

FIFTEEN PROBLEMS CONCERNING THE  
REPRESENTATIONAL THEORY OF MEASUREMENT

1. INTRODUCTION

The type of theory of measurement we discuss is the representational one that is treated in great detail in the three volume treatise *Foundations of Measurement*.<sup>1</sup> We attempt to make the present treatment self-contained, including definitions of major concepts, but these volumes may be consulted for greater detail.

Our aim is to discuss two classes of, probably difficult, issues that need clarification. The first type, covered in Section 3, has to do with principled arguments to justify the representational perspective and possible modifications of it to deal with error and with limiting properties of passing from finite, through denumerable, to continuum representations. These issues are, at least partially, philosophical in character. The second type, covered in Sections 5 and 6, is more technical and focuses primarily on the issues of relating the useful, but abstract, ideas (to be defined below) of scale type, Archimedeaness, and Dedekind completeness to observable properties of qualitative structures.

Many of the problems we describe are primarily conceptual and their resolution probably requires something of an intellectual break-through. Once that is achieved, however, some may well prove to be technically easy to solve. Others that we cite are well defined within existing mathematical ideas, and some of these appear to be difficult.

2. THE REPRESENTATIONAL THEORY OF MEASUREMENT

2.1. *Historical Background*

Mathematicians and scientists, among them Helmholtz (1887), Hölder (1901) and von Neumann (in von Neumann and Morgenstern, 1947),

clearly had the idea of axiomatizing ordered qualitative structures as possible models of empirical attributes and in the latter two cases of establishing, as mathematical theorems, the existence and uniqueness of numerical representations. The power and generality of this approach, however, was not widely appreciated by empirical scientists or philosophers of science during the first half of the 20th century, as evidenced by the discussions of measurement in, for example, Bridgman (1922, 1931), Campbell (1920, 1928), Cohen and Nagel (1934), and Ellis (1966). More than anyone else, Suppes brought to the attention of non-mathematicians this axiomatic style of studying the measurement of attributes. Particularly important were Scott and Suppes (1958), Suppes (1951), and Suppes and Zinnes (1963). The latter was especially influential among mathematical psychologists, and it was the forerunner of the *Foundations of Measurement*.

For over 30 years, representation and uniqueness theorems<sup>2</sup> were the mode of research. New ordered mathematical structures that appeared to have relevance to the measurement of certain attributes were isolated, and two theorems were established: the existence of a representation into or onto some prescribed numerical structure, and the uniqueness of that representation in the sense of formulating the class of transformations relating equally good representations into or onto the same numerical structure. This has come to be called the Representational Theory of Measurement, abbreviated RTM.

Many key concepts and methods of RTM, as it is now formulated, had analogues in 19th century geometry, particularly in (i) the assignment of coordinate systems to qualitative geometries (e.g., Hilbert, 1899), (ii) the idea that invariants of structure preserving mappings had geometrical significance (Klein, 1893), and (iii) the idea that the geometric structure of space is the same at each point, i.e., that space is homogeneous (Helmholtz, 1868; Lie, 1886).

## 2.2. *The Underlying Principles*

The formal part of RTM can be reduced to five main ideas:

1. *A qualitative situation is specified by a (usually ordered) relational structure  $\mathcal{X}$  consisting of a domain  $X$ , finitely many relations on  $X$  and finitely many special elements of  $X$ .*

These relations, subsets, and elements are called the *primitives* of  $\mathcal{X}$ . Measurement axioms are then stated in terms of primitives of  $\mathcal{X}$ . These axioms are intended to be true statements about  $\mathcal{X}$  for some empirical identification and are intended to capture important empirical properties of  $\mathcal{X}$ , usually ones that prove useful in constructing measurements of its domain,  $X$ .

2. *The representational theory requires that the primitives of  $\mathcal{X}$  can be given an empirical identification.*

In particular, if  $R$  is an  $n$ -ary primitive relation on  $X$ , then it is required that the truth or falsity of  $R(x_1, \dots, x_n)$ , for any particular choice of the  $n$ -tuples  $x_i$ , be empirically decidable.

3. *As much as possible, measurement axioms, stated in terms of the empirically identified primitives, should be empirically testable.*

Next, a numerically-based, representing structure  $\mathcal{N}$  is selected, and for each structure  $\mathcal{X}$  that satisfies the measurement axioms, the set  $\mathcal{S}(\mathcal{X})$  of homomorphisms of  $\mathcal{X}$  into  $\mathcal{N}$  is considered.

4. *Measurement of  $\mathcal{X}$  is said to take place if and only if the following two theorems can be shown:*

- (i) *(Existence Theorem).  $\mathcal{S}(\mathcal{X})$  is non-empty for each  $\mathcal{X}$  that satisfies the measurement axioms.*
- (ii) *(Uniqueness Theorem). An explicit description is provided about how the elements of  $\mathcal{S}(\mathcal{X})$  relate to one another. In practice this description usually consists of specifying a group of functions  $G$  such that for each  $\phi$  in  $\mathcal{S}(\mathcal{X})$*

$$\mathcal{S}(\mathcal{X}) = \{g * \phi : g \text{ is in } G\},$$

where  $*$  denotes function composition.

A concept of meaningfulness similar to that proposed by Stevens (1946) is used for judging the empirical significance of quantitative statements: an  $n$ -ary relation  $S$  on the domain of the numerically based structure  $\mathcal{N}$  is said to be *meaningful* about  $\mathcal{X}$  if and only if for all  $x_1, \dots, x_n$  and all  $\phi$  and  $\psi$  in  $\mathcal{S}(\mathcal{X})$ ,

$$S[\phi(x_1), \dots, \phi(x_n)] \quad \text{iff} \quad S[\psi(x_1), \dots, \psi(x_n)].$$

Following Suppes and Zinnes (1963),

5. *RTM identifies empirical significance with meaningfulness.*

The concept of meaningfulness with respect to  $\mathcal{S}(\mathcal{X})$  is easily extended to numerical statements involving measurements of elements of the domain: meaningful statements are those whose truth value is unaffected by the particular representation in  $\mathcal{S}(\mathcal{X})$  used to measure  $\mathcal{X}$ .

It should be noted that the concepts of empirical significance and meaningfulness as embodied in RTM are independent of truth; both true and false statements can be meaningful and empirically significant.

Although the following two features are not part of the formal theory, they hold in many important measurement situations with infinite domains:

6. *Using the measurement axioms, sequences of equal spaced elements – standard sequences – are constructed.*

These standard sequences are then used to establish constructively the set of homomorphisms of  $\mathcal{X}$  into  $\mathcal{N}$ . To construct a specific one, one proceeds as follows: a first standard sequence  $s_1^1, \dots, s_i^1, \dots$  is chosen to produce the first level of approximation  $\phi^1$  of a homomorphism  $\phi$ . For each element  $x$  in  $X$ , find the  $n$  such that  $s_n^1 \lesssim x < s_{n+1}^1$ , where  $\lesssim$  denotes the primitive of  $\mathcal{X}$  that orders  $X$ . Set

$$\phi_1(x) = n/1.$$

A second level of approximation is carried out by choosing the sequence  $s_1^2, \dots, s_i^2, \dots$ , where for each positive integer  $i$ ,  $s_{2i}^2 = s_i^1$ . The second approximation to  $\phi$  is obtained by setting

$$\phi_2(x) = k/2,$$

where  $k$  is such that  $s_k^2 \lesssim x < s_{k+1}^2$ . Continuing in this way, a sequence of approximations  $\phi_m$  is constructed, and for many measurement axiom systems it can be shown that as  $m \rightarrow \infty$ , the limit of  $\phi_m$  exists and is the desired homomorphism  $\phi$  of  $\mathcal{X}$  into  $\mathcal{N}$ .

Beginning with a different standard sequence will usually yield a different homomorphism.

The accuracy of measurement, in practice, is controlled by how many terms in this approximation are used.

The ancient geometer Eudoxes used standard sequences in much the same way as in RTM. They also played a critical role in the approaches taken by Helmholtz (1887) and Hölder (1901). Later Luce and Tukey

(1964) used related kinds of standard sequences to obtain their representation and uniqueness results for additive conjoint measurement.<sup>3</sup> Krantz (1964) and, in a more useful way, Holman (1971) recognized that the standard sequence approach of Luce and Tukey closely related to the algebraic structure described by Hölder for physical measurement – an *Archimedean*,<sup>4</sup> *totally ordered group*. Basically, Krantz and Holman defined an operation on one component that captured completely the trade-off structure between the components. Under the axioms, the operation was shown to be associative and so the problem was reduced to an application of Hölder's theory.

Most of FM1 is devoted to recasting various measurement situations, often using the ideas inherent in Krantz and Holman's approach, so that Archimedean ordered groups (or large semigroup portions of them) come into play. It has been remarked, with some justice, that the theory of measurement is largely an application of Hölder's theorem, i.e., of standard sequences in the guise of Archimedean ordered groups. Moreover, as we shall see in Section 4.2, this continues to be the case even for non-additive structures.

*7. The representational theory offers an abstract theory of the kinds of well-behaved scales that one encounters in science.*

This theory is based on a concept of homogeneity that abstractly is much like the geometric one mentioned above and provides qualitative and empirical criteria for empirical structures to be homogeneous. (Issues involving homogeneity are discussed more fully in Sections 4 and 5.)

Until the 1980s, the major emphasis of RTM has been on producing existence and uniqueness theorems for various kinds of empirical situations. Scott and Suppes (1958) remarked:

A primary aim of measurement is to provide a means of convenient computation. Practical control or prediction of empirical phenomena requires that unified, widely applicable methods of analyzing the important relationships between the phenomena be developed. Imbedding the discovered relations in various numerical relational systems is the most important such unifying method that has yet been found (pp. 116–117).

### 3. PROBLEMS CONCERNING THE FORMULATION OF THE REPRESENTATION THEORY

Although most critics accept that RTM, via its axiomatizations of important empirical situations and the various representation and uniqueness theorems, is a major contribution to our understanding of measurement in science, they criticize it as a *theory* of measurement. The criticisms are of four major types: (1) questions about the definition of measurement in terms of scales of homomorphisms, (2) objections to the lack of a theory of error for RTM, (3) concerns about exactly what invariance concepts have to do with the concept of meaningfulness, and (4) doubts about the heavy use of infinite structures and continuum mathematics when, after all, the universe is, according to contemporary physics, composed of only finitely many particles. A number of these concerns are expressed by Kyburg (1984) and in papers in Savage and Ehrlich (1992).

#### 3.1. *Homomorphism Definition of Measurement*

The criticisms of the homomorphism definition of measurement are of two types: those arguing that the definition is too narrow in the sense that it does not account for certain kinds of measurement; and those arguing that it is too broad in that it permits too many kinds of measurement. The former usually contain elements of the following criticism of Adams (1966):

It seems to me that in characterizing measurement as the assignment of numbers to objects according to rule, the proponents of the representational theory have fastened on something which is undoubtedly of great importance in modern science, but which is not by any means an essential feature of measurement. What is important is that the real numbers provide a very sophisticated and convenient conceptual framework which can be employed in describing the results of making measurements: but, what can be conveniently described with numbers can be less conveniently described in other ways, and these alternative descriptions no less 'give the measure' of a thing than do the numerical descriptions. . . . Note, too, that the ancient Greeks did not have our concepts of rational, much less real numbers, yet it seems absurd to say that they could not measure because they did not assign numbers to objects. In sum, I would say that the employment of numbers in describing the results of measurement is not essentially different from their employment in other numerical descriptions, and that this employment is neither a necessary nor a sufficient condition for making or describing measurements (pp. 129–130).

Even granting the assumption that measurement necessarily involves assigning numbers, it seems to me to be far from true that in making these assignments it is always the

case that mathematical operations and relations are made to correspond to or represent empirical relations and operations. . . . The situation is worse with most of the widely used measures in the behavioral sciences, like I.Q.s and aptitude test scores. It may be claimed, of course, that these are not really measurements at all, but to justify this [claim], some argument would have to be given, unless the theory of representational measurement is not to degenerate into a mere definition (I.Q.s are not measurements because they do not establish numerical representations of empirical operations and relations) (p. 130).

Niederée (1987, 1992) and others criticize RTM as being too broad. He notes that requiring a structure to be numerically representable into structures based on the real numbers places little restriction on the structure beyond cardinality and even that has not been justified. This kind of restriction is too liberal because

. . . it does not involve any concept of measurement whatsoever. Indeed measurement theorists would hardly be prepared to accept [it] as a *sufficient* criterion for a structure to be called representable in terms of fundamental measurement. What seems to be lacking here is an analysis of what it should mean that a 'number' expresses an ideal value of measurement (Niederée, 1992, p. 245).

The above criticisms of RTM suggest the following problem:

**PROBLEM 1.** *Justify in a philosophically principled fashion RTM (or a large part of it) as a general theory of measurement without severely restricting its positive uses.*<sup>5</sup>

A few comments are in order. First, theories of ability measurement alluded to by Adams, although based on quantitative data, are not in principle ruled out by RTM. Empirical structures can be based on numerical information just as long as those numbers arise from empirical means – which is the case for ability measurement. What is lacking from the RTM perspective are axiomatic theories for the various kinds of ability measurement. We do not see any in principle impediments to the development of such axiomatic theories, although in practice none has been devised and it does not appear to be easy to do so.

Second, either directly or indirectly standard sequences are used to establish scales in almost all<sup>6</sup> of the major results of FM. Thus, since the process of measurement through standard sequences is usually taken as paradigmatic of 'measurement processes', almost all of the representational results of FM are valid not only from the RTM viewpoint but from a number of different perspectives about what measurement is.

Third, some attempts at solving Problem 1 already exist, including Michell (1990) and Niederée (1987, 1992). The present authors are not yet persuaded that their attempts are useful beyond the additive cases. In particular, as we shall see in Sections 4 and 5, the greatest progress to date in understanding non-additive structures involves mapping certain general types of structures into a particular subgroup of their automorphisms, showing that this subgroup is Archimedean ordered, and then using Hölder's theorem to map these into the multiplicative positive real numbers. Because we do not know, in general, direct structural descriptions of individual automorphisms, it is not obvious that the procedure proposed by Niederée for constructing a suitable representation will capture the known results. Considerably more work is needed to establish how these approaches lead to these general measurement representations.

Fourth, much of FM is concerned with issues that are not limited to measurement per se, but are closely related to measurement. The most prominent of these is the testing of scientific theories through the use of RTM. An example of this is subjective expected utility (SEU) theory, which holds that a subject's preference ordering over lotteries is explainable as if the person is trying to maximize a numerical SEU, which is computed in the obvious way using a simultaneously constructed subjective probability over the family of uncertain events and a utility over the set of consequences. One way to test this scientific theory is to formulate it as a quantitative model, gather data, and test the model using standard statistical methods to determine the degree to which the model accounts for most of the variance in the data. This approach often involves approximate construction of standard sequences. A different way is to formulate the scientific theory as a qualitative, axiomatic measurement model that through RTM is equivalent to the quantitative SEU, and then test the axioms' qualitative theory through direct empirical observations of samples of the stimuli.

Three additional examples of theories that are closely related to measurement issues are the axioms for distributive triples, which can be used to construct the space of physical quantities (see Section 6.1); a variety of psychophysical models involving more than one sensory attribute (one of which is described in Section 6.2); and a theory of certainty equivalents to uncertain monetary gambles (Section 6.3).



### 3.2. *Error in Measurement Theory*

The second class of criticisms concern error and how it should be treated *vis-à-vis* measurement models. While ‘error’ considerations for measurement give rise to many different kinds of problems, we cite three treatments of error that are intimately connected with key concepts of RTM. The first is based on the idea that probabilities underlie the observations that are made. Although some progress has been made on this general approach (Falmagne, 1979, 1980; Falmagne and Iverson, 1979; Iverson and Falmagne, 1985; Michell, 1986), little effective use has yet been made of it to test measurement axiomatizations in detail.

**PROBLEM 2.** *Specify a probabilistic version of measurement theory and the related statistical methods for evaluating whether or not a data set supports or refutes specific measurement axioms.*

One approach, taken by a number of people, is to assume the data base consists of probabilities of binary or more complex choices, with the major question being the conditions on these probabilities that correspond to the existence of an underlying random variable representation in which the largest value observed determines the choice. A survey of such work is found in Chapter 17 of FM2. One of the most interesting recent contributions is Heyer and Niederée (1992) who study when a probabilistic structure, e.g., of choices, can be considered to arise from a probability distribution over a family of conventional measurement structures that all satisfy the same axioms.

From a fundamental measurement perspective, this approach is not fully satisfactory because it assumes as primitive a numerical structure of probabilities and thus places the description of randomness at a numerical, rather than qualitative, level. One would like to formulate the qualitative primitives so as simultaneously to capture at a qualitative level both the structural and the random qualities of the situation.

The following is a reasonably concrete instance of what we have in mind. Consider the typical extensive situation where both judgments of order can be made and entities can be concatenated to form new ones exhibiting the same attribute. One would like axioms that lead to a random variable representation that specifies the random variables with respect both to their structural relations with each other and to the nature of their distributions. For example, if  $\oplus$  denotes concatenation within

the extensive structure, it would be interesting to arrive at qualitative axioms sufficient to guarantee that the representing family can be taken to be the gamma family. Mathematically, the problem undoubtedly entails finding a functional equation characterization of the gamma family which, to our knowledge, has never been given. More generally, one would expect the distribution of the random variable associated to  $x \oplus y$  to be the convolution of the distributions of those associated to  $x$  and to  $y$ . With such a representation, the expected value would behave as a traditional measurement representation, in particular

$$x \oplus y = z \quad \text{iff} \quad E[\phi(x)] + E[\phi(y)] = E[\phi(z)].$$

Thus, we state the problem as:

**PROBLEM 3.** *Extend the qualitative primitives of RTM in such a way that the objects of the domain are represented by random variables (instead of by numbers = constant random variables).*

The idea that randomness can be captured qualitatively within a relational structure is not totally idle because, in a certain sense, that is exactly what has been done in theories of subjective utility theory (Savage, 1954, and much subsequent literature). In such theories, choices among uncertain alternatives are used to infer a random variable – utility – and probability distributions over families of events underlying the uncertain alternatives. The problem is to do it in contexts more analogous to extensive and conjoint measurement, not preference among uncertain alternatives.

Progress on Problem 3 has been made for the special case of finite, ordinal empirical structures, i.e., there is just one primitive, an ordering relation (Cohen and Falmagne, 1990). Suppes and Zanotti (1992) have followed a different tack in attempting to axiomatize qualitative moment information and to use that to characterize a random variable representation.

Alternative approaches to random variable ideas of error may be appropriate for extensions of RTM. In particular, applications of Boolean-valued and other multi-valued logics and fuzzy logics seem potentially interesting.

PROBLEM 4. *Extend the RTM approach to include axioms formulated in terms of multi-valued logics.*

Efforts in this direction should be carried out so as to produce either new kinds of results – not just translations of known random variable representation results – or new insights into measurement through concepts not available in the standard approaches to random variables. Some progress has been made by Heyer and Niederée (1989), but much more research on the topic is needed.<sup>7</sup>

### 3.3. *Meaningfulness*

The meaningfulness part of RTM has not received as much attention as the existence and uniqueness parts and so it is less fully developed. In particular, as with the definition of a measurement scale, the criterion for meaningfulness invoked in RTM has not been adequately justified. Although FM3 describes methods for linking qualitative correlates to meaningful quantitative relations, no justification is provided for why these qualitative correlates are indeed empirical.

Narens (1988) showed that these qualitative correlates are definable in terms of the primitives through a very powerful higher-order logical language that includes individual constant symbols for purely mathematical entities. Because empirical definitions require only much weaker logical languages, Narens' results establish that the qualitative correlates of non-meaningful qualitative relations cannot be defined empirically in terms of the primitives of the empirical structure. Thus, the correlates of non-meaningful relations are non-empirical with respect to the primitives. These results can also be used to show that there exist qualitative correlates of meaningful quantitative relations that are necessarily non-empirical. The conclusion to be drawn from this is that the RTM concept of meaningfulness gives a necessary but not sufficient condition for empirical significance. It should be remarked that most of the applications of the meaningfulness concept, such as dimensional analysis (Bridgman, 1922, 1931; Luce, 1971, 1978), use it only as a necessary condition for empirical significance, i.e., use it as a condition for eliminating from consideration non-meaningful relationships.

PROBLEM 5. *Amend RTM's concept of meaningfulness so that it captures in a more appropriate fashion the concept of empirical significance (with respect to the qualitative structure). Obviously, a major part of this problem is giving a coherent formulation of 'empirical significance'.*

It may well be the case that the solutions to Problems 1 and 5 are intimately connected.

### 3.4. Continuum Representations

Although there are some results on finite measurement structures, some of which are quite useful in applications, and on denumerable structures, the general consensus is that the strongest, most elegant results are about structures that map onto a continuum. Examples were cited in Section 2.1 and additional ones are given below.

The results about structures on finite sets are of two distinct types. One in essence axiomatizes a finite standard sequence, which leads in the usual way to an integer representation and a simple uniqueness theorem. The other establishes a set of inequalities that must be satisfied and the existence of a numerical representation is established but we usually are unable to give a compact *and* useful characterization of its uniqueness.

Assuming a universe of finitely many particles, which many believe is implied by current physical knowledge, why does one ever need to look at denumerable let alone continuum results; yet it is these, especially the latter, that seem to be of greatest import for scientific measurement. The problem divides naturally into two phases: from finite to denumerable and from denumerable to continuous. One can envisage some sort of theory concerning a nesting of finite systems that leads, asymptotically, to a denumerable structure from which the finite systems can be thought of as samples. But it is well known that the step from denumerable systems, even the rational numbers, to the continuum is delicate. For example, the results on scale type to be described in Section 4 become vastly more complex in the denumerable case (Cameron, 1989). Considerable research exists on the Dedekind completions<sup>8</sup> of certain classes of structures (see Narens, 1985; Ch. 19 of FM3), but this has not yet been given an adequate justification for using continuum models.

PROBLEM 6. *Provide a principled account of why it is scientifically useful to replace finite structures by continuum ones. In particular, it is important to make clear just what limiting processes give rise to Dedekind completeness or Archimedeaness in the continuum, and also what gives rise to the properties of homogeneity and finite uniqueness that are discussed next.*

This problem may well be closed related to Problems 1 and 5.

#### 4. CLASSIFICATION OF STRUCTURES BY SCALE TYPE

Coexisting with the representational approach has been another theme which, beginning in 1981, began to flourish as a major alternative. During the earlier debates over the existence of psychological measures of any sort, as distinct from physical ones, Stevens (1946, 1951) placed great emphasis on the uniqueness of representations. His list of scale types – ordinal, interval, ratio, and absolute – is famous, and most empirical examples fell within it. But not all. For example, Narens and Luce (1976) showed that one could simply drop the associativity axiom from classical extensive measurement and still show the existence of an (inherently non-additive) numerical representation. But they failed to provide a satisfactory description of its uniqueness beyond the fact that specifying a single point rendered it unique. The reason for their failure did not become apparent until the work of Cohen and Narens (1979) in which the uniqueness problem was first treated as essentially equivalent to understanding the group of symmetries (= automorphisms) of the structure, i.e., isomorphisms of the structure onto itself. They showed that for non-associative concatenation structures the group of automorphisms is surprisingly simple: an Archimedean ordered group. Hölder (1901) had completely characterized such structures by showing that each is isomorphic to some subgroup of the multiplicative positive real numbers (MPRN). Ratio scales onto the positive reals are those cases where the automorphism group is isomorphic to the entire MPRN group.

##### 4.1. *Homogeneity, Finite Uniqueness, and Scale Type*

The results just described were a precursor to Narens' (1981a, b) proposed classification of the automorphism groups of ordered relational

structures in terms of two major properties, called the degree of homogeneity and degree of uniqueness. The automorphism group is said to be *M-point homogeneous* if for any two ordered sequences of  $M$  distinct elements, there exists an automorphism that takes the first sequence into the second. That group is said to be *N-point unique* if whenever two automorphisms agree at  $N$  distinct points, they are necessarily identical. For an  $M$ -point homogeneous structure with more than  $M$  points, it is easy to see that  $M \leq \min N$ . The *scale type* of the structure is the ordered pair of numbers  $(\max M, \min N)$ . Ratio scale structures are of type  $(1, 1)$ ; interval scales ones, of type  $(2, 2)$ .

The question posed by Narens was: what scale types are possible? A number of general, but still partial, answers are known for (simply) ordered relational structures in which the primitive relations are of finite order.

#### 4.2. *A Recipe for Constructing Numerical Representations*

The major results rest on properties of a subset of the automorphisms called *translations*: the identity map together with all automorphisms that do not have any fixed points. Three major questions about them are:

- (i) Do the translations form a mathematical group? The only real problem is in showing that they are closed under function composition which is equivalent to showing that they are 1-point unique.
- (ii) Under the ordering of the translations naturally induced from the order of the given structure,<sup>9</sup> are the translations Archimedean?<sup>10</sup>
- (iii) Are the translations 1-point homogeneous?

Once these facts are established, one can construct a numerical representation in which the translations appear as multiplication by constants (i.e., the similarity group) as follows: using homogeneity, map the structure isomorphically onto the group of translations; using Hölder's theorem, map the translation group, and so the structure, into MPRN (Alper, 1987; Luce, 1986, 1987). That leaves unanswered the question about the rest of the automorphisms. A simple answer is known when the underlying ordered relational structure is order dense:<sup>11</sup> the automorphisms are a subgroup of the power group  $x \rightarrow rx^s$ , which means that  $\min N \leq 2$  (see FM3, Theorem 20.7, Corollary 2).

#### 4.3. *Some Structures with Translations Satisfying the Recipe*

So a key problem is to try to understand what structural properties give rise to these three properties of translations. So far, no one has derived any necessary structural properties from these three properties of translations. All that is known are certain sufficient conditions. We cite two partial results.

First, if the ordered relational structure is Dedekind complete and order dense, then (i) implies (ii) (see Theorem 20.6, FM3). Second, and this is the most general result known at present, if the ordered relational structure can be mapped *onto* the real numbers, is homogeneous ( $\max M \geq 1$ ), and is finitely unique ( $\min N < \infty$ ), then the three conditions are satisfied and the possible scale types are (1, 1), (1, 2) and (2, 2), with the first being a ratio scale, the third an interval scale, and the (1, 2) case falling between the two. An example of the (1, 2) case is the group of real transformations  $x \rightarrow k^n x + s$ , where  $k > 0$  is fixed,  $n$  ranges over all integers, and  $s$  ranges over all real numbers. Narens (1981b) proved the part of this result for which  $\max M = \min N$ , and Alper (1987), using a very different approach, proved the result without that restriction. Alper's methods first made very clear the importance of the three properties of the translations. Surprisingly, the most difficult property to establish is that the translations form a group.

#### 4.4. *Structural Equivalents to the Key Properties of Translations*

Given the results just mentioned, it is clear that we still have much to learn about the conditions under which the translations form a homogeneous, Archimedean-ordered group.

**PROBLEM 7.** *What structural properties are implied by each of the translation properties – group, Archimedean, and homogeneity – separately or together? As none are currently known, a simpler problem may be: what structural properties are sufficient for each of the three properties of translations, either separately or jointly? Ideally, one would like to find distinct structural features that correspond separately to each property, although that may well prove infeasible.*

For any homogeneous and finitely-unique ordered structure having a binary, monotonic operation, not necessarily on a continuum, one can

prove directly that  $\min N \leq 2$  (Luce and Narens, 1985). Little is known in general about the automorphism groups in this case, except, as was noted above, when the operation is also positive, solvable, and Archimedean, one can prove, without assuming homogeneity, that the automorphism group is Archimedean ordered. We do not know of comparable results in the remaining homogeneous case in which the operation is necessarily idempotent.<sup>12</sup> In particular, we do not know of conditions that result in the translations forming an Archimedean ordered group.

**PROBLEM 8.** *What algebraic properties on homogeneous, finitely unique relational structures are 'just sufficient' to prove  $\min N \leq 2$ ? The two known results should be special cases. When  $\min N \leq 2$  can be proved, what can be said about the set of translations?*

(It strikes us as unlikely that a reasonable set of necessary and sufficient conditions will be found, and so the criteria for 'just sufficient' should be considered to be flexible.)

## 5. SCALE TYPE IN SPECIFIC STRUCTURAL CONTEXTS

### 5.1. Combining Scale Type and Structural Conditions

Once the results on scale types began to be discovered, the possibility of applying them to specific measurement problems began to be explored. Basically the strategy has been to combine the homogeneity and finite uniqueness conditions with more specific structural assumptions and to characterize in greater detail the classes of structures that can arise. One example of this is found in Luce and Narens (1985) in which homogeneous, finitely unique concatenation structures on the continuum are described quite fully. Closely related is the fact that for the class of concatenation structures that are positive, solvable, Archimedean, and Dedekind complete, if the ordered automorphism group is order dense, then it is also homogeneous. A third example is continuous semiorders; they have a  $(1, \infty)$  group of automorphisms with a subgroup of translations that is homogeneous and can be ordered so that it is Archimedean. We do not know of other cases where the numerical representation of a structural axiomatization has been studied by proving that its transla-



tions form a homogeneous, Archimedean ordered group.

**PROBLEM 9.** *Explore the possibility of using the recipe described above to construct numerical representations of particular ordered structures that do not fall under the scope of the theorem of Alper and Narens mentioned in Section 4.3.*

### 5.2. Structures That Are Not Finitely Unique

A structure may fail either homogeneity or finite uniqueness; the two failures are very different.

Ordinally scalable structures are not finitely unique; indeed, it takes at least an order dense subset of the domain to fix an automorphism. Moreover, they are also  $M$ -point homogeneous for each finite  $M$ . We say they are of scale type  $(\infty, \infty)$ . Roberts and Rosenbaum (1985) established that for an ordered relational structure that is  $M$ -point homogeneous, if  $M$  does not exceed the cardinality of the domain and if the order of each defining relation of the structure is not greater than  $M$ , then the automorphism group of that structure is identical to that of just the ordered domain. Thus, if these conditions are met and the ordered domain is isomorphic to the ordered real numbers, then the structure is of scale type  $(\infty, \infty)$ . Droste (1987) has characterized in a convenient form the automorphism groups of such structures.

This leaves the  $(M, \infty)$  cases, about which relatively little is known. Although it has been shown that continuous semiorders are examples of  $(1, \infty)$  structures (Narens, 1994), we know of no explicit measurement structures with  $1 < M < \infty$ . Thus, many questions remain unanswered.

**PROBLEM 10.** *What can be said about structures and their automorphism groups of scalar type  $(M, \infty)$  with  $1 < M < \infty$ ?*

### 5.3. Structures That Are Finitely Unique But Not Homogeneous

For general finitely unique, non-homogeneous structures on the continuum, Alper (1987) provided a description of the possible automorphism groups, but so far his classification has not been used successfully to describe the corresponding structures. Luce (1992a) followed a far

more restricted tack, but one that seems highly relevant to some measurement applications. He defined a point in a structure to be *singular* if it remains fixed (or invariant) under every automorphism of the structure. The concept of a *translation* is generalized to be either the identity or any automorphism whose only fixed points are singular ones. Clearly, if the structure is finitely unique, then it has only finitely many singular points and so it is meaningful to speak of such a structure as being *translational homogeneous between adjacent singular points*. For a class of structures that he calls *generalized concatenation structures* which have a monotonic<sup>13</sup>  $n$ -ary operation, he gave a fairly complete description of the possibilities when the structure is both finitely unique and translation homogeneous between adjacent singular points. What makes matters simple is that such structures can have at most three singular points: a maximum, a minimum, and an interior one. An example of such a structure is the multiplicative, positive real numbers augmented by 0 and  $\infty$  with translations  $x \rightarrow x^r$ , where  $r > 0$ . The singular points are the two extreme ones, 0 and  $\infty$ , and one interior one, 1.

If, in addition, the group of translations commute,<sup>14</sup> then an interior singular point, call it  $e$ , acts like a generalized zero<sup>15</sup> in the following sense: if  $F$  denotes the operation, then  $F(e, \dots, e, x_i, e, \dots, e) = \theta_i(x_i)$ , where on either side of  $e$  the function  $\theta_i$  agrees with a translation of the structure. Moreover, if any singular point is a generalized zero, then any other singular point,  $e'$ , acts like an infinity in the sense that if  $e'$  is an argument of the function, then the value of  $F$  is  $e'$ . Finally, for structures on a continuum, Alper's results can be used to derive a numerical representation in which translations on each side of the interior singularity are multiplication by a constant, the two constants being simply related by a power relation. As we shall see in Section 6.3, these and related results have been applied effectively in devising a theory of certainty equivalents for gambles (Luce, 1992b).

**PROBLEM 11.** *What structures are there that are useful for applied measurement, beyond those based on a general operation, for which homogeneity fails at selected points? And what structures, although failing homogeneity more globally, still have a fairly rich automorphism group, such as  $x \rightarrow k^n x$ , where  $k > 0$  is a fixed constant and  $n$  ranges over the integers?*

Important non-homogeneous structures lie outside this framework. The most notable examples are qualitative probability structures. They are non-homogeneous not only because of their extreme points – the null and universal events – but because two events that are qualitatively equally probable need not exhibit the same relational patterns to other events. Also, the only automorphisms of such a structure are ones, like the identity, that take an event into an equally probable one. Obviously, the previous tacks we have taken in structures that are more or less homogeneous are completely useless in such contexts. Yet, clearly the probability case exhibits a great deal of regularity.

**PROBLEM 12.** *What kind of useful classification can be given for structures that exhibit a great deal of regularity, such as is seen in probability structures, but that only have automorphisms  $\alpha$  for which  $\alpha(x)$  is equally probable to  $x$ ?*

## 6. APPLICATIONS OF RESULTS ABOUT SCALE TYPES

### 6.1. *Product-of-Powers Compatibility in Bounded Cases*

One feature of physical measurement is the existence of two distinct kinds of attributes: those having an internal structure, like mass, time, length, and charge, and those having a trade-off structure between components, like energy, momentum, density, etc. A major feature of classical physical measures is that the conjoint ones can be represented as products of powers of the extensive ones. For example, kinetic energy is given by  $\frac{1}{2}mv^2$  and density by  $m/V$ . This aspect of the representation is reflected in the fact that the units of the conjoint measures are always products of powers of the units of extensive measures. How this arises from measurement considerations is discussed in Ch. 10 of FM1 and again, much more generally, in parts of Chs. 20 and 22 of FM3.

The most general result to date involves two major properties: a qualitative concept of how a structure on one component of a solvable conjoint structure distributes<sup>16</sup> in the latter structure (Definition 20.6, FM3), and the assumption that the component structure is such that its translations form a homogeneous Archimedean ordered group. These two properties force the conjoint structure to have a multiplicative representation involving a power of the representation of the component attribute (Theorem 20.7, FM3).

This model of interrelated measures covers much of classical measurement and makes clear exactly which generalization of extensive structures – basically any ratio scale structure – can be added to the physical structure without disrupting the product-of-powers feature. Unfortunately, it fails to cover everything of importance. The most notable exception is relativistic velocity, which is bounded from above by the speed of light.

As is well known, physicists have elected to keep the multiplicative conjoint relation  $s = vt$  among distance, velocity, and time, and to use a non-additive and bounded representation of the associative concatenation of velocities, namely,  $u \oplus v = (u + v)/(1 + uv/c^2)$ , where  $c$  denotes the maximum velocity, that of light. (In principle, they could have adopted an additive representation of  $\oplus$ , but at the very considerable expense of foregoing the simple relation  $s = vt$ ). It is important to realize that  $\oplus$  is not really an operation on the velocity component of the defining conjoint structure. The reason is that  $u$  and  $u \oplus v$  are velocities in a single frame of reference whereas  $v$  is in a different frame, one in which the distances and times are modified relative to the first frame. Thus, from the perspective of relativistic measurement, it is inappropriate to think of  $\oplus$  as an operation on the conjoint structure. But even if one does, it fails to be distributive in the conjoint structure. Within the context of a binary operation, distribution comes to: if  $ut = u't'$  and  $vt = v't'$ , then  $(u \oplus v)t = (u' \oplus v')t'$ . It is easy to verify that this fails. More deeply, when distribution holds, the automorphism structures of the conjoint and component structures agree in the sense that every translation of the component structure is that component's contribution to an automorphism of the conjoint structure, and conversely. This is not true in the case of the velocity component. Yet, at the same time multiplication by a positive constant  $r$ , which is an automorphism of the conjoint structure, simply maps one velocity representation into another, with  $c$  becoming  $rc$ . This lack of a connection between the automorphism groups of the two structures and yet their tight product of powers representation is not understood in terms of measurement theory.

This problem is important not only to complete our understanding of how relativistic velocity ties into the classical measures, but also to allow the introduction of other bounded measures. The most obvious of these is probability which, in computing expectations, has a product relation with other measures, as in (subjective) expected utility theory

(Savage, 1954; von Neumann and Morgenstern, 1947). Aside from the boundedness, the two cases are very different. Other examples may arise in the behavioral sciences where measures of subjective intensity are almost surely best modelled as bounded from above.

**PROBLEM 13.** *What links the bounded component structures of a conjoint structure and the conjoint structure itself so as, again, to lead to a product-of-powers representation? Possibly one should initially assume the components are homogeneous between bounds, as in the velocity case, but ultimately that restriction must be removed to deal with the probability case.*

### 6.2. Compatibility of Psychological and Physical Theory

Matching is a psychological procedure in which a subject ‘matches’ a stimulus in one intensity domain, such as brightness, to a given stimulus in another intensity domain, such as loudness. Formally, there are two physical structures  $\mathcal{X}$  and  $\mathcal{S}$ , with domains  $X$  and  $S$ , that each have ratio scale representations; classically, they are simply extensive structures but the theory applies to any with a ratio scale representation. The psychological data may be summarized as follows: for each  $x$  in  $X$ , there is some  $s$  in  $S$  such that, according to the subject,  $s$  matches  $x$ , which we symbolize as  $xMs$ . In this situation, there are two physical relational structures and a purely psychological connecting relation  $M$  between them. Empirically, to a first approximation at least, the matching relation can be described as a power relation between the two physical ratio-scale measures. The question is to what does this correspond.

Luce (1990) suggested a principle of compatibility between the two domains that may be described as follows. To each translation  $\tau$  of  $\mathcal{X}$ , there is a corresponding translation  $\sigma_\tau$  of  $\mathcal{S}$  such that for all  $x \in X$  and all  $s \in S$ :

$$xMs \quad \text{if and only if} \quad \tau(x)M\sigma_\tau(s).$$

This is easily shown to be equivalent to a power relation between the ratio scale representations, and a number of other relationships are explored. The multiplicative constant of the relation has, of course, dimensions that depend upon the exponent involved.

Clearly, the application just described is highly special to a particular situation, but the general idea of asking which psychological laws

relating physical variables are in fact compatible with the translation structure of these variables is a far more general principle. The failure of such compatibility means non-physical variables are required to describe the phenomenon.

**PROBLEM 14.** *Is there any deep scientific or philosophical grounding for the supposition that psychological laws should be compatible with the automorphism structures of the physical variables that they relate? Are there other examples in which application of this principle can be illustrated? And are there examples where it clearly is violated?*

### 6.3. Applications of Results about Structures with Singular Points

To date just one new application (beyond relativistic velocity) has been made in the measurement literature of the results about structures with general operations that are finitely unique, have singular points, and are translation homogeneous between adjacent singular points (see Section 5.3). It concerns certainty equivalents to uncertain monetary alternatives.<sup>17</sup> For a fixed event partition, a certainty equivalent can be viewed as a monotonic function of money arguments – the pay-offs associated with the several subevents – into an amount of money that is indifferent to the uncertain alternative. A sharp distinction is maintained between gains and losses, making 0 an interior singular point. Assuming the structure is homogeneous on either side of 0 and finitely unique, which has been typical of utility theories, and defining utility to be the isomorphism that represents the translations as multiplication by positive constants yields a linear weighted average utility representation. The fact that 0 must be a generalized zero is very important in constructing the weighting functions of rank- and sign-dependent character. We do not go into any of the details, for the only point in mentioning it here is as evidence that such general results do have applications.

**PROBLEM 15.** *Given that we gain some results about structures with various forms of non-homogeneity (see Problems 11 and 12) including ones with interior singular points, are there applications that up to now have been overlooked because previously we did not know how to deal with such situations?*

## 7. CONCLUSIONS

The representational theory of measurement, despite its positive contributions, has been subjected to attack on a number of fronts. One basic issue is how to formulate clearly what one means by measurement in such a way that the representing structure is derived from the axioms of the qualitative structure. Of considerable interest are structures that do not map into the real number system, but rather into families of random variables or into structures based on multi-valued logics (Problems 1–5). Further, even in the case of numerical representations, one can wonder why anything as idealized as the continuum is relevant to science (Problem 6). The remaining problems are all considerably less philosophical and involve somewhat complex issues in the theory of scale types. We know a lot more about scale type than we did 12 years ago, but much about structures remains shrouded; hard work and new ideas are probably needed to gain a deeper understanding. Some of these problems are listed as 7–15.

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## NOTES

<sup>1</sup> Krantz, Luce, Suppes, and Tversky (1971) will be referred to as FM1; Suppes, Krantz, Luce, and Tversky (1989) as FM2; and Luce, Krantz, Suppes, and Tversky (1990) as FM3.

<sup>2</sup> Some of the subtleties involved in formulating uniqueness results are dealt with by Roberts and Franke (1976).

<sup>3</sup> A conjoint structure is a weak ordering of a Cartesian product; it is additive if it admits an additive representation over its components.

<sup>4</sup> Archimedeaness simply means that any bounded standard sequence is finite. Put another way, no positive element is infinitesimal relative to another element of the structure. In practice, the impact of Archimedeaness is to permit homomorphisms into the real numbers rather than ordered extensions of the real numbers such as the non-standard reals.

<sup>5</sup> Among the positive uses we include the testing of mathematically formulated theories relating several variables, although we recognize that others may not want to take such a consideration into account in finding a solution to this problem. Subjective expected utility, mentioned later, is one example; three more are given in Section 6.

<sup>6</sup> The primary exceptions are purely ordinal cases including variants such as interval orders and semiorders.

<sup>7</sup> See Heyer and Mausfeld (1987) for a discussion of some conceptual problems connected with Boolean-valued approaches.

<sup>8</sup> A structure is Dedekind complete if every subset of elements that is bounded from above has a least upper bound in the domain. A structure that can be embedded in a Dedekind complete one of the same algebraic form is said to have a Dedekind completion.

<sup>9</sup> For automorphisms  $\alpha, \beta$ ,  $\alpha \lesssim' \beta$  if and only if for every element  $x$  of  $X$ ,  $\alpha(x) \lesssim \beta(x)$ .

<sup>10</sup> For any two translations one of which is greater than the identity, finitely many applications of the positive one will exceed the other.

<sup>11</sup> If  $x < y$ , there exists  $z$  in  $X$  such that  $x < z < y$ .

<sup>12</sup> Homogeneity implies, for all elements  $x$ , either weak positivity,  $x \circ x > x$ , idempotence,  $x \circ x \sim x$ , or weak negativity,  $x \circ x < x$ . The first and third are formally identical if  $\succsim$  is replaced by  $\lesssim$ .

<sup>13</sup> The definition of monotonic is the usual one except that some care is needed in dealing with extreme points, if such exist.

<sup>14</sup> This is true if they can be represented in the multiplicative real numbers. The assumption may be redundant, but Luce failed to derive it from the other assumptions.

<sup>15</sup> The term 'zero' is appropriate when one thinks of homomorphisms to the real numbers where 0 is the interior singular point and the translations are  $x \rightarrow rx$ .

<sup>16</sup> Let  $\mathcal{C} = \langle A \times X, \succsim \rangle$  be the conjoint structure with  $\mathcal{A}$  a relational structure on  $A$  whose order is that induced from  $\mathcal{C}$ . For fixed  $x, y \in X$ , let  $\alpha$  be the function defined by all solutions to  $(a, x) \sim (\alpha(a), y)$ . Then,  $\mathcal{A}$  distributes in  $\mathcal{C}$  if every such  $\alpha$  is an automorphism of  $\mathcal{A}$ .

<sup>17</sup> This is closely related to the prospect theory of Kahneman and Tversky (1979) and to such extensions of it as Luce and Fishburn (1991) and Tversky and Kahneman (1992).

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