2 What Is a Ratio in Ratio Scaling?

R. Duncan Luce
Irvine Research Unit in Mathematical Behavioral Science
University of California, Irvine

THE PROBLEM

Ratio Scales and Ratio Scaling

The terms *ratio scale*, apparently first introduced by Stevens (1946, 1951), and *ratio scaling*, apparently first introduced by Krantz (1972) but closely related to previous phrasing, denote different but interrelated things. Although Krantz was very clear on the matter,[1] the similarity of the terms seems to invite confusion and confounding. My goal here is to explicate some aspects of the differences and relations.

A ratio scale concerns one aspect of numerical measurement representations[2] for a certain class of one-dimensional, empirical structures. In particular, it refers to those cases where the numerical representation of a qualitative structure of stimuli is uniquely specified up to multiplication by a positive constant. The most familiar physical examples of ratio scales are length, mass, and time intervals. They are all examples of what are called *extensive structures*, and they have in common two primitives. The first is a binary ordering relation $\succsim$ that reflects the ordering induced on

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[1] On p. 169, he wrote: "I use the term '*ratio scaling*' to refer to a family of interrelated psychophysical methods, discussed and classified by Stevens [1975]. The quotation marks are used to emphasize the distinction between the term '*ratio scaling*' and the term *ratio scale*, the latter having its usual technical meaning in measurement theory . . . . 'Ratio scaling' need not lead to ratio scales."

[2] That is, isomophisms between the empirical structure and a numerically based structure.

the objects of X by the attribute in question. The second is a binary operation $\circ$ of combining that has the property that when "a" and "b" each exhibit the attribute being measured, so does $a \circ b$. Together, $\gtrsim$ and $\circ$ satisfy four major properties. The operation is *associative* relative to the ordering: for objects a, b, and c of the system, then

$$(a \circ b) \circ c \sim a \circ (b \circ c), \tag{1}$$

where $\sim$ denotes equivalence in the attribute being measured. It is *commutative*, that is, for each a and b,

$$a \circ b \sim b \circ a. \tag{2}$$

It is *monotonic* relative to the ordering, that is, for all a, b, and c,

$$a \gtrsim b \text{ if and only if } a \circ c \gtrsim b \circ c. \tag{3}$$

And it is *positive*, that is, for each a and b,

$$a \circ b > \max(a,b). \tag{4}$$

For a general discussion of such structures, see Krantz, Luce, Suppes, and Tversky, (1971, Chapter 3).

Ratio scaling refers to a class of experimental procedures and results that are inherently psychological in character, namely ones in which a person relates two distinct empirical—usually physical or mathematical—structures so that, in some sense, *stimulus ratios* are preserved. Examples of such matching between structures abound: cross-modal matches between one-dimensional continua of physical intensity in which the subject matches stimuli from one modality to those from another so as "to preserve" stimulus ratios; matches between stimuli of the same type in which more than one physical dimension is varied but a psychological attribute such as loudness, brightness, or color is maintained; and procedures, such as constructing jnds, that are not normally thought of as matching but can be so construed (see Eq. 8).

The results described here pertain only to those one-dimensional cases, such as the first and third examples, where it makes sense to speak of stimulus ratios. Sometimes the idea that stimulus ratios should be preserved is invoked through experimental instructions, other times in the interpretation of the data.

The question I wish to address is: What does it mean for a subject to preserve ratios? The answer usually proffered is: The stimuli are chosen so that numerical ratios of stimulus representations are preserved. Although I ultimately make some sense of that statement, I question its intellectual adequacy as the basic formulation of ratio scaling. One problem with it is how does one know which representation to use? This is a real issue because any strictly increasing function of a representation generates another, equally good, numerical representation. Why do we invoke it for intensity and not dB? Another problem is why should our method of representing physical information have anything whatsoever to do with the empirical issue of a subject's choice of stimuli? The representation is merely a scientific convenience; presumably whatever the subject is doing is concerned with the stimuli as such, not with scientists' representation of them.

Any proposed answer should, I believe, exhibit three features: It should be formulated in qualitative terms about the stimuli; it should be in the form of a psychological theory of matching; and it should account for the importance of numerical ratios when the usual numerical representations are invoked.

Krantz's Five Empirical Generalizations

To amplify on the third point just made and to make clear what needs to be accounted for, let me quote five empirical generalizations that Krantz (1972, p. 171) listed that need to be accounted for by a psychophysical matching theory:

"(i) **Magnitude consistency**. Magnitude estimation functions with different moduli differ only by similarity transformations. . . .

"(ii) **Pair consistency**. Pair estimates [= ratio judgments] behave like ratios. . . .

"(iii) **Magnitude-pair consistency**. The pair estimate is equal to the ratio of magnitude estimates for the members of the pair. . . .

"(iv) **Consistency of magnitude estimates and cross-modality matching**. If y_j is matched with y_i, in the cross-modality matching function with modulus (x_j,x_i), then the ratio of magnitude estimates of y_j to x_j equals the ratio of the magnitude estimates of y_i to x_i . . .

"(v) **Power law**. Magnitude estimates are approximately power functions of stimulus energy. . ."

He argued that the kind of *direct mapping theory* proposed by Stevens was inadequate to account for all of this systematically, and he proposed instead a formalization of some ideas of Shepard (1978, 1981) that he called *relation theory*.[3] It is characterized by the fact that within the theory all judgments are assumed to be of ratios, even if the ratios are implicit when the subject is matching stimuli. The theory I describe is a matching theory that in a sense explains ratio matches without assuming them directly.

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[3] Shepard's thoughts were circulated in manuscript many years before their ultimate publication.

QUALITATIVE REPRESENTATION OF PHYSICAL STIMULI

Ordered Relational Structures

As was mentioned earlier, the usual ratio scale representations arise from physical concatenation structures consisting of a set X, a binary ordering relation $\succsim$, and a concatenation (or combining) operation $\circ$. In this structure, $x \succsim y$ is interpreted to mean that x exhibits at least as much of the attribute in question as does y. And $x \circ y$ denotes an object composed from x and y that also exhibits the attribute. Well-known assumptions are sufficient for such a structure to have a ratio scale representation into (or onto) the additive positive real numbers (see Krantz et al., 1971). Such structures are called *extensive*.

Recent research has extended considerably our understanding of the scope of the structures that have ratio scale representations. There are many such structures aside from extensive ones, and some of these generalizations may prove to be of use in the behavioral sciences where extensive structures have played a limited role. The relevant references are Alper (1987); Luce (1986, 1987); Luce, Krantz, Suppes, and Tversky (1990); Luce and Narens (1985); and Narens (1981a, 1981b, 1985). As in the standard physical case, we assume a set X of objects or events and an ordering $\succsim$ by the attribute in question. But instead of additional structure as specific as a binary operation (which is a special type of trinary relation), we simply assume that the structure satisfies a finite number of constraints that are captured as relations X_j of finite order, where j is in a finite index set J. The collection of these together with $\succsim$ is called an *ordered relational structure*, and it is typically denoted $\mathscr{X} = \langle X, \succsim, X_j \rangle_{j \in J}$.

Translations

Were one to follow the traditional approach to measurement, one would provide an axiomatization characterizing these relations and then prove or quote a theorem to the effect that there exists an isomorphism between the qualitative structure and some structure of real numbers and that representation is unique up to similarity transformations (i.e., multiplication by positive constants). The new developments depart from this pattern in a novel way, which was first proposed by Narens (1981a, 1981b). They focus axiomatically not on the structure itself but rather on what physicists call the symmetries and mathematicians call the automorphisms of the structure. To be precise, an *automorphism* (= *symmetry*) of the structure $\mathscr{X} = \langle X, \succsim, X_j \rangle_{j \in J}$ is any isomorphism of the structure with itself, that is, a one-to-one map of X onto X that preserves the order and all the other relations in the sense that if $(x_1, \ldots x_{n(j)})$ is in X_j and if α is the mapping, then $(\alpha(x_1), \ldots, \alpha(x_{n(j)})$ is also in X_j.

The set of automorphisms exhibits a good deal of algebraic structure. Automorphisms can be composed with one another as functions, which forms an associative but not, in general, commutative operation. The set can also be ordered indirectly using the order from the structure (I do not give the details here). And the order so induced and the operation of composition is monotonic. There is an identity, namely the identity map, and each automorphism has an inverse relative to it. All this is summarized by saying that the automorphisms form an ordered group.

In the case of extensive structures, which we recall are isomorphic to the additive real numbers, the automorphisms are easily characterized. For example, suppose an extensive structure is isomorphic to the positive additive reals with ϕ one of the isomorphisms. Let r be any positive number, then the function $\phi^{-1}r\phi$ from X onto X can readily be shown to be an automorphism of the extensive structure.

Among the automorphisms of any structure, one subset has turned out to be especially important, namely those for which no point is fixed (i.e., no point maps into itself under the automorphism). These along with the identify, for which every point is fixed, are called *translations*. In the case of an extensive structure, all the automorphisms are translations, but that is not true for interval scales.

It turns out that if the translations exhibit some nice properties, which I elucidate, then the empirical structure giving rise to these translations is isomorphic to a numerical structure and the translations map into multiplication by positive constants, just as in the extensive case. In particular, denote by ϕ the isomorphism of the structure into the reals and denote by τ a translation, then there exists real $k_\tau > 0$ such that for all x in X,

$$\phi[\tau(x)] = k_\tau \phi(x). \tag{5}$$

Such structures are called *unit structures* (Cohen & Narens, 1979; Luce, 1987).

Properties of the Translations

So, the question is, what are those nice properties? One is that given any two points in the structure, there is a translation that takes the one point into the other. This property is known as *homogeneity* of the translations. It fails when one or more of the points exhibit inherently different properties from the others. Examples are maxima, minima, and identity elements. Another nice property is that the translations are *closed* under function composition: A translation applied to another translation results in a translation. And a third is the somewhat technical property of being *Archimedean*, which I do not explain beyond saying that it means that given any translations β and any positive translation[4] α, then sufficiently many repeated applications of α will exceed β according to the induced order. The upshot of closure and Archimedeanness is that the translations form what is known as an Archimedean ordered group, and by a classical theorem of Hölder the translations in this case are isomorphic to a subgroup of the ordered additive reals. Using this isomorphism of the translations into the additive real numbers together with the homogeneity of the translations allows one to construct a numerical image of the entire structure (Luce, 1987).

Proposed Answer

Thus, the translations, which are defined entirely within the qualitative structure, capture what corresponds to multiplication by a constant, and so a fixed ratio, within the numerical representing structure. Note that what I am describing has nothing to do with changes of units in the representation, but only with the representation of those symmetries of the structure that are translations. It is, of course, the case that such translations effect a possible change in the representation, but the important fact is that the translations of such a qualitative structure correspond to ratios in its unit representation.

So the answer I propose to the question of my title—What is a ratio in ratio scaling?—is simple a *translation*. I try to defend this answer by describing a theory of matching that is formulated in terms of translations. (For details, see Luce, 1990.)

SIMPLE MATCHING RELATIONS

Suppose that $\mathscr{X} = \langle X, \succsim_X ; X_j \rangle_{j \in J}$ and $\mathscr{S} = \langle S, \succsim_S, S_k \rangle_{k \in K}$ are two ordered relational structures. When a subject establishes a matching relation between the two domains, we simply observe a function from one to the other that, presumably, has the property of maintaining order: If $y \succsim_X x$ and s is matched to x and t to y, then we anticipate $t \succsim_S s$. That seems a minimal requirement. So I define a *matching relation* to be a strictly increasing function M from X onto S.

Now, among all possible matching relations, I wish to single out those that in some reasonable sense can be said to maintain ratios. If, indeed, the intuitive idea of a ratio corresponds mathematically to a translation, then for ratios to be preserved one expects a translation in the one domain to correspond to a translation in the matched domain. To that end, we

[4] That is, one that exceeds the identity in the induced order.

define a matching relation to be *translation consistent* if and only if for each translation τ of $\mathscr{X}$ there is a translation σ_τ of $\mathscr{S}$ such that for all x in X, s in S,

$$xMs \Leftrightarrow \tau(x)M\sigma_\tau(s). \tag{6}$$

The first result shows that this qualitative definition of preserving ratios does in fact correspond to preserving numerical ratios:

THEOREM 1: *Suppose that* (i) $\mathscr{X}$ and $\mathscr{S}$ *are unit structures with real-unit representations* ϕ *and* ψ *onto* Re^+, and (ii) M *is a matching relation between them. Then* M *is translation consistent if and only if there exist positive constants* K *and* ρ *such that for all* x *in* X, s *in* S

$$xMs \Leftrightarrow \psi(s) = K^{\phi}(x)^{\rho}. \tag{7}$$

Theorem 1 is, in essence, a qualitative reformulation of the widely misunderstood and controversial result of Luce (1959, 1962) which was commented on critically by Rozeboom (1962a, 1962b). It is clear from the current formulation that the issue has nothing to do with changes of unit in the representation, which was how I originally phrased it, but rather with stimulus changes that correspond to translations. The two formulations are identical at the representational level, but from an empirical point of view there is an enormous difference. The concept of translation consistency is empirically testable, whereas statements about changes of unit are decidedly not empirical—whether one uses inches or centimeters is a matter of convention and convenience, not of empirical substance.

As an example of translation consistency, let $\mathscr{X} = \langle X, \succsim, \circ \rangle$ (e.g., pure tone intensities) and $\mathscr{R} = \langle Re^+, \geq, + \rangle$, with I: $\mathscr{X} \to \mathscr{R}$. Thus, I is the usual measure of intensity. Let $\Delta(I)$ denote the Weber (or jnd) function. $\Delta(I)$ induces a matching relation M from $\mathscr{X}$ to $\mathscr{X}$, namely,

$$xMz \text{ provided } I(x) + \Delta[I(x)] = I(x \circ z). \tag{8}$$

By Theorem 1, the "near-miss" to Weber's law, $\Delta(I) = cI^{1-\beta}$, which is an empirical generalization found in the literature on pure tones (Jesteadt, Wier, & Green, 1977; McGill & Goldberg, 1968), holds if and only if the qualitative Weber function, M, is translation consistent.

SIMILAR MATCHING RELATIONS

Given a matching relation, other matching relations that are, in a sense, similar to it can be constructed from it simply by transforming every pair of matching elements by unrelated translations in the two unit structure domains $\mathscr{X}$ and $\mathscr{S}$. So, formally, we say that any two matching relations M and N between the structures are *similar* if and only if there exist translations τ of $\mathscr{X}$ and σ of $\mathscr{S}$ such that for all x in X and s in S, if xMs then $\tau(x)N\sigma(s)$.

THEOREM 2: *Suppose that* (i) $\mathscr{X}$ *and* $\mathscr{S}$ *are unit structures with real-unit representations onto* Re^+, and (ii) M *and* N *are translation consistent matching relations between* $\mathscr{X}$ *and* $\mathscr{S}$. *Then the following statements are equivalent:* (a) M *and* N *are similar.* (b) *The power function representations of* M *and* N *have the same exponent.*

RATIO RELATIONS

Our next task is to consider procedures in which pairs of stimuli are matched so as to maintain ratios. To that end, suppose M is a matching relation between ordered relational structures $\mathscr{X}$ *and* $\mathscr{S}$. A function R from $X \times X$ onto $S \times S$ is said to be a *ratio relation relative to* M provided that it is strictly increasing in each component and for every x,y in X and r,s in S, $(x,y)R(r,s) \Leftrightarrow$ for some translation θ of $\mathscr{X}$ both $\theta(x)Mr$ and $\theta(y)Ms$.

THEOREM 3: *Suppose that* M *is a translation-consistent matching relation between unit structures* $\mathscr{X}$ *and* $\mathscr{S}$ *with real-unit representations* ϕ *and* ψ *onto* Re^+.

(i) *Then these exists a unique ratio relation* R *relative to* M, *namely,*

$$(x,y)R(r,s) \Leftrightarrow \psi(r)/\psi(s) = [\phi(x)/\phi(y)]^{\rho}, \tag{9}$$

where ρ *is the exponent of the power representation of* M.

(ii) *If* M *and* N *are similar, translation-consistent matching relations, then they have the same ratio relation* R.

(iii) *If* R *is any relation for which Eq. (9) holds for some exponent* ρ *and* M *is any relation from* $\mathscr{X}$ *to* $\mathscr{S}$ *that satisfies Eq. (7) with the same exponent* ρ, *then* R *is a ratio relation relative to* M.

It is easy to verify that Eq. (9) implies that R is dimensionally invariant in the following sense: If

(i) $\mathscr{X}$ and $\mathscr{S}$ are unit structures,
(ii) $R: X \times X \rightarrow S \times S$ is strictly monotonic in each component, and
(iii) R satisfies Eq. (9),

then for any translations τ of $\mathscr{X}$ and σ of $\mathscr{S}$,

$$(x,y)R(r,s) \Leftrightarrow (\tau(x),\tau(y))R(\sigma(r),\sigma(s)). \tag{10}$$

However, Eq. (10) does not imply Eq. (9), but only that there is a strictly increasing function F such that:

$$(x,y)R(r,s) \Leftrightarrow \psi(r)/\psi(s) = F[\phi(x)/\phi(y)]. \tag{11}$$

As Krantz (1972) showed, if ratio consistency is demanded among ratio relations between more than two ordered systems, then except for a single strictly increasing function, everything is related as powers of ratios of scale values. The only way the unknown function was eliminated was to insist that one of the ratios had a special status. In accounting for the regularity found with single stimulus procedures, he was forced to assume an implicit standard stimulus with respect to which ratios are computed. This formulation of Shepard's (1978, 1981) earlier ideas is called *relation theory*.

CONCLUSION

Krantz (1972) concluded: "The principal reason for favoring relation theory over the others is that it gives a satisfactory account of generalization (iv): that magnitude estimates predict cross-modality matches, independent of the choice of the moduli in the cross-modality matching" (p. 174). His theory has the failing that ratios are preserved only up to an unknown function.

The present theory, which identifies stimulus ratios with translations in the qualitative structure describing the stimulus domains, meets all his criteria listed at the beginning (see the section "Krantz's Five Empirical Generalizations") without introducing a free function. It does this by introducing the idea of preserving stimulus ratios in the form of translation consistency, which is an empirically testable property. This property embodies the intuitive idea that the psychology of the situation (i.e., the matching relation) should be completely consistent with the physics of the situation (i.e., the two qualitative relational structures being matched). Should translation consistency fail, then the laws of matching simply cannot be formulated in a manner similar to the laws of physics. Put another way, the psychophysics of matching would not be an extension of physics.

The fact that approximate power functions are found empirically for certain attributes having to do with stimulus intensity suggests that translation consistency may be valid there, which has led to a search for sources of bias to account for discrepancies from power functions. What does not fit at all well into this view of psychophysics is the evidence that the exponents are subject to easy manipulation (King & Lockhead, 1981).

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