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When Four Distinct Ways to Measure Utility Are the Same*

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Assuming that binary rank- and sign-dependent utility holds for gambles and that riskless choice utility can be constructed using the binary operation of joint receipt of consequences, there are four distinct measures of utility over gains: the two, one from gambles and the other from joint receipt, that arise by working with the trade-offs of just gains and the two that arise by working with the trade-offs of gains with losses. The problem is to discover testable behavioral properties that make the various pairs of measures identical. This is worked out completely, leading to six distinct properties. Three have previously been studied both theoretically and empirically: event commutativity, segregation, and duplex decomposition. Three are new and have not been studied empirically: joint receipt decomposition (for gains and for the mixed case) and joint receipt consistency. Similar issues and results hold for the utility of losses, where the mathematics is identical except for parameter values. The paper concludes with a discussion of how the results extend to general finite gambles and of some possible difficulties. © 1996 Academic Press

1. BACKGROUND

The recent literature on utility theory includes an approach that attempts to combine within a single framework the concepts of the utility of gambles (uncertain alternatives) and the utility of sure consequences. The general purpose of this paper is to study their relations more fully.

1.1. *Utility Based on Gambles*

The approach based on preferences among gambles evolved from Ramsey (1931), through von Neumann and Morgenstern (1947) and Savage (1954), and summarized by Fishburn (1982), to the more recent developments of rank-dependent utility, summarized by Quiggin (1993) and Wakker (1989), and the rank- and sign-dependent models of Kahneman and Tversky (1979), called *prospect theory*, of Luce and Fishburn (1991, 1995), and of Tversky and Kahneman (1992), called *cumulative prospect theory*. The modifications were motivated in large part by a mix of thought and empirical experiments that called into question the descriptive quality of their subjective expected utility theory predecessors. For a summary of some of the relevant empirical results see Kahneman and Tversky (1979) and Schoemaker (1982, 1990), and for a summary of many of the issues and ideas see Edwards (1992).

One feature of the newer work on the utility of gambles, whose importance was first noted by Edwards (1962) and Markowitz (1952), is that people make a significant distinction between gains and losses relative to a status quo position. In some cases, including some experiments that are cited later, it is reasonable to treat the status quo as no exchange of money, but in other cases it appears that some form of context-dependent aspiration level may be more appropriate (Fishburn, 1977; Fishburn & Kochenberger, 1979; Libby & Fishburn, 1977; Lopes, 1984, 1987; Luce, Mellers, & Chang, 1993). The first model of gambles that both placed significant emphasis on the status quo and attracted widespread attention was Kahneman's and Tversky's (1979) prospect theory.

1.2. *Utility Based on Joint Receipts*

An alternate approach to utility, much less fully developed, focuses only on the consequences without regard to chance events. The major structure that has evolved rests on the quite natural idea of *joint receipt*—of receiving two (or more) entities at once. Although there was a little early empirical work on such an operation in psychology (Slovic & Lichtenstein, 1968; Payne & Braunstein, 1971) and later some in economics (Linville & Fischer, 1991; Thaler, 1985; Thaler & Johnson, 1990), the main theoretical references are quite recent: Fishburn and Luce (1995), Luce (1991, 1995), Luce and Fishburn (1991, 1995), and Thaler (1985). Several empirical studies arising out of Luce and Fishburn's work are cited below.

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1.3. *The Four Measures of Utility*

Consider only gains for the moment. Then four distinct ways to measure their utilities arise depending on whether binary gambles or binary joint receipts are used and on whether the trade-off studied is of gains versus gains or of gains versus losses (but focusing only on the gains aspect). An exactly parallel situation exists for losses, but because its mathematics is unchanged, except for parameters, I do not spell it out explicitly.

Now, if utility really is a measurable concept—some economists and many psychologists have strong doubts—it seems unlikely that there should be more than one such measure. This issue is analogous to one that recurred in physical measurement where often one can measure the same physical attribute in more than one way. There one usually finds that there are linking laws showing that the several, apparently distinct, ways of measuring the attribute really are basically the same measure. A familiar example is mass. It can be measured using a pan balance by studying either how concatenations of masses behave or how the conjoint structure formed by varying volumes and homogeneous substances behaves. A distribution law links the concatenations to the conjoint structure, and one shows the latter measure of mass must be a power function of the former one, which is the best one can expect to do since multiplicative representations of conjoint structures are only unique up to positive power transformations (see Krantz, Luce, Suppes, & Tversky, 1971, Section 10.7; Luce, Krantz, Suppes, & Tversky, 1990, Sections 20.2.6, 20.2.7 and 22.7).

The basic issue of this paper is to uncover comparable linking relations—putative "laws"—that lead the above four measures of utility to reduce to a single measure.

1.4. *Issues of Empirical Evaluation*

Throughout the paper, we will also confront questions of empirical evaluation. It should be recognized that the issues are subtle, controversial, and not fully resolved. Most of the assertions we will focus on here concern indifference of preference: That two different formulations involving gambles and joint receipts are seen by subjects as indifferent to one another. A major difficulty is that, despite the fact that the theory stated here is algebraic, the behavior appears better described as probabilistic.[1] So single observations of preferences or indifferences cannot be taken at face value. That being the case, repeated observations of the same choice pair are required. Thus, if there are n stimuli and we wish to have m observations of each of the n^2 stimulus pairs, the size of the experiment can be formidable (e.g., with $m = 100$ and $n = 50$, a quarter of a million observations must be made which at a pace of 200 per hour session means 1,250 sessions!). One device to make experimentation practical is to work with certainty equivalents (CEs)—to seek for each stimulus (gamble or joint receipt) in the experiment the sum of money that is seen as indifferent to it. Then the size drops to mn instead of mn^2 (for the above example, 25 sessions rather than 1,250).

The question then becomes one of accurately estimating CEs with as small a value for m as possible. Asking the subject to judge his or her CE surely keeps m small, namely, $m = 1$ (in which case, all 50 CEs are easily obtained in a single session). But considerable evidence (Bostic, Herrnstein, & Luce, 1990; Birnbaum, 1992; Mellers, Weiss, & Birnbaum, 1992; Tversky, Slovic, & Kahneman, 1990) has convinced me that this method (and any modified choice method that invites the subject to use a judged CE) is not equivalent to a truly choice-based CE. In particular, establishing buying or selling prices is not the same as a choice CE. So I have been led to use a modified psychophysical method called PEST[2] which is considerably more efficient than collecting full psychometric function data. As we will see in Section 8.3, there are now reasons to be uneasy about the way PEST has been implemented in a number of experiments.

It should be added that because buying and selling are clearly of great social importance, it is essential to devise theories about how these prices are established, but that is not the purpose of this paper. Rather, I am focused on a single issue: If choice preferences can be represented numerically at all, then they can be represented in at least four distinct ways. When are these four ways basically the same?

1.5. *Outline*

The paper is structured as follows. Section 2 describes the four numerical representations that arise from gambles of gains, gambles of mixed consequences, joint receipt of gains, and joint receipt of mixed consequences. Section 3 examines the behavioral "law" underlying a special property of many representations of gambles that is called "separability" and that plays a significant role throughout. Section 4 amplifies some aspects of the rank- and sign-dependent representations that have arisen for gambles. Section 5 does the same thing for joint receipts and establishes the first pair of

[1] If so, why do we bother with algebraic theories? One reason is that we are able to formulate structural properties of interest algebraically that have, so far, eluded those attempting to construct probabilistic theories. One can hope that the algebraic models will prove to be the expected value version of the currently unknown random utility theories that ultimately should evolve.

[2] PEST is an up–down method in which a gamble is repeatedly compared—but separated by many trials of other judgments—with sums of money. The amount of money is increased or diminished depending upon whether the gamble or the money was selected as preferable. The step size is initially large, but each time the direction changes it is reduced, with the stopping rule being 1/50 of the range of consequences in the gamble.

linking relations, "segregation" and "joint receipt decomposition" for gains, that lead to the utilities based on gains alone to be the same whether gambles or joint receipts are used. Section 6 treats comparably the utilities for mixed consequences, leading again to two linking relations, "duplex decomposition" and a mixed-case version of "joint receipt decomposition." Section 7 raises the question of the conditions under which the two joint-receipt measures, that based entirely on gains and that based on mixed gains and losses, are the same. Here the linking law is known as "joint receipt consistency." Section 8 is devoted to extensions from the binary case to general finite gambles, to a possible connection between joint receipt of lotteries and their convolution, and to some vexing empirical issues having to do with the monotonicity of joint receipt and whether our estimated CEs are order preserving. Section 9 provides a summary and lists some open problems.

2. THE REPRESENTATIONS IN QUESTION

2.1. *Notation*

Denote by $\mathscr{C}$ the set of consequences—often money in the experimental literature—and by $\mathscr{E}$ the set of chance events, where $\mathscr{C} \cap \mathscr{E} = \varnothing$—often realized as a simple probabilistic device, such as a die or a pie-like disk with a spinner, with known probabilities. A distinguished consequence $e \in \mathscr{C}$ is the status quo; distinguished events are the universal event $\Omega \in \mathscr{E}$ and the empty event $\varnothing \in \mathscr{E}$. From any $x, y \in \mathscr{C}$ and any $D, E \in \mathscr{E}$, with $D \cap E = \varnothing$, we generate a *binary gamble* $(x, E; y, D)$, based on the occurrence of $D \cup E$. This gamble is interpreted to mean that when the "experiment" corresponding to $D \cup E$ is conducted, the consequence is x if E occurs and y if D occurs. When $D \cup E = \Omega$, so $D = \neg E$, the notation is often abbreviated to $(x, E; y)$. When the consequences are money and the event probability is known to the subject, we speak of a *lottery* rather than a gamble. As an illustration, suppose that the experiment is generated by a toss of a fair die, $E = \{1, 2\}$, $D = \{3, 4, 5, 6\}$, $x = \$10$, and $y = -\$5$, then $(x, E; y) = (\$10, \{1, 2\}; -\$5)$ means the lottery in which one wins \$10 if either face 1 or 2 comes up on the toss, which happens with probability 1/3, and loses \$5 for any other face.

Let $\mathscr{G}$ denote the union of $\mathscr{C}$ and the set of all binary gambles that can be generated recursively from $\mathscr{C}$ and $\mathscr{E}$. Thus, for example, if $x, y, z \in \mathscr{C}$, then $(x, E; y)$ is a binary gamble and $((x, E; y), D; z)$ is a compound binary gamble. The convention followed here and later is that each occurrence of an event symbol denotes the outcome of an independent realization of the chance "experiment" underlying the gambles. This can be somewhat misleading when an event, such as E, recurs several times: In $((x, E; y), E; z)$ the consequence is x if E occurs on both of two independent realizations of the underlying experiment, y if E occurs on the first and $\neg E$ on the second, and z if $\neg E$ occurs on the first. In cases where I think added specificity would help, I write E' and E'' to mean E occurs on independent replications of the underlying experiment.

Let $\oplus$ be a binary operation on $\mathscr{G}$, where for $g, h \in \mathscr{G}$ we interpret $g \oplus h$ as the receipt of both g and h—their *joint receipt* (JR). Examples are gifts on a holiday, checks and bills in the day' s mail, and sets of purchases. We let $\mathscr{G}^{c}$ denote the closure of $\mathscr{G}$ under $\oplus$, and $\mathscr{G}^{c}$ the closure of $\mathscr{G}$ under $\oplus$. Clearly, $\mathscr{C}^{c} \subseteq \mathscr{G}^{c}$.

Let $\succsim$ denote a binary preference relation on $\mathscr{G}^{c}$ with its converse denoted $\precsim$. Also, let $\sim \, = \, \succsim \cap \precsim$ and $\succ \, = \, \succsim \cap$ (not $\precsim$). For any $\mathscr{H} \subseteq \mathscr{G}$, define

$$\mathscr{H}^{+} = \{g \mid g \in \mathscr{H} \text{ and } g \succsim e\}$$

$$\mathscr{H}^{-} = \{g \mid g \in \mathscr{H} \text{ and } g \precsim e\}$$

$$\mathscr{H}^{++} = \{g \mid g \in \mathscr{H},\ g = (x, E; y),\ x \succsim e,\ y \succsim e\}$$

$$\mathscr{H}^{+-} = \{g \mid g \in \mathscr{H},\ g = (x, E; y),\ x \succsim e,\ y \precsim e\}$$

$$\mathscr{H}^{--} = \{g \mid g \in \mathscr{H},\ g = (x, E; y),\ x \precsim e,\ y \precsim e\}.$$

2.2. *Elementary Assumptions*

We make several assumptions which seem natural under the foregoing interpretations (Luce, 1992b, 1995):

A1. (i) *Ordering*: $\succsim$ is a weak order. (Hence $\sim$ is an equivalence relation.)

(ii) *Certainty equivalents*: $\forall g \in \mathscr{G}^{c} \exists x \in \mathscr{G}$ such that $x \sim g$. Such an x, known as the *certainty equivalent* of g, is denoted by CE(g).

(iii) *Non-triviality*: $\exists x, y \in \mathscr{C}$ such that $x \succ e \succ y$.

A2. $\forall f, g, h \in \mathscr{G}^{c}$,

(i) *Commutativity of* $\oplus$: $g \oplus h \sim h \oplus g$.

(ii) *Identity of* $\oplus$: $e \oplus g \sim g \oplus e \sim g$.

(iii) *Monotonicity of* $\oplus$ *relative to* $\succsim$: $g \succsim h$ iff $f \oplus g \succsim f \oplus h$.

Note that by A1(ii) and A2(iii), it is sufficient to study $\oplus$ over $\mathscr{C}^{c}$ because a gamble can be replaced by its certainty equivalent. This property, which rests primarily on the monotonicity of $\oplus$, may well be stronger than it seems and, what is worse, it may be wrong. Cho and Luce (1995) have reported data where it appears that either our estimates of CEs are not order preserving or $\oplus$ is not monotonic. More discussion on this point appears in Section 8.3. For now, we assume methods for which there is no problem in these assumptions.

A3. $\forall x, y \in \mathscr{C}$, $\forall D, E \in \mathscr{E}$,

(i) *Idempotence of gambles*: $x \sim (x, E; x)$.

(ii) *Null event in gambles*: $y \sim (x, \varnothing; y)$.

(iii) *Complementarity of gambles*: $(x, E; y) \sim (y, \neg E; x)$.

(iv) *Monotonicity of consequences*: $(x, E; e) \succsim (y, E; e)$ iff $x \succsim y$.

(v) *Monotonicity of events*: $\forall x, y \succsim e$ or $\forall x, y \precsim e$,

$$(x, D; e) \succsim (x, E; e) \qquad \text{iff} \quad (y, D; e) \succsim (y, E; e).$$

(vi) *Restricted solvability*:

(a) If $\bar{x}, \underline{x} \in \mathscr{C}$ are such that $(\bar{x}, E; e) \succsim g \succsim (\underline{x}, E; e)$, then $\exists x \in \mathscr{C}$ such that $(x, E; e) \sim g$.

(b) If $\bar{E}, \underline{E} \in \mathscr{E}$ are such that $(x, \bar{E}; e) \succsim g \succsim (x, \underline{E}; e)$, then $\exists E \in \mathscr{E}$ such that $(x, E; e) \sim g$.

A controversy has arisen over the monotonicity of consequences (A3(iv)) when the order among gambles is judged by estimated CEs with Birnbaum (1992) and Mellers *et al.* (1992) claiming evidence of trouble when one of the consequences is e and von Winterfeldt, Chung, Luce, and Cho (1996) claiming that with different estimates of CEs there is no problem.

2.3. *The Several Representations*

The following order preserving representations have arisen in the literature:

DEFINITION 1. A function $U_1: \mathscr{G}^{++} \to \mathbb{R}^+$ is a (*binary*) *rank-dependent* (*weighted*) *utility* (*for gambles of gains*) iff there exists a weighting function $W_1^+: \mathscr{E} \to [0, 1]$ such that for all $x, y \in \mathscr{C}^+$ and $E \in \mathscr{E}$,

$$U_1(x, E; y) = \begin{cases} U_1(x)\, W_1^+(E) + U_1(y)[1 - W_1^+(E)], \\ \qquad x \succsim y \\ U_1(x)[1 - W_1^+(\neg E)] + U_1(y)\, W_1^+(\neg E), \\ \qquad x \precsim y. \end{cases} \tag{1a}$$

$$U_1(e) = 0. \tag{1b}$$

It is well known that, for suitably dense $\mathscr{C}$ and $\mathscr{E}$, the constraint (1a) means that U_1 is unique up to a positive linear transformation and that adding (1b) forces it to be unique up to multiplication by a positive constant—U_1 forms a ratio scale. The W_1^+ is absolutely unique.

DEFINITION 2. A function $U_2: \mathscr{G}^{+-} \to \mathbb{R}$ is a (*binary*) *sign-dependent* (*weighted*) *utility* (*for mixed consequences*) iff there exist weighting functions $W_2^k: \mathscr{E} \to [0, 1]$, $k = +, -$, such that for all $x^+ \in \mathscr{C}^+$ and $y^- \in \mathscr{C}^-$ and $E \in \mathscr{E}$,

$$U_2(x^+, E; y^-) = U_2(x^+)\, W_2^+(E) + U_2(y^-)\, W_2^-(\neg E). \tag{2a}$$

$$U_2(e) = 0. \tag{2b}$$

Note that if, for some E, $W_2^+(E) + W_2^-(\neg E) \neq 1$, then by A3(i), Eq. (2b) follows from Eq. (2a). The uniqueness of U_2 is also a ratio scale and W_2^+ is an absolute one.

DEFINITION 3. A function $U_3: \mathscr{G}^{+c} \to \mathbb{R}^+$ is a *negative exponential utility* (*on gains JR*) iff for some $C > 0$ and all $g, h \in \mathscr{G}^{+c}$,

$$U_3(g \oplus h) = U_3(g) + U_3(h) - \frac{U_3(g)\, U_3(h)}{C}. \tag{3a}$$

$$U_3(e) = 0. \tag{3b}$$

Note by the monotonicity of $\oplus$ (A2(iii)) that U_3 is bounded from above by C. Equation (3b) follows from Eq. (3a) by setting $g = e$ and using A1(iii) and A2(ii).

The reason for the name "negative exponential utility" is that the transformation $V_a = -\ln(1 - U_3/C)$ is additive, i.e.,

$$V_a(g \oplus h) = V_a(g) + V_a(h). \tag{3c}$$

Thus, $U_3 = C[1 - e^{-V_a}]$ is the negative exponential function of an additive representation of $\langle \mathscr{G}^{+c}, \succsim, \oplus, e \rangle$. Thus, U_3 and C have the same unit, and so the representation is unique up to multiplication by a positive constant.[3]

It may seem unusual to consider a non-additive representation when an additive one exists, but it is not unprecedented to do so when the non-additive one relates more simply to the representations of other related structures. We shall see that this is true in the present context. The best-known example of using a non-additive representation when an additive one exists is the relativistic velocity "addition" formula:

$$u \oplus v = \frac{u + v}{1 + uv/c^2}.$$

This velocity measure exhibits the expected relation $s = vt$, where s is distance traversed in time t at velocity v. One of the several drawbacks of the corresponding additive representation, called "rapidity," is its failure to have a simple relation to distance and time.

[3] For those familiar with measurement theory, it should be noted that the ratio scale transformation of U_3 does not correspond to an automorphism of the underlying structure. It is a change of scale that is not an automorphism. The ratio scale changes of V_a do correspond to automorphisms. The situation is completely analogous to the non-additive representation used for relativistic velocity concatenation that is invariant under multiplication by positive constants, but these are not velocity automorphisms.

TABLE 1

Summary of Four Different Utility Representations of Gains. In All Cases $U_i(e)=0$

	Gamble representations	Joint receipt representations
Gains $(x \succsim e,\ y \succsim e)$	Rank-dependent utility $U_1(x, E; y) = \begin{cases} U_1(x)\, W_1^+(E) + U_1(y)[1 - W_1^+(E)], & x \succsim y \\ U_1(x)[1 - W_1^+(\neg E)] + U_1(y)\, W_1^+(\neg E), & x \precsim y \end{cases}$	Negative exponential Extensive $U_3(x \oplus y) = U_3(x) + U_3(y) - \dfrac{U_3(x)\, U_3(y)}{C}$ [$V_a = -\ln(1 - U_3/C)$ is additive]
Mixed $(x^+ \succsim e \succsim y^-)$	Sign-dependent utility $U_2(x^+, E; y^-) = U_2(x)\, W_2^+(E) + U_2(y^-)\, W_2^-(\neg E)$	Additive mixed JR Additive conjoint $U_4(x^+ \oplus y^-) = U_4(x^+) + U_4(y^-)$
	Separable $U(x, E; e) = U(x)\, W^{(1)}(E), \quad i = \begin{cases} +, & x \succsim e \\ -, & x \prec e \end{cases}$	

Definition 4. A function $U_4\colon \mathscr{G}^c \to \mathbb{R}$ is *additive* (*on mixed* JR) iff for all $g^+ \in \mathscr{G}^+$ and $h^- \in \mathscr{G}^-$,

$$U_4(g^+ \oplus h^-) = U_4(g^+) + U_4(h^-). \tag{4a}$$

$$U_4(e) = 0. \tag{4b}$$

By A2(ii), Eq. (4b) follows from (4a). Once again, under suitably density restrictions Eq. (4a) limits one to an interval scale and Eq. (4b) narrows it to a ratio scale.

These four representations are summarized in Table 1 for handy reference.

The aim of the paper is to understand the conditions under which these four utility measures on gains are the same measure. The case for losses is similar. In the process, it will become apparent how to axiomatize the common measure. Note that once this is done, it is easy to see that except for A1(ii) and (iii) and A3(vi), which are existential statements, Assumptions A1–A3 follow from the representations.

3. SEPARABLE UTILITY OF BINARY GAMBLES

We begin with the general concept of a separable utility function. It serves as an important condition throughout the paper, but unlike the other measures that are tied down to one-parameter (ratio scale) transformations, separable utility is unique only up to two-parameter (power) transforms of the form $U \to \alpha U^\beta$.

Definition 5. A utility function $U\colon \mathscr{G} \to \mathbb{R}$ is *separable* iff there exist weighting functions $W^k\colon \mathscr{E} \to [0, 1]$, $k = +, -$, such that for all $x \in \mathscr{C}$ and $E \in \mathscr{E}$,

$$U(x, E; e) = U(x)\, W^k(E), \qquad \text{where} \quad k = + \text{ if } x \in \mathscr{C}^+ \text{ and } k = - \text{ if } x \in \mathscr{C}^-. \tag{5a}$$

$$U(e) = 0. \tag{5b}$$

Separability is a property of many utility theories, including not only the weighted averages of Eqs. (1a) and (2a) but also many that are not simple weighted averages. See the broad class of models in Luce (1992a).

3.1. *Separability and the Thomsen Condition*

The existence of a separable representation (Definition 5) of gambles of the form $(x, E; e)$ simply means that the conjoint structure $\langle \mathscr{C} \times \mathscr{E}, \succsim' \rangle$, where $(x, D) \succsim' (y, E) \Leftrightarrow (x, D; e) \succsim (y, E; e)$, has an additive representation. We know a good deal about the existence of such representations (see Krantz *et al.*, 1971, Chap. 6). In addition to what we have assumed in A3, there are two important additional necessary properties. The first is a version of the *Thomsen condition*: for $x, y, z \in \mathscr{C}$ and $D, E, F \in \mathscr{E}$,

$$(x, F; e) \sim (z, E; e) \quad \text{and} \quad (z, D; e) \sim (y, F; e) \Rightarrow (x, D; e) \sim (y, E; e). \tag{6}$$

The second is a form of Archimedeaness which, although easily formulated, is not really needed here. My goal here is to find a simpler form than Eq. (6) for testing the Thomsen condition.

3.2. *Status-quo Event Commutativity*

The relevant property turns out to be one whose interest and significance was noted at least as early as Ronen (1971, 1973).

DEFINITION 6. $\langle \mathscr{G}, \succsim \rangle$ satisfies *event commutativity* iff for all $x, y \in \mathscr{C}$ and $D, E \in \mathscr{E}$,

$$((x, D; y), E; y) \sim ((x, E; y), D; y). \tag{7}$$

The special case with $y = e$ in Eq. (7) is called *status-quo event commutativity*. (Recall the convention that D and E are from independent realizations of an experiment.)

Equation (7) formulates a condition that is clearly rational in the sense that both sides have the same "bottom line": one receives x if both D and E occur on independent realizations of the underlying chance experiment, and receives y otherwise. The two compound gambles are actually the same gamble if the order of occurrence of D and E is ignored.

THEOREM 1. *Given Assumptions* A1–A3, *if* $\langle \mathscr{C} \times \mathscr{E}, \succsim \rangle$ *satisfies status-quo event commutativity and Archimedeaness,*[4] *then there is a separable representation* (*Eq.* 5) *of* $\mathscr{G}$ *that is unique up to power transformations.*

Proofs of all theorems can be found in the Appendix.

3.3. *Relevant Data*

Tversky (1967) collected data on gambles of the form $(x, p; e)$ from prisoners where consequences $\mathscr{C}$ were numbers of cigarettes. He showed by construction that a separable representation existed with, however, a weighting function W that definitely was not finitely additive.

The property of event commutativity has been studied several times. Ronen (1971, 1973)[5] studied status-quo event commutativity with $x \succ e$ where subjects made direct choices between the two forms; in Experiment 1 many of the subjects recognized that the pairs of gambles had the same expected value and in Experiment 2 they were informed of this. The choice was forced and a substantial fraction of subjects selected the alternative with the greater probability of advancing to the second gamble. Forcing a choice, of course, invites the subjects to seek a distinguishing feature, and the order of the events really is the only one available. Brothers (1990) studied general ($y \neq e$) event commutativity in several different ways: direct comparisons, judged certainty equivalents (CEs), and PEST-determined CEs for lotteries. His results were mixed, but provided the greatest support for Eq. (7) in the third experiments that used both PEST CEs and directly judged CEs. He studied both $x > y$ and $y > x$ and found no difference. Chung, von Winterfeldt, and Luce (1994) carried out a more elaborate version of Brother's PEST approach and found event commutativity well confirmed for 22 out of 25 subjects.

[4] For an exact definition, see p. 253 of Krantz *et al.* (1971).

[5] These two papers report, without cross reference, the same data sets.

3.4. *Reduction of Compound Lotteries*

Observe that, for all $x \in \mathscr{C}^+$ and $D, E \in \mathscr{E}$,

$$\begin{aligned}
(x, D; e) &\sim ((x, D; e), E; (x, D; e)) && \text{(A3(i))}\\
&\succsim ((x, D; e), E; e) && \text{(A3(iii))}\\
&\succsim (e, E; e) && \text{(A3(iii))}\\
&\sim e && \text{(A3(i))}\\
&\sim (x, \varnothing; e), && \text{(A3(ii))}
\end{aligned}$$

so by restricted solvability (A3(vi)), $F(x, D, E)$ exists such that

$$(x, F(x, D, E); e) \sim ((x, D; e), E; e). \tag{8a}$$

COROLLARY 1. *Under the assumptions of Theorem* 1, $F(x, D, E)$ *depends on* x *only to the extent of its sign, and indeed in terms of Eq.* (5)

$$F^k(x, D, E) = (W^k)^{-1} [W^k(D)\, W^k(E)], \quad k = +, -. \tag{8b}$$

When the gambles are described only in terms of event probabilities with $p = \Pr(E)$, then we write $(x, p; e)$ for $(x, E; e)$.

COROLLARY 2. *Under the assumptions of Theorem* 1, *consider gambles of the form* $(x, p; e)$. *Suppose that, for every* $p \in [0, 1]$, *there exists* $E \in \mathscr{E}$ *with* $\Pr(E) = p$ *and that*

$$(x, p; e) \succsim (x, q; e) \quad \text{iff}$$
$$\text{either } x \succ e \text{ and } p \geqslant q \text{ or } x \prec e \text{ and } p \leqslant q.$$

Let $F^k(p, q)$ *solve*

$$(x, F^k(p, q); e) = ((x, p; e), q; e)), \qquad k = +, -.$$

If $F^k(p, q) = pq$, *then for some* $\alpha(k) > 0$,

$$W^k(p) = p^{\alpha(k)}, \qquad k = +, -. \tag{9}$$

Note that the property $F^k(p, q) = pq$ is a rationality condition prescribed by classical probability theory.

4. RANK- AND SIGN-DEPENDENT UTILITY OF BINARY GAMBLES

4.1. *Some Background on the Rank-Dependent Representation*

As noted in Section 1, there is a half century of history of seeking order-preserving utility representations of gambles that involve weighted sums of the utilities of the consequences. Initially these weights were taken to be the objective probabilities of the events yielding expected utility (von Neumann & Morgenstern, 1947; for a number of subsequent variants see, e.g., Fishburn, 1982). In the next line of development, (subjective) probabilities of events were constructed from the preference behavior (Savage, 1954) Again, there were a number of subsequent variants. In the 1980s a variety of rank-dependent models were developed with the notable feature that the weight assigned to an event depended not only on that event but on the rank-order position of the associated consequence relative to the other consequences that can arise in the gamble. A number of these models are summarized by Quiggin (1993) and some by Wakker (1989). Unlike the earlier models, these weights were not probabilities. Indeed, in the simplest case where the events have associated probabilities p_i and the numbering i runs from the most preferred to the least preferred consequence, the weight on event j is of the following form:

$$f\left[\sum_{k=1}^{j} p_k\right] - f\left[\sum_{k=1}^{j-1} p_k\right],$$

which gives, under f, the incremental impact of cumulative probabilities.

All models mentioned above have the property that U forms an interval scale in the sense that for some constants $k > 0$ and c, U^* is another representation iff $U^* = kU + c$. Indeed, Luce and Narens (1985) showed that among the class of binary models that are unique up to interval scale transformations, the most general form is rank dependent. Much greater freedom exists for theories in the ratio scale case.[6]

4.2. *Some Features of Rank-Dependent Utility*

First, it is obvious that rank-dependent utility (RDU) is separable. Of course, this means that status-quo event commutativity must hold.

Second, RDU implies general event commutativity. The proof is a simple calculation: Assume $x \succsim y$, then using A3(i and iv) $(x, E; y) \succsim (y, E, y) \sim y$,

$$\begin{aligned} &U[((x, E; y), D; y)] \\ &\quad = U(x, E; y)\, W^+(D) + U(y)[1 - W^+(D)] \\ &\quad = U(x)\, W^+(E)\, W^+(D) + U(y)[1 - W^+(E)\, W^+(D)] \\ &\quad = U(x, D; y)\, W^+(E) + U(y)[1 - W^+(E)] \\ &\quad = U[((x, D; y), E; y)]. \end{aligned}$$

For $x \prec y$, one uses the second part of Eq. (1a).

Third, consider what, in a sense, is the next more complex (possible) indifference[7] among compound gambles after event commutativity,

DEFINITION 7. $\langle \mathscr{G}, \succsim \rangle$ is *right autodistributive* iff for all $x, y, z \in \mathscr{C}$ and $E \in \mathscr{E}$,

$$((x, E; y), E; z) \sim ((x, E; z), E; (y, E; z)). \tag{10}$$

(Recall the convention about repeated events mentioned in Section 2.1.) Note that Eq. (10) is a rational indifference in the sense that both sides have the same bottom line: x if E occurs in both experiments, y if E occurs in one experiment and $\neg E$ in the other, and z otherwise. The only difference is the order in which E and $\neg E$ occur. As Luce and von Winterfeldt (1994) demonstrated, complementarity (A3(iii)), rank dependence, and right autodistributivity imply that the weights are finitely additive. Thus, in particular, $1 - W^+(\neg E)$ can be replaced by $W^+(E)$, thereby losing rank dependence and reducing the model to binary subjective expected utility (SEU). This means that event commutativity (Eq. 7) and right autodistributivity (Eq. 10) define a sharp division between RDU and SEU. As noted earlier, the latest data obtained under carefully controlled experimental conditions seems to support event commutativity, whereas Brothers, (1990) data exhibited major violations of right autodistributivity.

4.3. *Rank- and Sign-Dependent Utility*

Slightly prior to the flurry of work on RDU models, Kahneman and Tversky (1979) proposed a model of restricted scope that was both rank and sign dependent in the sense of Table 1 and for which, over gains and over losses, separately, $U_1 = U_2$. This model has attracted a good deal of attention, and it was later generalized to arbitrary finite gambles by Luce and Fishburn (1991, 1995) and Tversky and Kahneman (1992). The former called it

[6] At first reading, this assertion seems paradoxical. Interval scales are more general than ratio ones, so the former seem to have more flexibility than the latter. That of course is true for the transformations, but in terms of theories that are invariant under these transformations, exactly the opposite holds. Thus, for example, for any f that is strictly increasing and $f(z)/z$ that is strictly decreasing, the numerical operation defined by $x \circ y = yf(x/y)$ is monotonic and invariant under ratio scale transformations. To be invariant also under interval scale limits f to the following form:

$$f(z) = \begin{cases} z^{c}, & z \geqslant 1, \quad 0 < c < 1 \\ z^{d}, & z < 1, \quad 0 < d < 1. \end{cases}$$

[7] It has three consequences and one event, instead of two and two. The reason for considering it more complex is both the fact that people find it appreciably more difficult to understand and an empirical result cited below.

rank- and sign-dependent utility (RSDU); the latter called it cumulative prospect theory.[8]

Wakker and Tversky (1993) provided an axiomatization for their version of the model that followed traditional lines of working only with a preference order over gambles. As with Wakker's (1989) earlier work on RDU models, this axiomatization has the drawback that the axioms make sense only if one is aware of the rank-dependent nature of the intended representation, and even then the axioms seem somewhat contrived. Luce and Fishburn (1991, 1995) approached the issue of axiomatization differently by introducing joint receipt as a primitive, axiomatizing $\langle \mathscr{C}^{c}, \succsim, \oplus, e\rangle$ and then showing how to go from that to the RSDU form. In various ways their treatment was not fully satisfactory, and part of the motive for this paper is to fill in some gaps while making clear the entire structure of ideas.

4.4. *Some Features of RSDU*

The unusual feature of RSDU is the representation for binary gambles with both a gain and a loss, namely, Eq. (2a), in which the weights do not necessarily add to 1. This is a decided non-rationality of the theory. We see that the first term on the right, $U_2(x^+)\ W_2^+(E)$, is of the same form as that which arises in the gains RDU model, Eq. (1), when $y^- = e$, namely, $U_1(x^+)\ W_1^+(E)$. In like manner, the second term, $U_2(y^-)\ W_2^-(\neg E)$, is of the same form as that which arises from the (not explicitly stated) rank-dependent model for losses when $x^+ = e$, $U_1(y^-)\ W_1^-(\neg E)$. This observation leads to two conclusions:

First, the sign-dependent expression also implies that U_2 is separable, as was U_1.

Second, monotonicity of preference across e implies $U_1 = U_2$.

The remainder of the paper focuses on the joint receipt primitive: its axiomatization, its relation to RSDU, and its providing a fairly natural axiomatization of RSDU.

5. THE NEGATIVE EXPONENTIAL REPRESENTATION OF GAINS JOINT RECEIPT

5.1. *Evidence in Support of Joint Receipt of Gains Being an Extensive Structure*

As was noted, the negative exponential representation can always be transformed into an additive one, Eq. (3c), which means that, in addition to satisfying A2, $\{\mathscr{G}^{+c}, \succsim, \oplus, e\rangle$ satisfies associativity, $(f \oplus g) \oplus h \sim f \oplus (g \oplus h)$; positiveness, $g \oplus h \succ g$ and h; and Archimedeaness, namely, for each $g, h \in \mathscr{G}^+$, with $h \succ e$, $\exists$ integer n such that $nh \succsim g$, where $1h = h$ and $nh = (n-1)\,h \oplus h$. It is well known that adding a condition of solvability yields a sufficient condition for an additive representation (Krantz *et al.*, 1971, Theorem 3.3, p. 85, restricted to a closed operation). Such qualitative structures are called *extensive*.

Note that although I state the theory in terms of $\langle \mathscr{G}^{+c}, \succsim, \oplus, e\rangle$ it is really sufficient to study $\langle \mathscr{C}^{+c}, \succsim, \oplus, e\rangle$ because, by assumption, a gamble can always be replaced by its CE (A1(ii) and A2(iii)).

One question is the degree to which the above properties are supported by data. For money consequences, at least, the solvability condition seems approximately correct. One's experience with joint receipts intuitively supports the idea that $\oplus$ is commutative and associative, at least for gains and losses separately. The following data exist. Thaler (1985) had subjects answer a questionnaire about how a "typical person" would react to certain schematic, real-world scenarios. From these group data, he argued, not entirely convincingly in my opinion, that subjects either integrate two monetary consequences as $x + y$ and evaluate $U(x + y)$ or segregate[9] them as $U(x) + U(y)$, whichever is better. That is,

$$U(x \oplus y) = \max\{U(x) + U(y),\ U(x + y)\}.$$

His data suggested segregation for gains, integration for losses, and a more complex pattern for mixed gains and losses. Fishburn and Luce (1995) worked out a description of the boundaries in the mixed case on the assumption that U is concave for gains and either concave or convex for losses. Later empirical work by Linville and Fisher (1991) and Thaler and Johnson (1990) paint a more complex picture, but the scenarios used involve time delays that make the interpretation problematic within a timeless framework.

Cho and Luce (1995) used the experimental procedure PEST[10] to determine the CEs of money lotteries and their joint receipts when presented as pie diagrams on a computer. Our data were unambiguous: for the joint receipt of sums of money with either both gains or both losses, $x \oplus y = x + y$. (Unfortunately, we did not explore the mixed case.)

Given that $\oplus = +$ for gains, it is easy to see from Eq. (3c) that for some $c > 0$, $V_a(x) = cx$, and so

$$U(x) = C[1 - e^{-cx}], \qquad x > 0. \tag{11}$$

This form plays a role in Section 5.4.

[8] Does the term "cumulative prospect theory" refer just to the general rank and sign dependent form or does it include the quite specific forms for U and W found in some of their accounts of phenomena and analyses of data? Some subsequent authors seem to take the more restrictive view, in which case the two terms are not synonymous, but judging by comments of Tversky and Kahneman (1992, pp. 300–301) they appear to mean rank and sign dependence and not specific functional forms.

[9] This term has a different meaning here from that introduced in Definition 8 below.

[10] See Footnote 2.

5.2. *Segregation, Separability, Weak Concavity, and* $U_1 = U_3$

One argument against using joint receipt (e.g., by Tversky and Kahneman, 1992) is that if the joint receipt structure is extensive with $\oplus = +$, then the additive representation V_a is proportional to money, and this flies in the face of all sorts of evidence of diminishing marginal utility for gains. The question is whether one should think of V_a as a utility, especially the utility that arises from studying gambles. We turn to this now.

The ideas and results of this subsection are adapted from Luce and Fishburn (1991, 1995).

DEFINITION 8. $\langle \mathcal{G}^{+c}, \succsim, \oplus, e \rangle$ satisfies *segregation* iff for all $g, h \in \mathcal{G}^+$ and $E \in \mathcal{E}$,

$$(g, E; e) \oplus h \sim (g \oplus h, E; h). \tag{12}$$

This property is rational in the sense that the bottom line on both sides is the same: One receives $g \oplus h$ if E occurs and $h \sim e \oplus h$ if E fails to occur.

We note that the apparent asymmetry of the definition is illusory:

$$\begin{aligned} h \oplus (e, E; g) &\sim (e, E; g) \oplus h && [\text{A2(i)}] \\ &\sim (g, \neg E; e) \oplus h && [\text{A3(iii)}] \\ &\sim (g \oplus h, \neg E; h) && (\text{Eq. 12}) \\ &\sim (h, E; g \oplus h) && [\text{A3(iii)}] \\ &\sim (h, E; h \oplus g). && [\text{A2(i) and A3(iv)}] \end{aligned}$$

Segregation has been examined experimentally by Cho. Luce, and von Winterfeldt (1994) and by Cho and Luce (1995) and has, within the noise level of their data, been sustained.

The following property follows immediately from Eq. (3a) and it captures, in part at least, the idea that the utility function for gains should exhibit diminishing marginal utility for money.

DEFINITION 9. A utility function $U: \mathcal{G} \to \mathbb{R}$ is *weakly concave for gains* iff for all $x \in \mathcal{G}^+$,

$$U(x \oplus x) < 2U(x). \tag{13}$$

THEOREM 2 (Adapted from Luce & Fishburn, 1991, 1995). *Suppose* $\langle \mathcal{G}^{+c}, \succsim, \oplus, e \rangle$ *satisfies assumptions* A1–A3. *Then any two of the following three conditions implies the third*:

(i) *The structure satisfies segregation* (*Eq.* 12).

(ii) *The substructure* $\langle \mathcal{G}^+, \succsim \rangle$ *has a weakly concave* (*Eq.* 13), *binary*, *rank-dependent representation* (*Eq.* 1) U *with* $W^+: \mathcal{E} \xrightarrow{\text{onto}} [0, 1]$.

(iii) *The structure* $\langle \mathcal{G}^{+c}, \succsim, \oplus, e \rangle$ *has a separable* (*Eq.* 5), *negative exponential representation* (*Eq.* 3) U *with* $W^+: \mathcal{E} \xrightarrow{\text{onto}} [0, 1]$.

Three major implications of this theorem are as follows: First, within this framework, the boundedness of U is tied up with weak concavity. For U to be unbounded, it must not be weakly concave, which can mean either weakly convex or mixed concave and convex intervals. Second, segregation, weak concavity, plus RDU (including, of course, SEU as a special case) imply the negative exponential representation, so $U_1 = U_3$ and this common utility function is separable. Third, segregation plus $\langle \mathcal{G}^+, \succsim, \oplus \rangle$ being an extensive structure yields a fairly natural axiomatization of the rank-dependent utility representation for binary gambles *provided* that we have reason to believe there is a negative exponential representation of the extensive structure that is also separable. So, an important task, which is taken up next, is to seek empirical conditions that insure this connection.

5.3. *Separable Negative Exponential Utility and Joint Receipt Decomposable Gains*

We seek an empirically testable property that assures us that some utility function over joint receipts having the form of Eq. (3) is also separable. To that end, consider:

DEFINITION 10. $\langle \mathcal{G}^{+c}, \succsim, \oplus, e \rangle$ is said to be *joint receipt decomposable* (*for gains*) iff for all $x \in \mathcal{C}^+$ and $E \in \mathcal{E}$, $\exists D = D(x, E) \in \mathcal{E}$ such that for for all $y \in \mathcal{C}^+$,

$$(x \oplus y, E; e) \sim (x, E; e) \oplus (y, D; e). \tag{14}$$

THEOREM 3. *If* U *is separable* (*Eq.* 5), *with* W^+: $\mathcal{E} \xrightarrow{\text{onto}} [0, 1]$, *and negative exponential* (*Eq.* 3), *then the structure satisfies joint receipt decomposability* (*Eq.* 14).

The event D relates to x and E in the representation as follows. With no loss of generality set $C = 1$ in Eq. (3), then in the proof we show that D is defined by

$$W^+(D) = W^+(E)\,\frac{1 - U(x)}{1 - U(x)\,W^+(E)}, \tag{15}$$

which exists because W^+ is onto $[0, 1]$ and so $1 - U(x) \leqslant 1 - U(x)\,W^+(E)$.

Going in the other—more interesting—direction is somewhat subtle for the following reason. Suppose U and U' both satisfy Eq. (3a); how do they relate? By the remark following Eq. (3c) we know that there are additive representations V_a and V'_a such that

$$U = 1 - \exp(-V_a) \qquad \text{and} \qquad U' = 1 - \exp(-V'_a).$$

By the well known uniqueness theorem for additive representations of an extensive structure, we know that for some $\alpha > 0$, $V_a = \alpha V'_a$, and so

$$U' = 1 - (1 - U)^{\alpha}.$$

Now it is reasonably obvious (see proof) that for $\alpha \neq 1$, both U and U' can not be separable. So the question we must face is this: Suppose there is a separable V and a U that satisfies the negative exponential property, Eq. (3a); is there then a V^* that both is separable and satisfies Eq. (3a)? By the uniqueness theorem for additive conjoint structures, we know that the only transformations of V that maintain separability and monotonicity are V^{β}, $\beta > 0$, and so if such a V^* exists it must be of the form V^{β}. The gist of the next theorem is to show that such a β exists when the structure satisfies Eq. (14).

THEOREM 4. *Suppose that a bounded separable utility function V exists, that U satisfies Eq.* (3*a*), *and that Eq.* (14) *holds. Assume U and V are normalized so that, over gains, they span* $[0, 1[$. *Define F by $U = F(V)$ and suppose that F and F^{-1} are differentiable.*[11] *Then there is a $\beta > 0$ such that V^{β} satisfies Eq.* (3*a*) *and is separable.*

We continue to seek a proof that requires only the strict monotonicity (and so continuity) of F, but not its differentiability which seems extraneous to the issues being addressed.

5.4. *Tversky and Kahneman's Estimate of Weighting Functions*

Tversky and Kahneman (1992) pointed out that data from lotteries of the form $(x, p; e)$ plus separability allow one to estimate $W^k(p)$ as a function of p and k, where $k = +, -$ according to whether $x \in \mathscr{C}^+$ or $x \in \mathscr{C}^-$. In particular, they noted that by separability and A1(ii), $U[\mathrm{CE}(x, p; e)]/U(x) = W^k(p)$. This yields an estimate of W^k if one knows U. To that end, they assumed U is a power function, estimated the powers $\alpha(k)$, and used $[\mathrm{CE}(x, p; e)/x]^{\alpha(k)}$ as the estimate[12] of $W^k(p)$.

[11] Because U and V are each order preserving, F must be strictly increasing and so F^{-1} exists.

[12] It is worth noting that their Fig. 3 which claims to be a plot of $W^k(p)$ versus p is, in reality, a plot of $\mathrm{CE}(g)/x$ versus p, uncorrected by the power. When that correction is made, the plots are rather more symmetric. Indeed, using slightly different estimates of the power for the utility function than they did, one can easily fit to them the perfectly symmetric Karmarkar (1978, 1979) function.

TABLE 2

Suppose That Half the Population Has $W(p) = p^{1/2}$ and Half Has $W(p) = p^{3/2}$ and That $U(x) = C(1 - e^{-ax})$, $a = 0.01$, for $x = 100$

p	0.05	0.25	0.50	0.75	0.95
CE(1/2)	15.23	37.99	59.24	79.27	95.70
CE(3/2)	0.71	8.23	25.29	52.86	88.02
avg CE	7.97	23.11	42.27	66.06	91.86
CE/x	0.08	0.23	0.42	0.66	0.92
Inferred $W(p)$	0.12	0.33	0.55	0.73	0.94
T-K data CE/x	0.10	0.30	0.44	0.57	0.83
T-K $W(p)$	0.13	0.33	0.49	0.59	0.87

The flaw in this approach is that if one accepts RDU (which they do), segregation (which has some empirical support), weak concavity (which is rarely questioned[13]), and $\oplus = +$ (which they clearly believe and which also has empirical support), then by Theorem 2 the correct form for the utility function is not a power but the negative exponential of Eq. (11). Simple calculations make clear that the difference in estimates of W^k can be considerable, and hence we may have been significantly misled about the form of W^k by their estimates.

For example, Table 2 provides a calculation for the average of two subjects both of whom satisfy the rational model[14] in which p^{β} is the form of the weighting function, one with $\beta = 1/2$ and the other with $\beta = 3/2$. It is clear that the average is unlike either subject and that it overshoots small probabilities and undershoots large ones, as reported by Tversky and Kahneman. This problem bears careful reexamination, and at this point one should be very cautious about what form to assume for $W^k(p)$. I believe that whatever family of functions is assumed for a descriptive W^k, it should[15] include as a special case the rational form of Eq. (9), namely, a power of p. Neither Tversky and Kahneman's (1992) nor Karmarkar's (1978, 1979) ad hoc formulas include Eq. (9) as a special case.

5.5. *Segregation and Event Commutativity*

Earlier I showed that status quo event commutativity was key to separability and that the rank-dependent form implied general event commutativity. Here I show still another tie.

[13] One exception is the empirical study of Chechile and Cooke (1996) which, taken at face value, strongly suggests a convex utility function for gains. This may be misleading, however, because the experimental method involved obtaining judged probabilities that made one lottery indifferent to another one. This method may, as judged CEs seem to, provide biased estimates as compared to choice-based ones. Until some methodological work is carried out, one must be cautious.

[14] See Section 3.4 and Eq. (9).

[15] The rule I try to adhere to is that descriptive theories should not preclude the rational ones, but only generalize them.

THEOREM 5. *If* $\langle \mathscr{G}^{+c}, \succsim, \oplus, e\rangle$ *satisfies status quo event commutativity* (*Eq.* 7 *with* $y=e$) *and segregation* (*Eq.* 12), *then it satisfies event commutativity for any* $x, y \in \mathscr{C}^+$ *with* $x \succsim y$.

It is unclear whether we should or should not expect event commutativity to hold for $y \succ x$. The data of Chung *et al.* (1994) and Ronen (1971, 1973) were run only with $x \succ y$. In the best of Brothers, (1990) experiments, 3, both $x \succ y$ and $x \prec y$ were studied and appeared little different.

6. THE ADDITIVE REPRESENTATION FOR MIXED JOINT RECEIPTS

6.1. *Axiomatization of Mixed Joint Receipts*

The existence of the additive representation of Eq. (4) can be understood if we replace $\langle \mathscr{G}, \succsim, \oplus, e\rangle$ by the conjoint structure $\langle \mathscr{G}^+ \times \mathscr{G}^-, \succsim'\rangle$, where $(g^+, g^-) \succsim' (h^+, h^-)$ iff $g^+ \oplus g^- \succsim h^+ \oplus h^-$. In particular, monotonicity is assumed in A2(iii). The Thomsen condition asserts that $(g^+, f^-) \sim' (f^+, h^-)$ and $(f^+, g^-) \sim' (h^+, f^-)$ imply $(g^+, g^-) \sim' (h^+, h^-)$. This, in turn, is equivalent to $g^+ \oplus f^- \sim f^+ \oplus h^-$ and $f^+ \oplus g^- \sim h^+ \oplus f^-$ implying $g^+ \oplus g^- \sim h^+ \oplus h^-$. Clearly, this condition should be checked empirically; it has not been. Non-triviality holds because of A1(iii). Solvability takes the form that if $\bar{g}^+ \oplus g^- \succsim h^+ \oplus h^- \succsim \underline{g}^+ \oplus g^-$, then there exists g^+ such that $g^+ \oplus g^- \sim h^+ \oplus \bar{h}^-$. This condition certainly seems plausible for money gambles. Finally, a form of Archimedeaness is assumed, which we need not spell out. Thus the main experimental test is of the Thomsen condition.

We next explore how this additive representation relates to the sign-dependent one for gambles with mixed consequences, Eq. (3). And following that we look into the issue of when the additive representation of joint receipts is separable.

An alternative approach, studied in Luce (submitted), should be mentioned. One could assume that associativity holds for mixed consequences as well as for all gains and all losses, i.e., that the entire joint receipt structure forms an Archimedean ordered group and so has an additive representation V_a throughout. The relation to gambles is unchanged for gains, but it is substantially different for the mixed case. I do not attempt to describe the differences here. Suffice it to say that we do not currently know which is the better model for mixed consequences.

6.2. *Duplex Decomposition, Separability, and* $U_2 = U_4$

The major empirical property linking additive mixed joint receipts and sign-dependent utility of binary gambles with mixed consequences that was used by Luce and Fishburn (1991) is:

DEFINITION 11. The structure $\langle \mathscr{G}^c, \succsim, \oplus, e\rangle$ is said to satisfy *duplex decomposition* iff for all $x^+ \in \mathscr{C}^+$, $y^- \in \mathscr{C}^-$, and $E \in \mathscr{E}$,

$$(x^+, E; y^-) \sim (x^+, E'; e) \oplus (e, E''; y^-), \tag{16}$$

where E' and E'' are both occurrences of the event E in independent realizations of the underlying chance experiment.

In contrast to the other empirical properties that have been introduced, this one lacks a rational basis. The bottom line on the left, in which either x^+ or y^- but not both must occur, differs from the bottom line on the right where, in addition, $x^+ \oplus y^-$ and $e \oplus e \sim e$ can also occur. Duplex decomposition is a rule of thumb that says: replace a mixed gamble by the joint receipt of the gains part and of the losses part, independently evaluated. Despite its non-rationality, it has found empirical support in experiments. The first to note it, and name it, were Slovic and Lichtenstein (1968), who found it supported when the two gambles were displayed as pie diagrams that were presented simultaneously. Payne and Braunstein (1971) also studied it. More recently Cho *et al.* (1994) explored it fairly systematically, using PEST-determined CEs, and again found support for it.

THEOREM 6. *Suppose a structure of binary gambles satisfies Assumptions* A1–A3. *Then any two of the following properties imply the third*:

(i) *Duplex decomposition* (*Eq.* 16).

(ii) *U has a binary sign-dependent utility representation* (*Eq.* 2).

(iii) *U is separable* (*Eq.* 5) *and additivity of mixed joint receipts* (*Eq.* 4) *holds for all* $g^+ \in \mathscr{G}^+$ *and* $h^- \in \mathscr{G}^-$ *such that* $\exists E \in \mathscr{E}$, $x^+ \in \mathscr{C}^+$, *and* $y^- \in \mathscr{C}^-$ *with* $g^+ \sim (x^+, E; e)$ *and* $h^- \sim (e, E; y^-)$.

This interesting result has two weakness. One weakness, which we deal with, concerns the issue of the additive U being separable. The other, which is left as an open problem, is how to prove the additivity of U over all of $\mathscr{G}^+ \times \mathscr{G}^-$, not just a restricted part of it where the consequences are not too extreme. The difficulty arises primarily because of the boundedness of U. For suppose $U(g^+) > \alpha C$ and $U(h^-) < -\alpha C'$, where $\alpha \in\,]0, 1[$. Then for x^+, y^-, and E to exist we must have

$$CW^+(E) > U(x^+)\, W^+(E) = U(g^+) > \alpha C$$

$$\text{and } -C'W(\neg E) < U(y^-)\, W^-(E) = U(h^-) < -\alpha C'.$$

So both $W^+(E) > \alpha$ and $W^-(\neg E) > \alpha$. Clearly, this is not to be expected when α is near 1.

6.3. *Separable Additivity and Joint Receipt Decomposition in the Mixed Case*

Just as in the case of gains, we need to understand in the mixed case what empirical condition corresponds to the utility function being both additive and separable. The relevant condition is very similar to, although a bit stronger than, joint receipt decomposability for gains (Definition 10).

DEFINITION 12. $\langle \mathscr{G}^c, \succsim, \oplus, e\rangle$ is said to be *joint receipt decomposable* (*for mixed consequences*) iff for all $E \in \mathscr{E}$, there exist $D = D(E) \in \mathscr{E}$ and $D^* = D^*(E) \in \mathscr{E}$ such that for all $x^+ \in \mathscr{C}^+$ and $y^- \in \mathscr{C}^-$,

$$(x^+ \oplus y^-, E; e) \sim \begin{cases} (x^+, E; e) \oplus (e, D; y^-), \\ \qquad \text{if} \quad x \oplus -y \succsim e \\ (x^+, D^*; e) \oplus (e, \neg E; y^-), \\ \qquad \text{if} \quad x \oplus -y \prec e. \end{cases} \tag{17}$$

It should be noted that D and D^* depend only on E and are independent of x^+ and y^-. Indeed, if U is both additive and separable it is easy to see that D is defined in the first case by $W^-(\neg D) = W^+(E)$ and in the second case by $W^+(D^*) = W^-(E)$.

THEOREM 7. *Suppose U is additive over the joint receipt $\oplus$ of mixed consequences* (*Eq.* 4), *V is separable* (*Eq.* 5) *with weighting functions $W^k\colon \mathscr{E} \xrightarrow{\text{onto}} [0, 1]$, and both U and V have least upper bounds and greater tower bounds. Then U is separable over binary gambles if and only if the gambles are joint receipt decomposable* (*Eq.* 17) *with $W^-(\neg D) = W^+(E)$ and $W^+(D^*) = W^-(E)$.*

7. THE EQUIVALENCE OF THE TWO JOINT RECEIPT MEASURES

From what has been proved, we know that if A1–A3, segregation (Eq. 12), and duplex decomposition (Eq. 16) are satisfied, then U is both separable (Eq. 5) and a negative exponential function for gains (Eq. 3) iff U is separable (Eq. 5) and additive over mixed joint receipts (Eq. 4). This follows from Theorems 2 and 5 along with the consequence monotonicity of Assumption A3(iv). The question considered here is what empirical condition involving just joint receipts and not gambles is equivalent to U being both negative exponential for gains and additive for mixed consequences. Basically, the key property was formulated as (J7′) in Luce and Fishburn (1991). Using a functional equation argument, we show here that this is indeed exactly what is needed.

DEFINITION 13. The structure $\langle \mathscr{G}^c, \succsim, \oplus\rangle$ is said to be *joint receipt consistent* if the following condition is satisfied. Suppose for some $x, y \in \mathscr{C}^+$ and $r, s \in \mathscr{C}^-$ with $x \oplus r \in \mathscr{C}^+$ that $(x \oplus r) \oplus y \sim (x \oplus y) \oplus s$. Then for all $x' \in \mathscr{C}^+$ with $x' \oplus r \in \mathscr{C}^+$, $(x' \oplus r) \oplus y \sim (x' \oplus y) \oplus s$.

THEOREM 8. *Suppose that Assumptions* A1–A2 *hold, U is negative exponential* (*Eq.* 3) *for gains, and V is additive for mixed joint receipt* (*Eq.* 4). *Then, the following two statements are equivalent*:

(i) *Joint receipt consistency is satisfied.*

(ii) *Over gains and with r, s, and y as in Definition* 13, *either*

(a) *V is additive over the entire structure and $s = r$ or*

(b) *V is negative exponential and for some $B > 0$,*

$$V(s) = V(r)[1 - V(y)/B]. \tag{18}$$

In (ii)(a), *if V is separable, then U cannot be separable*; *and in* (ii)(b) *if U and V are both separable and agree at one point, then $U = V$.*

Note that in the case of (ii)(a), the operation $\oplus$ is associative throughout, not just for gains and losses separately. Thus, this solution could be rejected empirically if it is shown that, for mixed gains and losses, associativity fails. As noted earlier, the purely associative case has been studied in Luce (submitted) and a student is currently developing an empirical study.

For losses, an exactly parallel property is defined and the relations between U and V are exactly parallel.

Joint receipt consistency has not been examined empirically.

8. RELATED ISSUES

8.1. *Extensions to Finite Gambles*

The work described above has the limitation that it does not apply to general gambles with more than two consequences. (Of course, the same was true of Kahneman and Tversky, 1979.) One of the goals of Luce and Fishburn (1991, 1995), Tversky and Kahneman (1992), and Wakker and Tversky (1993) was to provide a generalization to a general rank- and sign-dependent form. The version given by Luce and Fishburn (1995) has been considerably simplified and improved by Liu (1995). This approach is easily outlined.

Any general gamble can be partitioned into three subgambles: g^+ consists of all gains and so is defined on E^+, the union of all events E_k with $g_k \succ e$; g^- which consists of all losses defined on E^-; and the status quo e defined on E^0. One behavioral postulate is that such a partition is indifferent to the original gamble,

$$g \sim (g^+, E^+; e, E^0; g^-, E^-). \tag{19}$$

Assuming Eq. (19) and consequence monotonicity (A3v), then two applications of Eq. (2)—first to the partition $\{E^+, E^0 \cup E^-\}$ and then to $\{E^0, E^-\}$—yields

$$U(g) = U(g^+)\ W^+(E^+) + U(g^-)\ W^-(E^-). \tag{20}$$

The term involving e as a consequence does not appear because $U(e) = 0$.

The remaining postulates concern the terms g^+ and g^-. We deal explicitly only with g^+; the discussion for g^- is similar. The fundamental idea is first to order the events of g^+ starting with the one that gives rise to the most preferred consequence and ending with the one giving rise to the least preferred consequence, i.e.,

$$g^+ = (x_1, E_1; x_2, E_2, ...; x_n, E_n),$$

where $E_1 \cup E_2 \cup \cdots \cup E_n = E^+$ and $x_1 \succ x_2 \succ \cdots \succ x_n \succ e$. Next, one assumes that

$$g^+ \sim (g^+_{n-1}, E(n-1); x_n, E_n), \tag{21}$$

where $E(n-1) = E_1 \cup E_2 \cup \cdots \cup E_{n-1}$ and g^+_{n-1} the subgamble of g^+ restricted to $E(n-1)$.

The empirical validity of Eq. (21) has never been checked experimentally.

Assuming Eq. (21), one then applies Eq. (1a), thereby reducing the analysis of $U(g^+_n)$ to that of $U(g^+_{n-1})$ Care must be taken to normalize the weights according to the conditioning event, first E^+, and then $E(n-1)$, etc. One proceeds inductively and by doing a bit of algebra determines the form of the resulting weighting function. It is rank dependent because the order of the induction matters and it is determined by the ordering of the consequences.

So, assuming that Eqs. (19) and (21) are empirically valid, the theory can be extended to general finite gambles.

8.2. *Joint Receipt and Convolution*

Consider the lotteries

$$g = (x_1, p_1; x_2, p_2; ...; x_m, p_m)$$

and

$$h = (y_1, q_1; y_2, q_2; ...; y_n, q_n),$$

where $\sum p_j = \sum q_k = 1$ and, in general, $m \neq n$. The equivalent (but not automatically indifferent) first order lottery to $g \oplus h$ takes the form $(x_1 \oplus y_1, p_1 q_1; ...; x_j \oplus y_k, p_j q_k; ...; x_m \oplus y_n, p_m q_n)$. We note that segregation (Def. 8, Eq. 12) is the special case of asserting that indifference holds when $m = 2$, $x_2 = e$, and $n = 1$, so $h = y$. If, as seems to be true, $\oplus = +$ over money gains, then this first-order form reduces to the convolution

$$g * h = (x_1 + y_1, p_1 q_1; ...; x_j + y_k, p_j q_k; ...; x_m + y_n, p_m q_n). \tag{22}$$

So, one natural hypothesis to investigate is whether

$$g \oplus h \sim g * h. \tag{23}$$

Luce (1995) argued that we cannot expect to find anyone who satisfies Eq. (23) and is monotonic in $*$ who also finds attractive a lottery g with a negative expected value, i.e., $EV(g) < 0$ and $g \succ e$. The reason is this: Let $g^{(n)}$ denote the convolution of g with itself n times; then if $\oplus = *$ is monotonic, a simple induction shows $g^{(n)} \succ e$. But because $EV(g^{(n)}) = nEV(g)$ and $SD(g^{(n)}) = \sqrt{n}\, SD(g)$, the standard error decreases as $1/\sqrt{n}$, and so it is difficult not to believe that, for sufficiently large n, $g^{(n)} \prec e$.

Because people can be expected to differ on the desirability of lotteries with small negative expected values, Cho and Luce (1995) partitioned their 40 subjects according to the certainty equivalents that they gave to 10 screening gambles with small negative expected values. Those whose CEs were positive on 7 or more[16] of the 10 were classed as "gamblers" and the remainder as "non-gamblers." There were, respectively, 16 and 24 subjects. Then in another part of the experiment we established for several $\{g, h\}$ pairs certainty equivalents for $g \oplus h$ and for $g * h$. We concluded that for the set of gamblers, Eq. (23) is not rejected but that the common operation $\oplus = *$ fails monotonicity in the sense that the condition

$$\mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h) \tag{24}$$

was rejected. Luce (1995) showed under fairly weak conditions that Eq. (24) is equivalent to $*$ being monotonic. For non-gamblers, we rejected $\oplus = *$ but found that $*$ appeared to be monotonic in the sense that Eq. (24) was not rejected.

8.3. *Is Joint Receipt Non-monotonic or Is PEST Not Order Preserving*?

The above empirical finding is both interesting for the distinction between gamblers and non-gamblers and disturbing because it suggests that joint receipt may not be monotonic, at least for gamblers. Luce (1995) showed that under fairly weak assumptions the monotonicity of $\oplus$ is equivalent to

$$g \oplus h = \mathrm{CE}(g) \oplus \mathrm{CE}(h). \tag{25}$$

[16] The results do not change when using either 6, 7, or 8 as the cutpoint; of course, the sample sizes do change.

TABLE 3

Summary of Behavioral Properties and Their Empirical Status

Empirical indifferences	References	Status
Status quo event commutativity		
$\forall(x, D, E)$	Tversky (1967)	OK (indirect)
$((x, E; e), D; e) \sim ((x, D; e), E; e)$	Ronen (1971, 1973)	No (forced choice)
	Brothers (1990)	OK (PEST, direct)
	Chung *et al.* (1994)	OK (22/25 PEST)
Segregation		
$\forall(x, y \succsim e)$	Cho *et al.* (1994)	OK (PEST)
$(x, E; e) \oplus y \sim (x \oplus y, E; y)$	Cho & Luce (1995)	OK (PEST)
Duplex decomposition		
$\forall(x^+ \succsim e \succsim y^-)$	Slovic & Lichtenstein (1968)	OK
$(x^+, E; y^-) \sim (x^+, E'; e) \oplus (e, E''; y')$	Payne & Braunstein (1971)	OK
	Cho *et al.* (1994)	OK (PEST)
Joint receipt decomposable		
$\forall(x, y \succsim e)\ \exists$ event $D = D(x, E)$ s.t.	No empirical studies at present	
$(x \oplus y, E; e) \sim (x, E; e) \oplus (y, D; e)$		
$\forall(x^+ \succsim e \succsim y^-)\ \exists$ events $D = D(E)$ and $D' = D'(E)$ s.t.		
$(x^+ \oplus y^-, E; e) \sim \begin{cases} (x^+, E; e) \oplus (y^-, D; e), & \text{if } x^+ \oplus y^- \succsim e \\ (x^+, D'; e) \oplus (y^-, E; e), & \text{if } x^+ \oplus y^- \prec e \end{cases}$		
Joint receipt consistency		
$\forall(x, y, x' \succsim e, r, s \precsim e, x \oplus r \succsim e, x' \oplus r \succsim e)$	No empirical studies at present	
$(x \oplus r) \oplus y \sim (x \oplus y) \oplus s \Rightarrow (x' \oplus r) \oplus y \sim (x' \oplus y) \oplus s.$		

In other words, a gamble can be replaced in a joint receipt by its certainty equivalent. Cho and Luce (1995) studied this by establishing for both sides of Eq. (25) their certainty equivalents, and they found that it apparently does not hold for either gamblers or non-gamblers. The conclusion we were forced to was that either $\oplus$ is not monotonic or that PEST, as currently implemented, is not order preserving. The former conclusion is certainly disturbing, if true, because all of the theory described here involving joint receipt presupposes $\oplus$ to be a monotonic operation, and I know of no theory of binary operations where monotonicity is not assumed. The latter conclusion means that we need to examine the PEST procedure with care.

Cho and Fisher (in preparation) have used choice repetitions together with PEST-determined CEs to separate the causes. Without going into the details, monotonicity seems sustained and PEST is implicated as the source of the problem. Sneddon and Luce (in preparation) have shown by simulations that PEST, as implemented, is biased in estimating the CE when the gambles are badly skewed and the psychometric function is not symmetric. Moreover, our truncation of the PEST procedure to include only sums of money within the range of consequences of a gamble, which seemed only rational, inadvertently makes all psychometric functions, including symmetric ones, into ones that are functionally asymmetric. Additional methodological studies are underway in an attempt to improve our estimates of CEs.

9. CONCLUSIONS

9.1. *Summary of Results*

Our major results are various behavioral linking conditions that force the four measures of Table 1 to be the same. These linking conditions are summarized in Table 3 along with the relevant empirical references. And Table 4 summarizes the general nature of the theoretical results.

The key links between the joint receipt and the gambling measures are segregation (Definition 8, Eq. 12) for gains and duplex decomposition (Def. 11, Eq. 16) for the mixed case. Both of these properties have been subjected to empirical study, and they appear to be sustained within the somewhat large noise levels of experiments in this area. What complicates the theoretical picture considerably is the need for the joint receipt measure also to be separable (Definition 5, Eq. 5). A new existential property called joint receipt decomposability (Definition 10, Eq. 14, for gains and Definition 12, Eq. 17 for mixed cases) was shown to be

TABLE 4

Summary of the Main Theoretical Connections. The Definitions Are Given in Tables 1 and 3. Note That the Same Laws Hold for Losses as for Gains

Gambling representation	Linking property	Joint receipt representation	Theorem
Separable	Status quo event commutativity		1
Separable	Joint receipt decomposable (gains)	Negative exponential	3, 4
Rank dependent, weakly concave	Segregation (gains)	Negative exponential, separable	2
Separable	Joint receipt decomposable (mixed)	Additive	7
Sign dependent	Duplex decomposition (mixed)	Additive, separable	6
	Joint receipt consistent (mixed)	Negative exponential (gains), additive (mixed)	8

just what is needed. A final result concerns conditions when the two joint receipt measures are the same, namely, joint receipt consistency (Definition 13). This equality is proved on the assumption that each measure is separable.

9.2. *Open Problems*

Recall that if the joint receipt operation turns out to be associative throughout, rather than only for gains and losses separately as assumed in this paper, then the empirical and theoretical situation is much simpler (Luce, submitted). Whether or not associativity holds in the mixed case can be decided directly and does not entail any existence issues. If it turns out to hold, then only joint receipt decomposability for gains (and for losses separately) needs to be checked because the mixed case is completely determined by the gains and losses cases.

If, however, associativity fails, then a number of theoretical and empirical problems need to be considered.

At the theoretical level the most obvious question concerns the linkage between joint receipt and gambles in the mixed case, Theorem 6, which applies only to a subset of $\mathscr{G}^+ \times \mathscr{G}^-$. One wonders what is needed to get additivity, Eq. (4), to hold over all of $\mathscr{G}^+ \times \mathscr{G}^-$.

At the empirical level, there is much to do. In reducing the general theory to the binary one, we assumed Eqs. (19) and (21), but no attempt has yet been made to verify either property. Several studies attempt to explore rank dependence directly. Perhaps the most convincing study in favor of rank dependence is Birnbaum and McIntosh (1995). Turning to the binary situation, joint receipt decomposability (Eqs. 14 and 17) and joint receipt consistency (Definition 13) need to be investigated. Because these are conditions that involve finding elements that meet an indifference condition and then verifying that they continue to hold when one or more of the variables is varied, there is a double source for error and so it will be difficult to make these tests at all sensitive. Cho and Luce (1995) found some reassurance about our method of testing indifference in the fact that some tests were rejected while others were accepted, and these acceptances and rejections formed a consistent pattern. Achieving a similar level of confidence is likely to be considerably more difficult in these types of tests.

Finally, a major uncertainty in the approach so far taken by me and my students arises from some evidence, both indirect and direct, that PEST as currently implemented may not be order preserving for skewed gambles. This is currently under investigation.

APPENDIX: PROOFS

A1. *Proof of Theorem* 1.

LEMMA 1. *Assuming* A3(iv), *status quo event commutativity implies the Thomsen condition* (*Eq.* 6) *in* $\langle \mathscr{C} \times \mathscr{E}, \succsim' \rangle$.

Proof. Suppose $(x, F; e) \sim (z, E; e)$ and $(z, D; e) \sim (y, F; e)$. Using status quo event commutativity and consequence monotonicity freely,

$$\begin{aligned}((x, D; e), F; e) &\sim ((x, F; e), D; e)\\ &\sim ((z, E; e), D; e)\\ &\sim ((z, D; e), E; e)\\ &\sim ((y, F; e), E; e)\\ &\sim ((y, E; e), F; e),\end{aligned}$$

and so $(x, D; e) \sim (y, E; e)$ follows by consequence monotonicity. ∎

Turning now to the proof of Theorem 1 itself, monotonicity[17] is assumed in A3(iv and v) and restricted solvability in A3(vi). Essentialness—that both coordinates make a difference—is assured by A1(iii) and A3(vi). Lemma 1 proves that the Thomsen condition holds, and

[17] Also called *independence* in the conjoint measurement literature.

Archimedeaness is assumed. So by a well known theorem (Krantz *et al.*, 1971, Theorem 6.2, p. 257) the representation exists, and it is unique up to a power transformation. ▮

Proof of Corollary 1. By Theorem 1,

$$U(x)\; W^k[F(x, D, E)] = U(x)\; W^k(D)\; W^k(E).$$

Dividing by $U(x)$, we see that the right side is independent of x except as W^k depends on whether $x \succsim e$ or $x \precsim e$, so F must depend only on the sign of x. Taking inverses yields the form. ▮

Proof of Corollary 2. Because the assumed monotonicity in p implies that W^k is strictly increasing and

$$W^k(pq) = W^k[F(p, q)] = W^k(p)\; W^k(q),$$

it is immediately obvious that W^k is a power function. ▮

A2. *Proof of Theorem* 2. Luce and Fishburn (1991) showed that segregation together with the binary rank-dependent representation imply

$$U(x \oplus y) = AU(x) + BU(y) + C'U(x)\; U(y).$$

It follows immediately from A2(ii) and Eq. (1a) that $A = B = 1$. Applying this and Eq. (1a) we see that

$$\begin{aligned} U(x \oplus x, E; x) &= [U(x \oplus x) - U(x)]\; W^+(E) + U(x) \\ &= U(x, E; e) + U(x) + C'U(x, E; e)\; U(x) \\ &= U(x)\; W^+(E) + U(x) + C'U(x)^2\; W^+(E). \end{aligned}$$

Subtracting,

$$[U(x \oplus x) - 2U(x)]\; W^+(E) = C'U(x)^2\; W^+(E),$$

and so by weak concavity (Eq. 13) $C' < 0$. So we set $C = -1/C'$, yielding Eq. (3a).

They also showed the other two implications. ▮

A3. *Proof of Theorem* 3. With $C = 1$, choose $D \in \mathscr{E}$ as defined by Eq. (15). Then,

$$\begin{aligned} &U[(x, E; e) \oplus (y, D; e)] \\ &\quad = U(x, E; e) + U(y, D; e) \\ &\qquad - U(x, E; e)\; U(y, D; e) && \text{(Eq. 3a)} \\ &\quad = U(x)\; W^+(E) + U(y)\; W^+(D) \\ &\qquad \times [1 - U(x)\; W^+(E)] && \text{(Eq. 5)} \\ &\quad = U(x)\; W^+(E) + U(y)\; W^+(E) \\ &\qquad \times [1 - U(x)] && \text{(Eq. 15)} \\ &\quad = U(x \oplus y)\; W^+(E) && \text{(Eq. 3a)} \\ &\quad = U(x \oplus y, E; e). && \text{(Eq. 5)} \end{aligned}$$

Because U is order preserving, Eq. (14) follows. ▮

A4. *Proof of Theorem* 4. Let V be any bounded separable utility function with weighting function W^+ and let U satisfy Eq. (3a). With no loss of generality choose the bounds of U and V to be 1. Let Eq. (14) hold and let $X = V(x)$, $Y = V(y)$, $Z = W^+(E)$, and $P = W^+(D) = P(X, Z)$. Then

$$\begin{aligned} &F^{-1}[F(X) + F(Y) - F(X)\; F(Y)]\; Z \\ &\quad = F^{-1}[U(x) + U(y) - U(x)\; U(y)]\; Z \\ &&& \text{(Definitions)} \\ &\quad = F^{-1}[U(x \oplus y)]\; Z && \text{(Eq. 3a)} \\ &\quad = V(x \oplus y)\; W^+(E) && \text{(Definitions)} \\ &\quad = V(x \oplus y, E; e) && \text{(Eq. 5 on } V\text{)} \\ &\quad = V[(x, E; e) \oplus (y, D; e)] && \text{(Eq. 14)} \\ &\quad = F^{-1}U[(x, E; e) \oplus (y, D; e)] && \text{(Definition of } F\text{)} \\ &\quad = F^{-1}[U(x, E; e) + U(y, D; e) \\ &\qquad - U(x, E; e)U(y, D; e)] && \text{(Eq. 3a)} \\ &\quad = F^{-1}[F[V(x, E; e)] + F[V(u, D; e)] \\ &\qquad - F[V(x, E; e)]\; F[V(u, D; e)]] \\ &&& \text{(Definition of } F\text{)} \\ &\quad = F^{-1}[F(XZ) + F[YP(X, Z)] \\ &\qquad - F(XZ)\; F[YP(X, Z)]] && \text{(Eq. 5 and defs.).} \end{aligned}$$

For $X, Y \in [0, 1[$, define

$$\begin{aligned} H(X, Y) &= F^{-1}[F(X) + F(Y) - F(X)\; F(Y)] \\ &= G^{-1}[G(X)\; G(Y)], \end{aligned}$$

where $G(X) = 1 - F(X)$. Substituting this into the above functional equation, we see that, for $X, Y \in [0, 1[$, $Z \in [0, 1]$,

$$H(X, Y)\; Z = H[XZ, YP(X, Z)].$$

Using the differentiability of F and F^{-1}, Aczél *et al.* (1996) show that for some $\alpha > 0$ and $\beta > 0$,

$$F(X) = 1 - (1 - X^{\beta})^{\alpha}$$

and

$$P^{\beta} = Z^{\beta} \frac{1 - X^{\beta}}{1 - X^{\beta} Z^{\beta}}.$$

It is easy to verify that these do indeed solve the above functional equation.

Noting that $1 - F(X) = (1 - X^{\beta})^{\alpha}$ and that U satisfies Eq. (3a), we have

$$\begin{aligned} &1 - U(x \oplus y) = [1 - U(x)][1 - U(y)] \\ &\quad \Leftrightarrow 1 - F[V(x \oplus y)] = [1 - F[V(x)]][1 - F[V(y)]] \\ &\quad \Leftrightarrow [1 - V(x \oplus y)^{\beta}]^{\alpha} = [1 - V(x)^{\beta}]^{\alpha}\,[1 - V(y)^{\beta}]^{\alpha} \\ &\quad \Leftrightarrow [1 - V(x \oplus y)^{\beta}] = [1 - V(x)^{\beta}][1 - V(y)^{\beta}], \end{aligned}$$

and so V^{β} satisfies Eq. (3a) and is separable. ▮

A5. *Proof of Theorem* 5. For $x, y \in \mathscr{C}^{+}$ with $x \succsim y$, define $x \ominus y = z$ iff $z \oplus y \sim x$. Then, we have

$$\begin{aligned} &((x, D; y), E; y) \\ &\quad \sim ((x \ominus y, D; e) \oplus y, E; y) \qquad [\text{Eq. 12 and A3(iv)}] \\ &\quad \sim ((x \ominus y, D; e), E; e) \oplus y \qquad (\text{Eq. 12}) \\ &\quad \sim ((x \ominus y, E; e), D; e) \oplus y \\ &\qquad\qquad (\text{status quo event commutativity and A2(iii)} \\ &\quad \sim ((x, E; y), D; y) \qquad [\text{Eq. 12 and A3(iv)}]. \end{aligned}$$ ▮

A6. *Proof of Theorem* 6.

(i) & (iii) $\Rightarrow$ (ii):

$$\begin{aligned} &U(x^{+}, E; y^{-}) \\ &\quad = U[(x^{-}, E; e) \oplus (e, E; y^{-})] \qquad (\text{i}) \\ &\quad = U(x^{+}, E; e) + U(e, E; y^{-}) \qquad (\text{iii}) \\ &\quad = U(x^{+})\, W^{+}(E) + U(y^{-})\, W^{-}(\neg E), \qquad (\text{Eq. 5}) \end{aligned}$$

which is property (ii).

(i) & (ii) $\Rightarrow$ (iii): The separability of U is immediate from (ii). So we focus on the additivity. Suppose $g^{+} \in \mathscr{G}^{+}$ and $g^{-} \in \mathscr{G}^{-}$ are such that $\exists E \in \mathscr{E}$ and $x^{+} \in \mathscr{C}^{+}$, $y^{-} \in \mathscr{C}^{-}$ are such that $g^{+} \sim (x^{+}, E; e)$ and $g^{-} \sim (e, E; y^{-})$. So, denoting independent realizations of E by E' and E'',

$$\begin{aligned} &U(g^{+} \oplus g^{-}) \\ &\quad = U[(x^{+}, E'; e) \oplus (e, E''; y^{-})] \qquad [\text{A2(iii)}] \\ &\quad = U(x^{+}, E; y^{-}) \qquad (\text{i}) \\ &\quad = U(x^{+})\, W^{+}(E) + U(y^{-})\, W^{-}(\neg E) \\ &\qquad\qquad (\text{ii}) \\ &\quad = U(x^{+}, E; e) + U(e, E; y^{-}) \qquad [(\text{ii}) \text{ and } U(e) = 0] \\ &\quad = U(g^{+}) + U(g^{-}) \qquad (\text{Definitions}), \end{aligned}$$

which is the additivity of Eq. (4a). So (iii) holds.

(ii) & (iii) $\Rightarrow$ (i). Suppose $g^{+} \in \mathscr{G}^{+}$, $h^{-} \in \mathscr{G}^{-}$, $x^{+} \in \mathscr{C}^{+}$, $y^{-} \in \mathscr{C}^{-}$, and $E \in \mathscr{E}$ are such that $g^{+} = (x^{+}, E'; e)$ and $g^{-} = (e, E''; y^{-})$, where E' and E'' are independent realizations of E. Then,

$$\begin{aligned} &U(x^{+}, E; y^{-}) \\ &\quad = U(x^{+})\, W^{+}(E) + U(y^{-})\, W^{-}(\neg E) \\ &\qquad\qquad (\text{ii}) \\ &\quad = U(x^{+}, E'; e) + U(e, E''; y^{-}) \qquad [(\text{ii}) \text{ and } U(e) = e] \\ &\quad = U(g^{+}) + U(g^{-}) \qquad (\text{Definitions}) \\ &\quad = U(g^{+} \oplus g^{-}) \qquad (\text{iii}) \\ &\quad = U[(x^{+}, E'; e) \oplus (e, E''; y^{-})] \qquad [\text{A2(iii)}] \end{aligned}$$

which, since utility is order preserving, yields (i). ▮

A7. *Proof of Theorem* 7. Suppose U is separable and $x^{+} \oplus y^{-} \succsim e$, then choosing D such that $W^{-}(\neg D) = W^{+}(E)$, which is possible since the weighting functions span $[0, 1]$,

$$\begin{aligned} &U(x^{+} \oplus y^{-}, E; e) \\ &\quad = U(x^{+} \oplus y^{-})\, W^{+}(E) \qquad (\text{Eq. 5}) \\ &\quad = [U(x^{+}) + U(y^{-})]\, W^{+}(E) \qquad (\text{Eq. 4}) \\ &\quad = U(x^{+})\, W^{+}(E) + U(y^{-})\, W^{-}(\neg D) \\ &\qquad\qquad (\text{Definition of D}) \\ &\quad = U(x^{+}, E; e) + U(e, D; y^{-}) \qquad (\text{Eq. 5}) \\ &\quad = U[(x^{+}, E; e) \oplus (e, D; y^{-})], \qquad (\text{Eq. 4}) \end{aligned}$$

whence by the fact that U is order preserving, the top part of Eq. (17) obtains. The proof for $x^{+} \oplus y^{-} \precsim e$ is similar.

Conversely, suppose the binary gambles are joint receipt decomposable with $W^{-}(\neg D) = W^{+}(E)$ and $W^{+}(D^{*}) = W^{-}(E)$. Because U and V are both order preserving there exists a strictly increasing function F mapping $U = F(V)$. With no loss of generality, we may normalize U and V so their least upper bounds are 1. Let the greatest lower bounds of these normalized functions be $-k$ and $-K$, respectively. So F: $]-k, 1[\xrightarrow{\text{onto}}]-K, 1[$, $F(0)=0$, and F is strictly increasing. So for $x^+ \oplus y^- \succsim e$,

$$\begin{aligned}
&V(x^+ \oplus y^-, E; e)\\
&\quad = V(x^+ \oplus y^-)\, W^+(E) && (V \text{ satisfies Eq. 5})\\
&\quad = F^{-1}[U(x^+ \oplus y^-)]\, W^+(E) && (\text{Definition of } F)\\
&\quad = F^{-1}[U(x^+) + U(y^-)]\, W^+(E) && (U \text{ satisfies Eq. 4})\\
&\quad = F^{-1}[F[V(x^+)] + F[V(y^-)]]\, W^+(E)\\
& && (\text{Definition of } F)
\end{aligned}$$

and from the right side of Eq. (17) we have

$$\begin{aligned}
&V[(x^+, E; e) \oplus (e, D; y^-)]\\
&\quad = F^{-1}[U[x^+, E; e) \oplus (e, D; y^-)]]\\
& && (\text{Definition of } F)\\
&\quad = F^{-1}[U(x^+, E; e) + U(e, D; y^-)]\\
& && (U \text{ satisfies Eq. 4})\\
&\quad = F^{-1}[F[V(x^+, E; e)] + F[V(e, D; y^-)]]\\
& && (\text{Definition of } F)\\
&\quad = F^{-1}[F[V(x^+)\, W^+(E)] + F[V(y^-)\, W^-(\neg D)]].\\
& && (V \text{ satisfies Eq. 5})
\end{aligned}$$

Setting $X = V(x^+)$, $-Y = V(y^-)$, $Z = W^+(E) = W^-(\neg D)$, and using the assumption of joint receipt decomposable, F is characterized by the following functional equation:

$$F^{-1}[F(X) + F(-Y)]\, Z = F^{-1}[F(XZ) + F(-YZ)]. \tag{A1}$$

The case with $x^+ \oplus y^- \precsim e$ is similar and leads to the same functional equation. It is shown in Aczél *et al.* (1996) that the monotonic solutions to Eq. (A1) are power functions, whence U is separable. ∎

A8. *Proof of Theorem* 8. We show that joint receipt consistency holds iff, for $x \in \mathscr{C}^+$, either there exists a constant $B > 0$ such that

$$U(x) - C[1 - \exp[-V(x)/B]], \tag{A2a}$$

or equivalently,

$$V(x) = -B \ln[1 - U(x)/C]; \tag{A2b}$$

or there are constants B and β, $B\beta > 0$, such that

$$U(x) = C[1 - (1 - V(x)/B)^{\beta}], \tag{A3a}$$

or equivalently,

$$V(x) = B[1 - (1 - U(x)/C)^{1/\beta}]. \tag{A3b}$$

First, suppose that Eq. (A2) and the hypotheses of joint receipt consistency hold, then the left side yields

$$\begin{aligned}
&V[(x \oplus r) \oplus y]\\
&\quad = -B \ln[1 - U[x \oplus r) \oplus y]/C] && (\text{Eq. A2b})\\
&\quad = -B \ln[1 - U(x \oplus r)/C][1 - U(y)/C] && (\text{Eq. 3})\\
&\quad = V(x \oplus r) + V(y) && (\text{Eq. A2a})\\
&\quad = V(x) + V(r) + V(s). && (\text{Eq. 4})
\end{aligned}$$

The right side yields

$$\begin{aligned}
&V[(x \oplus y) \oplus s]\\
&\quad = V(x \oplus y) + V(s) && (\text{Eq. 4})\\
&\quad = -B \ln[1 - U(x \oplus y)/C] + V(s) && (\text{Eq. A2b})\\
&\quad = -B \ln[1 - U(x)/C][1 - U(y)/C] + V(s)\\
& && (\text{Eq. 3})\\
&\quad = V(x) + V(y) + V(s) && (\text{Eq. A2a}).
\end{aligned}$$

Clearly, the equality holds provided that

$$V(s) = V(r),$$

whence joint receipt consistency holds trivially with $s = r$.

Next, suppose that Eq. (A3) and the hypotheses of joint receipt consistency both hold, then we have from the left side of that hypothesis that

$$\begin{aligned}
&V[(x \oplus r) \oplus y]\\
&\quad = B[1 - (1 - U[(x \oplus r) \oplus y]/C)^{1/\beta}] && (\text{Eq. A3b})\\
&\quad = B[1 - [1 - U(x \oplus r)/C]^{1/\beta}\, [1 - U(y)/C]^{1/\beta}]\\
& && (\text{Eq. 3})\\
&\quad = B[1 - [1 - V(x \oplus r)/B][1 - V(y)/B)]]\\
& && (\text{Eq. A3a})\\
&\quad = B\left[1 - \left[1 - \frac{V(x) + V(r)}{B}\right]\left[1 - \frac{V(y)}{B}\right]\right]\\
& && (\text{Eq. 4})\\
&\quad = B[1 - [1 - V(x)/B][1 - V(y)/B]]\\
&\qquad + V(r)[1 - V(y)/B].
\end{aligned}$$

For the right side,

$$
\begin{aligned}
&V[(x \oplus y) \oplus s] \\
&\quad = V(x \oplus y) + V(s) && \text{(Eq. 4)} \\
&\quad = B[1 - [1 - U(x \oplus y)/C]^{1/\beta}] + V(s) && \text{(Eq. A3b)} \\
&\quad = B[1 - [1 - U(x)/C]^{1/\beta}\,[1 - U(y)/C]^{1/\beta}] + V(s) && \text{(Eq. 3)} \\
&\quad = B[1 - [1 - V(x)/B][1 - V(y)/B)]] + V(s). && \text{(Eq. A3a)}
\end{aligned}
$$

Equating, we see that

$$V(s) = V(r)[1 - V(y)/B],$$

which is Eq. (18) and is independent of x, and so the same condition holds for any $x' \in \mathscr{C}^+$ such that $x' \oplus r \in \mathscr{C}^+$, proving joint receipt consistency.

Conversely, suppose joint receipt consistency holds. By the fact that U and V are both increasing functions over gains, there exists some strictly increasing function F such that for $x \in \mathscr{C}^+$, $U(x) = F[V(x)]$ and $F(0) = 0$. So,

$$
\begin{aligned}
&U[(x \oplus r) \oplus y] \\
&\quad = U(x \oplus r)[1 - U(y)/C] + U(y) && \text{(Eq. 3)} \\
&\quad = F[V(x \oplus r)][1 - U(y)/C] + U(y) && \text{(Definition of } F) \\
&\quad = F[V(x) + V(r)][1 - F[V(y)]/C] + F[V(y)]. && \text{(Eq. 4)}
\end{aligned}
$$

And,

$$
\begin{aligned}
&U[(x \oplus y) \oplus s] \\
&\quad = F[V[(x \oplus y) \oplus s]] && \text{(Def. of } F) \\
&\quad = F[V(x \oplus y) + V(s)] && \text{(Eq. 4)} \\
&\quad = F[F^{-1}[U(x \oplus y)] + V(s)] && \text{(Def. of } F) \\
&\quad = F[F^{-1}[U(x)[1 - U(y)/C] + U(y)] + V(s) && \text{(Eq. 3)} \\
&\quad = F[F^{-1}[F[V(x)][1 - F[V(y)]/C] \\
&\qquad + F[V(y)]] + V(s). && \text{(Def. of } F)
\end{aligned}
$$

By the hypotheses of joint receipt consistency, these two expressions are equal. Setting $X = V(x)$, $Y = V(y)$, $-R = V(r)$, $-S = V(s)$, and noting that since $x \oplus r \in \mathscr{C}^+$, $X \geqslant R$,

$$
\begin{aligned}
&F(X - R)[1 - F(Y)/C] + F(Y) \\
&\quad = F[F^{-1}[F(X)[1 - F(Y)/C] + F(Y)] - S]. \qquad \text{(A4)}
\end{aligned}
$$

Joint receipt consistency then asserts that this equality holds for all $X \geqslant R$. Aczél *et al.* (1996) show that the solutions to Eq. (A4) are either Eq. (A2) or (A3).

The solution Eq. (A2) of Eq. (A4) means that V is additive over both gains and mixed gains and losses. Thus $\oplus$ must be associative throughout.

We next show that the solution of Eq. (A3), i.e.,

$$1 - U/C = (1 - V/B)^{\beta},$$

means that V as well as U has the negative exponential form over $\mathscr{C}^+$: Let $x, y \in \mathscr{C}^+$,

$$
\begin{aligned}
[1 - V(x \oplus y)/B]^{\beta} &= 1 - U(x \oplus y)/C \\
&= [1 - U(x)/C][1 - U(y)/C] \\
&= [1 - V(x)/B]^{\beta}[1 - V(y)/B]^{\beta}.
\end{aligned}
$$

Thus, taking the $1/\beta$ root,

$$1 - V(x \oplus y)/B = [1 - V(x)/B][1 - V(y)/B],$$

which implies the negative exponential form.

Suppose Eq. (A2) holds and that V is separable. We show that assuming U is also separable leads to a contradiction. Using the assumed separability of both U and V we have for $x \in \mathscr{C}^+$

$$
\begin{aligned}
U(x, E; e) &= U(x)\, W^+(E) \\
&= W^+(E)\, B[1 - \exp[-\alpha V(x)]],
\end{aligned}
$$

and

$$
\begin{aligned}
U(x, E; e) &= B[1 - \exp[-\alpha V(x, E; e)]] \\
&= B[1 - \exp[-\alpha V(x)\, Z^+(E)]].
\end{aligned}
$$

Omitting the E arguments and setting $X = \alpha V(x) > 0$,

$$1 - \exp(-XZ^+) = W^+(1 - \exp(-X)].$$

Taking the derivative with respect to X,

$$Z^+ \exp(-XZ^+) = W^+ \exp(-X),$$

and taking logarithms and solving either $Z^+ = W^+ = 1$ or

$$X = \frac{\log(W^+/Z^+)}{1 - Z^+}.$$

The latter is impossible because X is independent of W^+ and Z^+ which both depend only on E.

Turning to Eq. (A3) and assuming U and V are both separable, we may rewrite Eq. (A3) for $(x, E; e)$, $x \in \mathscr{C}^+$, as

$$1 - \frac{U(x, E; e)}{C} = \left[1 - \frac{V(x, E; e)}{B}\right]^{\beta}. \tag{A5}$$

Applying separability on both sides, $U(x, E; e) = U(x)\, W^+(E)$ and $V(x, E; e) = V(x)\, Z^+(E)$, where $Z^+ = G(W^+)$ for some strictly increasing G, both $W^+, Z^+ \in [0, 1]$, and, replacing $U(x)/C$ by Eq. (A2a), $V(x)/B$ by X, and rewriting,

$$W^+[1 - (1 - X)^{\beta}] = 1 - [1 - XG(W^+)]^{\beta},$$

Note that both Z^+ and X are in $[0, 1]$. Let

$$H(X) = 1 - (1 - X)^{\beta},$$

then we have the functional equation

$$W^+ H(X) = H[G(W^+)\, X].$$

Since $H(1) = 1$, setting $P = G(W^+)$ we have

$$H(P)\, H(X) = H(PX),$$

and so, for some $\gamma > 0$, $H(X) = X^{\gamma}$. Thus, by the definition of H we see

$$1 - X^{\gamma} = (1 - X)^{\beta},$$

which can only be satisfied for all X in $[0, 1]$ by $\beta = \gamma = 1$. From $\beta = 1$ in Eq. (A5), it is easy to see that if U and V agree at a point, then $B = C$ and $U = V$. ∎

REFERENCES

Aczél, J., Luce, R. D., & Maksa, Gy. (1996). Solutions to three functional equations arising from different ways of measuring utility. *Journal of Mathematical Analysis and Applications*, in press.

Birnbaum, M. H. (1992). Violations of monotonicity and contextual effects in choice-based certainty equivalents. *Psychological Science*, **3**, 310–314.

Birnbaum, M. H., & McIntosh, W. R. (1995). *Violations of branch independence in choices between gambles*, manuscipt.

Bostic, R., Herrnstein, R. J., & Luce, R. D. (1990). The effect on the preference-reversal phenomenon of using choice indifferences. *Journal of Economic Behavior and Organization*, **13**, 193–212.

Brothers, A. J. (1990). *An empirical investigation of some properties that are relevant to generalized expected-utility theory*. Unpublished doctoral disseration, University of California, Irvine.

Chechile, R., & Cooke, A. (1996). *An experimental test of a general class of utility models: Evidence for context dependency*, manuscript.

Cho, Y., & Fisher, G. (in preparation). *Test of joint receipt assumptions: Monotonicity, order preservation, and scale invariance*.

Cho, Y., & Luce, R. D. (1995). Tests of hypotheses about certainty equivalents and joint receipt of gambles. *Organizational Behavior and Human Decision Processes*, **64**, 229–248.

Cho, Y., Luce, R. D., & von Winterfeldt, D. (1994). Tests of assumptions about the joint receipt of gambles in rank- and sign-dependent utility theory. *Journal of Experimental Psychology: Human Perception and Performance*, **20**, 931–943.

Chung, N-K., von Winterfeldt, D., & Luce, R. D. (1994). An experimental test of event commutativity in decision making under uncertainty. *Psychological Science*, **5**, 394–400.

Edwards, W. (1962). Subjective probabilities inferred from decisions. *Psychological Review*, **69**, 109–135.

Edwards, W. (1992). *Utility theories: Measurements and applications.* Boston: Kluwer Academic.

Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *American Economic Review*, **67**, 116–126.

Fishburn, P. C. (1982). *The foundations of expected utility*. Dordrecht: Reidel.

Fisburn, P. C., & Kochenberger, G. A. (1979). Two-piece von Neumann–Morgenstern utility functions. *Journal for the American Institute for Decision Sciences*, **10**, 503–518.

Fishburn, P. C., & Luce, R. D. (1995). Joint receipt and Thaler's hedonic editing rule. *Mathematical Social Sciences*, **29**, 33–76.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, **47**, 263–291.

Karmarkar, U. S. (1978). Subjectively weighted utility: A descriptive extension of the expected utility model. *Organizational Behavior and Human Performance*, **21**, 61–72.

Karmarkar, U. S. (1979). Subjectively weighted utility and the Allais paradox. *Organizational Behavior and Human Performance*, **24**, 67–72.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement* (Vol. 1). New York: Academic Press.

Libby, R., & Fishburn, P. C. (1977). Behavioral models of risk taking in business decisions: A survey and evaluation. *Journal of Accounting Research*, **15**, 272–292.

Linville, P. W., & Fischer, G. W. (1991). Preferences for separating or combining events. *Journal of Personality and Social Psychology*, **60**, 5–23.

Liu, L. (1995). *A theory of coarse utility and its application to portfolio analysis*. Unpublished Ph.D. dissertation, University of Kansas.

Lopes, L. L. (1984). Risk and distributional inequality. *Journal of Experimental Psychology: Human Perception and Performance*, **10**, 465–485.

Lopes, L. L. (1987). Between hope and fear: The psychology of risk. *Advances in Experimental Social Psychology*, **20**, 255–295.

Luce, R. D. (1991). Rank- and sign-dependent linear utility models for binary gambles. *Journal of Economic Theory*, **53**, 75–100.

Luce, R. D. (1992a). Singular points in generalized concatenation structures that otherwise are homogeneous. *Mathematical Social Sciences*, **24**, 79–103.

Luce, R. D. (1992b). A theory of certainty equivalents for uncertain alternatives. *Journal of Behavioral Decision Making*, **5**, 201–216.

Luce, R. D. (1995). Joint receipt and certainty equivalents of gambles. *Journal of Mathematical Psychology*, **39**, 73–81.

Luce, R. D. (submitted). *Associative joint receipts*.

Luce, R. D., & Fishburn, P. C. (1991). Rank- and sign-dependent linear utility models for finite first-order gambles. *Journal of Risk and Uncertainty*, **4**, 29–59.

Luce, R. D., & Fishburn, P.C. (1995). A note on deriving rank-dependent utility using additive joint receipts. *Journal of Risk and Uncertainty*, **11**, 5–16.

Luce, R. D., Krantz, D. H., Suppes, P., & Tversky, A. (1990). *Foundations of measurement* (Vol. 3). San Diego: Academic Press.

Luce, R. D., Mellers, B., & Chang, S.-J. (1993). Is choice the correct primitive? On using certainty equivalents and reference levels to predict choices among gambles. *Journal of Risk and Uncertainty*, **6**, 115–143.

Luce, R. D., & Narens, L. (1985). Classification of concatenation measurement structures according to scale type. *Journal of Mathematical Psychology*, **29**, 1–72.

Luce, R. D., & von Winterfeldt, D. (1994). What common ground exists for descriptive, prescriptive and normative utility theories? *Management Science*, **40**, 263–279.

Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy*, **60**, 151–158.

Mellers, B., Weiss, R., & Birnbaum, M. H. (1992). Violations of dominance in pricing judgments. *Journal of Risk and Uncertainty*, **5**, 73–90.

Payne, J. W., & Braunstein, M. L. (1971). Preferences among gambles with equal underlying distributions. *Journal of Experimental Psychology*, **87**, 13–18.

Quiggin, J. (1993). *Generalized expected utility theory*: *The rank-dependent model*. Boston: Kluwer Academic.

Ramsey, F. P. (1931). *The Foundations of mathematics and other logical essays* (Chap. 7). New York: Harcourt, Brace & Co.

Ronen, J. (1971). Some effects of sequential aggregation in accounting on decision-making. *Journal of Accounting Research*, **9**, 307–332.

Ronen, J. (1973). Effects of some probability displays on choices. *Organizational Behavior and Human Performance*, **9**, 1–15.

Savage, L. J. (1954). *The Foundations of Statistics*. New York: Wiley.

Schoemaker, P. J. H. (1982). The expected utility model: Its variants, purposes, evidence and limitations. *Journal of Economic Literature*, **20**, 529–563.

Schoemaker, P. J. H. (1990). Are risk-attitudes related across domain and response modes? *Management Science*, **36**, 1451–1463.

Slovic, P., & Lichtenstein, S. (1968). Importance of variance preferences in gambling decisions. *Journal of Experimental Psychology*, **78**, 646–654.

Sneddon, R., & Luce, R. D. (in preparation). *Bias in a PEST procedure*.

Thaler, R. H. (1985). Mental accounting and consumer choice. *Marketing Science*, **4**, 199–214.

Thaler, R. H., & Johnson, E. (1990). Gambling with the house money or trying to break even: The effects of prior outcomes on risky choice. *Management Science*, **36**, 643–660.

Tversky, A. (1967). Additivity, utility, and subjective probability. *Journal of Mathematical Psychology*, **4**, 175–201.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, **5**, 204–217.

Tversky, A., Slovic, P., & Kahneman, D. (1990). The causes of preference reversals. *The American Economic Review*, **80**, 204–217.

von Neumann, J., & Morgenstern, O. (1944/1947/1953). *Theory of Games and Economic Behavior*. Princeton, NJ: Princeton Univ. Press.

von Winterfeldt, D., Chung, N.-K., Luce, R. D., & Cho, Y. (1996). Tests of consequence monotonicity in decision making under uncertainty. *Journal of Experimental Psychology*: *Learning*, *Memory*, *and Cognition*, in press.

Wakker, P. P. (1989). *Additive representations of preferences*: *A new foundation of decision analysis*. Dordrecht, The Netherlands: Kluwer Academic.

Wakker, P. P., & Tversky, A. (1993). An axiomatization of cumulative prospect theory. *Journal of Risk and Uncertainty*, **7**, 147–176.

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