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Coalescing, Event Commutativity, and Theories of Utility

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Abstract

Preferences satisfying rank-dependent utility exhibit three necessary properties: coalescing (forming the union of events having the same consequence), status-quo event commutativity, and rank-dependent additivity. The major result is that, under a few additional, relatively non-controversial, necessary conditions on binary gambles and assuming mappings are onto intervals, the converse is true. A number of other utility representations are checked for each of these three properties (see Table 2, Section 7).

Key words: Coalescing, event commutativity, event splitting, rank-dependent additivity, rank-dependent utility, status-quo event commutativity.

There is an important behavioral property, which I call coalescing, of uncertain alternatives that seems not to have received the theoretical attention it may deserve. It concerns alternative framings (Kahneman & Tversky, 1979; Tversky & Kahneman, 1986) of the same gamble that occur when two potentially distinct consequences are in fact the same one.

To describe it fully and its connections to other concepts, I must first make clear the framework within which I am working. It is neither that first formulated by Savage (1954), and adopted by many others, of acts treated as functions from a fixed state space into a set of consequences nor the formulation of risky alternatives or lotteries (i.e., money gambles with known probabilities) as random variables so that one can think of choices as between distribution functions. Rather, I follow a generalization of the algebraic approach taken by von Neumann and Morgenstern (1947), who originally formulated expected utility theory, and which was first generalized to binary uncertain alternatives by Pfanzagl (1959; 1968, Ch. 12). In this view, a gamble g may be described as follows: An "experiment"[1] E is carried out and g is a function from a partition[2] $\{E_i\}$ of E into the set $\mathscr{X}$ of possible consequence with x_i assigned to event E_i. In this paper, such a gamble is denoted $g = (x_1, E_1;\ldots; x_i, E_i;\ldots; x_n, E_n)$, where $\bigcup_{i=1}^{n} E_i = E$.

A second difference, also found in von Neumann and Morgenstern, is the inclusion of compound gambles in which the consequences of a gamble may themselves be gambles based on independently run experiments. So, in the binary case, one might have the compound gamble

$$((x_1, E_1; x_2, E_2), D_1; (y_1, F_1; y_2, F_2), D_2)$$

which is interpreted as follows. An experiment $D = D_1 \cup D_2$ is run. If the outcome lies in D_1, then an independent experiment $E = E_1 \cup E_2$ is run to determine if the payoff is x_1 or x_2; and if the outcome of the first experiment lies in D_2, then an independent experiment $F = F_1 \cup F_2$ is run to determine if the payoff is y_1 or y_2.

A limitation of using compound gambles is that many uncertain events in the real world cannot be repeated, whereas in my formulation this is presumed to be a possibility at least once. For the kinds of experiments that are typically run to test utility theories, compound gambles are easily realized.

From the set of pure consequences, $\mathscr{X}$, and the set of experiments one constructs all possible first-order gambles, $\mathscr{G}_1$. By identifying x with the gambles in which x occurs for any outcome of the experiment E, we see that $\mathscr{X} \subseteq \mathscr{G}_1$. From the first order gambles one generates all second order ones, $\mathscr{G}_2$, by letting $\mathscr{G}_1$ play the role of $\mathscr{X}$. I shall write the events of a gamble, i.e., a partition from an algebra[3] $\mathscr{E}_E$ of subsets of the experiment E and their corresponding consequences in order from the most preferred to the least preferred consequence, i.e., $x_1 \succsim \ldots \succsim x_n$. Denote by $\vec{E}_n = (E_1, E_2, \ldots, E_n)$ a corresponding ordered partition of the chance event $E = \bigcup_{i=1}^{n} E_i$ corresponding to the experiment under consideration.

It is usual in this literature to treat all the events under consideration as elements of an algebra of subsets of some universal event. When we think of different experiments giving rise to the events, such a model is rather impractical. As an example, consider traveling from New York to Boston. The options are to drive one's automobile, which is experiment A; to take a bus, which is experiment B; to fly, which is experiment F; or to take a train, which is experiment T. Each experiment has many possible outcomes. It does not, however, make a lot of sense to talk about the union of driving and flying, i.e., $A \cup F$. The only way to make this situation into an algebra is to form the Cartesian product $A \times B \times F \times T$ and to treat this as the universal experiment. Basically, this construction is implicit in Savage's approach. One drawback with such a construction is that A cannot be realized if any of the other options are selected. So, I shall treat events as just subsets of the universal event of a single experiment and use notation to distinguish different experiments underlying different gambles. Thus, it is possible to compare A with F which have wholly different underlying experiments.

Finally, let $\succsim$ denote an individual's weak preference ordering over $\mathscr{G}_2$ and so over $\mathscr{G}_1$ and $\mathscr{X}$.

The purpose of the paper is to provide, under structural conditions that insure adequate density of events and consequences plus some well accepted elementary necessary conditions, another set of necessary and sufficient conditions for the existence of the well known rank-dependent representation (also called a Choquet or a cumulative representation). The formal definition of rank dependence is given in Definition 4 of Section 3.3.

The three conditions can be described informally as follows. First, "rank-dependent additivity," which as been independently axiomatized by Wakker (1991, 1993), simply says that with $\vec{E}_n$ held fixed and the consequences are varied subject to maintaining the ordering constraint $x_1 \succsim \ldots \succsim x_n$, there is an additive representation over the consequences. Second, if in a gamble of order n the identical consequence is attached to E_i and E_{i+1}, then the gamble is indifferent to the one of order $n - 1$ in which the common consequence is attached to $E_i \cup E_{i+1}$. I call this "coalescing."[4] And third, consider binary gambles in which the second consequence is the status quo e and the first consequence of one such gamble is another one of the same type, i.e., $((x, C; e), D; e)$. "Status-quo event commutativity" asserts that this compound gamble is indifferent to the one in which C and D are interchanged. These are easily seen to be necessary conditions of rank dependent utility. The thrust of Theorem 7 is that, in the presence of the richness and background assumptions, the three are also sufficient for the rank-dependent utility representation.

One drawback of an approach such as Savage's, in which acts are functions from the state space into consequences, is that coalescing and event commutativity are automatically satisfied by that formulation, and so are not seen as testable properties. This is discussed more fully in Section 1.2.

Finally, I examine a number of representations that have been proposed in terms of whether or not they satisfy each of these properties. The results are tabulated in Table 2.

1. Coalescing

1.1. The definition

Definition 1 The property of **coalescing** (see note 4) is said to hold if, for all ordered partitions $\vec{E}_n$ and all consequences x_i such that $x_1 \succsim \ldots \succsim x_n$, whenever $x_j = x_{j+1}$, then

$$(x_1, E_1; \ldots; x_j, E_j; x_j, E_{j+1}; \ldots; x_n, E_n) \\ \sim (x_1, E_1; \ldots; x_j, E_j \cup E_{j+1}; \ldots; x_n, E_n). \tag{1}$$

Note that for $n = 2$ coalescing implies

$$(x, E_1; x, E_2) \sim (x, E).$$

Later (Axiom 3) I explicitly assume $(x, E) \sim x$, which means that coalescing of binary gambles is the same as idempotence.

1.2. Comments

It is not difficult to see that, because the subjective probability of subjective expected utility (SEU) is finitely additive, SEU implies coalescing. Indeed, provided that consequences and events are independent, coalescing is a highly rational "accounting" property for which the "bottom lines" of the two sides of Eq. (1) are identical in the following sense: Each x_i arises under exactly the same outcomes of the experiment (Luce, 1990). In particular, x_j arises if the outcome of the experiment lies in either E_j or in E_{j+1} and so in $E_j \cup E_{j+1}$.

As a referee has emphasized, the condition that events and consequences are independent is crucial in arguing for coalescing. The example suggested was: Let x be "possessing an umbrella" and y "not possessing an umbrella." Let E_1 be the event of rain locally and E_2 the event of no rain locally on a particular day. Assume that $x \sim y$ and so, as above, $(x, E) \sim (y, E)$, where $E = E_1 \cup E_2$. However, it seems plausible that having the umbrella when it rains and not having it when it does not is preferable to always having it or to never having it, i.e., $(x, E_1; y, E_2) \succ (y, E_1 \cup E_2)$. But because $x \sim y$, coalescing and consequence monotonicity implies $(x, E_1; y, E_2) \sim (y, E_1 \cup E_2)$, a contradiction. The caution here is that one must be very careful exactly how one formulates gambles so as to keep independent the consequences and events. Of course, no such problem arises with the gambles usually studied in experiments.

Looked at in terms of numerical utility theory, coalescing is a kind of continuity property. Suppose that the consequences are money and utility is onto a real interval, then the utility of the order n gamble with distinct x_j and x_{j+1} changes continuously with x_{j+1}, in particular as it approaches x_j. Coalescing simply requires continuity at the limit when $x_{j+1} = x_j$ between the utility assigned to the n-order gamble and the $(n - 1)$-order one to which it coalesces. Stated more formally for $n = 3$, a failure of coalescing would mean

$$\lim_{x \to y} U(x, C; y, D; z, \overline{C \cup D}) = U(x, C; x, D; z, \overline{C \cup D}) \neq U(x, C \cup D; z, \overline{C \cup D}).$$

Within either the Savage framework or in any model of the domain for which gambles are seen as probability distributions over the ordered set $\mathscr{X}$ of consequences and for which first-order stochastic dominance holds, coalescing *must* be satisfied. For example, in the risky context where there are probabilities rather than events, Wakker (1994, p. 9) remarked[5]

> "For example, $(\alpha, p_1; \alpha, p_2; x_3, p_3; \ldots; x_n, p_n)$ is identical to $(\alpha, p_1 + p_2; x_3, p_3; \ldots; x_n, p_n)$. This identity is not an assumption, but a logical necessity, these notations merely being two different ways of writing the same probability distribution."

To my view, this is a decided disadvantage of these traditional approaches. They force identities that simply may not be valid empirically. Although a distribution representation of risky alternatives forces coalescing, it is really an empirical matter whether or not these risky alternatives are actually perceived as indifferent by decision makers.

1.3. Empirical evidence

The empirical literature on directly testing coalescing is small, but what there is has been interpreted as not favoring it. In Humphrey (1995) and Starmer and Sugden (1993) the focus is mostly on trying to disentangle failures of coalescing, which they call "event-splitting effects" from what they call "juxtaposition effects," and so only a part of their data examine coalescing in pure form. In those cases, conditions (7, 7*) through (11, 11*) of Starmer and Sugden, the basic idea was to present pairs of the form (R, S) and (R, S^*), where R denotes what is called the risky gamble (higher variance) and S the safe one in coalesced form and S^* in an uncoalesced form. (Starmer and Sugden referred to these as undivided and divided, respectively). If coalescing holds, then $R \succ S$ if and only if $R \succ S^*$. The stimulus presentation of the (9, 9*) pair shown in Fig. 1 is typical, where the horizontal intervals are the number of tickets receiving the indicated amount in British £s. The numerical values for all cases are shown in Table 1 along with the proportion of S and S^* choices made by 90 subjects.

Unfortunately the data from the coalesced and uncoalesced cases were from two non-overlapping sets of subjects, so it is not possible to do a two-by-two cross tabulation for these cases. Because the proportions are significant at the 5% level in four of the seven cases, the authors conclude there is evidence against coalescing. Given that distinct groups are being compared and we know there are substantial individual preference differences among people, I view this as a decidedly weak test of the property.

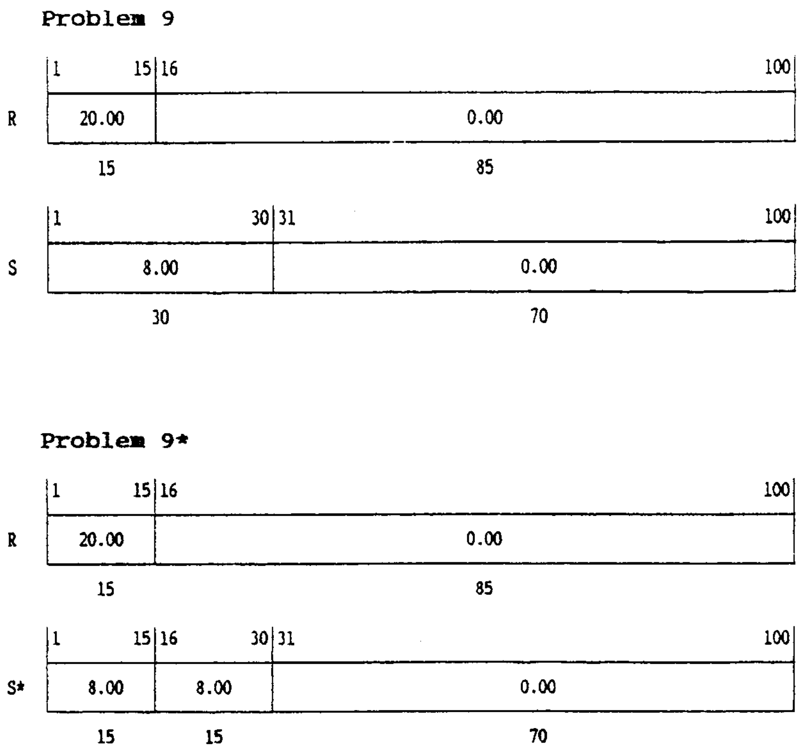

Figure 1. An example of the displays used by Starmer and Sugden (1993) in testing the coalescing property. This is their Figure 4 and it is reproduced with permission.

Table 1. Stimuli and proportions of *S* and *S** stimuli chosen over *R* in the coalescing part of the Starmer and Sugden (1993) study

Condition	Lotteries	Proportion
7	*R* = (20, 1–20; 0, 21–100)	
	S = (10, 1–30; 0, 31–100)	.400
	*S** = (10, 1–20; 10, 21–30; 0, 31–100)	.444
8	*R* = (14, 1–20; 0, 21–100)	
	S = (8, 1–30; 0, 31–100)	.422
	*S** = (8, 1–15; 8, 16–30; 0, 31–100)	.711
9	*R* = (20, 1–15; 0, 21–100)	
	S = (8, 1–30; 0, 21–100)	.567
	*S** = (8, 1–15; 8, 16–30; 0, 31–100)	.767
10	*R* = (25, 1–15; 0, 16–100)	
	S = (10, 1–30; 0, 31–100)	.544
	*S** = (10, 1–20; 10, 21–30; 0, 31–100)	.544
11	*R* = (16, 1–20; 0, 21–100)	
	S = (9, 1–30; 0, 31–100)	.500
	*S** = (9, 1–20; 0, 21–90; 9, 91–100)	.567

A literature that might be (mis)interpreted as against coalescing is "support theory" (Tversky & Fox, 1995; Tversky & Koehler, 1994). In that work it has been shown that when events are defined implicitly, often they are assigned less (subjective) probability of occurring than when at least some of the potential outcomes are enumerated and evaluated explicitly. In the typical gambling context, the underlying experiment is usually completely characterized in terms of all of its possible outcomes, and so the several events of the gamble are also enumerated. The only difference between the two listings in the coalescing property is whether or not they are also partitioned. So, I do not believe that the results of support theory bear directly on the current issue.

The most striking data casting possible doubt on coalescing were first reported by Tversky and Kahneman (1986, problem 8) and have been elaborated on by Birnbaum and Navarrete (submitted). Consider the following two gambles:

($12, .05; $14, .05; $96, .90) and ($12, .10; $90, .05; $96, .85).

Using transitivity, consequence monotonicity, and coalescing, it is easy to verify that the former dominates the latter. Yet in a series of choices like this, about 70% of the responses favored the latter. Since evidence exists in favor of consequence monotonicity for choices (von Winterfeldt, Chung, Luce, & Cho, 1997) and transitivity is usually sustained, these data cast doubt on the remaining property of the argument, coalescing.

The present paper, however, is not empirical. It addresses only theoretical issues involving coalescing and other properties, one of which we turn to next.

2. Event commutativity

2.1. The definition

A simple property of compound binary gambles, which has recently attracted some attention (Luce, 1990, 1996; Chung, von Winterfeldt, & Luce, 1994), is the following:

Definition 2 **Event commutativity** is said to hold if for all consequences x and y and all independently realized experiments D and E,

$$((x, E_1; y, E_2), D_1; y, D_2) \sim ((x, D_1; y, D_2), E_1; y, E_2), \tag{2}$$

where $D = D_1 \cup D_2$ and $E = E_1 \cup E_2$. If e denotes the status quo (neither a gain nor a loss) and Eq. (2) holds only for $y = e$, then it is called **status-quo event commutativity**.

Note that this too is a highly rational accounting equivalence in that the bottom line is the same on both sides, namely, x is the consequence if D_1 and E_1 both occur and y otherwise. The only distinction between the two sides is the order in which the experiments are realized. Again, the property holds automatically whenever the domain is modeled as probability distributions. Some theories satisfy status-quo event commutativity without satisfying the general property, and so we must maintain that distinction.

2.2. Empirical evidence

Event commutativity has received some empirical attention in the literature (for a summary, see Chung, von Winterfeldt, & Luce 1994). The earliest studies gave it, at best, limited support, but the reason may have been, at least in part, an artifact of a design that forced choices and did not permit subjects to assert indifference. The later study avoided this problem by establishing certainty equivalents for gambles from which the order was inferred, and these data provided fairly strong empirical support—namely, 22 out of 25 subjects—for event commutativity. The primary failures tended to occur with gambles of mixed gains and losses.

3. Rank-dependent representations

3.1. Rank-dependent additivity

A third major property to be considered is that, for each event partition, the structure satisfies axioms sufficient to prove the existence of an additive conjoint representation of the consequences when their order according to $\succsim$ is maintained. Wakker (1991, 1993) in, respectively, algebraic and topological contexts has formulated suitable axioms which I do not repeat here. As he explained in detail, some delicacy is required at maximum and minimum elements.

As above, let $\mathscr{X}$ denote the set of consequences, $e \in \mathscr{X}$ the status quo, $\mathscr{E}_E$ the algebra of events when experiment E is conducted, $\mathscr{G}$ the set of finite gambles including $\mathscr{X}$ and generated recursively from $\mathscr{X}$ and $\mathscr{E}_E$, and $\succsim$ the preference order on $\mathscr{G}$. Let $\mathscr{E}_n$ denote the set of ordered lists of n disjoint, non-empty events.

Definition 3 **Rank-Dependent Additivity (RDA)** is said to hold if for each $n \geq 2$, for each ordered partition $\vec{E}_n \in \mathscr{E}_n$, and for all $x_i \in \mathscr{X}$, $i = 1, 2, \ldots, n$, with $x_1 \succsim \ldots \succsim x_n$, there are functions $U_{n,i}: \mathscr{X} \times \mathscr{E}_n \xrightarrow{into} \mathbb{R}$, the real numbers, with $U_{n,i}$ strictly increasing in the first argument and with $U_{n,i}(e, \vec{E}_n) = 0$, such that

$$U_n(x_1, E_1; \ldots; x_n\ E_n) = \textstyle\sum_{i=1}^{n} U_{n,i}(x_i, \vec{E}_n) \tag{3}$$

is an order preserving representation. For losses, a similar definition may be formulated. RDA is said to be **onto an interval** if the image of each $U_{n,i}$ is a real interval $R_{n,i}$ that includes 0.

Axiom 1 For each integer n, preferences over the ordered first-order gambles satisfy axioms of additive conjoint measurement (e.g., Wakker, 1991) that are sufficient for the existence of a rank-dependent, additive representation onto an interval.

Assuming that $\sim$ has more than two equivalence classes and excluding any maximal or minimal gambles and without requiring that e map into 0, Wakker shows that these $U_{n,i}$ functions are unique up to interval scale transformations. With the added requirement that e map into 0, they are ratio scales. So, with no loss of generality, we may select a unit for U_n such that $1 \in R_{n,i}$, $i = 1, \ldots, n$.

Note that Axiom 1 implies the property of restricted solvability (see Krantz, Suppes, & Tversky, 1971, p. 256) and that consequence monotonicity holds, in particular, $\forall x, y \in \mathscr{X}$, $E_1, E_2 \in \mathscr{E}$, with $E_1 \cap E_2 = \varnothing$,

$$x \succsim y \Leftrightarrow (x, E_1; e, E_2) \succsim (y, E_1; e, E_2).$$

3.2. Evidence about additivity

The basic underlying property of rank-dependent additivity is the property of *co-monotonic independence* which in essence says that if two gambles have the same consequence-event pair in the same ranked position, i.e., (x_i, E_i) is common to both gambles, then the preference ordering is unchanged if x_i is replaced by any x_i' provided that the rank ordering among consequences is unchanged. The distinction between the rank-dependent theories and ones like subjective expected utility that are not rank dependent is whether invariance holds beyond co-monotonic changes or not. When it is independent of ranked positions, Birnbaum and McIntosh (1996) call it *branch independence*.

There are several relevant empirical studies all based on the same simple idea involving order 3 gambles. Two pairs of consequences, (x, y) and (x', y') and the probability distribution are held fixed and a third consequence is varied so that it falls in any of the intervals defined by the order of these consequences. Suppose, for example, $x' \prec x \prec y \prec y'$, then the co-monotonic cases are when $z \prec x'$, $x \prec z \prec y$, and $y' \prec z$. Wakker, Erev, and Weber (1994) ran such an experiment and found no evidence favoring co-monotonic independence over branch independence. However, many (e.g., Birnbaum & McIntosh, 1996, and Weber & Kirsner, 1997, and in correspondence) have criticized the design as one that strongly invited subjects simply to drop the common z terms in a kind of pre-editing before evaluating the gambles. Both Birnbaum and McIntosh (1996) and Weber and Kirsner (1997) provide evidence against branch independence and in favor of co-monotonic independence. For example, the former authors say (p. 102) "The present results are quite compatible with generic rank-dependent utility theory." This is true, but misleading. They are compatible with rank-dependent additive theory, but they do not test other aspects of rank-dependent utility, such as coalescing (see Section 3.4). They do go on to note that if in addition to rank dependence one assumes the special forms of utility and weighting functions proposed by Tversky and Kahneman (1992), the model fails to account for the data.

One thing seems very clear: if choices are offered and if the subjects see ways to simplify the response, such as ignoring common terms, they will. This form of editing prior to evaluation can appear to be evidence against a theory of evaluation of the unedited gambles. In my opinion, one should be careful to maintain this distinction when evaluating theories such a rank-dependent utility.

It should be noted that the above results were all for money gains. For the mixed case of both gains and losses Chechile and Cooke (1997) report data that appear to reject decisively the additivity property. This important issue needs considerable further investigation.

At a theoretical level, Luce (1997) showed that if the binary operation of joint receipt—of getting two things at once—is everywhere associative and if joint receipts are related to gambles by properties[6] called "segregation" for gains (and losses) and "duplex decomposition" for the mixed case, then rank-dependent additivity holds for gambles of gains (and losses) but fails to hold in the mixed case. Cho, Luce, and von Winterfeldt (1994) report data that are consistent with both segregation for gains and losses separately and duplex decomposition for mixed cases. The associativity of joint receipt has not yet been studied directly. Assuming that it is sustained, then we have a good theoretical reason as well as an empirical one to doubt whether additivity holds in the mixed case. For that reason I state the results here only for gains and, implicitly, for losses separately.

3.3. Rank-dependent utility

The following is a well known generalization of subjective expected utility (SEU). The history is complex and is recounted in Quiggin (1993) and Wakker (1989). Suffice it to say that the first version, stated for known probabilities, is Quiggin (1982), and more general formulations were given in a short period of time by Gilboa (1987), Luce (1988), and Schmeidler (1989). Schmeidler is generally given priority because the first version of his paper circulated in 1982.

Definition 4 **Rank-dependent utility (RDU) representation** is said to hold if there is an order-preserving utility function U over gambles (including pure consequences as a special case) and a weighting function W over events in ξ_E that is monotonic increasing with event inclusion and such that for $x_1 \succsim \dots \succsim x_n$

$$U(x_1, E_1; x_2, E_2; \dots; x_n E_n) = \sum_{i=1}^{n} U(x_i) W_i(\vec{E}_n), \tag{4}$$

where W_i is defined in terms of W by

$$W_i(\vec{E}_n) = W(E_1 \cup \dots \cup E_i) - W(E_1 \cup \dots \cup E_{i-1}), \text{ and } W(E) = 1. \tag{5}$$

RDU is said to be **onto an interval** if the image of U is a real interval including 0 and that of W is [0, 1] with $W(\varnothing) = 0$.

The assumption that W is monotonic increasing with event inclusion insures that $W_i(\vec{E}_n) \geq 0$, $i = 1, 2, \dots, n$.

The special case where W is finitely additive, i.e., for disjoint events C, D, $W(C \cup D) = W(C) + W(D)$, is called *subjective expected utility (SEU)*. The further special case where $p_i = \Pr(E_i)$ is given and $W(E_i) = p_i$ is known as *expected utility (EU)*. And finally, if $\mathscr{X}$ consists of sums of money and $U(x) = x$, it is called *expected value (EV)*.

It is easy to see that RDU, and so SEU and EU, satisfies coalescing: In Eqs. (4) and (5), consider just the j and $j + 1$ terms with $x_j = x_{j+1}$, then the weights combine as:

$$\begin{aligned}[W(E_1 \cup \dots \cup E_j \cup E_{j+1}) - W(E_1 \cup \dots \cup E_j)]& \\ + [W(E_1 \cup \dots \cup E_j) - W(E_1 \cup \dots \cup E_{j-1})]& \\ = [E_1 \cup \dots \cup E_{j-1} \cup (E_j \cup E_{j+1})] - W(E_1 \cup \dots \cup E_{j-1}),&\end{aligned}$$

which obviously yields coalescing.

Clearly rank-dependent utility (onto an interval) is rank-dependent additive (onto an interval). And a routine calculation shows that the binary model satisfies event commutativity.

One reason that event commutativity is an important property is that it lies on one side of the dividing line between RDU and SEU. The latter representation implies all conceivable accounting indifferences generated by compounding gambles, whereas RDU satisfies event commutativity but not the next more complex binary accounting indifference called "autodistributivity." Adding that property can be shown to force the weights to be finitely additive, thereby reducing it to SEU (Luce & Narens, 1985; Luce & von Winterfeldt, 1994). What empirical evidence there is, Brothers (1990), rejects autodistributivity.

Thus, the distinction between RDU and SEU provides a very specific meaning to Simon's (1956) somewhat vague concept of "bounded rationality." To repeat, RDU does not require that the decision maker exhibit indifference between two formally equivalent gambles that are any more complicated than those entering into event commutativity. My experience with students suggests that many, even with tutoring, fail to see through accounting indifferences more complicated than event commutativity.

3.4. Evidence about RDU

The major data concern cases of order 3 or more. Because RDU implies additivity, the evidence favoring additivity is favorable. But RDU also implies coalescing, and any evidence against that property is of course evidence against RDU. A direct study that seems clearly to reject RDU is Wu (1994). It is based on a property that he calls ordinal independence which is a mix of co-monotonic independence and coalescing. As an example, consider the choice between

$$\begin{pmatrix} .32 & .01 & .77 \\ \$3600 & \$3500 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} .32 & .02 & .76 \\ \$3600 & \$2000 & 0 \end{pmatrix}$$

and between

$$\begin{pmatrix} .33 & .77 \\ \$3500 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} .32 & .02 & .76 \\ \$3500 & \$2000 & 0 \end{pmatrix}$$

In the first choice he found 60% of the subjects selected the left option whereas in the second 78% selected the right one.

Using coalescing, we see that the first choice can be transformed into

$$\begin{pmatrix} .32 & .01 & .01 & .76 \\ \$3600 & \$3500 & 0 & 0 \end{pmatrix} \text{ vs. } \begin{pmatrix} .32 & .02 & .76 \\ \$3600 & \$2000 & 0 \end{pmatrix}$$

By rank dependent additivity the choice is reduced to

$$\begin{pmatrix} .01 & .01 \\ \$3500 & 0 \end{pmatrix} \text{ vs. } \begin{pmatrix} .02 \\ \$2000 \end{pmatrix}$$

Applying coalescing to the second choice

$$\begin{pmatrix} .32 & .01 & .01 & .76 \\ \$3500 & \$3500 & 0 & 0 \end{pmatrix} \text{vs.} \begin{pmatrix} .32 & .02 & .76 \\ \$3500 & \$2000 & 0 \end{pmatrix}$$

and once again we see by rank-dependent additivity that the choice devolves to the same as the first one.

The experimental data explored similar cases carefully, and it is unambiguous that rank-dependent additivity and coalescing do not both hold. Given that there is some other support for additivity and some against coalescing, one has to be skeptical of coalescing despite its great normative appeal.

4. When coalescing, event commutativity, and rank-dependent additivity imply rank-dependent utility

Given that RDU implies coalescing, event commutativity, and rank-dependent additivity, a natural question to ask is: Under what conditions do these conditions imply RDU? Under the usual constraints of adequate densities of events and consequences, several additional necessary binary conditions suffice to prove the converse. This axiomatization differs both from the co-monotonicity arguments of Wakker (1989) and Wakker and Tversky (1993) and from the use of a joint receipt operation found in Luce and Fishburn (1991, 1995).

The result is stated in two parts. The first uses status-quo event commutativity and other assumptions, stated below, to construct a binary rank dependent representation. [Recall that one can show in the binary case that the most general interval scale utility theory must be rank dependent (Luce & Narens, 1985).] The second theorem assumes the binary rank-dependent representation, coalescing, and RDA to construct the general RDU. The role of coalescing is a fairly obvious inductive one of being able to reduce some cases of nth-order gambles to $(n - 1)$th-order ones.

4.1. Status-quo event commutativity and binary rank dependence

To state the additional properties needed, let us simplify the notation in the binary case by writing $(x, E_1; y, E_2)$ as $(x, E_1; y)$, thereby making the underlying event $E_2 = E \backslash E_1$ implicit, and then drop the subscript by writing $D = E_1$ and $\bar{D} = E \backslash D$.

Axiom 2 **Idempotence**: $\forall x \in \mathscr{X}, \forall D \in \mathscr{E}, (x, D; x) \sim x$.

Axiom 3 **Certainty**: $\forall x, y \in \mathscr{X}, (x, E) \sim x$, where E is the event of the underlying experiment.

Note that if Axiom 3 holds, then idempotence, Axiom 2, is equivalent to binary coalescing:

$$(x, D; x, \bar{D}) \sim x \sim (x, E) = (x, D \cup \bar{D}).$$

Axiom 4 **Complementarity**: $\forall x, y \in \mathcal{X}, \forall D \in \mathcal{E}, (x, D; y) \sim (y, \bar{D}; x)$.

Definition 5 Suppose $C, D \in \mathcal{E}$ are such that it is not the case that for all $x \in \mathcal{X}$, $(x, C; e) \sim (x, D; e)$. For $x_i \in \mathcal{X}$, $i \in I$, where I is an interval of consecutive integers, $\{x_i\}$ is a **standard sequence of consequences** if $(x_{i+1}, C; e) \sim (x_i, D; e)$ where $i, i + 1 \in I$. A **standard sequence of events** is defined similarly.

Axiom 5 **Archimedean**: Every bounded standard sequence is finite.

Note that this Archimedean property does not follow from the one that is implicit in Axiom 1 and which is based on trade-offs among consequences for a fixed event partition.

Axiom 6 $\forall x \in X, \forall C, D \in \mathcal{E}$, if $x \succ e$ and $D \supset C$, then $(x, D; e) \succsim (x, C; e)$.

Theorem 6 Over gains (or over losses)[7] a binary structure has a rank-dependent representation onto an interval with $U(e) = 0$ if and only if the structure satisfies Axioms 1–6 and status-quo event commutativity [Eq. (2) with $y = e$].

All proofs are presented in the Appendix.

4.2. Coalescing and general rank dependence

Having shown the conditions under which binary RDU occurs, we now assume that the binary RDU representation holds and show that it together with coalescing and Axiom 1 are necessary and sufficient for general RDU onto an interval to hold. The notation for consequences, events, gambles, etc. is as above.

Theorem 7 Over gains, the structure $\langle \mathcal{D}, \succsim, e \rangle$ has a rank-dependent representation satisfying Eqs. (4) and (5) onto an interval of the non-negative real numbers if and only if:

(i) Axiom 1 is satisfied.
(ii) There is an order preserving utility function U over $\mathcal{D}$ onto a real interval R with $0 \in R$ and $U(e) = 0$, and a weighting function W over $\mathcal{E}$ onto [0, 1] that is monotonic increasing with event inclusion such that the subdomain of binary gambles has a RDU representation with

$$W_1(\vec{E}_2) = W(E_1),\ W_2(\vec{E}_2) = 1 - W(E_1). \tag{6}$$

(iii) Coalescing, Eq. (1), holds.

So, the upshot of these two theorems is that if Axioms 1–6 hold, then RDU onto an interval is equivalent to status-quo event commutativity and coalescing both holding.

5. Other representations that satisfy coalescing

Definition 8 **Rank- and sign-dependent utility (RSDU)** (Luce & Fishburn, 1991, 1995; Starmer & Sugden, 1989; also called cumulative prospect theory,[8] Tversky & Kahneman, 1992) is said to hold if there is an order-preserving utility function U and two weighting functions W^+ for gains and W^- for loses such that (i) in each domain separately the utility representation is rank dependent and (ii) for $x_1 \succsim \ldots \succsim x_k \succsim e\ x_{k+1} \succsim \ldots \succsim x_n$ where e denotes the status quo, then $U(e) = 0$ and

$$U(x_1, E_1;\ldots x_n, E_n) = U(x_1, E_1;\ldots; x_k, E_k)W^+(E_1 \cup \ldots \cup E_k) + U(x_{k+1}, E_{k+1};\ldots; x_n, E_n)W^-(E_{k+1} \cup \ldots \cup E_n). \quad (7)$$

Here, and especially in some subsequent models, some theorists will reject totally the application of the theory to compound gambles, in which case any remarks about event commutativity are irrelevant for them.

Proposition 9 Assuming certainty [Axiom 3], rank- and sign-dependent utility theory [Eq. (7)] implies coalescing [Eq. (1)], status quo event commutativity [Eq. (2) with $y = e$], and rank-dependent additivity [Eq. (3)]. For gambles of just gains or of just losses, general event commutativity [Eq. (2)] holds, but with mixed gains and losses, it holds only if $W^+ = W^-$, i.e., the purely rank dependent case.

Note that in the following statements if status-quo event commutativity fails, then so too does general event commutativity; and if the general case holds, so too does the status-quo one. The issue of rank-dependent additivity, although obvious in each case, is explicitly stated.

For the rest of this section and the following one, matters are simplified considerably by dealing just with the $n = 3$ cases and altering the notation so as to avoid subscripts: the consequences will be denoted x, y, z, and the events by B, C, D. The proofs for general finite gambles are conceptually easy but require somewhat more complex notation.

Definition 10 (Chew, 1983) **Weighted utility** is said to hold in cases with known probabilities of events if there is an order-preserving utility function U on gambles and a weighting function W on consequences (not probabilities) such that for $\forall x, y \in \mathscr{X}, p, q \in [0, 1]$

$$U(x, p; y, q) = \frac{pW(x)U(x) + qW(y)U(y)}{pW(x) + qW(y)}, \qquad (p + q = 1) \quad (8)$$

$$U(x, p; y, q; z, r) = \frac{pW(x)U(x) + qW(y)U(y) + rW(z)U(z)}{pW(x) + qW(y) + rW(z)}. \quad (p + q + r = 1) \quad (9)$$

Proposition 11 Weighted utility satisfies coalescing but neither status-quo event commutativity nor rank-dependent additivity.

Definition 12 (Chew, Epstein, & Segal, 1991) **Quadratic utility** is said to hold in the trinary case (the binary is similar) if there is an order-preserving utility function U on gambles and a symmetric function φ on $\mathscr{X} \times \mathscr{X}$, where $\mathscr{X}$ is the set of consequences such that for $\forall x_i \in \mathscr{X}$ and $p_i \in [0, 1]$, $i = 1, 2, 3$, $p_1 + p_2 + p_3 = 1$,

$$U(x_1, p_1; x_2, p_2; x_3, p_3) = \sum_{i,j=1}^{3} \varphi(x_i, x_j)p_i p_j. \quad (10)$$

Proposition 13 Quadratic utility satisfies coalescing but neither status-quo event commutativity nor rank-dependent additivity.

6. Theories that do not satisfy coalescing

Actually what we will discover is that for three theories, coalescing holds only for special cases that also happen to be special cases of RDU.

Definition 14 (Edwards, 1962) **Linear weighted utility theory** is said to hold (in the trinary case) if there is an order-preserving utility function U over gambles and a weighting function W over events such that $W(\varnothing) = 0$ and for $\forall x, y, z \in \mathscr{X}$ and $B, C, D \in \mathscr{E}$,

$$U(x, B; y, C; z, D) = U(x)W(B) + U(y)W(C) + U(z)W(D). \quad (11)$$

Note that W is not assumed to be additive over disjoint events, which is the only difference from SEU.

Comment: Some authors, e.g., Fennema and Wakker (1997) and Starmer and Sugden (1993), have claimed that Kahneman and Tversky (1979) intended Eq. (11), and its trivial generalization to gambles of order n, as the natural generalization for gains and losses separately of their prospect theory. Although I find it difficult to draw this inference from their paper, it certainly is the case that others have. One reason I do not think this is a reasonable inference is that Kahneman and Tversky did not explicitly write down Eq. (11). Why not?

A second pair of facts can be used to argue the issue either way. They invoked coalescing, which they called “combination,” as a preliminary editing step in their 1979 paper. But, as is shown in the next proposition, Eq. (11) is not in general consistent with coalescing. Thus, one can say that Eq. (11) would not be a suitable extension of prospect theory. But one can also argue that because they wanted both Eq. (11) and coalescing to hold, it was necessary to invoke coalescing prior to using Eq. (11).

And, of course, there is the fact that 13 years later when they did explicitly generalize prospect theory to arbitrary gambles, they postulated the rank-dependent, not the linear weighted, representation (Tversky & Kahneman, 1992).

Proposition 15 For gambles satisfying the linear weighted utility representation, coalescing holds if and only if the linear weighted utility is the special case of SEU, i.e., for $C \cap D = \varnothing$,

$$W(C \cup D) = W(C) + W(D). \tag{12}$$

Except when $W(C) + W(D) = 1$, linear weighted utility fails general event commutativity, but does satisfy status-quo event commutativity and rank-dependent additivity.

The first part of this proposition was clearly recognized by Starmer and Sugden (1993).

Proposition 16 For gambles with known probabilities satisfying linear weighted utility theory, coalescing holds if and only if the linear weighted utility is the special case of EU. General event commutativity fails except in the special case of EU or when $W(p) + W(1 - p) = 1$.

Definition 17 (Viscusi, 1989). **Prospective reference utility** is said to hold if there is an order-preserving utility function U and a constant $\alpha \in [0, 1]$ such that the utility of a gamble g of order n is given by:

$$U(g) = \alpha EU(g) + (1 - \alpha)\frac{1}{n}\textstyle\sum_{i=1}^{n} U(x_i). \tag{13}$$

Proposition 18 For preferences satisfying prospective reference theory, coalescing holds if and only if $\alpha = 1$ and so it reduces to EU. General event commutativity fails except in the case of EU. The representation satisfies rank-dependent additivity.

The next concept really is a family of related representations that have been proposed by M. Birnbaum and his colleagues. Each one differs in some detail, but in general spirit they are all similar. They all have an underlying weighting function S on events, and they all take into account the ordering of the consequences by having different coefficients applied to some function of the $S(p_i)$ where the coefficient depends upon the rank-order position i of the consequence. $U(x_i)$ is then multiplied by its appropriate modified weight normalized by the sum of all the modified weights. The following definition formulates one version. The others lead to similar propositions.

Definition 19 (Birnbaum, Coffey, Mellers, & Weiss, 1992) **Configural weight theory (1992 model)** is said to hold if there is an order-preserving utility function U, a strictly increasing weighting function S from [0, 1] onto [0, 1], and a constant a such that in the binary and trinary cases with $x, y, z \in \mathscr{X}$, $x \succsim y \succsim z$,

$$U(x, p; y, q) = \frac{U(x)(1-a)[1-S(q)] + U(y)aS(q)}{(1-a)[1-S(q)] + aS(q)}. \qquad (p + q = 1) \tag{14}$$

$$U(x, p; y, q; z, r) = \frac{U(x)(1-a)[1-S(1-p)] + U(y)(1-a)[1-S(1-q)] + U(z)aS(r)}{(1-a)[2-S(1-p)-S(1-q)] + aS(r)}. \qquad (p + q + r = 1) \tag{15}$$

Proposition 20 Configural weight theory satisfies coalescing if and only if it is the special case of EU theory. The representation satisfies event commutativity and rank-dependent additivity.

7. Conclusions

Table 2 summarizes for each of the theories discussed which of the three properties—coalescing, status-quo event commutative, and rank-dependent additivity—hold.

General event commutativity has been studied empirically and there is some evidence that it holds, thereby ruling out all of these theories except rank-dependent and rank- and sign-dependent utility and configural weight theory. However, because some authors are wary of compound gambles and because of the simplicity of coalescing and its sharp partitioning of theories, it would be useful to explore thoroughly coalescing empirically. The existing data, which seem to be against it, are in one case group results from two separate groups and in another an indirect test that involves other assumptions, and so I

Table 2. Representations and properties.

Utility Representation	Coalescing	Status-Quo Event Commutativity	Rank-Dependent Additivity
RDU	Yes	Yes	Yes
RSDU	Yes	Yes	Yes
Weighted	Yes	No	No
Quadratic	Yes	No	No
Linear Weighted	No[(a)]	Yes[(b)]	Yes
Prospective Reference	No[(a)]	No	Yes
Configural Weight	No[(a)]	Yes	Yes

(a) Except in special cases that reduce to one of RDU, SEU, EU, or EV.
(b) But idempotence and general event commutativity fail except when $W(C) + W(D) = 1$.

do not think the conclusion is yet clear. Studies should be done on individual subjects in which other properties are not also invoked. Should results about coalescing be unambiguous, they would serve to eliminate classes of theories that have been seriously considered.

The existing data favor rank-dependent additivity, although some studies appear to support the appreciably stronger branch independence, for gambles of just gains or of just losses. One study, Chechile and Cooke (1997), raises substantial doubts about additivity in the case of mixed gains and losses, and I have provided a theoretical reason for also doubting it in that case.

Despite these questions about rank-dependent additivity in the mixed case, an interesting question for gambles of exclusively gains or exclusively losses is the degree to which these three properties characterize the rank-dependent representation. As we saw in Section 4, in the case of mappings on to real intervals including 0, basically, all one needs to add are some fairly well accepted necessary conditions for binary gambles.

Among the problems remaining open are the question of what together with coalescing imply the Chew (1983) weighted utility and the Chew, Epstein, and Segal (1991) quadratic utility and what together with event commutativity and additivity imply configural weight theory.

Appendix: Proofs

Proof of Theorem 6 It is trivial that binary RDU onto an interval implies Axioms 1–6 and status quo event commutativity.

Assume these properties. By the remarks just prior to Section 3.2, we know that restricted solvability and consequence monotonicity hold. These coupled with Axioms 1–5 and status quo (i.e., $y = e$) event commutativity is equivalent (Luce, 1996) to the following concept of *separability*: If $\mathscr{X}^+ = \{x: x \in \mathscr{X}, x \succsim e\}$, then there exists $V^*: \mathscr{X}^+ \overset{onto}{\rightarrow} [0, k]$, a half open interval including [0, 1], and $W^*: \mathscr{E} \overset{onto}{\rightarrow} [0, k]$ such that, over the gambles of the form $(x, D; e)$, the function $V^*(x)W^*(D)$ is order preserving and that it is unique up to positive powers. But by Axiom 1 we also know that $U_2(x, D; e) = U_{2,1}(x, D)$ is order preserving over these same gambles. Thus there is a strictly increasing function f such that $U_{2,1}(x, D) = f[V^*(x)W^*(D)]$. Note also that by Axioms 1 and 4 and setting $\bar{D} = E\backslash D$,

$$U_{2,2}(y, D) = U_2(e, D; y) = U_2(y, \bar{D}; e) = U_{2,1}(y, \bar{D}) = f[V^*(y)W^*(\bar{D})].$$

Thus, again by Axiom 1, for $x \succsim y$,

$$\begin{aligned} U_2(x, D; y) &= U_{2,1}(x, D) + U_{2,2}(y, D) \\ &= f[V^*(x)W^*(D)] + f[V^*(y)W^*(\bar{D})]. \end{aligned} \tag{A.1}$$

The task is to determine f. Clearly, $f(0) = 0$, and one may choose the units of U_2 and V^* so that $f(1) = 1$. In Eq. (A.1) set $y = x$ and use Axioms 2 and 3 to yield

$$\begin{aligned} f[V^*(x)W^*(D)] + f[V^*(x)W^*(\bar{D})] &= U_2(x, D; x) \\ &= U_2(x) \\ &= U_2(x, E) \\ &= f[V^*(x)]. \end{aligned}$$

To simplify the notation let $v = V^*(x)$, $w = W^*(D)$, and $Q(w) = W^*(\bar{D})$. Note that by the assumption that W^* is onto [0, 1] and Axiom 6, one knows that $Q(0) = 1$ and $Q(1) = 0$. So f satisfies

$$f(vw) + f[vQ(w)] = f(v), v \in [0, k[, w \in [0, 1]. \tag{A.2}$$

If we can show that $f(v) = v^{\beta}$, for some $\beta < 0$, then it follows that

$$1 = w^{\beta} + Q(w)^{\beta} = W^*(D)^{\beta} + W^*(\bar{D})^{\beta},$$

and so setting $V = V^{*\beta}$ and $W = W^{*\beta}$, it follows that the binary RDU model holds and we are done. □

When I did not see how to solve Eq. (A.2), I asked for help from János Aczél, an expert on functional equations, and he in collaboration with two other mathematicians have solved it under even more general conditions than I needed.

Lemma A.1 (Aczél, Ger, & Járai, in press) If Eq. (A.2) holds, f: $[0, k] \to \mathbb{R}_+$ (= the set of non-negative real numbers), $k > 1$, and Q: $[0, 1] \to \mathbb{R}_+$, then either f is a constant function or for some $\alpha > 0$, $\beta > 0$

$$f(v) = \alpha v^{\beta},$$

and

$$Q(w)^{\beta} = 1 - w^{\beta}.$$

Because the strict monotonicity of f rules out the trivial solution and $f(1) = 1$ implies $\alpha = 1$, we have the desired result. □

Proof of Theorem 7. First, it is obvious that Eqs. (1), (3), and (6) necessarily hold if rank–dependent utility holds.

To show sufficiency, first consider the case $n = 2$. One has two additive representations, that of the assumed binary rank-dependent models of Eq. (3),

$$U_2(x_1, E_1; x_2, E_2) = U_{2,1}(x_1, \vec{E}_2) + U_{2,2}(x_2, \vec{E}_2)$$

and of Eq. (6)

$$U(x_1, E_1; x_2, E_2) = U(x_1)W_1(E_1) + U(x_2)[1 - W_1(E_1)].$$

The lemma following the completion of the rest of the proof establishes that if the units are chosen so that the ranges agree, these two additive representations are the same, i.e.,

$$U_{2,1}(x_1, \vec{E}_2) = U(x_1)W_1(E_1), U_{2,2}(x_2, \vec{E}_2) = U(x_2)[1 - W_1(E_1)].$$

Consider next $n = 3$. By a lemma analogous to the one proved and using coalescing, one can choose the unit of U_3 to agree with that of U_2. Assume that $x_1 \succsim x_2 \succsim x_3$, then by Eq. (1) we have two cases of coalescing corresponding to setting $x_1 = x_2$ and to setting $x_3 = x_2$ leading to, respectively,

$$(x_2, E_1; x_2, E_2; x_3, E_3) \sim (x_2, E_1 \cup E_2; x_3, E_3) \tag{A.3}$$

$$(x_1, E_1; x_2, E_2; x_2, E_3) \sim (x_1, E_1; x_2, E_2 \cup E_3). \tag{A.4}$$

Applying Axiom 1 to the equivalences (A.3) and (A.4) and using the fact that U_3 has been chosen to agree with U_2, one has:

$$\begin{aligned} U_{3,1}(x_2, \vec{E}_3) &+ U_{3,2}(x_2, \vec{E}_3) + U_{3,3}(x_3, \vec{E}_3) \\ &= U_3(x_2, E_1; x_2, E_2; x_3, E_3) \\ &= U_2(x_2, E_1 \cup E_2; x_3, E_3) \\ &= U_{2,1}[x_2, (E_1 \cup E_2, E_3)] + U_{2,2}[x_3, (E_1 \cup E_2, E_3)] \\ &= U(x_2)W(E_1 \cup E_2) + U(x_3)[1 - W(E_1 \cup E)]. \end{aligned}$$

$$\begin{aligned} U_{3,1}(x_1, \vec{E}_3) &+ U_{3,2}(x_2, \vec{E}_3) + U_{3,3}(x_2, \vec{E}_3) \\ &= U_3(x_1, E_1; x_2, E_2; x_2, E_3) \\ &= U_2(x_1, E_1; x_2, E_2 \cup E_3) \\ &= U_{2,1}[x_1, (E_1, E_2 \cup E_3)] + U_{2,2}[x_2, (E_1, E_2 \cup E_3)] \\ &= U(x_1)W(E_1) + U(x_2)\,[1 - W(E_1)]. \end{aligned}$$

Taking into account that the x_i can be chosen independently, subject to the inequalities among the x_i's being maintained, one obtains the following four equalities:

$$U_{3,1}(x_2, \vec{E}_3) + U_{3,2}(x_2, \vec{E}_3) = U(x_2)W(E_1 \cup E_2), \tag{A.5}$$

$$U_{3,3}(x_3, \vec{E}_3) = U(x_3)[1 - W(E_1 \cup E_2)], \tag{A.6}$$

$$U_{3,1}(x_1, \vec{E}_3) = U(x_1)W(E_1), \tag{A.7}$$

$$U_{3,2}(x_2, \vec{E}_3) + U_{3,3}(x_2, \vec{E}_3) = U(x_3)[1 - W(E_1)]. \tag{A.8}$$

Equations (A.6) and (A.7) yield two of the three weights. The third is obtained by substituting Eq. (A.7) into Eq. (A.5) yielding

$$U_{3,2}(x_2, \vec{E}_3) = U(x_2)[W(E_1 \cup E_2) - W(E_1)]. \tag{A.9}$$

Equation (A.8) is obviously satisfied by substituting Eq. (A.6) and (A.9) into the left side. Observe that with these binary values substituted, the utilities for $n = 3$ satisfy the rank dependent form of Eqs. (4.1) and (4.2).

The induction for $n > 3$ is routine. □

Left unproved is the assertion that the two binary additive representations of Eqs. (3) and Eq. (6) are the same. I do that next.

Lemma A.2 The utilities of the binary additive model of Eq. (3) and the binary rank-dependent model of Eq. (6), where the utilities are 0 when $x_i = e$ and the units are chosen so the ranges are the same, are equal.

Proof. Because both binary representations are, by definition, order preserving and the U's are onto R, there exists a strictly increasing function $h: R \overset{onto}{\rightarrow} R$ such that

$$U_{2,1}(x_1, \vec{E}_2) + U_{2,2}(x_2, \vec{E}_2) = h(U(x_1)W_1(E_1) + U(x_2)[1 - W(E_1)]).$$

Let $w = W_1(E_1)$, $u_i = U(x_i)$, where $u_1 \geq u_2$. Observe that by first setting $x_2 = e$ and then $x_1 = e$ and using the assumptions about the U's, we have

$$U_{2,1}(x_1, \vec{E}_2) = h(u_1 w), \quad u_1 \geq 0; \quad U_{2,2}(x_2, \vec{E}_2) = h[u_2(1 - w), \quad u_2 \leq 0.$$

So for all $u_1 \geq 0 \geq u_2$ one has the functional equation

$$h(u_1 w) + h[u_2(1 - w)] = h[u_1 w + u_2(1 - w)]. \tag{A.10}$$

We next show that h is symmetric around 0. Choose $u_1 w + u_2(1 - w) = 0$ and set $y = u_1 w \geq 0$, then since $h(0) = 0$, we see from Eq. (A.10) that $h(-y) = -h(y)$. Now, for $u_1 w + u_2(1 - w) > 0$ and using symmetry we see that over the positive real numbers

$$h(u_1 w) - h[-u_2(1 - w)] = h[u_1 w + u_2(1 - w)]$$

and letting $y = u_1 w > 0$, $z = -u_2(1 - w) > 0$

$$h(y) = h(y - z) + h(z), \quad y - z > 0.$$

It is well known (Aczél, 1966) that the only strictly increasing solutions to this Cauchy equation for $y > 0$ are $h(y) = \alpha y$, where α is a constant and >0. Because the ranges of y and h are the same, $\alpha = 1$. For negative values of y, we have

$$h(y) = -h(-y) = -(-y) = y.$$

□

In proving the propositions of Sections 5 and 6, the issue of whether or not rank-dependent additivity holds is omitted because in each case it is completely obvious.

Proof of Proposition 9. Because rank- and sign-dependent theory involves rank dependence for subgambles of gains and subgambles of losses separately, we know coalescing holds in these regions. So we must consider the case where $x \succsim e \succsim y$. There are two possible ways for the collapse to occur, either with $x' \succsim x \succsim e \succsim y$ as $x' \to x$ or with $x \succsim e \succsim y \succsim y'$ as $y' \to y$. But because each case lies in the domain of rank dependence, we know coalescing holds.

To show that event commutativity need not hold in the case of mixed gains and losses, consider a case where $x \succ e \succ y$, $(x, C; y) \succ e \succ (x, D; y)$. Routine calculations of $U[(x, C; y), D; y]$ and $U[(x, D; y), C; y]$ yield as coefficients of the $U(x)$ term, respectively, $W^{+}(C)W^{+}(D)$ and $W^{-}(C)W^{+}(D)$. Thus since C is arbitrary, $W^{+} = W^{-}$, and so sign dependence is lost. □

Proof of Proposition 11. Setting $y = x$ and $z = y$ in Eq. (9) yields, according to Eq. (8),

$$\begin{aligned} U(x, p; x, q; y, r) &= \frac{(p + q)W(x)U(x) + rW(y)U(y)}{(p + q)W(x) + rW(y)} \\ &= U(x, p + q; y, r), \end{aligned}$$

which, because U is order preserving, yields the coalescing property.

Turning to event commutativity, either the theory is not defined for compound binary gambles because $W(x, p; y, q)$ has no meaning or it can be treated as $W[CE(x, p; y, q)]$, where $CE(g)$ denotes the monetary certainty equivalent of g in the sense that $CE(g) \sim g$. In the latter case it is easy to verify that in general Eq. (2) fails to hold because of a lack of symmetry in interchanging p and q. □

Proof of Proposition 13. Setting $x_2 = x_1$ in Eq. (10) yields

$$
\begin{aligned}
U(x_1, p_1; x_1, p_2; x_3, p_3) \\
&= \varphi(x_1, x_1)(p_1 p_1 + 2p_1 p_2 + p_2 p_2) + \varphi(x_1, x_3)2(p_1 + p_2)p_3 \\
&\quad + \varphi(x_3, x_3)p_3 p_3 \\
&= U(x_1, p_1 + p_2; x_3, p_3),
\end{aligned}
$$

which, again because U is order preserving, implies the coalescing property.

In checking event commutativity for quadratic utility, a term such as $x_i x_j$ arises, and when one of the consequences, x_i, is a gamble $(x, p; y, q)$, either the product is meaningless or it must be treated as $CE(x, p; y, q)x_j$, in which case event commutativity fails because of the lack of symmetry in interchanging p and q. □

Proof of Proposition 15. Setting $x = y$ in Eq. (11)

$$
\begin{aligned}
U(y)W(D) + U(y)W(E) + U(z)W(F) &= U(y, D; y, E; z, F) \\
&= U(y, D \cup E; z, F) \\
&= U(y)W(D \cup E) + U(z)W(M).
\end{aligned}
$$

Collecting terms, and dividing by $U(y)$ yields Eq. (12). The converse is trivial.

For event commutativity, we have

$$
\begin{aligned}
U[(x, D_1; y, D_2), E_1; y, E_2] &= U(x, D_1; y, D_2)W(E_1) + U(y)W(E_1) \\
&= U(x)W(D_1)W(E_1) \\
&\quad + U(y)[W(D_2)W(E_1) + W(E_2)],
\end{aligned}
$$

which clearly is not, in general, symmetric in D and E. It is not difficult to show that symmetry holds if and only if $W(E_2) = 1 - W(E_1)$, which reduces it to the binary RDU model.

Event commutativity obviously holds. □

Proof of Proposition 16. As in the proof of Proposition 15, one shows that for all p, q, $p + q \in [0, [1]]$

$$W(p + q) = W(p) + W(q). \tag{A.11}$$

Because W is strictly increasing over the unit interval, Theorem 3 on p.48 of Aczél (1966) implies $W(p) = cp$, $c > 0$, which with $W(1) = 1$ implies

$$W(p) = p,$$

which is EU. Again, the converse is trivial.

The argument against event commutativity holding is as in the proof of Proposition 15. □

Proof of Proposition 18. It is clear that with $x_j = x_{j+1}$,

$$\frac{1}{n}\sum_{i=1}^{n} U(x_i) \neq \frac{1}{n-1}\sum_{i=1}^{n-1} U(x_i),$$

and so since EU does satisfy coalescing, the only way prospective reference theory can is for $\alpha = 1$.

A routine calculation shows that except for $\alpha = 1$ event commutativity fails. □

Proof of Proposition 20. Setting $x = y$ in Eq. (15) and using Eq. (14), one sees that coalescing holds if and only if

$$\frac{U(x)(1-a)[2 - S(1-p) - S(1-q)] + U(z)aS(r)}{(1-a)[2 - S(1-p) - S(1-q)] + aS(r)} = \frac{U(x)(1-a)[1 - S(r)] + U(z)aS(r)}{(1-a)[1-S(r)] + aS(r)}$$

which holds if and only if

$$(1-a)\,[2 - S(1-p) - S(1-q)] + aS(1-p-q) = (1-a)[1 - S(1-p-q)] + aS(1-p-q)$$

if and only if

$$1 + S(1-p-q) = S(1-p) + S(1-q). \tag{A.12}$$

Let $T(p) = S(1-p) - 1$, then

$$T(p+q) = T(p) + T(q), \qquad (0 \leq p, q, p+q \leq 1) \tag{A.13}$$

which is the same as Eq. (A.11). Because S is strictly increasing, T is strictly decreasing so for some $c > 0$, $T(p) = -cp$, and so $S(p) = cp + 1 - c$. Because $S(0) = 0$, then $c = 1$, and so

$$S(p) = p. \tag{A.14}$$

There is a second possible coalescing, namely to set $z = y$ in Eq. (15). Following an exactly parallel argument one shows that S must also satisfy

$$1 - a + aS(1 - p - q) = aS(1 - p) + (1 - a)S(1 - q). \tag{A.15}$$

By substituting Eq. (A.14), $a = 1 / 2$, which then reduces Eqs. (14) and (15) to EU.

To demonstrate event commutativity note that Eq. (14) is of the form $U(x)W(p) + U(y)[1 - W(p)]$, where

$$W(p) = \frac{(1 - a)[1 - S(1 - p)]}{(1 - a)[1 - S(1 - p)] + aS(1 - p)}.$$

□

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Notes

1. This terminology is familiar from statistics; it has little to do with what an "experiment" means to an empirical scientist. In many laboratory studies such a gamble is presented as a toss of a die, as a spin of a pointer over a partitioned circle, or as a draw of a ball from an urn of colored balls. Consequences are assigned to the several possible outcomes to form gambles. Subjects are asked to judge which gamble of a pair is preferred. For example, consider the choice between two spinners, one with 6 pie regions and the other with 5. To cast such a decision in the Savage framework, one works with the Cartesian product of the two event sets, making a space state of size 30. If the study included other chance experiments, the state space is multiplicatively enlarged. It can grow quite rapidly.
2. I use the same symbol E to denote both the name of the experiment and the set of all possible outcomes of the experiment. This double usage should not cause problems.
3. $\varnothing \in \mathscr{E}_E$ and $\mathscr{E}_E$ is closed under union and complementation. When there will be no ambiguity, I write just $\mathscr{E}$.
4. This concept is formally identical to that of "event splitting" (Starmer & Sugden, 1993), and so the question arises why introduce a new term. The reason is that, to me at least, event splitting suggests partitioning that which is unitary whereas coalescing suggests uniting that which is separate. If one thinks of this situation from the perspective of continuity of utility, which is described below, coalescing seems the better term.

5. I have changed his notation to mine.
6. If $\oplus$ denotes the joint receipt operation on $\mathscr{X}$, *segregation* means that for all gains (or all losses) x, y and all events $D \subset E$,

   $(x, D; e) \oplus y \sim (x \oplus y, D; y)$.

   Duplex decomposition means for $x \succ e \succ y$,

   $(x, D; y) \sim (x, D'; e) \oplus (e, D''; y)$,

   where D' and D'' denote the occurrence of D on two independent replications of the underlying experiment E.
7. The restriction of the domain to exclude gambles with mixed gains and losses is primarily because I do not believe the model holds in that case; see Chechile and Cooke (1997). Mathematically, nothing is changed if we drop the domain restriction.
8. Some authors seem to restrict the phrase "cumulative prospect theory" to include additional restrictions on the forms of U, W^+, and W^- within the RSD framework. I do not read Tversky and Kahneman (1992) that way.

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