COMMENTS BY PATRICK SUPPES

Luce and Narens give a substantial list of problems that are central to the representational theory of measurement, a topic on which Duncan and I have worked together with other colleagues for more than a quarter of a century. Duncan and Louis bring out nicely the range of philosophical and scientific problems still to be faced in the theory of measurement. Unfortunately, most of these problems as well as the ones that have been solved in the past have not attracted the interest of philosophers of science in the way one might have thought they would. It has turned out that in spite of the philosophical roots of much of the work in the theory of measurement, the current developments have mainly been due to scientists and mathematicians.

The problems of measurement in the behavioral and social sciences present foundational and conceptual issues of considerable subtlety which now have a large literature, especially those surrounding the measurement of subjective probability and utility. Without question the problems formulated by Luce and Narens all deserve attention, although of course some are of more general interest than others. It is a reflection of the general nature of the theory of measurement that my own list of problems would overlap but still be rather different from that presented by them. Saying something about my own list is not meant to be a criticism of theirs, but is a way of emphasizing the range of philosophical issues still open in the theory of measurement. The two large topics that I would organize problems around and that are not directly mentioned by Luce and Narens are geometrical problems of measurement – what are also called multidimensional scaling problems – and secondly, computation problems.

Geometrical Problems. As Luce and Narens point out in various places briefly and casually, there is a long history of representation and uniqueness problems in geometry. There is a substantial review of classical work in *Foundations of Measurement, Vol. II* but there are ways in which the modern representational theory of measurement calls for new developments in geometrical representation theory. The most important direction, in my own judgment, is to develop representation theories like those of measurement that are embeddings and not isomorphisms to standard analytic representations. Typical examples would be sufficient, and where possible, necessary and sufficient conditions to embed a bounded fragment of Euclidean geometry in a Euclidean space of the same dimension and with the usual results on uniqueness for that embedding. Another kind of example of interest in current physics is a qualitative axiomatization of a discrete lattice of points for special relativity to provide a qualitative geometrical framework for lattice computations and other discrete geometrical reasoning characteristic of much modern physics.

In a similar vein, but with undoubtedly a somewhat different conceptual apparatus, we could much benefit from deeper qualitative analysis of the geometrical structures that arise in multidimensional scaling.

The close relation between topology and measurement in many contexts is perhaps the part of geometry most neglected in the *Foundations of Measurement*, but deliberately so for reasons given in Volume I. Topology provides a general set of concepts for formulating axioms on qualitative structures of a different sort than have been mainly exploited in the theory of measurement thus far, although there has been a certain body of work in economics using topological conditions rather than algebraic ones in the theory of utility. On the other hand, problems of the relation between topology and measurement in the theory of perception have as yet been little explored.

Some of the most interesting recent work of Luce and Narens has been on characterizing the automorphism group of a suitable measurement structure as an Archimedean ordered group. There is a similar but deeper and more far-ranging set of problems in geometry on the connection between various transformation groups for geometry and the characterization of these transformation groups as themselves being instances of manifolds. For example, the group of rotations of Euclidean three-space is a nonsingular surface with a system of local coordinates provided by familiar Euler angles. Exploitation of this kind of relationship has scarcely begun as yet in the theory of measurement.

Computation and the Continuum. Another important aspect of recent work by Duncan and Louis in the theory of measurement is that of scale type. Here the work has depended almost exclusively on assuming the measurement structure is isomorphic to the continuum of real numbers. These results naturally raise philosophical questions that call for a deeper analysis of why the continuum is important in the theory of measurement. Initially we think of measurement as a very finitistic constructive procedure used in almost all domains of science to assign numerical quantities to various empirical properties. It would be surprising if highly nonconstructive aspects of the entire continuum of real numbers really did play some essential philosophical role in our conception of measurement. In saying this, of course, I am expressing philosophical disagreement with Luce and Narens, a disagreement that we have discussed on various occasions.

As I have stated in comments on other articles, I look upon classical analysis and continuum mathematics as being mainly computationally important. The differential and integral calculus is just that, a calculus, not a foundational view of how the universe is really put together. I as much as Luce and Narens, perhaps even more so, cherish infinitesimals and what they can do for providing efficient methods of computation, especially in physics and engineering. I do not for a moment necessarily believe that infinitesimals are really out there in the real world. I would be quite prepared to accept the fact that space is ultimately discrete and we cannot go below a certain smallest measurement of length or of other quantities. This would not for a moment shake my confidence in the importance of infinitesimal methods in science which have been used so successfully for over two centuries, but it is rather to insist that they provide wonderfully efficient methods of computation, not a fundamental view of the world. I like very much the derivation of standard diffusion equations from taking the limit of very discrete random walks. I am quite happy to look upon the limit operation as an ideal one abstracted from the real detail of particles and the spaces in which they operate.

It is this kind of philosophical view of mine that has led me to be a much stronger supporter of finite structures and finitistic or constructive measurement procedures than are Duncan and Louis. That debate will continue and probably no end is in sight. That they remain unconstructed continuum advocates is clear from their statement of problems at the beginning of their paper. But I found puzzling their statement in Section 3.4 that we could have "doubts about the heavy use of infinite structures and continuum mathematics when, after all, the universe is, according to contemporary physics, composed of only finitely many particles." This is not the real problem of using the continuum. We could very well require the continuum if there were only three particles but they were moving along continuous paths. The real commitment is to there being only discrete space and discrete properties or, if you wish, only a finite number of spatial points, at least in any bounded region, and a finite number of values of any property of any particle. This kind of constructivism is clearly very far removed from a large number of their Problems.

In raising and pursuing once again this dialogue with them, I am not suggesting that I have any strong commitment that the way I am suggesting is the only way to proceed. In fact, it may well turn out that the line of attack they have taken will be more fruitful than a more constructive finitistic approach. Actually, from a philosophical standpoint I now favor a view that has as yet not been developed very far technically. This is the view that in the framework of current physical theory we cannot empirically determine whether space is continuous or discrete, and possibly a decision that was empirically supported could only be for discreteness. In any case, given current physics, the choice is transcendental, i.e., beyond experience. Consequently it is of interest to develop theories whose fundamental concepts are invariant under appropriate mappings from the infinite to the finite.

As a simple example, let (X, d) and (Y, d') be two metric spaces with X being a bounded infinite set and Y a finite set. Then (X, d) is ϵ-homomorphic to (Y, d') if and only if there is a mapping f from X to Y such that $\forall x, y$ in X

$$|x - y| < \epsilon \quad \text{iff} \quad |f(x) - f(y)| < \epsilon.$$

More to the point, rather than this simple abstract example, are the methods currently used to approximate continuous processes by the discrete ones used in computer simulations, or the classical approximation of the discrete by the continuous, as in approximating the binomial by the normal distribution. Studies of invariance and meaningfulness in these contexts would in all likelihood be conceptually enlightening not only in terms of theories of measurement.