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Solutions to Three Functional Equations Arising from Different Ways of Measuring Utility*

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Utility of gains (losses) can be measured in four distinct ways: riskless vs risky choices and gains (losses) alone vs the gain-loss trade-off. Conditions forcing these measures all to be the same lead to functional equations, three of which are

$$F^{-1}[F(X)+F(-Y)]Z = F^{-1}[F(XZ)+F(-YZ)]$$
$$(F\colon\ ]-k,k'[\ \to\ ]-K,K'[;\ k,k',K,K'>0)\quad \text{(i)}$$

$$F(X-R)[1-F(Y)]+F(Y) = F\big[F^{-1}(F(X)[1-F(Y)]+F(Y))-S\big]$$
$$(F\colon [0,1[\ \to [0,1[)\quad \text{(ii)}$$

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$$F^{-1}[F(X)+F(Y)-F(X)F(Y)]Z$$
$$=F^{-1}[F(XZ)+F[YP(X,Z)]-F(XY)F[YP(X,Z)]]$$
$$\big(F\colon [0,1[\to[0,1[,\ P\colon [0,1[\times[0,1]\to[0,1]\big). \quad \text{(iii)}$$

We determine all strictly increasing, surjective (and thus continuous) solutions of (i) and (ii) and all strictly increasing, surjective solutions of (iii) that are differentiable on $[0,1[$ as are their inverses (thus, $F' \neq 0$ on $]0,1[$).

1. INTRODUCTION

Modern theories of utility of valued entities have been developed both for risky and riskless choices. The former rests on studying trade-offs between the valued entities and the chance of receiving them. The latter is based on how the joint receipt of valued alternatives is valued and is somewhat analogous to, for example, mass measurement. These theories also have the feature of making sharp distinctions between gains and losses. Thus, the utility of gains can be measured in four distinct ways: risky or riskless choices, and gains alone or the trade-offs between gains and losses. A natural question to investigate is the conditions under which these four measures are all the same. This is done in Luce [4] and results in a number of functional equations to be solved. Of these, three are somewhat intriguing and not quite routine to solve; they are studied in this paper.

We begin by outlining the nature of these utility theories, the nature of the conditions that render the measures to be the same, and the resulting functional equations. After that, we solve them. One is solved under the added condition of differentiability of the strictly increasing, surjective function relating the two utility measures and of its inverse; an interesting open problem is to eliminate this differentiability assumption.

2. BACKGROUND OF THE FUNCTIONAL EQUATIONS

Consider the problem of finding numerical representations of a structure $\langle \mathscr{G}, \succsim, \oplus, e \rangle$, where $\mathscr{G}$ is a set, $\succsim$ a binary weak order on $\mathscr{G}$, $\oplus$ a closed binary operation on $\mathscr{G}$, and $e \in \mathscr{G}$ is an identity of $\oplus$. The intended interpretations are as follows. A gamble, i.e., uncertain alternative, is a function from a finite partition of an event from an algebra $\mathscr{E}$ of events into a set $\mathscr{C}$ of consequences, where $\mathscr{C} \cap \mathscr{E} = \varnothing$. The set $\mathscr{G}$ is generated from $\mathscr{C}$ and the set of gambles as the recursive closure under $\oplus$ (i.e., if $x \in \mathscr{C}$, then $x \in \mathscr{G}$; if g is a gamble, then $g \in \mathscr{G}$; and if $g, h \in \mathscr{G}$, then $g \oplus h \in \mathscr{G}$); $\succsim$ is a preference ordering on $\mathscr{G}$; $\oplus$ is the operation of the *joint receipt* of pairs of elements from $\mathscr{G}$; and e represents the status quo relative to which gains and losses are defined. Assume that $\oplus$ is weakly commutative, i.e., for all $g, h \in \mathscr{G}$, $g \oplus h \sim h \oplus g$, where $\sim$ denotes $\succsim \cap \precsim$; $\oplus$ is monotonic relative to $\succsim$; for all $g \in \mathscr{G}$, $g \oplus e \sim g$; and if g is a gamble each of whose values are $x \in \mathscr{C}$, then $g \sim x$.

Luce and Fishburn [5, 6] were led from certain theoretical and empirical conditions to consider the following order preserving utility representation U of $\langle \mathscr{G}, \succsim \ \oplus, e \rangle$ where C and C' are positive constants and U is from $\mathscr{G}$ onto the real interval $[-C', C[$:

$$U(g \oplus h) = \begin{cases} U(g) + U(h) - U(g)U(h)/C, & g \succsim e, h \succsim e, & \text{(1a)} \\ U(g) + U(h), & g \succsim e \succsim h \text{ or } h \succsim e \succsim g, & \text{(1b)} \\ U(g) + U(h) + U(g)U(h)/C' & e \succsim g, e \succsim h, & \text{(1c)} \end{cases}$$

$$U(e) = 0. \tag{1d}$$

Note that the monotonicity of $\oplus$ forces U to be bounded by $-C'$ and C.

Let $\mathscr{G}^+ = \{g^+ : g^+ \in \mathscr{G} \,\&\, g^+ \succsim e\}$ and $\mathscr{G}^- = \{g^- : g^- \in \mathscr{G} \,\&\, e \succsim g^-\}$. Note, first, the transformation $V = -\ln(1 - U/C)$ and (1a) imply that $\langle \mathscr{G}^+, \oplus, \succsim, e \rangle$, with $\oplus$ and $\succsim$ appropriately restricted, has an additive representation on the non-negative reals, i.e., $V(g \oplus h) = V(g) + V(h)$, where $V(e) = 0$. Second, from (1b), $\langle \mathscr{G}^+ \times \mathscr{G}^+, \succsim' \rangle$, with the ordering defined by $(g^+, g^-) \succsim' (h^+, h^-)$ iff $g^+ \oplus g^- \succsim h^+ \oplus h^-$, also has an additive representation of the form $U(g^+, g^-) = U(g^+) + U(g^-)$ where $U(e) = 0$. And, finally, from (1c), we see that $\langle \mathscr{G}^-, \oplus, \succsim^*, e \rangle$, where $\succsim^*$ is the converse of $\succsim$, also has an additive representation.

So, the task of finding axioms on $\langle \mathscr{G}, \succsim, \oplus, e \rangle$ leading to the representation (1) can be divided into two steps. The first is to axiomatize separately $\langle \mathscr{G}^+, \oplus, \succsim, e \rangle$, $\langle \mathscr{G}^+ \times \mathscr{G}^-, \succsim' \rangle$, and $\langle \mathscr{G}^-, \oplus, \succsim^*, e \rangle$ leading in all three cases to additive representations. This problem is very well understood (see, e.g., Krantz, Luce, Suppes, and Tversky [3, Chaps. 3 and 6]). Considering only gains for the moment, let $U_+ : \mathscr{G}^+ \to [0, 1[$ denote the representation of $\langle \mathscr{G}^+, \oplus, \succsim, e \rangle$ satisfying

$$U_+(g^+ \oplus h^+) = U_+(g^+) + U_+(h^+) - U_+(g^+)U_+(h^+), \tag{1a$'$}$$

and let U_m (m for mixed consequences) denote an additive representation of $\langle \mathscr{G}^+ \times \mathscr{G}^-, \succsim' \rangle$, i.e.,

$$U_m(g^+) + U_m(g^-) \text{ preserves the order } \succsim', \tag{1b$'$}$$

that is normalized to have l.u.b. 1 and g.l.b. $-k$, where $k > 0$. The second task is to achieve a united axiomatization by providing a condition for gains that shows $U_+ = U_m$ over $\mathscr{G}^+$.

A necessary behavioral condition implied by $U_+ = U_m$ can be shown (Luce [4]; Luce and Fishburn [5]) to be: Suppose $x, y \succsim e$, $r, x \precsim e$, $x \oplus r \succsim e$, and $(x \oplus r) \oplus y \sim (x \oplus y) \oplus s$. Then, for any $x' \succsim e$ with $x' \oplus r \succsim e$, it follows that $(x' \oplus r) \oplus y \sim (x' \oplus y) \oplus s$. Conversely, by the monotonicity of $\oplus$, both U_+ and U_m are order preserving, so there exists a strictly increasing F: $[0,1[\to [0,1[$ such that $U_+ = F(U_m)$. Assuming the condition just stated, one shows that F must solve the following functional equation for all $X, Y \in [0,1[$, $R \in [0, X]$, and for all $S \geq 0$ (independent of X) for which the equation makes sense:

$$F(X-R)[1-F(Y)] + F(Y) = F\left[F^{-1}\left(F(X)[1-F(Y)] + F(Y)\right) - S\right]. \tag{2}$$

Note that $F(0) = 0$ follows from the fact that F is increasing and maps $[0,1[$ onto $[0,1[$. Equation (2) is the first of three equations to be solved. Although (2) has solutions other than the identity, an additional argument in Luce [4] assuming the separation property defined by (3) below shows the non-identity solutions can be ruled out.

A similar result is found relating U_m to U_- over $\mathscr{G}^-$, but we need not formulate it explicitly.

Next, consider the special gamble $g = (x, E; e, D)$, $D \cap E \neq \emptyset$, which means that if a chance device with possible outcomes in $E \cup D$ is run, then x is the consequence if E occurs and it is e if D occurs. E and D exhaust the possibilities. Usually $(x, E; e, D)$ is abbreviated as $(x, E; e)$, where the domain of the gamble is implicit. A wide variety of utility theories imply for this case the existence of functions U from $\mathscr{G}$ onto an open interval of the reals and W^k from $\mathscr{E}$ onto $[0,1]$, $k = +, -$, such that This property is called *separability*. For some of the results in Luce and Fishburn [5, 6] leading to what is known as the rank- and sign-dependent utility theory for gambles (also called cumulative prospect theory, see Tversky and Kahneman [8]), it is necessary that the U of (1) be separable.

There are two natural behavioral questions: When do functions U_+ satisfying (1a$'$) and U_m satisfying (1b$'$) also satisfy (3)? Again, necessary conditions on $\langle \mathscr{G}^+, \succsim, \oplus, e \rangle$ and $\langle \mathscr{G}^+ \times \mathscr{G}^-, \succsim' \rangle$ follow from these two requirements: For each $x^+ \succsim e$ and event E there exists an event $D =$

$D(x, E)$ such that for all $y^+ \succsim e$

$$(x^+ \oplus y^+, E; e) \sim (x^+, E; e) \oplus (y^+, D; e). \tag{4a}$$

For each x^+ and y^- with $x^+ \succsim e \succsim y^-$ and each event E, there exist events $D = D(E)$ and $D^* = D^*(E)$ such that

$$(x^+ \oplus y^-, E; e) \sim \begin{cases} (x^+, E; e) \oplus (y^-, D; e), & \text{if } x^+ \oplus y^- \succsim e \\ (x^+, D^*; e) \oplus (y^-, E; e), & \text{if } x^+ \oplus y^- \prec e. \end{cases} \tag{4b}$$

The converse problem leads to two functional equations, one for gains and one for the mixed case. Let U_+ be the representation of (1a′), let V be any separable function (whose existence is assumed) normalized to $[0, 1[$, and let F be defined by $U_+ = F(V)$, where F is strictly increasing. Equation (4a) results in the functional equation (see Luce [4])

$$F^{-1}[F(X) + F(Y) - F(X)F(Y)]Z$$
$$= F^{-1}[F(XZ) + F[YP(X, Z)] - F(XZ)F[YP(X, Z)]], \tag{5}$$

where F is strictly increasing on $[0, 1[$ onto $[0, 1[$, $X, Y \in [0, 1[$, $Z \in [0, 1]$, and P: $[0, 1[\times [0, 1] \to [0, 1]$. Again, as in (2), $F(0) = 0$ follows. This is the second functional equation to be solved. We do so under the added assumption that F and F^{-1} are differentiable; we do not know of a proof under the weaker condition of continuity.

The third equation arises from (4b) with $U_m = F(V)$,

$$F^{-1}[F(X) + F(-Y)]Z = F^{-1}[F(XZ) + F(-YZ)], \tag{6}$$

where F is strictly increasing from $[-k, 1[$ onto $]-K, 1[$, $k > 0$, $K > 0$, $X \in [0, 1[$, $Y \in [0, k[$, and $Z \in [0, 1]$. Note that $F(0) = 0$ follows by setting $Z = 0$ in (6). Equation (6) makes sense because $F(X) + F(-Y) \in]-K, 1[$ whenever $X \in [0, 1[$, $Y \in [0, k[$.

By symmetry, the case of losses is similar to that of gains and leads to analogues of (5) and (6).

It is convenient to solve the equations in the order (6), (2), and (5).

3. SOLUTION TO EQ. (6)

We take more general domains and ranges for F, which maps from $]-k, k'[$ onto $]-K, K'[$ (of course, we may choose $k' = K' = 1$) and search for strictly increasing solutions F to (6) with $X \in [0, k'[$. Because F is a strictly monotonic and surjective mapping from an interval onto an interval, it is also continuous.

For the time being, fix $Z > 0$ in (6) and write

$$G(W) = F(WZ) \qquad (W \in]-k, k'[, Z \in]0,1]). \tag{7}$$

So (6) becomes

$$G^{-1}[G(X) + G(-Y)] = F^{-1}[F(X) + F(-Y)]$$
$$\big(X \in [0, k'[, Y \in [0, k[\big).$$

Because F is an increasing surjection and $F(0) = 0$, so for every $S \in [0, K'[$ and $T \in]-K, 0]$ there exist unique $X \in [0, k'[$ and $Y \in [0, k[$ such that

$$S = F(X) \qquad \text{and} \qquad T = F(-Y), \tag{8a}$$

or equivalently

$$X = F^{-1}(S) \qquad \text{and} \qquad Y = -F^{-1}(T). \tag{8b}$$

Setting

$$f(V) = G\big[F^{-1}(V)\big], \qquad V \in]-K, K'[, \tag{9}$$

we get from (7)

$$f(S + T) = f(S) + f(T). \tag{10}$$

In view of (8) and the preceding remark, this Cauchy equation holds on $[0, K'[\times]-K, 0]$. Since 0 is in each of the three intervals $[0, K'[$, $]-K, 0]$, and

$$\{S + T: S \in [0, K'[, T \in]-K, 0]\} =]-K, K'[,$$

(10) can be extended to $\mathbb{R}^2$ (see Daróczy and Losonczi [2], Radó and Baker [7], and Aczél [1, p. 82]), that is, there exists an $\tilde{f}: \mathbb{R} \to \mathbb{R}$ that satisfies (10) on $\mathbb{R}^2$ and whose restriction to $[-K, K'[$ is f. Since, moreover, F is monotonic, so are both G and f [see (7) and (9)]. Therefore,

$$f(V) = AV \qquad (V \in]-K, K'[),$$

and so by (9)

$$G(W) = AF(W).$$

Taking (7) into consideration and recalling that we have held Z temporarily constant,

$$F(WZ) = A(Z)F(W) \qquad (W \in]-k, k'[, Z \in]0,1]). \tag{11}$$

We distinguish two cases:

(i) If $k' > 1$, then we substitute $W = 1$ into (11) and, with $F(1) = \gamma$, we get

$$F(Z) = \gamma A(Z) \qquad (Z \in]0, 1]). \tag{12}$$

Because F is strictly increasing, $\gamma \neq 0$ and so A is strictly monotonic. Putting (12) back into (11) yields that A is *multiplicative*, that is,

$$A(ZW) = A(Z)A(W) \qquad (Z \in]0,1], W \in]0,1]). \tag{13}$$

[In contrast to (11), (13) holds only for $W \in]0, 1]$ because we obtained it by using (12) for W in place of Z.] It is clear from (13) (by choosing $Z = W = \sqrt{z}$) that $A(z) \geq 0$. So, because F maps $[0, k'[$ onto $[0, K'[$ and is strictly increasing in (12) $\gamma > 0$ and so A is strictly increasing. Moreover, the general strictly increasing solution of (13) is (see Aczél [1, pp. 27–28, 81])

$$A(Z) = Z^{\beta} \qquad (Z \in]0, 1], \beta > 0). \tag{14}$$

By (12) we get that

$$F(Z) = \gamma Z^{\beta} \qquad (Z \in]0, 1], \beta > 0, \gamma > 0). \tag{15}$$

We will extend this to $[0, k'[$ a bit later.

(ii) The other case is $k' \leq 1$. Then substitute $W = C \in]0, k'[$ into (11) and get, with $\alpha = F(C) \neq 0$,

$$F(CZ) = \alpha A(Z) \qquad (Z \in]0, 1])$$

or, with $x = CZ \in]0, C]$,

$$F(x) = \alpha A(x/C) \qquad (x \in]0, C]). \tag{16}$$

Putting this back into (11), we obtain

$$A(WZ/C) = A(Z)A(W/C) \qquad (Z \in]0,1], W \in]0, C]).$$

With $w = W/C \in]0, 1]$ and $WZ/C = wZ \in]0, 1]$, this becomes

$$A(wZ) = A(w)A(Z) \qquad (w \in]0,1], Z \in]0,1]),$$

that is, we have (13) and thus (14) again. However, instead of (15), we only get from (16) that

$$F(x) = \alpha (x/C)^{\beta} = \gamma x^{\beta} \qquad \text{for } x \in]0, C]. \tag{17}$$

Now we unite the two cases and extend both (15) and (17) to $[0, k'[$. For any $X \in]0, k'[$ there exists $Z \in]0, 1]$ such that XZ is in $]0, 1]$ or in $]0, C]$, respectively. Applying (11), (14), and either (15) or (17), we obtain

$$\gamma(XZ)^{\beta} = Z^{\beta}F(X),$$

and so, taking into account that $F(0) = 0$, as asserted,

$$F(X) = \gamma X^{\beta} \qquad (X \in [0, k']). \tag{18}$$

For the interval $]-k, 0]$, we just put $W = -Y \in]-k, 0]$ into (11), and so $Y \in [0, k[$. Then the function defined by

$$H(Y) = F(-Y) \qquad (Y \in [0, k[), \tag{19}$$

satisfies

$$H(YZ) = A(Z)H(Y) \qquad (Y \in [0, k[, Z \in [0, 1]).$$

This is completely analogous to the case discussed above and so, as before,

$$H(Y) = \delta Y^{\beta} \qquad (Y \in [0, k[).$$

(The function A is the same so the exponent β is the same as in (18), but δ can be different from γ—indeed, it must be as we will see shortly.) From (19) with $W = -Y$

$$F(W) = \delta(-W)^{\beta} \qquad (W \in]-k, 0]).$$

Because F is strictly increasing and $\beta > 0$, necessarily $\delta < 0$ (and so $\gamma \neq \delta$). So, with $\gamma' = -\delta > 0$,

$$F(W) = \begin{cases} \gamma W^{\beta}, & \text{for } W \in [0, k'[\\ -\gamma'(-W)^{\beta}, & \text{for } W \in]-k, 0]. \end{cases} \tag{20}$$

Substituting (20) and its inverse

$$F^{-1}(T) = \begin{cases} (T/\gamma)^{1/\beta}, & \text{for } T \in [0, K'[\\ -(-T/\gamma')^{1/\beta}, & \text{for } T \in]-K, 0] \end{cases}$$

into (6) demonstrates this is a solution, and so we have proved the following.

THEOREM 1. *The general strictly monotonic increasing surjection* F: $]-k, k'[\to]-K, K'[$ *satisfying* (6) *for* $X \in [0, k'[$, $Y \in [0, k[$, $Z \in [0, 1]$ *is given by* (20), *where* β, γ, *and* γ' *are positive constants satisfying*

$$\gamma k'^{\beta} = K' \qquad \textit{and} \qquad \gamma' k^{\beta} = K.$$

(*If* $k' = K' = 1$, *then* $\gamma = 1$.)

4. SOLUTIONS TO EQ. (2)

The next task is to find the strictly increasing solutions F to

$$F(X-R)[1-F(Y)]+F(Y)$$
$$=F\left[F^{-1}(F(X)[1-F(Y)]+F(Y))-S\right], \tag{2}$$

where $X, Y \in [0,1[$, $R \in [0, X]$, and $S \geq 0$ is compatible with (2), and $F(0)=0$. A peculiarity of this problem is that (2) turns out to have a strictly monotonic solution only if a certain equation holds among S, R, and Y (S was supposed to be independent of X); specifically S is a bilinear function of R and Y. This is unusual, but not unheard of. Consider, for instance, the variant

$$f(2R-Y+S)=f(R)+f(Y) \tag{21}$$

of (10). If S were an independent variable, then with $S=R=Y=0$ or just $Y=0$, we would get

$$f(0)=0 \qquad \text{and} \qquad f(2R+S)=f(R),$$

respectively, so $f=0$ would be the only solution. If, however, S is some function of Y and R, say $S=2Y-R$, then (21) becomes $f(R+Y)=f(R)+f(Y)$, which is (10), and so $f(V)=AV$ $(A>0)$ is the general strictly increasing solution.

We solve (2) by first introducing the notation

$$H(X,Y)=F^{-1}[F(X)+F(Y)-F(X)F(Y)]$$
$$=F^{-1}(1-[1-F(X)][1-F(Y)]), \tag{22}$$

yielding the equivalent functional equation

$$H(X-R,Y)=H(X,Y)-S. \tag{23}$$

Since $F(0)=0$, we obtain from (22) and (23) with $X=0$ or $X=R$

$$H(0,Y)=Y$$

and

$$S=H(R,Y)-Y=f(R,Y), \tag{24}$$

respectively. Equation (24) describes the announced dependence of S on Y and R. In terms of f, (23) becomes

$$f(X-R,Y)=f(X,Y)-f(R,Y) \qquad (0 \leq R \leq X < 1, 0 \leq Y < 1),$$

or, with $Z = X - R \in [0,1[$,

$$f(Z+R,Y) = f(Z,Y) + f(R,Y) \qquad (R,Z,R+Z,Y \in [0,1[).$$

This Cauchy equation [compare with (10)], on

$$\{(R,Z)\colon R,Z,R+Z \in [0,1[\}$$

with Y kept fixed (as was Z in Section 2), can again be extended to $\mathbb{R}^2$ (see, e.g., Aczél [1, p. 82]). Because F is strictly increasing, (22) and (24) show that H and f both increase strictly in their first variable, so again we have that $f(R,Y) = A(Y)R$, and by (24)

$$H(X,Y) = A(Y)X + Y. \tag{25}$$

Substituting in (22) yields

$$F[A(Y)X + Y] = 1 - [1 - F(X)][1 - F(Y)]. \tag{26}$$

Because the right hand side is symmetric in X and Y, the left must be also. We have two cases:

(i) There exists an $X_0 \in [0,1[$ such that $A(X_0) \neq 1$. Because F is strictly monotonic, we get

$$A(Y)X_0 + Y = A(X_0)Y + X_0.$$

and so

$$A(Y) = 1 - \alpha Y, \qquad \text{where } \alpha = [1 - A(X_0)]/X_0 \neq 0.$$

By (25)

$$H(X,Y) = X + Y - \alpha XY = [1 - (1 - \alpha X)(1 - \alpha Y)]/\alpha. \tag{27}$$

On the other hand, by (24) and (27),

$$S = R(1 - \alpha Y), \tag{28}$$

which is the final form of the dependence of S on R and Y in this case. By (26) with

$$Z = 1 - \alpha X, \qquad W = 1 - \alpha Y, \qquad M(Z) = 1 - F\left(\frac{1 - Z}{\alpha}\right), \tag{29}$$

we get the equation [cf. (13)]

$$M(ZW) = M(Z)M(W). \tag{30}$$

This holds [see (29)] for all $W, Z \in]1-\alpha, 1]$ or $[1, 1-\alpha[$ depending on whether $\alpha > 0$ or < 0 [by (28) and $S \geq 0$ we have $1 - \alpha Y \geq 0$, so $1 - \alpha \geq 0$]. In both cases, (30) can be extended to $\mathbb{R}_+^2 =]0, \infty[^2$. Because F is strictly increasing, M defined by (29) is strictly monotonic, and because of (30) it is given by $M(Z) = Z^\beta$. Thus,

$$F(X) = 1 - (1 - \alpha X)^\beta \qquad (X \in [0,1[). \tag{31}$$

Because F is strictly increasing, it follows that $\alpha\beta > 0$. Substitution shows that (31) always satisfies (2) with (28).

(ii) In the remaining case $A(Y) \equiv 1$ on $[0, 1[$ and so, by (24) and (25), we have in this case $S = R$. With

$$G(X) = 1 - F(x), \tag{32}$$

(26) becomes

$$G(X + Y) = G(X)G(Y) \qquad (X, Y \in [0,1[).$$

In view of (32), G maps $[0, 1[$ onto $]0, 1]$ and is strictly decreasing so (see Aczél [1, pp. 68, 82]) $G(X) = e^{\gamma X}$, $\gamma < 0$. So by (32),

$$F(X) = 1 - e^{\gamma X} \qquad (X \in [0,1[), \tag{33}$$

which satisfies (2) with $S = R$ for any $\gamma < 0$. We have therefore proved the following:

THEOREM 2. *The general strictly increasing surjections* F: $[0,1[\to [0,1[$ *satisfying* (2) *are given by* (31) *and* (33) *where* $\alpha\beta > 0$ *and* $\gamma < 0$ *but otherwise* α, β *and* γ *are arbitrary constants*. *In* (2), $X, Y \in [0,1[$, $R \in [0, X]$ *are independent variables but* (2) *has strictly monotonic solutions only if* $S = R(1 - \alpha Y)$ *or* $S = R$, *respectively*.

5. SOLUTIONS TO EQ. (5)

The last and most difficult task is to find all strictly increasing functions F mapping $[0,1[$ onto $[0,1[$ for which there exists a function P: $[0,1[\times [0,1] \to [0,1]$ such that

$$\begin{aligned} &F^{-1}[F(X) + F(Y) - F(X)F(Y)]Z \\ &\quad = F^{-1}[F(XZ) + F[YP(X,Z)] - F(XZ)F[YP(X,Z)]], \end{aligned} \tag{5}$$

where $X, Y \in [0,1[$, $Z \in [0,1]$, and $F(0) = 0$. Actually, we only know how to determine the *differentiable* solutions. In order to simplify (5), we introduce the notations (of. (32), (22))

$$G(X) = 1 - F(X), \qquad H(X,Y) = G^{-1}[G(X)G(Y)] \ (X,Y \in [0,1[). \tag{34}$$

Because F is strictly increasing and maps $[0,1[$ onto $[0,1[$, G is strictly decreasing and maps $[0,1[$ onto $]0,1]$, and H: $[0,1[^2 \to [0,1[$ is onto and strictly increasing in each variable. Thus,

$$G(0) = 1 \qquad \text{and} \qquad \lim_{X \to 1-} G(X) = 0, \tag{35}$$

and

$$H(0,T) = H(T,0) = T \qquad \text{and} \qquad \lim_{S \to 1-} H(S,T) = \lim_{T \to 1-} H(S,T) = 1. \tag{36}$$

Because G is monotonic and maps an interval onto an interval, it is continuous, and by (34), H is also continuous on $[0,1[^2$.

Using (34), we may rewrite (5) as

$$H(X,Y)Z = H[XZ, YP(X,Z)] \qquad (X,Y \in [0,1[, Z \in [0,1]), \tag{37}$$

Repeated application of (37) yields

$$\begin{aligned} H[XZW, YP(X,ZW)] &= H(X,Y)ZW \\ &= H[XZ, YP(X,Z)]W \\ &= H[XZW, YP(X,Z)P(XZ,W)]. \end{aligned}$$

Because H is strictly increasing in its second variable, this implies

$$P(X,ZW) = P(X,Z)P(XZ,W) \qquad (X \in [0,1[, Z,W \in [0,1]). \tag{38}$$

If we could substitute $X = 1$ in (38), we would get

$$P(Z,W) = \tilde{g}(ZW)/\tilde{g}(Z),$$

where $\tilde{g}(Z) = P(1,Z)$, if $\tilde{g}(Z)$ is nowhere 0. But we cannot make that substitution, and so, as before, we work with an extension of (38); however, doing so here is far less simple than it was with the previous equations.

We first present a specific solution (which will turn out to be quite close to the general solution) of (5) or, what is the same, of (34) and (37), which is designed to suggest the caution needed to arrive at the general solution:

$$F(X) = X, \qquad G(X) = 1 - X, \qquad H(X,Y) = X + Y - XY,$$

$$P(X,Z) = Z\frac{1-X}{1-XZ}. \tag{39}$$

On the one hand, with $\tilde{g}(Z) = Z/(1 - Z)$ we indeed have $P(Z, W) = \tilde{g}(ZW)/\tilde{g}(Z)$. On the other hand, this $\tilde{g}\ (Z)$ is definitely not equal to $P(1, Z) = 0$. So, we must carry out the extension very carefully.

First, we dispose of the possibility that $P(X, Z) = 0$ for some X, Z. By (37) and (36), this would yield

$$H(X,Y)Z = H[XZ, YP(X,Z)] = H(XZ, 0) = XZ.$$

So, if $Z \neq 0$, then $H(X, Y) = X$ for all Y, which is impossible because H is strictly monotonic in Y. If $Z = 0$, then by (36) and (37) one has

$$0 = H[0, YP(X,0)] = YP(X,0),$$

and so $P(X, 0) = 0$. To summarize, for any $X \in [0, 1[$,

$$P(X,Z) = 0 \qquad \text{if and only if } Z = 0. \tag{40}$$

To avoid dividing by 0, we exclude for the time being $Z = 0$ (and also $W = 0$). [Note that in (39) we have $P(1, Z) = 0$; however, $X = 1$ is excluded from the domain.]

Because we cannot substitute $X = 1$ into (38), we choose $X = C < 1$,

$$P(C, ZW) = P(C,Z)P(CZ,W) \qquad (Z, W \in]0,1]).$$

With $T = CZ \leq C$ and letting

$$g_C(T) = \frac{1}{P(C, T/C)} \qquad (T \in]0, C]), \tag{41}$$

[$T = 0$ is excluded because of (40)], we get

$$P(T,W) = \frac{g_C(T)}{g_C(TW)} \qquad (T \in]0,C],\ W \in]0,1]). \tag{42}$$

This formula is similar to the one derived earlier from the incorrect substitution $X = 1$ (with $\tilde{g} = 1/g_C$) but only for $T \in]0, C]$, $C < 1$. As (39) and (41) show, $C \to 1-$ will do no good directly because, for (39),

$$\lim_{C \to 1-} g_C(T) = \lim_{C \to 1-} \left(\frac{1}{P(C, T/C)} \right) = \lim_{C \to 1-} \left(\frac{C(1-T)}{T(1-C)} \right) = \infty.$$

Rather than this approach, we extend g_C as follows. First, we check to what extent g_C is determined in (42) by P. If

$$P(T,W) = \frac{g_C(T)}{g_C(TW)} = \frac{\bar{g}_C(T)}{\bar{g}_C(TW)},$$

then for some constant γ

$$\frac{\bar{g}_C(T)}{g_C(T)} = \frac{\bar{g}_C(TW)}{g_C(TW)} = \gamma,$$

that is, $\bar{g}_C(T) = \gamma g_C(T)$. So g_C is determined up to a multiplicative constant. We pick one of these g_C and denote it by $g_{]0,C]}$ (because it is defined on $]0,C]$). Now take a $C' \in]C,1[$. Again,

$$P(T,W) = \frac{g_{C'}(T)}{g_{C'}(TW)} \qquad (T \in]0,C'], W \in]0,1]). \tag{43}$$

Now, let us use our freedom to attach a multiplicative constant to $g_{C'}$, choosing it so that the new $g_{]0,C']}$ coincides with $g_{]0,C]}$ at C:

$$g_{]0,C']}(C) = g_{]0,C]}(C) \qquad \left[\text{i.e., } \gamma = g_{]0,C]}(C)/g_{C'}(C)\right].$$

By (43), this $g_{]0,C']} = \gamma g_{C'}$ also satisfies

$$P(T,W) = \frac{g_{]0,C']}(T)}{g_{]0,C']}(TW)} \qquad (T \in]0,C'], W \in]0,1]). \tag{44}$$

In particular, (44) holds for all $T \in]0,C] \subset]0,C']$. However, g_C in (42) is determined up to a multiplicative constant, and so comparing (42) and (44) and using the fact that $g_{]0,C']}$ and $g_{]0,C]}$ are equal at C imply

$$g_{]0,C']}(T) = g_{]0,C]}(T) \qquad \text{on }]0,C].$$

So $g_{]0,C]}$ is a restriction of $g_{]0,C']}$ to $]0,C]$. Continuing this process, we get a function g, defined on all of $]0,1[$, whose restriction to $]0,C]$, $]0,C'], \ldots$ are $g_{]0,C]}$, $g_{]0,C']}, \ldots,$ respectively, and for which

$$P(T,W) = \frac{g(T)}{g(TW)} \quad (T \in]0,1[, W \in]0,1]), \quad P(T,0) = 0. \tag{45}$$

[The last equation follows from (40).] By (41), $g_{]0,C]} > 0$ and all subsequent multipliers γ are > 0, so $g > 0$.

By assumption, $P(X,Z) \in [0,1]$ and from (40) we know that $P(X,Z) = 0$ exactly when $Z = 0$. Observe from (45) that $P(X,1) = 1$ for $X \in]0,1[$. From (37), with $X = 0$, $Z = 1$, and using (36), we have $Y = H(0,Y) = H[0, YP(0,1)] = YP(0,1)$, and so $P(0,1) = 1$. Thus,

$$P(X,1) = 1 \qquad \text{for } X \in [0,1[.$$

Conversely, let $P(X,Z)=1$, then from (37), $H(X,Y)Z=H(XZ,Y)$. Let $Y\to 1-$ in this and take (36) into account, to get

$$P(X,Z)=1 \qquad \text{if and only if } Z=1 \quad (X\in[0,1[,\ Z\in[0,1]). \tag{46}$$

So (45) and (46) imply that g is strictly decreasing on $]0,1[$, and so $P(X,Z)$ is strictly increasing in Z.

To gain more information about P and g, let $X\to 1-$ in (37) and consider (36)

$$\begin{aligned} Z = \lim_{X\to 1-} H(X,Y)Z &= \lim_{X\to 1-} H[XZ, YP(X,Z)] \\ &= H\Big[Z, Y\lim_{X\to 1-} P(X,Z)\Big]. \end{aligned}$$

(Because H is continuous and increasing in each variable, the limits exist.) Observe that the left hand term does not depend on Y, so we must have

$$\lim_{X\to 1-} P(X,Z)=0. \tag{47}$$

Now, from (45)

$$g(X)=P(X,Z)g(XZ),$$

and if we let $X\to 1-$, we see that

$$\lim_{X\to 1-} g(X)=0. \tag{48}$$

Next, let $X\to 0+$ in (37). Of course, $H(0,Y)$ is defined and, by (36), $H(0,Y)=Y$. But we want to apply (45) subsequently and it does not hold for $T=0$ (for if it did, then we would have $P(0,Z)=1$, contrary to (46)). However,

$$\lim_{X\to 0+} H(X,Y)=H(0,Y)=Y,$$

and so, from (36) and (37),

$$YZ=\lim_{X\to 0+} H(X,Y)Z=\lim_{X\to 0+} H[XZ,YP(X,Z)]=Y\lim_{X\to 0+} P(X,Z).$$

Therefore,

$$\lim_{X\to 0+} P(X,Z)=Z.$$

The same proof with $X=0$ in place of $X\to 0+$ shows that $P(0,Z)=Z$. So

$$\lim_{X\to 0+} P(X,Z)=Z=P(0,Z) \qquad (Z\in[0,1[), \tag{49}$$

and P as a function of X is continuous at 0. Using (45), the first equality of (49) can hold only if

$$\lim_{X \to 0+} g(X) = \infty$$

(because g is strictly decreasing and positive, its limit at 0 cannot be 0).

We know that G and H are continuous and G is nowhere 0 on $[0, 1[$, so from (34) and (37) we see

$$P(X, Z) = \frac{1}{Y_0} G^{-1}\left(\frac{G[H(X, Y_0)Z]}{G(XZ)} \right) \tag{50}$$

($Y_0 \in]0, 1[$ an arbitrary constant) is also continuous and so [cf. (41)] is g on $]0, 1[$. We summarize: the continuous and strictly decreasing function g maps $]0, 1[$ onto $]0, \infty[$.

We see from (34) that H is symmetric so, from (34) and (37)

$$G(XZ)G[YP(X, Z)] = G(YZ)G[XP(Y, Z)]$$
$$(X, Y \in [0, 1[, Z \in [0, 1]). \tag{51}$$

At this point we begin differentiating. We will *suppose that* F *and* F^{-1} (and thus G and G^{-1}) *are differentiable on* $]0, 1[$. For the latter *we need that* $F' \neq 0$ (and thus $G' \neq 0$) on $]0, 1[$, where F' is the derivative of F. This condition does not follow from the differentiability and strict monotonicity of F or G [for instance, $F_0(X) = [(2X - 1)^3 + 1]/2$ is strictly increasing on $]0, 1[$, $F_0(0) = 0$, $F_0(1) = 1$, but $F_0'(1/2) = 0$]. But *if* F *is increasing and* $F' \neq 0$ *on* $]0, 1[$, *then* $F' > 0$ *on* $]0, 1[$. Conversely, *if* F' *exists and is positive on* $]0, 1[$, *then* F *is strictly increasing there*. Moreover, it then follows from (34) and (50) that H *is differentiable in either variable on* $]0, 1[$, whereas P *is differentiable in its first and second variable on* $]0, 1[$ or on $]0, 1]$, respectively. [Note that in (50) all expressions substituted into G stay in $]0, 1[$, even for $Z = 1$.] Although we wrote, as usual, G' for the derivative of G, we use $P_1'(\cdot, \cdot)$, $P_2'(\cdot, \cdot)$ for the derivatives of $P(\cdot, \cdot)$ with respect to the first and second variable, respectively. (The usual partial derivative notation turns out to be misleading here.)

First, differentiate (51) with respect to Z:

$$G'(XZ)XG[YP(X, Z)] + G(XZ)G'[YP(X, Z)]YP_2'(X, Z)$$
$$= G'(YZ)YG[XP(Y, Z)] + G(YZ)G'[XP(Y, Z)]XP_2'(Y, Z).$$

Substituting $Z = 1$ and recalling from (46) that $P(X, 1) = 1$, we get

$$G'(X)XG(Y) + G(X)G'(Y)YP_2'(X, 1)$$
$$= G'(Y)YG(X) + G(Y)G'(X)XP_2'(Y, 1), \tag{52}$$

for all $X, Y \in [0, 1[$. By (45) and keeping in mind that g is nowhere 0 on $]0, 1[$,

$$P_2'(X, Z) = -\frac{g(X)g'(XZ)X}{g(XZ)^2},$$

so

$$P_2'(X, 1) = -\frac{g'(X)X}{g(X)} \qquad (X \in]0, 1[).$$

Thus, because the derivative P_2' exists at $(X, 1)$, so does g'. Substituting into (52)

$$XG'(X)G(Y)\left(1 + \frac{g'(Y)}{g(Y)}Y\right) = YG'(Y)G(X)\left(1 + \frac{g'(X)}{g(X)}X\right)$$
$$(X, Y \in]0, 1[). \quad (53)$$

We know that G is nowhere 0 on $[0, 1[$. If $(1/Y) + (g'(Y)/g(Y)) = 0$ for all $Y \in]0, 1[$, then $g(Y) = \gamma/Y$, which contradicts (48). So

$$\text{for some } Y_0, \qquad \frac{1}{Y_0} + \frac{g'(Y_0)}{g(Y_0)} \neq 0, \quad (54)$$

whence from (53) we see that $G'(X)/G(X) = c[(1/X) + (g'(X)/g(X))] \neq 0$. Because G and g are everywhere positive, $\ln G(X) = c[\ln X + \ln g(X)] + \ln d'$, whence

$$G(X) = d'[Xg(X)]^c \qquad (X \in]0, 1[), \quad (55)$$

where c and d' are real constants with $c \neq 0$ and $d' > 0$.

Now let $X \to 1-$ in (51). Because of (35) and (47), we obtain

$$G(Z) = G(YZ)G[P(Y, Z)] \qquad (Y \in [0, 1[,\ Z \in [0, 1]).$$

By (55), and setting $(d')^{1/c} = d$,

$$dZg(Z) = d^2 YZg(YZ)P(Y, Z)g[P(Y, Z)],$$

which by (45) yields

$$g(Z) = dYg(Y)g[P(Y, Z)] \qquad (Y \in]0, 1[,\ Z \in]0, 1]).$$

Take the logarithms on both sides and differentiate with respect to Y or Z to get

$$0 = \frac{1}{Y} + \frac{g'(Y)}{g(Y)} + \frac{g'[P(Y, Z)]P_1'(Y, Z)}{g[P(Y, Z)]},$$

and

$$\frac{g'(Z)}{g(Z)} = \frac{g'[P(Y,Z)]P'_2(Y,Z)}{g[P(Y,Z)]},$$

respectively. Multiply the first of these two equations by $-P'_2(Y,Z)$, the second by $P'_1(Y,Z)$, and add them:

$$P'_1(Y,Z)\frac{g'(Z)}{g(Z)} + P'_2(Y,Z)\left(\frac{1}{Y} + \frac{g'(Y)}{g(Y)}\right) = 0. \qquad (56)$$

From (45)

$$P'_1(Y,Z) = \frac{g'(Y)g(YZ) - g(Y)g'(YZ)Z}{g(YZ)^2},$$

$$P'_2(Y,Z) = -\frac{g(Y)g'(YZ)Y}{g(YZ)^2},$$

so Eq. (56) becomes

$$g'(Y)g(YZ)g'(Z) - g(Y)g'(YZ)g'(Z)Z$$
$$= g(Z)[g(Y)g'(YZ) + g'(YZ)g'(Y)Y],$$

i.e.,

$$g(Y)g'(YZ)[g(Z) + g'(Z)Z]$$
$$= g'(Y)[g(YZ)g'(Z) - g'(YZ)g(Z)Y]. \qquad (57)$$

We must now exclude the possibility that g' is anywhere 0 on $]0,1[$. (As mentioned earlier in connection with F, the fact that g is strictly monotonic on $]0,1[$ is not sufficient for this.) Suppose, on the contrary, there is some $Y_1 \in]0,1[$ with $g'(Y_1) = 0$, then because $g(Y_1) > 0$ we see from (57) that

$$g'(Y_1 Z)[g(Z) + g'(Z)Z] = 0. \qquad (58)$$

Observe that if the second factor of (58) is nowhere 0, then $g'(Y_1 Z) = 0$ for $Z \in]0,1[$, i.e., $g' = 0$ on $]0, Y_1[$. But this *does* contradict the strict monotonicity of g, thus proving that $g'(Y) \neq 0$ for $Y \in]0,1[$.

So to complete the proof that g' is nowhere 0, it remains only to show that the second factor is nowhere 0. Suppose, on the contrary, there exists Z_0 with $g(Z_0) + g'(Z_0)Z_0 = 0$. Then if we set $X = Z_0$ in (53) we get

$$G'(Z_0)\left[1 + \frac{g'(Y)}{g(Y)}Y\right] = 0.$$

Because we assumed that $G' \neq 0$ on $]0,1]$, so $1 + (g'(Y)/g(Y))Y = 0$ for all $Y \in]0,1[$, which was excluded in (54) above. Thus, g' is not zero on $]0,1[$.

So we may divide (57) by $g'(YZ)g'(Y)g'(Z)$ to get

$$\frac{g(YZ)}{g'(YZ)} = \frac{g(Y)g(Z)}{g'(Y)g'(Z)} + Y\frac{g(Z)}{g'(Z)} + Z\frac{g(Y)}{g'(Y)}.$$

With the notation

$$h(Z) = \frac{g(Z)}{g'(Z)} \neq 0, \tag{59}$$

this becomes

$$h(YZ) = h(Y)h(Z) + Yh(Z) + Zh(Y) \qquad (Y, Z \in]0,1[).$$

Adding YZ to both sides we see that $h(Z) + Z$ is multiplicative on $]0,1[^2$ (see Section 3). Also, because g is positive and decreasing and $g' \neq 0$ on $]0,1[$, we have (compare the similar remark above concerning F) $g'(Z) < 0$, so $h(Z) < 0$ and $h(Z) + Z < Z < 1$ on $]0,1[$. The multiplicative $h(Z) + Z$ is thus bounded from above on $]0,1[$, so (see Aczél [1, pp. 13, 19–20, 27–28]) either $h(Z) + Z = 0$ on $]0,1[$, which in view of (59) would lead to the case excluded by (54), or

$$h(Z) + Z = Z^{\beta} \qquad (Z \in]0,1[),$$

where β is a constant. Clearly, $\beta = 1$ would give $h(Z) = 0$ which, by (59), is impossible. So $\beta \neq 1$. For convenience we write $\beta = 1 + b$, so

$$\frac{g(Z)}{g'(Z)} = h(Z) = Z^{1+b} - Z \qquad (Z \in]0,1[),$$

where $b < 0$ because $h(Z) < 0$. Equivalently,

$$\frac{g'(Z)}{g(Z)} = \frac{1}{Z(Z^b - 1)} = -\frac{1}{Z} - \frac{Z^{b-1}}{1 - Z^b} \qquad (Z \in]0,1[).$$

Because all denominators are positive, taking antiderivatives yields

$$\ln[g(Z)] = -\ln Z + \frac{1}{b}\ln(1 - Z^b) + \ln \delta,$$

with $\delta > 0$ constant, i.e.,

$$g(Z) = \delta\frac{(1 - Z^b)^{1/b}}{Z} \qquad (Z \in]0,1[). \tag{60}$$

[We are grateful to Nicole Brillouët-Belluot for this proof of (60) which is more direct than our original version.] In view of (45) and (60), which covers the case of $X \in]0,1[$, and (49), which treats $X = 0$, we have

$$P(X,Z) = Z\frac{(1-X^b)^{1/b}}{(1-X^bZ^b)^{1/b}} \qquad (X \in [0,1[, Z \in [0,1]). \qquad (61)$$

From (55) and (60)

$$G(X) = d'\delta^c(1-X^b)^{c/b} = \alpha(1-X^b)^a \qquad (X \in]0,1[),$$

where $\alpha = d'\delta^c$ and $a = c/b$. We know, however, that

$$1 = G(0) = \lim_{X\to 0+} G(X) = \alpha,$$

so $\alpha = 1$; therefore

$$G(X) = (1-X^b)^a \qquad (X \in]0,1[),$$

and so

$$G^{-1}(T) = (1-T^{1/a})^{1/b},$$

where $b > 0$ and, because G is strictly decreasing, $a > 0$. Substituting in (34) we get

$$F(X) = 1-(1-X^b)^a \qquad (X \in]0,1[), \qquad (62)$$

$$H(X,Y) = (X^b + Y^b - X^bY^b)^{1/b} \qquad (X,Y \in [0,1[). \qquad (63)$$

Substitution shows that (61), (62), and (63) satisfy (5), (34), and (37), so the third problem is solved under the above conditions, including differentiability. We have proved the following.

THEOREM 3. *Suppose the function F maps* $[0,1[$ *onto* $[0,1[$, *is strictly increasing, and both F and F^{-1} are differentiable on* $]0,1[$. *Then F satisfies* (5) *if and only if* (61) *and* (62) *hold, where a and b are arbitrary positive constants.*

Of course, (39) is the case of $b = 1$ in (61), (62), and (63).

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