$ and if consequence monotonicity were to hold in second component as well as the first.)

Using Eq. (18), holding x fixed, and following essentially the same argument as used by Luce (1996) for establishing the Thomsen condition for the (x, C) pairs, one shows that under the usual assumptions of monotonicity, restricted solvability, and Archimedeanness there is a separable representation of the (y, C) pairs,

$$u_x(y)w_x(C), \quad x \succsim y \succsim e.$$

This is not yet fully satisfactory because of the dependence of the functions on the choice of x and on the fact that for any specific choice of x it will in general exclude many possible values of y. Note, however, that for $x' \succ x$, $u_{x'}(y)w_{x'}(C)$ and $u_x(y)w_x(C)$ are two multiplicative representations of the region where $y \precsim x$, and so by the well known uniqueness result of conjoint measurement, for some $\beta > 0$, $u_{x'}^{\beta}(y)w_{x'}^{\beta}(C) = u_x(y)w_x(C)$. Thus, by a countable induction one can span the entire region with a separable representation $u_2(y)w_2(C)$.

This does not, however, justify Eq. (17) because we have no argument showing that $u_2(y)w_2(C)$ and $\Psi(y,C,E)$ preserve the same order. To find the order preserved by $\Psi(y,C,E)$, one needs to make the first term of Eq. (5), $\varphi[u_1(x)w_1(C)]$, a constant. Because $x \succsim y$, we cannot make it 0, and so x must be some function of C. But that order is not that of $u_2(y)w_2(C)$, which was derived on the assumption of a fixed x. We do not see how to overcome this problem, and we leave it for future research.

A second approach was suggested by a referee. It is motivated by the idea if we can hold the term $u_1(x)w_1(C)$ constant as we vary y and C, then in the additive expression the first terms cancel and we are left with relations that only depend on pairs of terms, (y, C) and (y', C'). To do this, one defines

$$(y, C) \succsim^* (y'C')$$

if and only if

$$\exists x, x' \in \mathscr{B}^+ (x, C; e) \sim (x', C'; e) \;\&\; (x, C; y) \succsim (x', C'; y').$$

For this to be well defined, the obvious consistency condition is

$$\text{If } (x, C; e) \sim (x', C'; e) \text{ and } (z, C; e) \sim (z', C'; e), \text{ then}$$

$$(x, C; y) \succsim (x', C'; y') \Leftrightarrow (z, C; y) \succsim (z', C'; y').$$

[4] We thank the referee for suggesting how this should be done.

This assumption does not strike us as particularly compelling. Accepting it, then one simply assumes that $\succsim^*$ satisfies suitable conjoint axioms so as to get the multiplicative representation, which means primarily the Thomsen condition. These axioms, when translated back into the original ordering $\succsim$, are exceedingly complex, and we have seen no behavioral property of comparable simplicity to status quo event commutativity that gives rise to them. Thus, as yet, we are not greatly taken by this approach.

Nonetheless, it seems interesting to know where Eq. (17) leads.

Theorem 2. *Assume that Eq. (17) and event commutativity both hold. Suppose* u_i *have as their image the real interval* $[0, k[$ *and the* w_i *are onto* $[0, 1]$, $i = 1, 2$. *Then there exist constants* q *and* $\mu \geq -1/k^q$ *and functions* U *over gambles onto* $[0, k^q[$ *and* W *over events onto* $[0, 1]$ *such that the representation Eq. (16) holds.*

The proof of this result rests upon the solution to Eq. (17) worked out by (Aczél et al., 2000) (see Section 7.4).

5. Properties that force RDU

It is certainly interesting to understand what properties will force RRDU to be RDU, i.e., force $\mu = 0$. Of course, assuming $u_2 = u_1$ does the job in Theorem 2, but that is not relevant to Theorem 1 and it is not a behavioral property that can be studied in isolation empirically. At the present time, we know of three conditions that do force $\mu = 0$.

5.1. Co-monotonic consistency

A behavioral property having much to do with RDU was suggested and used in deriving that representation by Wakker (1989) and Wakker and Tversky (1993). We say that binary gambles exhibit *co-monotonic trade-off consistency* if the following condition is satisfied. Let

$$\mathbf{f} = (f, C; y), \quad \mathbf{f}' = (f', C; y)$$

$$\mathbf{g} = (g, C; z), \quad \mathbf{g}' = (g', C; z)$$

$$\mathbf{h} = (f, C; u), \quad \mathbf{h}' = (f', C; u)$$

$$\mathbf{k} = (g, C; v), \quad \mathbf{k}' = (g', C; v),$$

where in each gamble the first consequence $\succsim$ the second consequence. Then not both $\{\mathbf{f} \succsim \mathbf{g},\ \mathbf{f}' \prec \mathbf{g}'\}$ and $\{\mathbf{h} \prec \mathbf{k},\ \mathbf{h}' \succsim \mathbf{k}'\}$.

We do not find co-monotonic trade-off consistency to be a very compelling behavioral property; no one that we know of has an ''of course'' reaction to it. So, although as we shall see it forces $\mu = 0$, it does not provide a particular satisfactory defense of RDU.

5.2. Rank-dependent bisymmetry

A second behavioral property, closely related to a mathematical one that has been widely studied (Aczél, 1966), can be called *rank-dependent bisymmetry:* Consider consequences $x \succsim y \succsim s \succsim e$, $x \succsim r \succsim s \succsim e$ and events C, C', and C'' which are the event C on two independent replications of the underlying experiment **E.** Then

$$((x, C'; y), C; (r, C''; s)) \sim ((x, C'; r), C; (y, C''; s)). \tag{19}$$

This property has been used by Chew (1989) and Nakamura (1990, 1992).

Note that the assumed conditions on the consequences, plus consequence monotonicity and complementarity, imply

$$(x, C'; y) \succsim (r, C''; s), \quad (x, C'; r) \succsim (y, C''; s).$$

Rank-dependent bisymmetry seems rational in the sense that the two sides of Eq. (19) have the same bottom line; however, there does not seem to be any a priori reason for imposing rank dependence.

5.3. Joint receipt and segregation

Luce and Fishburn (1991, 1995) have suggested that it is interesting to add as a primitive to the system a binary relation $\oplus$ of joint receipt. The interpretation of $x \oplus y$ is having (or receiving) both x and y at the same time. As they did, we assume that $\oplus$ is commutative. They suggested the following highly rational property linking gambles and joint receipt of consequences: *Segregation* is said to hold iff for all x, $y \succsim e$ and all events $C \subseteqq E$

$$(x, C; e) \oplus y \sim (x \oplus y, C; y). \tag{20}$$

For money gambles and interpreting $\oplus = +$, Kahneman and Tversky (1979) had previously invoked this property as a form of pre-editing. Luce and Fishburn (1991) showed that RDU and segregation imply that for some constant $\delta \in \mathbb{R}$ with the dimensions of $1/U$,

$$U(x \oplus y) = U(x) + U(y) - \delta U(x)U(y). \tag{21}$$

From this one easily shows that:

1. There is an additive representation V of $\oplus$, i.e., for all $x,y \succsim e$

$$V(x \oplus y) = V(x) + V(y).$$

2. If $\delta = 0$, then for some $\alpha > 0$,

$$U = \alpha V. \tag{22}$$

3. If $\delta > 0$, then U is subadditive, i.e., $U(f \oplus g) < U(f) + U(g)$, bounded by $\Delta =$

$(1/|\delta|) > 0$, and for some $\kappa > 0$

$$U(f) = \Delta\left(1 - \mathrm{e}^{-\kappa V(f)}\right), \tag{23}$$

4. If $\delta < 0$, then U is superadditive, i.e., $U(f \oplus g) > U(f) + U(g)$, unbounded, and for $\Delta = 1/|\delta|$ and some $\kappa > 0$

$$U(f) = \Delta(\mathrm{e}^{\kappa V(f)} - 1). \tag{24}$$

By a Theorem on p. 61 of Aczél (1966), one knows that Eq. (21) is the most general form of a polynomial that also has an additive representation with 0 representing the identity e. So we are led to call this the *polynomial additive,* or *p-additive* for short, representation of $\oplus$.

The above result shows that segregation together with RDU, i.e., RRDU with $\mu = 0$, are consistent, provided the utility function U is one of the forms 1–4 above. Part (ii) of the following result shows that this is the *only* case where RRDU and segregation both hold.

5.4. The result

Theorem 3. *Suppose the RRDU representation of Eq. (16) holds with the image of U a real interval* $[0, B[$ *and W onto* $[0, 1]$.

(i) *Then the following three statements are equivalent*:
  (a) $\mu = 0$, *i.e., RDU.*
  (b) *Co-monotonic trade-off consistency holds.*
  (c) *Rank-dependent bisymmetry holds.*

(ii) *If segregation of joint receipt holds, then* $\mu = 0$

5.5. Event commutativity and segregation

An interesting observation is that event commutativity, Eq. (8) follows from segregation and status-quo event commutativity, Eq. (3), provided that for $x \succsim y$, there exists z such that $x \sim y \oplus z$. In that case let $\ominus$ be defined by

$$x \ominus y \sim z \text{ iff } x \sim y \oplus z.$$

Then

$$\begin{aligned}((x, C; y), D; y) &\sim ((x\ominus y, C; e) \oplus y, D; y) \sim ((x\ominus y, C; e), D; e) \oplus y \\ &\sim ((x\ominus y, D; e), C; e) \oplus y \sim ((x\ominus y, D; e) \oplus y, C; y) \sim ((x, D; y), C; y).\end{aligned}$$

6. Conclusions

We began with two major features of the rank-dependent utility (RDU) representation for binary gambles $(x, C; y)$. First, with the event C held fixed, there is a ranked-additive representation of the consequence pair (x, y). This has been axiomatized by Wakker (1991, 1993). Second, with $y = e$, the status quo, there is a separable representation of the (x, C) pair. This has been axiomatized by Luce (1996). Furthermore, RDU exhibits the important property of event commutativity. The paper first investigated what assuming these three properties implies. The basic finding, Theorem 1, is that they give rise to what we called a ratio rank-dependent utility (RRDU) representation, of which RDU is a special case. The proof rests on solving a functional equation, which has only been done in the mathematical literature with the extra assumption that the unknown function is twice differentiable. We also examined the (unaxiomatized) assumption that in the additive representation the representation of the (y, C) term can be expressed as a function of a separable representation. We explained the nature of the difficulty in achieving an axiomatization of this representation. The resulting functional equation, which is solved in the literature without any additional assumptions, leads to the same RRDU form.

The derivation of RRDU is not fully satisfactory in either Theorem 1 or Theorem 2. Theorem 1 rests upon the extra assumption that the unknown function in Eq. (9) is twice differentiable. Theorem 2 rests upon the unaxiomatized assumption that the term $\Psi(y, C, E)$ of Eq. (5) can be expressed as $\psi[u_2(y)w_2(C)]$. We hope that future research will improve one or the other result.

We then investigated conditions that force RRDU to reduce to RDU. Working solely in terms of gambles, either of two conditions worked: co-monotonic trade-off consistency or ranked bisymmetry (Theorem 3(i)). However, we do not consider either of these properties as particularly satisfactory behaviorally. If, however, we add the operation of joint receipt and assume the (intuitively rational) property of segregation, then except for RDU, segregation and RRDU are inconsistent (Theorem 3(ii)).

7. Proofs

7.1. Proof of Proposition 1

Let

$$X = u_1(x), Y = u_1(y), W = w_1(C), Z = w_1(D), k = \limsup u_1$$

then for all X, Y, W, Z such that $k > X \geq Y \geq 0$ and $W, Z \in [0,1]$ Eqs. (4) and (6) yield

$$\begin{aligned}
&\varphi[u_1((x, C; y), D; y)] \\
&\quad = \varphi[u_1(x, C; y)w_1(D)] + \varphi[u_1(y)] - \varphi[u_1(y)w_1(D)] \\
&\quad = \varphi[\varphi^{-1}(\varphi[u(x, C; y)])w_1(D)] + \varphi[u_1(y)] - \varphi[u_1(y)w_1(D)]
\end{aligned}$$

$$= \varphi\big(\varphi^{-1}[\varphi[u_1(x)w_1(C)] + \varphi[u_1(y)] - \varphi[u_1(y)w_1(C)]]w_1(D)\big)$$

$$+ \varphi[u_1(y)] - \varphi[u_1(y)w_1(D)]$$

$$= \varphi\big(\varphi^{-1}[\varphi(XW) + \varphi(Y) - \varphi(YW)]Z\big) + \varphi(Y) - \varphi(YZ).$$

So event commutativity, Eq. (8) imposes Eq. (9). □

7.2. *Proof of Theorem 1*

Substituting Eq. (14) into Eq. (6) and using $U = u_1^q$, $W = w_1^q$, and recalling that $\beta\mu > 0$ and $1 + \mu u_1^q > 0$

$$\frac{1}{\beta q}\ln[1 + \mu U(x, C; y)] = \frac{1}{\beta q}\ln[1 + \mu U(x)W(C)] + \frac{1}{\beta q}\ln[1 + \mu U(y)] - \frac{1}{\beta q}\ln[1 + \mu U(y)W(C)]$$
$$= \frac{1}{\beta q}\ln\left(\frac{[1 + \mu U(x)W(C)][1 + \mu U(y)]}{[1 + \mu U(y)W(C)]}\right),$$

whence

$$[1 + \mu U(x, C; y)] = \frac{[1 + \mu U(x)W(C)][1 + \mu U(y)]}{1 + \mu U(y)W(C)},$$

and so

$$U(x, C; y) = \frac{U(x)W(C) + U(y)[1 - W(C)] + \mu U(x)U(y)W(C)}{1 + \mu U(y)W(C)}.$$

The case $e \precsim x \precsim y$ is obtained from the complementarity axiom. The condition $1 + \mu u_1^q > 0$ implies that $\mu > -1/k^q$.

From Eq. (15) we see that with the same substitutions

$$\frac{1}{\alpha q}U(x, C; y) = \frac{1}{\alpha q}U(x)W(C) + \frac{1}{\alpha q}U(y) - \frac{1}{\alpha q}U(y)W(C),$$

whence

$$U(x, C; y) = U(x)W(C) + U(y)[1 - W(C)],$$

which of course is the standard RDU model. □

7.3. Proof of Proposition 2

Using the brief notation introduced in Proposition 1, then from Eq. (16)

$$U((x, C; y), D; y) = \frac{U(x, C; y)Z(1+\mu Y) + Y(1-Z)}{1+\mu YZ}$$
$$= \frac{\left(\dfrac{XW + Y(1-W) + \mu XYW}{1+\mu YW}\right)Z(1+\mu Y) + Y(1-Z)}{1+\mu YZ}$$
$$= \frac{WZ\left[X(1+\mu Y)^2 - Y - 2\mu Y^2\right] + Y + (W+Z)\mu Y^2}{(1+\mu YW)(1+\mu YZ)},$$

which is clearly commutative in W and Z. □

7.4. Proof of Theorem 2

In Eq. (17) let $x = y$, $v = u_1(x)$, $w = w_1(C)$, $u_2(x) = g[u_1(x)] = g(v)$, and $w_2(C) = Q(w)$. Note that because of order preserving properties, g is strictly increasing and Q is strictly decreasing with $Q(0) = 1$ and $Q(1) = 0$. Setting $C = \emptyset$ in Eq. (17) yields

$$\varphi(v) = \Psi[g(v)], \tag{25}$$

and so Eq. (17) with $x = y$ implies

$$\varphi(v) = \varphi(vw) + \varphi\left(g^{-1}[g(v)Q(w)]\right), \quad v \in [0, k[, w, Q(w) \in [0, 1], \tag{26}$$

where the domain of φ, $[0, k[$, is the image of u_1, $\varphi(0) = 0$, $\varphi(1) = 1$, and φ is strictly increasing. (Aczél et al., 2000) have shown, on the assumption that g is strictly increasing, that the solutions to Eq. (26) are with $q > 0$, $\gamma > 0$, and $\mu \geq -1/k^q$

$$\varphi(v) = \frac{\ln(1+\mu v^q)}{\ln(1+\mu)} \tag{27}$$

$$g(x) = \left(\frac{(\mu+1)x^q}{1+\mu x^q}\right)^{1/\gamma} \tag{28}$$

$$Q(w) = (1-w^q)^{1/\gamma}, \tag{29}$$

including the limit as $\mu \to 0$, namely

$$\varphi(v) = v^q \tag{30}$$

$$g(v) = v^{q/\gamma} \tag{31}$$

$$Q(w) = (1-w^q)^{1/\gamma}. \tag{32}$$

By taking the derivative of φ in Eq. (27) we see that for $\mu \neq 0$, a necessary condition for φ to be strictly increasing is that $q\mu > 0$, and for Q to be in [0, 1] it is necessary that $q > 0$ and $\gamma > 0$.

It is easy to verify that

$$g^{-1}(v) = \left(\frac{v^{\gamma}}{(\mu+1) - \mu v^{\gamma}} \right)^{1/q}.$$

Let $U = u_1^q$ and $W = w_1^q$. From Eqs. (17) and (25) and the expressions for Φ, g, g^{-1}, Q we see that

$$1 + \mu U(x, C; y) = [1 + \mu U(x)W(C)]\left[1 + \mu g^{-1}(g[u_1(y)]Q(w_1(C))\right].$$

So

$$\begin{aligned} U(x, C; y) &= U(x)W(C) + \frac{U(y)[1 - W(C)]}{1 + \mu U(y)W(C)} + \mu \frac{U(x)W(C)U(y)[1 - W(C)]}{1 + \mu U(y)W(C)} \\ &= \frac{U(x)W(C) + U(y)[1 - W(C)] + \mu U(x)U(y)W(C)}{1 + \mu U(y)W(C)}. \quad \square \end{aligned}$$

7.5. Proof of Theorem 3

Part (i) (a) $\Rightarrow$ (b) and (c). These are easily checked.

(b) $\Rightarrow$ (a) In the gambles involved in co-monotonic trade-off consistency, use upper case letters for U of the lower case argument and $W = W(C)$ so, for example,

$$U(\mathbf{f}) = \frac{FW + Y(1 - W) + \mu FYW}{1 + \mu YW}.$$

Then the first pair of inequalities is equivalent to:

$$\frac{F - F'}{G - G'} > \frac{(1 + \mu Z)(1 + \mu YW)}{(1 + \mu Y)(1 + \mu ZW)}$$

and the second to:

$$\frac{F - F'}{G - G'} < \frac{(1 + \mu U)(1 + \mu VW)}{(1 + \mu V)(1 + \mu UW)}.$$

Clearly, with $\mu = 0$ this is impossible. Suppose $\mu > 0$. The latter right hand term is greater than the former one if $U > Z$ and $Y > V$ because both

$$\frac{1 + \mu U}{1 + \mu UW} > \frac{1 + \mu Z}{1 + \mu ZW}$$

and

$$\frac{1 + \mu VW}{1 + \mu V} > \frac{1 + \mu YW}{1 + \mu Y}$$

$$\Leftrightarrow \frac{1 + \mu V}{1 + \mu VW} < \frac{1 + \mu Y}{1 + \mu YW}.$$

Given that inequality, one chooses F, F', G, G' to be such that

$$\frac{F-F'}{G-G'}$$

lies between the limits. If $\mu<0$, the argument is similar with $U<Z$ and $Y<V$.

(c) ⇒ (a)[5] Using abbreviations $X=U(x)$, etc., and $W=W(C)$, $\overline{W}=1-W(C)$, $U[(x, C; y), C; (r, C, v)]$ can be written in the form

$$U[(x, C; y), C; (r, C, s)] = \frac{a(X, Y, R, S, W, \overline{W}, \mu)}{b(Y, R, S, W, \overline{W}, \mu)}$$

where

$$\begin{aligned} a(X, Y, R, S, W, \overline{W}, \mu) = {} & (XW + Y\overline{W} + \mu\, XYW)W(1+\mu SW) \\ & + (RW + S\overline{W} + \mu\, RVW)\overline{W}(1+\mu\, YW) \\ & + \mu\, W(XW + Y\overline{W} + \mu\, XYW)(RW + S\overline{W} + \mu\, RSW), \end{aligned}$$

and

$$b(Y, R, S, W, \overline{W}, \mu) = (1+\mu YW)[1+\mu SW + \mu W(RW + S\overline{W} + \mu RSW)].$$

Rank-dependent bisymmetry, i.e.,

$$U[(x, C; y), C; (r, C, s)] - U[(x, C; r), C; (y, C, s)] = 0,$$

then requires that

$$a(X, Y, R, S, W, \overline{W}, \mu)b(R, Y, S, W, \overline{W}, \mu) - a(X, R, Y, S, \mu)b(Y, R, S, W, \overline{W}, \mu) = 0 \tag{33}$$

If Eq. (33) holds for all the non-negative X, Y, R, S in the range $[0, B[$ of the utility function U that satisfy the conditions of rank-dependent bisymmetry, then it holds for all positive X in the range of U and for

$$Y=\frac{X}{2}, R=\frac{X}{3}, S=\frac{X}{4}, W=\frac{1}{5}, \overline{W}=1-\frac{1}{5}=\frac{4}{5}.$$

Substituting these values in the first term of Eq. (33), we have

$$\begin{aligned} & a\left(X, \frac{X}{2}, \frac{X}{3}, \frac{X}{4}, \frac{1}{5}, \frac{4}{5}, \mu\right) b\left(\frac{X}{3}, \frac{X}{2}, \frac{X}{4}, \frac{1}{5}, \frac{4}{5}, \mu\right) \\ & \quad = \Bigg[\left(X\frac{1}{5} + \frac{X}{2}\frac{4}{5} + \mu X \frac{X}{2}\frac{1}{5}\right)\frac{1}{5}\left(1+\mu\frac{X}{4}\frac{1}{5}\right) \\ & \qquad + \left(\frac{X}{3}\frac{1}{5} + \frac{X}{4}\frac{4}{5} + \mu\frac{X}{3}\frac{X}{4}\frac{1}{5}\right)\frac{4}{5}\left(1+\mu\frac{X}{2}\frac{1}{5}\right) \end{aligned}$$

[5] We thank Martin Chabot for this proof.

$$
\begin{aligned}
&+\mu \frac{1}{5}\left(X \frac{1}{5}+\frac{X}{2} \frac{4}{5}+\mu X \frac{X}{2} \frac{1}{5} \frac{X}{3} \frac{1}{5}+\frac{X}{4} \frac{4}{5}+\mu \frac{X}{3} \frac{X}{4} \frac{1}{5}\right)\Big] \\
&\cdot\left(\left[1+\mu \frac{X}{3} \frac{1}{5}\right]\left[1+\mu \frac{X}{4} \frac{1}{5}+\mu \frac{1}{5}\left(\frac{X}{2} \frac{1}{5}+\frac{X}{4} \frac{4}{5}+\mu \frac{X}{2} \frac{X}{4} \frac{1}{5}\right)\right]\right) \\
=&\left[\frac{1}{1000} X(6+\mu X)(20+\mu X)+\frac{1}{750} X(16+\mu X)(10+\mu X)\right. \\
&\left.+\frac{1}{3000} \mu X^{2}(6+\mu X)(16+\mu X)\right] \cdot \frac{1}{3000}(15+\mu X)(200+22 \mu X+\mu^{2} X^{2}) \\
=&\frac{1}{3000}\left(\frac{1}{3} X+\frac{139}{1500} \mu X^{2}+\frac{29}{3000} \mu^{2} X^{3}+\frac{1}{3000} \mu^{3} X^{4}\right) \\
&\cdot(15+\mu X)(200+22 \mu X+\mu^{2} X^{2}) \qquad (34)
\end{aligned}
$$

Similarly, substituting the values in the second term of Eq. (33), we have

$$
\begin{aligned}
&a\left(X, \frac{X}{3}, \frac{X}{2}, \frac{X}{4}, \frac{1}{5}, \frac{4}{5}, \mu\right) b\left(\frac{X}{2}, \frac{X}{3}, \frac{X}{4}, \frac{1}{5}, \frac{4}{5}, \mu\right) \\
=&\left[\left(X \frac{1}{5}+\frac{X}{3} \frac{4}{5}+\mu X \frac{X}{3} \frac{1}{5}\right) \frac{1}{5}\left(1+\mu \frac{X}{4} \frac{1}{5}\right)\right. \\
&+\left(\frac{X}{2} \frac{1}{5}+\frac{X}{4} \frac{4}{5}+\mu \frac{X}{2} \frac{X}{4} \frac{1}{5}\right) \frac{4}{5}\left(1+\mu \frac{X}{3} \frac{1}{5}\right) \\
&\left.+\mu \frac{1}{5}\left(X \frac{1}{5}+\frac{X}{3} \frac{4}{5}+\mu X \frac{X}{3} \frac{1}{5}\right)\left(\frac{X}{2} \frac{1}{5}+\frac{X}{4} \frac{4}{5}+\mu \frac{X}{2} \frac{X}{4} \frac{1}{5}\right)\right] \\
&\times\left(\left[1+\mu \frac{X}{2} \frac{1}{5}\right]\left[1+\mu \frac{X}{4} \frac{1}{5}+\mu \frac{1}{5}\left(\frac{X}{3} \frac{1}{5}+\frac{X}{4} \frac{4}{5}+\mu \frac{X}{3} \frac{X}{4} \frac{1}{5}\right)\right]\right) \\
=&\left[\frac{1}{1500} X(7+\mu X)(20+\mu X)+\frac{1}{750} X(12+\mu X)(15+\mu X)\right. \\
&\left.+\frac{1}{3000} \mu X^{2}(7+\mu X)(12+\mu X)\right] \cdot \frac{1}{3000}(10+\mu X)(300+31 \mu X+\mu^{2} X^{2}) \\
=&\frac{1}{3000}\left(\frac{1}{3} X+\frac{41}{500} \mu X^{2}+\frac{1}{120} \mu^{2} X^{3}+\frac{1}{3000} \mu^{3} X^{4}\right) \\
&\times(10+\mu X)(300+31 \mu X+\mu^{2} X^{2}). \qquad (35)
\end{aligned}
$$

Substituting the expressions given by Eqs. (34) and (35) into Eq. (33), and multiplying both sides of the equation by 3000, we obtain

$$
\begin{aligned}
&(1000X+278\mu X^{2}+29\mu^{2}X^{3}+\mu^{3}X^{4})(3000+530\mu X+37\mu^{2}X^{2}+\mu^{3}X^{3}) \\
&\quad-(1000X+246\mu X^{2}+25\mu^{2}X^{3}+\mu^{3}X^{4})(3000+610\mu X+41\mu^{2}X^{2}+\mu^{3}X^{3}) \\
=&3000000X+1364000\mu X^{2}+271340\mu^{2}X^{3}+29656\mu^{3}X^{4}+1881\mu^{4}X^{5} \\
&+66\mu^{5}X^{6}+\mu^{6}X^{7}-3000000X-1348000\mu X^{2}-266060\mu^{2}X^{3} \\
&-29336\mu^{3}X^{4}-1881\mu^{4}X^{5}-66\mu^{5}X^{6}-\mu^{6}X^{7}
\end{aligned}
$$

$$
\begin{aligned}
&= 16000\mu X^2 + 5280\mu^2 X^3 + 320\mu^3 X^4 \\
&= \mu X^2 (16000 + 5280\mu X + 320\mu^2 X^2)
\end{aligned}
$$

for all positive X in the range of U. Because both X and the numerical coefficients are positive, this can only be satisfied with $\mu = 0$.

Part (ii). Suppose that segregation and the RRDU representation with $\mu \neq 0$ both hold. Let $X = |\mu| U(x)$, $Y = |\mu| U(y)$, and $W = W(C)$. In the following proof, we require various quantities, such as YW and XY, to be in the image of $|\mu| U$, and various denominators, such as X, Y, XY, $1 \pm YW$, and $1 \pm XY$, to be nonzero. Since one of the conditions of the RRDU representation is $0 \leq |\mu| U \leq 1$ (see the statement of Theorem 1), the required conditions are satisfied provided we restrict the proof to X, Y, $W \in]0,1[\cap \{\text{range of } |\mu| U\}$. Now let

$$
G(X,Y) = |\mu| U(x \oplus y) = |\mu| U\left[U^{-1}\left(\frac{X}{|\mu|}\right) \oplus U^{-1}\left(\frac{Y}{|\mu|}\right) \right].
$$

Then segregation with the RRDU representation becomes with $\pm$ meaning use $+$ for $\mu > 0$ and $-$ for $\mu < 0$:]

$$
\begin{aligned}
G(XW, Y) &= |\mu| U\left[U^{-1}\left(\frac{XW}{|\mu|}\right) \oplus U^{-1}\left(\frac{Y}{|\mu|}\right) \right] \\
&= |\mu| U\left(U^{-1}[U(x)W(C)] \oplus y\right) \\
&= |\mu| U\left(U^{-1}[U(x, C; e)] \oplus y\right) \\
&= |\mu| U[(x, C; e) \oplus y] \\
&= |\mu| U[(x \oplus y, C; y] \\
&= \frac{|\mu| U(x \oplus y) W + |\mu| U(y)(1 - W) \pm \mu^2 U(x \oplus y) U(y) W}{1 \pm |\mu| U(y) W} \\
&= \frac{G(X,Y)W + Y(1 - W) \pm G(X,Y) YW}{1 \pm YW}.
\end{aligned}
\tag{36}
$$

Rewriting,

$$
\theta(X, Y) = \frac{G(X, Y)(1 \pm Y)}{Y} - 1 \tag{37}
$$

$$
= \frac{1}{W}\left(\frac{G(XW, Y)(1 \pm YW)}{Y} - 1\right). \tag{38}
$$

So from the last equation

$$
\begin{aligned}
G(XW, Y) &= \frac{Y[W\theta(X, Y) + 1]}{1 \pm YW} \\
&= G(WX, Y) \\
&= \frac{Y[X\theta(W, Y) + 1]}{1 \pm XY}.
\end{aligned}
$$

Canceling Y, cross multiplying, simplifying, and dividing by WX yields

$$\frac{1 \pm XY}{X}\,\theta(X, Y) \mp \frac{Y}{X} = \frac{1 \pm WY}{W}\,\theta(W, Y) \mp \frac{Y}{W}.$$

Because X and W are independent, it follows that this expression can only depend on Y, i.e., for some function φ,

$$\frac{1 \pm XY}{X}\,\theta(X, Y) \mp \frac{Y}{X} = \varphi(Y),$$

or solving

$$\theta(X, Y) = \frac{X\varphi(Y) \pm Y}{1 \pm XY}.$$

Substituting this back into Eq. (37)

$$\begin{aligned} G(X, Y) &= \frac{Y[\theta(X, Y) + 1]}{1 \pm Y} \\ &= \frac{1}{1 \pm XY}\left(\frac{XY}{1 \pm Y}[\varphi(Y) \pm Y] + Y\right). \end{aligned} \tag{39}$$

By the commutativity of $\oplus$, G is symmetric, so

$$\frac{XY}{1 \pm Y}[\varphi(Y) \pm Y] + Y = \frac{XY}{1 \pm X}[\varphi(X) \pm X] + X.$$

Subtracting $X + Y$ from both sides and simplifying

$$\frac{\varphi(Y) \pm Y}{1 \pm Y} - \frac{1}{Y} = \frac{\varphi(X) \pm X}{1 \pm X} - \frac{1}{X},$$

and so for some constant η

$$\frac{\varphi(Y) \pm Y}{1 \pm Y} = \eta + \frac{1}{Y}.$$

Substituting this into Eq. (36) yields

$$G(X, Y) = \frac{X + Y + \eta XY}{1 \pm XY}.$$

Substituting this equation into Eq. (36) and carrying out the necessary algebra, we see that

$$0 = \pm XYW(1 - W)(X - 1)[1 \mp Y^2 + \eta Y],$$

which clearly is impossible for general X, Y, W, so $\mu \neq 0$ is impossible. $\square$

Acknowledgements

This research has been supported in part by National Science Foundation grants SBR-9808057 (Luce) and SBR-9730076 (Grofman, Marley, Regenwetter) to the University of California, Irvine.

We are indebted to Drs. Aczél, Maksa, Ng, and Páles for solving the two functional equations we arrived at and to a referee for a number of suggestions including the one mentioned in footnote 3.

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