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Hyers–Ulam stability of functional equations with a square-symmetric operation

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ABSTRACT **The stability of the functional equation $f(x \circ y) = H(f(x), f(y))$ $(x, y \in S)$ is investigated, where H is a homogeneous function and $\circ$ is a square-symmetric operation on the set S. The results presented include and generalize the classical theorem of Hyers obtained in 1941 on the stability of the Cauchy functional equation.**

1. Introduction

In this paper we investigate the stability of the following family of functional equations:

$$f(x \circ y) = H(f(x), f(y)) \qquad (x, y \in S), \qquad \textbf{[1]}$$

where S is a nonempty set, $\circ$: $S \times S \to S$ is a binary operation, and H: $G \times G \to G$ is a G-homogeneous function of two variables, that is H satisfies

$$H(uv, uw) = uH(v, w) \qquad (u, v, w \in G), \qquad \textbf{[2]}$$

and G is a multiplicative subsemigroup of the real or complex field. The unknown function f is defined on S and takes values in G. For applications of this functional equation in the ratio scalability problem of unit structures, see refs. 1 and 2.

A particular case of **1** is the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \qquad (x, y \in S), \qquad \textbf{[3]}$$

where S is a semigroup with the operation $+$ and f: $S \to \mathbb{C}$.

The stability properties of this equation have attracted the attention of many mathematicians. A significant result is the so called Hyers–Ulam stability theorem obtained by Hyers (3). It states that *if S is an abelian semigroup and, for some $\varepsilon > 0$, g: $S \to \mathbb{C}$ satisfies the functional inequality*

$$|g(x + y) - g(x) - g(y)| \leq \varepsilon \qquad (x, y \in S)$$

then there exists a solution f: $S \to \mathbb{C}$ of **3** *such that*

$$|g(x) - f(x)| \leq \varepsilon \qquad (x \in S).$$

In other words, if g solves **3** approximately, then it is approximately equal to one of the solutions of **3**. We note that if S is the set of natural numbers, $\circ$ is the usual addition, and g is real valued, then the statement appeared as Exercise I 99 in the book of Pólya & Szegő (4).

For the noncommutative case, the first stability result was obtained by Rätz (5). The invariant mean technique was introduced and used by Székelyhidi (6). An account on the further progress and developments in this field can be found in recent survey papers of Forti (7) and Ger (8). A new approach and result unifying many of the known stability theorems for the Cauchy functional equation can be found in the publications of Borelli-Forti and Forti (9) and Páles (10).

In the next section we will study the operation $\circ$ appearing in **1**. We prove that if **1** has sufficiently many solutions, then $\circ$ satisfies the following identity:

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) \qquad (x, y \in S). \qquad \textbf{[4]}$$

In this case, the operation $\circ$ will be called *square symmetric*. As a consequence of this result, we describe a large class of square-symmetric operations. It is immediate that bisymmetric (e.g., commutative and associative) operations are automatically square symmetric. This algebraic property was considered already by Borelli-Forti and Forti (9) and also by Gajda and Kominek (11).

In the last section we investigate the stability of **1**. Assuming that $\circ$ is square symmetric and $|H(1, 1)| > 1$, we obtain the stability of **1** in the Hyers–Ulam sense. In the case $|H(1, 1)| < 1$ under divisibility assumptions another stability result is established. The case when $|H(1, 1)| = 1$ remains an open problem.

2. Square-Symmetric Operations

Let S be a nonempty set and $\circ$: $S \times S \to S$ be an arbitrary operation. In addition, let G be a multiplicative subsemigroup of $\mathbb{C}$, and let H: $G \times G \to G$ satisfy **2**.

In the following result we show that if **1** has sufficiently many solutions, then $\circ$ is necessarily square symmetric. A similar result concerning bisymmetric operations can be found in ref. 12.

THEOREM 1. *Assume that the set of solutions of the functional equation* **1** *separates the points of S, that is, if $u, v \in S$ and $u \neq v$, then there exists a solution f: $S \to G$ of* **1** *such that $f(u) \neq f(v)$. Then the operation $\circ$ is square symmetric.*

Proof: Let $x, y \in S$, and let f: $S \to G$ be an arbitrary solution of **1**. Then, using the homogeneity of H and **1** several times, we obtain

$$\begin{aligned} f((x \circ y) \circ (x \circ y)) &= H(f(x \circ y), f(x \circ y)) \\ &= f(x \circ y)H(1, 1) \\ &= H(f(x), f(y))H(1, 1) \\ &= H(f(x)H(1, 1), f(y)H(1, 1)) \\ &= H(H(f(x), f(x)), H(f(y), f(y))) \\ &= H(f(x \circ x), f(y \circ y)) \\ &= f((x \circ x) \circ (y \circ y)). \end{aligned}$$

By the separability assumptions of the theorem, the above identity yields that **4** is true, i.e., $\circ$ is square symmetric. Q.E.D.

The next result describes a set of square-symmetric operations.

COROLLARY 1. *Let G be a multiplicative subsemigroup of $\mathbb{C}$, let H: $G \times G \to G$ satisfy* **2**, *and let ϕ: $S \to G$ be an arbitrary*

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bijective function. Then the binary operation $\circ: S \times S \to S$ *defined by*

$$x \circ y := \phi^{-1}\big(H(\phi(x), \phi(y))\big) \qquad (x, y \in S) \qquad \textbf{[5]}$$

is square symmetric.

Proof: Clearly, ϕ is a solution of the functional equation **1** (with the operation $\circ$ defined in **5**). By its injectivity, it separates the points of S. Thus, because of the previous theorem, $\circ$ must be a square-symmetric operation. Q.E.D.

3. Hyers–Ulam Stability of Eq. 1

In the next lemma, we obtain a large family of endomorphisms of a square-symmetric structure. For analogous results, see ref. 1 and ref. 2 (theorem 20.14, p. 147).

LEMMA 1. *Let* $\circ$ *be a square-symmetric operation on S. Define, for* $x \in S$, *the sequence* $x[2^n]$ $(n = 0, 1, 2, \ldots)$ *by*

$$x[1] = x[2^0] := x, \qquad x[2^{n+1}] := x[2^n] \circ x[2^n], \qquad n \in \mathbb{N}. \quad \textbf{[6]}$$

Then, for each $n \in \mathbb{N}$, *the mapping* $x \mapsto x[2^n]$ *is an endomorphism of* $(S, \circ)$, *that is*

$$(x \circ y)[2^n] = x[2^n] \circ y[2^n] \qquad \textit{for } x, y \in S. \qquad \textbf{[7]}$$

Proof: We prove by induction on $n \in \mathbb{N}$. For $n = 1$, the statement is equivalent to the square-symmetry of the operation $\circ$. Assume that **7** is true for $n = k$. Then, by using **6** (three times), the square-symmetry, and the inductive hypothesis,

$$\begin{aligned}(x \circ y)[2^{k+1}] &= (x \circ y)[2^k] \circ (x \circ y)[2^k] \\ &= (x[2^k] \circ y[2^k]) \circ (x[2^k] \circ y[2^k]) \\ &= (x[2^k] \circ x[2^k]) \circ (y[2^k] \circ y[2^k]) \\ &= x[2^{k+1}] \circ y[2^{k+1}].\end{aligned}$$

Thus, **7** also holds for $n = k + 1$. Q.E.D.

Remark 1: If the assumptions of *Corollary 1* are satisfied and $\circ$ is defined via **5**, then it is possible to obtain a larger family of endomorphisms by defining, for $x \in S$, $h \in G$, the element x_h as

$$x_h := \phi^{-1}(h\phi(x)).$$

The endomorphism property is a consequence of the following direct computation.

$$\begin{aligned}(x \circ y)_h &= \phi^{-1}(h\phi(x \circ y)) \\ &= \phi^{-1}(hH(\phi(x), \phi(y))) \\ &= \phi^{-1}(H(h\phi(x), h\phi(y))) \\ &= \phi^{-1}(H(\phi(x_h), \phi(y_h))) \\ &= x_h \circ y_h.\end{aligned}$$

It is also easy to show (by using induction) that

$$x_{H(1,1)^n} = x[2^n] \qquad \text{for all } n \in \mathbb{N}.$$

Therefore, the class of endomorphism $\{x \mapsto x_h | h \in G\}$ is more general than that of described in the previous lemma.

Now we are ready to formulate the main result of this paper that offers a stability theorem for the functional equation **1** when $|H(1, 1)| > 1$. In *Theorem 3* below, we also consider the case $|H(1, 1)| < 1$.

THEOREM 2. *Let S be a nonempty set and* $\circ$ *be a square-symmetric operation on S. Let G be a closed multiplicative subsemigroup of* $\mathbb{C}$ *with* $1 \in G$ *and* $H: G \times G \to G$ *be a continuous G-homogeneous function such that* $|H(1, 1)| > 1$ *and* $\frac{1}{H(1,1)} \in G$. *Assume that, for some* $\varepsilon \geq 0$, *a function* $g: S \to G$ *satisfies the stability inequality*

$$|g(x \circ y) - H(g(x), g(y))| \leq \varepsilon \qquad (x, y \in S). \qquad \textbf{[8]}$$

Then there exists a function $f: S \to G$ *such that f is a solution of* **1** *and*

$$|g(x) - f(x)| \leq \frac{\varepsilon}{|H(1, 1)| - 1} \qquad (x \in S). \qquad \textbf{[9]}$$

Proof: Substituting $x = y$ into **8** and using the G-homogeneity of H, we get

$$|g(x \circ x) - g(x)H(1, 1)| \leq \varepsilon \qquad (x \in S). \qquad \textbf{[10]}$$

Let $x \in S$ be fixed, and replace x by $x[2^{n-1}]$ (defined in *Lemma 1*) in **10**. Then we obtain

$$\left|\frac{g(x[2^n])}{H(1, 1)^n} - \frac{g(x[2^{n-1}])}{H(1, 1)^{n-1}}\right| \leq \frac{\varepsilon}{|H(1, 1)|^n} \qquad \textbf{[11]}$$

for all $x \in S$ and $n \in \mathbb{N}$. Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := \frac{g(x[2^n])}{H(1, 1)^n} \qquad (x \in S).$$

Then $g_n: S \to G$ (because $\frac{1}{H(1,1)} \in G$) and, because of **11** and $|H(1, 1)| > 1$, we have that the series

$$\sum_{n=1}^{\infty}\big(g_n(x) - g_{n-1}(x)\big) \qquad (x \in S)$$

is absolutely convergent and therefore it is also convergent. Thus, the sequence $g_n(x)$ is convergent for all fixed $x \in S$. Let

$$f(x) := \lim_{n\to\infty} g_n(x) \qquad (x \in S).$$

The set G being closed, we have that f maps S into G. It follows from **11** that

$$\begin{aligned}|g_n(x) - g_0(x)| &\leq \sum_{k=1}^{n} \frac{\varepsilon}{|H(1, 1)|^k} \\ &\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{|H(1, 1)|^k} = \frac{\varepsilon}{|H(1, 1)| - 1}.\end{aligned}$$

Taking the limit $n \to \infty$, we obtain **9**.

To see that f satisfies **1**, let $x, y \in S$ and replace x, y by $x[2^n]$, $y[2^n]$ in **8**. By using *Lemma 1*, we get

$$|g((x \circ y)[2^n]) - H(g(x[2^n]), g(y[2^n]))| \leq \varepsilon.$$

Hence, by the G-homogeneity of H,

$$|g_n(x \circ y) - H(g_n(x), g_n(y))| \leq \frac{\varepsilon}{|H(1, 1)|^n}$$

for all $x, y \in S$ and $n \in \mathbb{N}$. Taking the limit $n \to \infty$, by the continuity of H it follows that

$$|f(x \circ y) - H(f(x), f(y))| = 0 \qquad (x, y \in S).$$

Thus **1** holds and the theorem is proved. Q.E.D.

THEOREM 3. *Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Assume that the operation $\circ$ has the following divisibility property: for each $x \in S$, there exists a unique element $y \in S$ such that $y \circ y = x$. In addition, let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and $H: G \times G \to G$ be a continuous G-homogeneous function such that $|H(1,1)| < 1$. Assume that, for some $\varepsilon \geq 0$, a function $g: S \to G$ satisfies the functional inequality* **8**. *Then there exists a function $f: S \to G$ such that f is a solution of* **1** *and*

$$|g(x) - f(x)| \leq \frac{\varepsilon}{1 - |H(1,1)|} \qquad (x \in S). \qquad \textbf{[12]}$$

Proof: The proof of this theorem is completely analogous to that of *Theorem 2*.

It follows from the divisibility assumption, that the equation $y[2^n] = x$ has a unique solution y for each fixed $x \in S$ and $n \in \mathbb{N}$. Denote this unique element y by $x[2^{-n}]$. Clearly, the mapping $x \to x[2^{-n}]$ is also an endomorphism of $(S, \circ)$.

Replacing x and y by $x[2^{-n}]$ in **8** and using the G-homogeneity of H, we get

$$|g(x[2^{1-n}]) - g(x[2^{-n}])H(1,1)| \leq \varepsilon \qquad (x \in S, n \in \mathbb{N}). \qquad \textbf{[13]}$$

Thus

$$|g(x[2^{1-n}])H(1,1)^{n-1} - g(x[2^{-n}])H(1,1)^n| \leq \varepsilon |H(1,1)|^{n-1} \qquad \textbf{[14]}$$

for $x \in S$, $n \in \mathbb{N}$. Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := g(x[2^{-n}])H(1,1)^n \qquad (x \in S).$$

Then $g_n: S \to G$ and, by **11**, exactly as in the proof of *Theorem 2*, we can deduce that the sequence $g_n(x)$ is convergent for all fixed $x \in S$. Define f as the pointwise limit function of the sequence g_n. It follows from **14** that

$$|g_n(x) - g_0(x)| \leq \sum_{k=1}^{n} \varepsilon |H(1,1)|^{k-1} \leq \sum_{k=1}^{\infty} \varepsilon |H(1,1)|^{k-1} = \frac{\varepsilon}{1 - |H(1,1)|}.$$

Taking the limit $n \to \infty$, we obtain **12**.

To see that f satisfies **1**, let $x, y \in S$ and replace x, y by $x[2^{-n}]$, $y[2^{-n}]$ in **8**. Then we get

$$|g((x \circ y)[2^{-n}]) - H(g(x[2^{-n}]), g(y[2^{-n}]))| \leq \varepsilon.$$

Hence, by the G-homogeneity of H,

$$|g_n(x \circ y) - H(g_n(x), g_n(y))| \leq \varepsilon |H(1,1)|^n$$

for all $x, y \in S$, $n \in \mathbb{N}$. By using the continuity of H, it follows that

$$|f(x \circ y) - H(f(x), f(y))| = 0 \qquad (x, y \in S).$$

Therefore, **1** holds and the theorem is proved. Q.E.D.

Remark 2: The statement of the above theorems can be extended to the case when G is a closed multiplicative subsemigroup of a Banach algebra with unit element. The details are left for the reader.

Now we mention an immediate corollary of the above theorems.

COROLLARY 2. *Let G be a closed multiplicative subsemigroup of $\mathbb{C}$ with $1 \in G$ and let $H: G \times G \to G$ be a continuous G-homogeneous function such that $|H(1,1)| \notin \{0, 1\}$ and $\frac{1}{H(1,1)} \in G$. Assume that, for some $\varepsilon \geq 0$, a function $g: G \to G$ satisfies the stability inequality*

$$|g(H(x,y)) - H(g(x), g(y))| \leq \varepsilon \qquad (x, y \in G). \qquad \textbf{[15]}$$

Then there exists a function $f: G \to G$ such that f is a solution of

$$f(H(x,y)) = H(f(x), f(y)) \qquad (x, y \in G). \qquad \textbf{[16]}$$

and

$$|g(x) - f(x)| \leq \frac{\varepsilon}{||H(1,1)| - 1|} \qquad (x \in S). \qquad \textbf{[17]}$$

Proof: To apply *Theorems 2* or *3*, let $S := G$ and $x \circ y := H(x,y)$. By *Corollary 1*, $\circ$ is a square-symmetric operation. Thus if $|H(1,1)| > 1$, then the conditions of *Theorem 2* are valid. The statement of *Theorem 2* is also equivalent to that of this corollary.

In the case $|H(1,1)| < 1$ it suffices to show that $\circ$ satisfies the divisibility assumption of *Theorem 3*. Let $x \in G$. Then the equation $y \circ y = x$ is equivalent to $yH(1,1) = x$. The element $\frac{1}{H(1,1)}$ being in G, we have that

$$y = \frac{x}{H(1,1)} \in G.$$

Thus, in this case, *Theorem 3* applies. Q.E.D.

Applying the above theorem, we easily obtain the stability, for instance, of the following functional equations:

$$f\left(\sqrt[3]{x^3 + y^3} + x + y\right) = \sqrt[3]{f^3(x) + f^3(y)} + f(x) + f(y)$$

$$f\left(\sqrt[3]{x^3 + x^2y + xy^2 + y^3}\right) = \sqrt[3]{f^3(x) + f^2(x)f(y) + f(x)f^2(y) + f^3(y)},$$

where $x, y \in G := \mathbb{R}$.

In the next results, if one takes the homogeneous function $H(x,y) = ax + by$ we obtain that **19** below is stable in Ulam's sense if $|a + b| \neq 1$. Clearly, Hyers' stability theorem is a special case of the following result. In the case when $X = \mathbb{C}$ or $X = \mathbb{R}$, these results are also corollaries of *Theorems 2* and *3*.

THEOREM 4. *Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of real or complex numbers. Let S be a nonempty set and $\circ$ be a square-symmetric operation on S. Let $a, b \in \mathbb{K}$ such that $|a + b| > 1$. Assume that, for some $\varepsilon \geq 0$, a function $g: S \to X$ satisfies the stability inequality*

$$|g(x \circ y) - ag(x) - bg(y)| \leq \varepsilon \qquad (x, y \in S). \qquad \textbf{[18]}$$

Then there exists a function $f: S \to X$ such that f is a solution of

$$f(x \circ y) = af(x) + bf(y) \qquad (x, y \in S) \qquad \textbf{[19]}$$

and

$$|g(x) - f(x)| \leq \frac{\varepsilon}{|a + b| - 1} \qquad (x \in S). \qquad \textbf{[20]}$$

Proof: Substituting $x = y$ into **18**,

$$|g(x \circ x) - (a + b)g(x)| \leq \varepsilon \qquad (x \in S). \qquad \textbf{[21]}$$

Let $x \in S$ be fixed, and replace x by $x[2^{n-1}]$ (defined in *Lemma 1*) in **10**. Then, for $x \in S$ and $n \in \mathbb{N}$,

$$\left| \frac{g(x[2^n])}{(a+b)^n} - \frac{g(x[2^{n-1}])}{(a+b)^{n-1}} \right| \leq \frac{\varepsilon}{|a+b|^n}. \qquad \textbf{[22]}$$

Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := \frac{g(x[2^n])}{(a+b)^n} \qquad (x \in S).$$

Then, arguing in the same way as in the proof of *Theorem 1*, we can see that the sequence $g_n(x)$ is convergent for all fixed $x \in S$. Defining $f: S \to X$ by

$$f(x) := \lim_{n\to\infty} g_n(x) \qquad (x \in S),$$

and making analogous steps, we can see that f satisfies **19** and the estimate **20**. Q.E.D.

THEOREM 5. *Let X be a Banach space over $\mathbb{K}$, S be a nonempty set, and $\circ$ be a square-symmetric operation on S with the divisibility property described in Theorem 3. Let $a, b \in \mathbb{K}$ such that $|a+b| < 1$. Assume that, for some $\varepsilon \geq 0$, a function $g: S \to X$ satisfies the stability inequality* **18**. *Then there exists a function $f: S \to X$ such that f is a solution of* **19** *and*

$$|g(x) - f(x)| \leq \frac{\varepsilon}{1 - |a+b|} \qquad (x \in S).$$

Proof: The proof of this theorem is completely analogous to that of *Theorem 3*. Define $x[2^{-n}]$ exactly as it was defined therein. Replacing x and y by $x[2^{-n}]$ in **18**, we get

$$|g(x[2^{1-n}]) - (a+b)g(x[2^{-n}])| \leq \varepsilon \qquad (x \in S, n \in \mathbb{N}). \quad \textbf{[23]}$$

Thus, for $x \in S$ and $n \in \mathbb{N}$,

$$|(a+b)^{n-1} g(x[2^{1-n}]) - (a+b)^n g(x[2^{-n}])| \leq \varepsilon |(a+b)|^{n-1}. \quad \textbf{[24]}$$

Let $g_0 := g$ and define, for $n \in \mathbb{N}$, the function g_n by

$$g_n(x) := (a+b)^n g(x[2^{-n}]) \qquad (x \in S).$$

Then, by using **24**, we can deduce that the sequence $g_n(x)$ is convergent for all fixed $x \in S$. Define f as the pointwise limit function of the sequence g_n. Now an analogous argument shows that f satisfies the desired conditions of this theorem. Q.E.D.

Remark 3: The results presented in *Theorem 2*, *3*, *4*, and *5* offer a generalization of Hyers stability theorem in a different way than that of Borelli-Forti and Forti (9).

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