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Binary Gambles of a Gain and a Loss: an Understudied Domain

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Assume that binary rank-dependent expected utility (subjective expected utility is a special case) holds for gains and losses separately and that a commutative binary operation of joint receipt on consequences and gambles is linked to binary gambles via the rational property of segregation. This implies that utility U of joint receipt is either itself additive or is an exponential transformation of an additive representation V. For joint receipt of mixed gains and losses two hypotheses are discussed: either U or V is additive over mixed joint receipts. Both hypotheses are linked back to gambles in two different ways: a generalization of segregation and an empirically sustained but nonrational property called duplex decomposition. The additive U model yields bilinear expressions like rank-dependent expected utility. Data favor the latter. The additive V models yield nonbilinear representations of mixed gambles.

Keywords: utility theory, non-expected-utility theory, decision making, binary gambles.

JEL classification: D81, D46, C91.

1 Introduction

Beginning in the 1970s it became increasingly widely recognized that subjective expected utility, no matter how compelling it may seem normatively, is systematically wrong descriptively. Edwards (1992) provides an appraisal. Moreover, for some of us, it even seemed questionable normatively because it admitted no real distinction between gains and losses. For subjective expected utility to make any reasonable sense, all alternatives had to be cast in terms of total wealth. Although little was said in print, almost everyone knew that this was a bit of a myth because hardly anyone formulates choices at that level. To be sure, one's state of wealth affects the domain of realistic choices, but it is an implicit background fact (if one can call something "a fact" when few of those who hold stocks and real estate know on a daily or weekly basis their current wealth to an accuracy of better than 10%). The alternatives actually contemplated always seem to be relatively precise increments and decrements in that wealth, whatever it is. Various theorists, and certainly all experimentalists, recognized this unreality of classical expected utility or subjective expected utility, but it was really Kahneman and Tversky's (1979) prospect theory that initiated a serious change in theorizing.

Since then and to the present, theorists have concentrated on variants of subjective expected utility. Most of the effort, both theoretical and experimental, has focused on risky or uncertain alternatives composed entirely of gains or entirely of losses. Although a number of proposals have been made, such as Birnbaum's (1992; Birnbaum et al., 1992; Birnbaum and Navarrete, 1998) configural weight theory, Chew's (1983) weighted utility, Chew et al.'s (1991) quadratic utility, Loomes' (1988; see his references for earlier papers with R. Sugden) disappointment-and-regret theories, and the various models by Fishburn (1988), the clearly dominant contender from 1982 until the present has been rank-dependent utility. Its history, from the perspective of one of its originators, is nicely summarized in Quiggin (1993). And many of the experiments run in the past ten years have been attempts to decide whether the properties underlying rank dependence hold descriptively. Among these studies are: Birnbaum and Chavez (1997), Birnbaum and McIntosh (1996), Birnbaum and Navarrete (1998), Brothers (1990), Camerer (1989, 1992), Cho and Fisher (unpubl.), Cho and Luce (1995), Cho et al. (1994), Chung et al. (1994), Fennema and Wakker (1997), Harless and Camerer (1994), Humphrey (1995), Ranyard (1977), Starmer and Sugden (1993), von Winterfeldt et al. (1997), Wakker et al. (1994), and Weber and Kirsner (1996).

One does find some theory concerned with mixed gambles (= uncertain alternatives) of gains and losses, but not many empirical phenomena involving mixed gambles have been described, and only a very few experiments about them have been run (e.g., Chechile and Cooke, 1997; Payne et al., 1980, 1981; and bits of some papers whose main focus is rank dependence). This is surprising considering that many important choices involve mixtures of gains and losses. Perhaps the most serious argument for reconsidering the mixed case very carefully are the several theories that suppose alternatives are always recast as gains and losses relative to a reference level and that, therefore, all choices actually concern the mixed case (Lopes, 1996, who gives references to a number of earlier papers; Luce et al., 1993). But even in these models, the postulates about the utility of the resulting mixed cases tend to be classical.

My goal here is to discuss why I think this lack of attention is inappropriate and to make clear that it is far from obvious what the utility expression should be for binary mixed gambles. I describe four "principled" arguments that lead to quite different predictions about mixed gambles, and the results of one experiment that appears to reject at least two of them, including the form postulated in both the old and new prospect theory (Tversky and Kahneman, 1992).

Although there are various ways to arrive at some of these forms, I know of only one way to encompass all four; it rests on working with both gambles and the binary operation of getting or having two (or more) gambles or sure consequences, an operation I call joint receipt.

2 Gambles and Joint Receipts

2.1 Gambles and Utility Representations

Suppose $\mathcal{C}$ is a set of pure consequences, including money that is modeled as the real numbers $\mathbf{R}$. The elements of $\mathcal{C}$ (and so $\mathbf{R}$) are denoted x, y, z. From $\mathcal{C}$ and a family $\mathcal{E}$ of chance "experiments"[1] one recursively constructs the set $\mathcal{G}$ of uncertain alternatives, often called gambles. It is useful to treat $\mathcal{C} \subset \mathcal{G}$. A typical first-order binary gamble is of the form $(x, C; y, E\backslash C)$, where E is an experiment in $\mathcal{E}$, and $C \subset E$ is called an event, and $x, y \in \mathcal{C}$. This notation is sometimes abbreviated $(x, C; y, \bar{C})$, where $\bar{C} = E\backslash C$, or even $(x, C; y)$ when the experiment E is not being varied. A compound binary gamble substitutes gambles $g, h \in \mathcal{G}$ for the consequences x, y.

The usual utility framework simply works with a preference order $\succsim$ over $\mathcal{G}$. One provides axioms for the preference order that in the presence of adequate structure are necessary and sufficient for the existence of a numerical representation of the following character.

1 I use the term "experiment" here in the sense of a statistician, not an empirical scientist.

A much stronger version of segregation was first invoked by Pfanzagl (1959) for monetary gambles; he called it the consistency axiom. He assumed one could replace a gamble g by its monetary certainty equivalent, $CE(g)$ and that $\oplus = +$. Thus, he wrote $(x+z, C; y+z) \sim CE(x, C; y) + z$ and its natural generalization to any finite number of consequences. Segregation is less restrictive in three ways. It does not assume that the consequences are money; when they are money, it does not assume that $\oplus$ is addition; and only the least gain is assumed to be segregated. Thaler (1985) provided evidence against $\oplus = +$ for gains, but sustained it for losses. Kahneman and Tversky (1979) invoked this limited form of consistency to money gambles with $y = 0$ as part of their pre-editing of gambles,[4] and they called it segregation.

Pfanzagl's conclusion about the form of U was generalized[5] by Luce and Fishburn (1995) to the more general context. If Eqs. (1) and (3) both hold and V is an additive representation of $\oplus$, then either U is proportional to V [which can be treated as a limiting case of Eq. (6)] or there is some constant $\Delta > 0$ such that if U is convex in V

$$U(g) = \Delta[e^{V(g)} - 1], \tag{4}$$

or if U is concave in V

$$U(g) = \Delta[1 - e^{-V(g)}]. \tag{5}$$

Note that in the concave case U is bounded by Δ whereas in the convex case it is unbounded. The concavity or convexity of U relative to V does not necessarily imply the same property relative to money unless $V(x) = \delta x$, $\delta > 0$, which is equivalent to assuming for money that $x \oplus y = x + y$.

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4 It is interesting that all of their pre-editing steps are now incorporated within the axiomatic rank-dependent theory itself and help give rise to the numerical representation. These steps were: "coding," which is the gain–loss distinction; "combination," which I think is better dubbed coalescing (Luce, 1998; see Sect. 3 below); "segregation," which was just stated; and "cancellation," which is now called the assumption of consequence monotonicity. The last three are all necessary conditions if the rank-dependent representation holds.

5 In fact, we had forgotten Pfanzagl's result and did not mention it.

An exactly parallel set of assumptions is made about losses. The only differences are that it is the least loss that is segregated and the constant in the analogue of Eqs. (4) and (5) may be different from Δ; call it Γ.

From the additivity of V over $\oplus$ and Eqs. (4) and (5) it follows readily that

$$U(g \oplus h) = \begin{cases} U(g) + U(h) + U(f)U(g)/\Delta, & \text{if convex in } V, \\ U(g) + U(h) - U(f)U(g)/\Delta, & \text{if convave in } V. \end{cases} \tag{6}$$

2.4 Empirical Evidence about Consistency and Segregation

Payne et al. (1980, 1981) explored what amounts to Pfanzagl's (1959) consistency property using three consequence gambles. They assumed $\oplus = +$ and used gambles of the form: for $x > x' > 0 > -y' > -y$,

$$g = \begin{pmatrix} p & 1-p-q & q \\ x & 0 & -y \end{pmatrix} \quad \text{and} \quad g' = \begin{pmatrix} p' & 1-p'-q' & q' \\ x' & 0 & -y' \end{pmatrix}.$$

Denote by $g \pm z$ the gamble with each entry of g augmented by $\pm z$. Choosing $z > 0$ such that $-y + z > 0$ and $x - z < 0$, they found empirically that $g + z \succ g' + Z$ if and only if $g - z \prec g' - z$. The comparison of the mixed gambles g and g' went either way, depending upon the exact numerical values of the consequences. Thus, the consistency property fails, as predicted by various sign-dependent theories, and the more limited property of segregation was not tested.

Using choice-based certainty equivalents Cho et al. (1994) and Cho and Luce (1995) explored segregation itself for gains and losses separately, and found it to be sustained.

3 Extensions to General Gambles

Given the above binary theory for gains and losses separately, one issue is how to extend it to cover finite gambles of all gains (and, separately, all losses) and also to mixed gains and losses. The extension to general, finite gambles of gains (losses) has been done in several ways. Wakker and Tversky (1993) did it using the comonotonicity approach in which, basically, one assumes what amounts to the usual independence axiom for all alternatives that maintain the same rank order of consequences on the event partition. Luce and Fishburn (1991, 1995) used an inductive argument based on the obvious generalization of segregation, Eq. (3), to arbitrary finite gambles of gains, which argument was significantly simplified by Liu (1995). And Luce (1998) has carried out a different inductive argument based on two assumptions that are necessary if rank dependence hold: (i) For a fixed event partition, the conjoint structure over consequences is additive; (ii) if in a gamble with n subevents two of the events give rise to the same consequence, then this gamble is indifferent to the gamble of order $n - 1$ in which the partition is the same except for coalescing the two relevant subevents into their union. The reason for calling this property coalescing is apparent; however, others have used other words. It was another of Kahneman and Tversky's (1979) "pre-editing" steps, and they called it "combination" (see footnote 5). Starmer and Sugden (1993) and Humphrey (1995), focusing on going from a gamble of order n to $n + 1$, called it "event splitting." Their term emphasizes partitioning an event into two subevents whereas ours emphasizes combining two events into one, but mathematically they are the same.

The weights in these general representations are related to the binary weights W_i, $i = +, -$, in a special cumulative fashion. Assuming that the event partition is ordered from the best to the worst consequence, then the weight associated with E_j is

$$W_i(E_1 \cup \cdots \cup E_{j-1} \cup E_j) - W_i(E_1 \cup \cdots \cup E_{j-1}), \quad i = +, -.$$

Although I do go into it in any detail in this paper, Birnbaum and Navarrete (1998) have provided evidence casting doubt on this model.

We focus on the next question, whose answer is not obvious, which is: How does one extend the theory from just gains and just losses to general mixed cases? In practice, this has been done in two steps.

So far, everyone has, in effect, assumed that any general finite gamble g is decomposed into the subgamble of gains, call it g_+ and the subgamble of losses, g_-, and that

$$g \sim (g_+, E(+); g_-, E(-)), \tag{7}$$

where $E(+)$ is the union of all subevents with consequences that are gains and $E(-)$ is the same for losses. If one of the consequences is the status quo e, it is simply dropped from consideration because $U(e) = 0$.

This assumption is a special case of what is often called the reduction of compound gambles (to first-order ones). Much data question the widespread use of the reduction of compound gambles, but this use is very special in the sense that it applies only to the partitioning into gains and losses. Nonetheless, given how essential Eq. (7) is to the general theory, it is surprising that no one has studied it directly in an experiment, but to my knowledge no one has. This needs attention.

Assuming Eq. (7) is correct, the theory thus devolves to understanding mixed binary gambles. Two major theoretical ideas have been developed, which are dealt with separately.

4 The Additive *U* Model of Mixed Alternatives

The first representation proposed, which Kahneman and Tversky (1979) stated without any real motivation aside from the fact that most of the rest of utility theory at the time was bilinear in consequences and events, is a fairly plausible mix of the two rank-dependent representations. In their generalization of prospect theory, cumulative prospect theory (Tversky and Kahneman, 1992), there are weighting functions W_+ and W_- over events, which arose in the respective rank-dependent theories for gains and losses such that for $g_+ \in \mathcal{G}_+$ and $h_- \in \mathcal{G}_-$

$$U(g_+, C; h_-, \bar{C}) = U(g_+)W_+(C) + U(h_-)W_-(\bar{C}). \tag{8}$$

In Kahneman and Tversky's (1979) paper the weights were not sign dependent[6], i.e., $W_+ = W_-$.

6 It should be noted that some of the lore in the field has been justified only for this stronger assumption. For example, it is often said that U must satisfy the following two properties: For $x \in \mathbf{R}_+$, $U(x) < -U(-x)$ and $U'(-x) > U'(x)$. In the 1979 paper this was derived from prospect theory and group data, but it does not follow from these data for their later, more general, sign-dependent theory. Many nontheorists have failed to distinguish carefully the model-dependent nature of some inferred properties.

4.1 *U Additive over Mixed Joint Receipt*

Luce (1991) and Luce and Fishburn (1991) pointed out that Eq. (8) is a reasonably natural consequence of assuming

$$U(g_+ \oplus h_-) = U(g_+) + U(h_-), \tag{9}$$

where U is the utility that arose in Eq. (1).

A natural question to raise is what qualitative properties give rise to Eq. (9). The answer may seem easier than in fact it is. Suppose we define

$$(g_+, h_-) \succsim' (g'_+, h'_-) \quad \text{iff } g_+ \oplus h_- \succsim g'_+ \oplus h'_-.$$

Then according to Eq. (9), $\langle \mathcal{G}_+ \times \mathcal{G}_-, \succsim' \rangle$ must satisfy the axioms of an additive conjoint structure, and the axiomatization of such structures is well understood (Krantz et al., 1971). Thus, we know qualitative conditions under which an additive representation $U_+^* + U_-^*$ exists. Because the gain and loss domains do not overlap except at e, we may simply denote the union of these functions as U^*. There is, at this point, no reason to assume that U^* is the same as (or linear with) the U obtained from the binary rank-dependent model over gains, Eq. (1), and, separately, over losses.

So, two questions must be addressed. First, what property permits one to assume that U and U^* agree? That issue was taken up by Luce (1996) and, as is usual in such cases, the answer rests upon finding a suitable qualitative property linking the extensive and conjoint structures, developing a functional equation from it, and solving that equation (which in this case turned out to be quite difficult to do). Second, what property involving joint receipt and gambles allows one to go from Eq. (9) to Eq. (8)? I take them up in that order.

4.2 *Joint-Receipt Consistency and the Extensive-Conjoint Model*

The necessary property that insures that U from gains and losses can also be assumed to be additive over mixed joint receipts is: For any gambles $f, f', g \succsim e \succsim h, k$ for which $f \oplus h, f' \oplus h \succsim e$,

$$(f \oplus h) \oplus g \sim (f \oplus g) \oplus k \quad \text{iff } (f' \oplus h) \oplus g \sim (f' \oplus g) \oplus k. \tag{10}$$

This property, which is called joint-receipt consistency, first arose in an axiomatic approach presented by Luce and Fishburn (1991), and Luce (1996, theorem 8) used it to derive a functional equation relating U and U^* that was then solved in Aczél et al. (1996). No one has tried to test the property empirically, and as we shall see it probably is not worth doing so.

The qualitative model that assumes $\oplus$ is extensive over gains and losses separately and conjoint over mixed alternatives with the two regions linked by joint-receipt consistency is called the extensive-conjoint model.

4.3 Duplex Decomposition

The second question is what property allows one to deduce Eq. (8) from Eq. (9). It is not difficult to see that it is

$$(g_+, C; h_-, E\backslash C) \sim (g_+, C_1; e, E_1\backslash C_1) \oplus (e, C_2; h_-, E_2\backslash C_2), \quad (11)$$

where E_1 and E_2 denote two independent replications of the experiment E and C_i denotes the realization of the event C in the replications E_i. This property, which is called duplex decomposition, was first suggested on the basis of experimental evidence by Slovic and Lichtenstein (1968). See below for additional empirical evidence. So, we have sufficient conditions for the prospect theory representation of Eq. (8): An additive conjoint structure for $\oplus$ on mixed alternatives, joint-receipt consistency, and duplex decomposition.

4.4 General Segregation

Luce (1997) also suggested a generalization of the segregation property of Eq. (3) for gains (or for losses) as a possible alternative "law" linking gambles. It can be formulated most easily by introducing the concept of "subtraction," namely,

$$g \ominus h \sim f \Leftrightarrow g \sim f \oplus h. \quad (12)$$

The idea continues to be that for mixed gambles that are perceived as a gain, one subtracts off the smaller term, h, which now may be a loss, and then re-adds it as a kind of "certainty"; whereas, if the gamble is seen as a loss one subtracts off the greater term, which now may be a gain, and then re-adds it, i.e.,

$$(g, C; h, \bar{C}) \sim \begin{cases} (g \ominus h, C; e, \bar{C}) \oplus h, & \text{if}(g, C; h, \bar{C}) \succ e, \\ (e, C; h \ominus g, \bar{C}) \oplus g, & \text{if}(g, C; h, \bar{C}) \prec e. \end{cases} \quad (13)$$

This is called general segregation because it agrees with segregation when g and h are both gains or both losses, and it extends it into the realm of mixed consequences.

Assuming Eqs. (9) and (13) one derives

$$U(g, C; h, \bar{C}) = \begin{cases} U(g)W_+(C) + U(h)[1 - W_+(C)], & \text{if } (g, C; h, \bar{C}) \succ e, \\ U(g)[1 - W_-(\bar{C})] + U(h)W_-(\bar{C}), & \text{if } (g, C; h, \bar{C}) \prec e. \end{cases} \tag{14}$$

Equations (8) and (14) are very similar in that the utilities of the consequences enter additively in both cases. The difference lies in what weights are assigned to complementary events. In the former equation, as in prospect theory, it is the weight for gains, W_+, and the weight for losses, W_-, which do not in general sum to 1. In the latter equation, whether we use the gain weight or loss weight depends on how the gamble is seen relative to the status quo, but in each case the weights of complementary events do add to 1.

4.5 Empirical Evidence

As was noted earlier, Slovic and Lichtenstein (1968) first suggested duplex decomposition on the basis of empirical evidence. After the theory described above was developed, Cho et al. (1994) studied it again by a choice-based method of certainty equivalents called PEST[7].

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7 PEST, which stands for parameter estimation by sequential testing, is a technique for homing in on the choice certainty equivalent of something, in this case a gamble g. The basic idea is this. A choice between an amount of money and g is presented. If the money is selected, its value is reduced in the next presentation of g; if g is selected, the money amount is increased. Successive presentations of a particular gamble are separated by presentations of many other similar money–gamble pairs. Once the direction of choice for g reverses, the size of the incremental change is halved. This pattern of reversals followed by halving is continued until the increment falls below a prescribed threshold, which we took to be 1/50 of the range of outcomes in g. The certainty equivalent is estimated to be the average of the last two money amounts prior to reaching threshold. When g drops out, it is replaced by a similar filler gamble whose only purpose is to maintain a constant expected recurrence time for the other gambles.

In all cases, duplex decomposition was not rejected at the noise level of the experiments – which, as is unfortunately true throughout this empirical literature, is rather high.

Although segregation was sustained for gains and losses separately, we do not know how it will fare experimentally in the mixed case. In particular, no one has yet run a study in the mixed domain in which duplex decomposition and general segregation are pitted against one another to see which is the more accurate.

Chechile and Cooke's (1997) is the only empirical study of which I am aware that has focused on the additive representation of the mixed case, and it appears to reject both models decisively. Their basic idea was as follows. The values of r used were seven probabilities spaced evenly from 0.05 to 0.95, and $g_r = (\$50, r; -\$50, 1 - r)$. For each r and for various pairs of consequences $(x, -y)$, they asked subjects to report[8] the value of p for which $(x, p; y, 1 - p) \sim g_r$.

Consider the class of utility models that Miyamoto (1988) called generic,

$$U(x, p; -y, 1 - p) = U(x)F(p) + U(-y)G(1 - p), \qquad (15)$$

which includes as special cases the two models above. Chechile and Cooke (1997) observed that if one writes $X = U(x)F(p)$ and $Y = U(-y)G(1 - p)$, then for each r one has the following linear prediction of how X and Y covary: $Y = -X + U(g_r)$. So, if one fits a linear regression to the data for each r separately, the slopes for all

8 The values for x and y in the pairs $(x, -y)$ were all combinations from $\{\$0, \$20, \$40, \$60, \$80, \$100\}$. That means that the published study included what must be spurious responses. For example, subjects were forced to make probability choices such as $(\$50, 0.95; -\$50, 0.05) \sim (\$0, p; \$0, 1 - p)$ and $(\$50, 0.95; -\$50, 0.05) \sim (\$20, p; -\$40, 1 - p)$. The former is senseless because the right side, independent of p, is nothing but \$0, which most people consider as decidedly inferior to the left gamble, which has an expected value of \$45. The latter case almost certainly runs into a ceiling effect. The largest one can go on the right is $p = 1$ in which case the consequence is \$20. For many of us $(\$50, 0.95; -\$50, 0.05)$ is also strictly preferred to \$20. A re-analysis, in which the cases that were either silly or may have encountered boundary effects were omitted, has been carried out (R. Chechile, pers. commun., 1997) and, although there are minor numerical differences, the conclusion is equally firm: The additive U models are rejected.

seven values of r should be -1. To test this, they calculated optimal parameters of nine combinations of models for U, F, and G. Contrary to prediction, they found, with few exceptions, the slope to decrease monotonically and appreciably with r – over the nine model variants, the ratio of the slope for $r = 0.05$ to the slope for $r = 0.95$ ranged between 41 and 700, which is not very consistent with the predicted ratio of 1. Thus, generic utility, and so in particular the two models described here and many others in the literature, are rejected for mixed gambles.

Somewhat surprisingly, the estimated average U functions seemed to be convex for gains and concave for losses, despite the conventional wisdom that for most people utility is concave for gains and convex for losses.

Is there any reason to be uneasy about the conclusion that these additive models are wrong in the mixed case? The only one of which I am aware is that we know that judged certainty equivalents and judged probability equivalents are not consistent with one another (Carbone and Hey, 1994, 1995; Delquié, 1993; Hershey and Schoemaker, 1985; Hey and Carbone, 1994; Hey and Orme, 1994). Indeed, Hershey et al. (1982) noted that in the collection of studies summarized by Fishburn and Kochenberger (1979), the 12 using certainty equivalents all estimated utility functions that were concave for gains and convex for losses whereas of the 8 based on probability equivalents 7 estimated utility functions that were convex for gains and concave for losses. That pattern is obviously systematic, but I suspect the effect is not sufficiently large to explain away the Chechile and Cooke finding. Nevertheless, the resulting conclusion is so important – after all, it rejects a huge class of models that many have been taking for granted – that additional studies are warranted. The most obvious approach is to check the major axioms of additive conjoint measurement over the consequences for fixed event partitions. The two most important axioms are monotonicity (called independence in the measurement literature, e.g., Krantz et al., 1971) and the Thomsen condition.

Monotonicity has been studied extensively and although rejected when appraised using judgment procedures it has been sustained in direct choices and using the PEST procedure for establishing certainty equivalents (von Winterfeldt et al., 1997; Birnbaum, 1997, surveys the literature).

The Thomsen condition for this situation is that if x, y, z are gains and u, v, w are losses, then with experiment E fixed and the notation $E \backslash C$ suppressed,

$$(x, C; v) \sim (y, C; w) \text{ and } (y, C; u) \sim (z, C; v)$$
$$\text{imply } (x, C; u) \sim (z, C; w).$$

This property has not been studied (in this context) but, given Chechile and Cooke's results, presumably it will fail in a major way.

From the perspective of a theorist, the question has to be: Where did we go wrong and what can be done about it? We turn to that next.

5 The Additive V Model of Mixed Alternatives

5.1 Joint Receipt as an Archimedean Ordered Group

Luce (1997) investigated what is, intuitively, a far more plausible model for joint receipt than the extensive-conjoint one, namely the assumption that $\langle \mathcal{D}, \succsim, \oplus, e \rangle$ forms an Archimedean, weakly ordered group and so V is additive throughout the structure (Hölder, 1901). For this model to hold, a primary qualitative property is associativity, i.e., for all $f, g, h \in \mathcal{D}$,

$$(f \oplus g) \oplus h \sim f \oplus (g \oplus h). \tag{16}$$

This simply means that the order of grouping does not matter in so far as preference among the alternatives is concerned. So, for example, when goods are packed in boxes one does not care, with respect to preference among the goods, if f and g are in one box and h is separate or if f is separate and g and h are in one box. In the mixed case, some of the goods are intact and others badly damaged, but for preference among the goods it is which are intact and which damaged, not their packaging, that matters.

In addition, the assumption being made means that each alternative has an "inverse" in the sense of a compensating element: For each $g \in \mathcal{G}$, there exists $g^{-1} \in \mathcal{G}$ such that $g \oplus g^{-1} \sim e$.

Assuming that V is additive has two advantages over assuming U is additive. First, the associativity of joint receipt seems very compelling both intuitively and normatively. It is currently being studied empirically. Second, we can deduce without any further assumptions how U behaves over mixed joint receipts because there is a single function V and we know how U behaves relative to it for both gains and losses. There are four combinations depending on whether U is concave or convex relative to V for both gains and losses. Suppose $f_+ \succsim e \succsim g_-$, and let ρ be 1 if $f_+ \oplus g_-$ is perceived as a gain and 0 if it is perceived as a loss. Then with a minor amount of algebra one shows for the concave-convex case

$$\frac{U(f_+ \oplus g_-)}{\rho\Delta + (1-\rho)\Gamma} = \frac{U(f_+)/\Delta + U(g_-)/\Gamma}{1 - (1-\rho)U(f_+)/\Delta + \rho U(g_-)/\Gamma}. \tag{17}$$

where Δ is the constant for gains in Eq. (5) and Γ is that for losses in the loss analogue of Eq. (4). For the convex-concave case, which apparently was typical of Chechile and Cooke's (1997) subjects, one gets the same formula except the signs in the right-hand denominator are both changed.

The concave-concave case is given by

$$\frac{U(f_+ \oplus g_-)}{\rho\Delta + (1-\rho)\Gamma} = \frac{U(f_+)}{\Delta} + \frac{U(g_-)}{\Gamma} - \frac{U(f_+)}{\Delta}\frac{U(g_-)}{\Gamma}. \tag{18}$$

The convex-convex formula is exactly the same except that the $-$ on the right becomes $+$.

5.2 *Mixed Binary Gambles*

To find out what happens with mixed gambles, one first combines Eq. (17) with duplex decomposition, Eq. (11), obtaining for the concave-convex case:

$$\frac{U(f_+, C; g_-)}{\rho\Delta + (1-\rho)\Gamma} = \frac{(U(f_+)/\Delta)W_+(C) + (U(g_-)/\Gamma)W_-(\bar{C})}{1 - (1-\rho)(U(f_+)/\Delta)W_+(C) + \rho(U(g_-)/\Gamma)W_-(\bar{C})}. \tag{19}$$

Alternatively, for general segregation, Eq. (13), one gets:

$$\frac{U(f_+, C; g_-)}{\Delta} = \frac{U(f_+)}{\Delta} W_+(C) + \frac{U(g_-)/\Gamma}{1 + U(g_-)/\Gamma}[1 - W_+(C)], \ \rho = 1,$$
$$\frac{U(f_+, C; g_-)}{\Gamma} = \frac{U(f_+)/\Delta}{1 - U(f_+)/\Delta}[1 - W_-(\bar{C})] + \frac{U(g_-)}{\Gamma} W_-(\bar{C}),$$
$$\rho = 0. \quad (20)$$

In the convex-concave case, the signs attached to the U terms in the denominators are interchanged.

The concave-concave and convex-convex cases are bilinear except for a multiplicative interaction term. Note that in all cases the general bilinear form of generic utility simply does not arise because of the denominators in the concave-convex and convex-concave cases and because of the interaction terms when both are either concave or convex.

5.3 Relation to Chechile and Cooke Data

The form based on duplex decomposition for the convex-concave case is not inconsistent with the Chechile and Cooke (1997) results according to the following argument. Let $X = (U(f_+)/\Delta)W_+(C)$, $Y = (U(g_-)/\Gamma)W_-(\bar{C})$, and $G_r = U(g_r)/(\rho\Delta + (1 - \rho)\Gamma)$. Then a little algebra on Eq. (19) for the convex-concave case yields the linear form $Y = [G_r - X(1 - (1 - \rho)G_r)]/(1 + \rho G_r)$. When r is small – the smallest value was 0.05 – G_r assumes its most negative value which, because $\rho = 0$ and U is concave, is bounded by -1. Thus, the slope, which is $-[1 - G_{0.05}]$, must lie between -1 and -2. When r is large – the largest was 0.95 – G_r assumes its maximum positive value which, because $\rho = 1$ and U is convex, is bounded between 0 and ∞. Thus, the slope, which is $-1/(1 + G_{0.95})$, must lie between -1 and 0. Therefor, the ratio of the two slopes, $[1 - G_{0.05}][1 + G_{0.95}]$, can be anything between 1 and ∞. This is not very restrictive, but certainly it includes all of the values calculated by Chechile and Cooke (1997) from their data and various models based on generic-utility theory.

It is not difficult to show that the rate of change of the slope with r is given by $[-1/(1 + \rho G_r)^2]\mathrm{d}G_r/\mathrm{d}r$. Because $\mathrm{d}G_r/\mathrm{d}r > 0$, this predicts a monotonic decrease in the slope with increasing r, as was seen in the data.

I do not see how to make a similar direct qualitative comparison for the model of Eq. (20) because the denominator terms do not have any weights multiplying the utilities and so it cannot be put in the X,Y format. So we cannot conclude without a detailed data analysis whether it is viable or not.

5.4 An Analysis of Individual Subjects

A severe test of these models is yet to be reported, but as I write a body of data involving a total of 144 subjects, collected for other reasons by Cho and Luce (1995), is being re-analyzed by R. Sneddon in terms of all four of the models described here using all four combinations of concave and convex exponential utility functions of money for gains and losses. The results are not yet final and so I do not comment on them.

6 Conclusions

The major point of the paper is simple. The field, in its concern with subjective expected utility and its generalization to rank-dependent models for gains and losses separately, has with a few exceptions lost sight of what I believe may in reality be the most significant case, namely, mixed binary consequences. I have attempted to make clear that this case is far more problematic and complicated than has been recognized.

The simplest class of representations from the perspective of gambles are those where U is additive over mixed joint receipts. Qualitatively, however, this model is not simple, and theoretically strong and untested assumptions (the Thomsen condition over consequences and joint-receipt consistency) are needed to justify it. Moreover, one study, based on group data, suggests that this class of additive-U models is grossly incorrect (Chechile and Cooke, 1997).

The axiomatization of the fully associative case of joint receipts leading to additive V is far simpler mathematically than the extensive-conjoint one because one does not need any special qualitative laws linking the representations for gains and losses separately to the mixed case. Despite that simplicity, the corresponding representations for the utility of mixed gambles are significantly more complicated than those of the traditional model. At least the one arising from duplex decomposition appears to be consistent with the data of Chechile and Cooke that clearly reject the additive-utility assumption.

These results, both empirical and theoretical, suggest that considerable additional attention needs to be focused on these mixed cases.

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