Three Aspects of the Effectiveness of Mathematics in Science*

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*This paper was supported, in part, by National Science Foundation Grant IRI-8996149 to the University of California, Irvine.

1. Introduction

Wigner (1960), in a widely read and cited article, articulated what had previously been recognized by many scientists, namely, the remarkable affinity between the basic physical sciences and mathematics, and he noted that it is by no means obvious why this should be the case. The remarkableness of this fact is obscured by the historical co-evolution of physics and mathematics, which makes their marriage appear to be natural and foreordained. But serious philosophical explanations for the underlying reasons are not many.

This paper dissects the normal use of mathematics in theoretical science into three aspects: idealizations of the scientific domain to continuum mathematics, empirical realizations of that mathematical structure, and the use in scientific arguments of mathematical constructions that have no empirical realizations. The purpose of such a dissection is to try to isolate gaps in our knowledge about the *justified* use of mathematics in science. We believe that the apparent "unreasonableness" referred to in Wigner's title "The unreasonable effectiveness of mathematics in the natural sciences" arises from such lack of knowledge, rather than to some principled difference in the nature of mathematics and that of science.

Of the three aspects mentioned, the second – empirical realizations of mathematical structure – is the one with the largest body of positive results. A number of these are recent, and a major portion of the paper is devoted to outlining some of them.[a]

2. Infinite Idealizations

In the sciences, the domains of interest are usually finite. But, from a human perspective, such domains are often "large" and "complex" and typically, scientists idealize them to infinite ones – to something ontologically much larger and more complex than the original domain. Paradoxically, the resulting models are frequently much more manageable mathematically than the more realistic, finite models. Although such idealized domains are necessarily not accurate descriptions of the ones of actual interest, these infinite, ideal descriptions are nevertheless useful in science. In the authors'

[a]The reader who wishes to know more about this and related research in the theory of measurement is referred to two expository papers by the authors, Luce and Narens (1987) and Narens and Luce (1986), and to two sets of books: Krantz *et al.* (1971), Suppes *et al.* (1989), and Luce *et al.* (1990); and Narens (1985; in preparation).

view, this is because the focus of science is generally not on giving exact descriptions of the entire domain under consideration – such descriptions would generally prove to be too unwieldy and too complex to be of use – but rather with the production of generalizations that capture particular, scientifically significant features of the domain and that interrelate nicely with other generalizations about related domains.

Many kinds of generalizations used in science require us to ignore certain properties that are inherent in finite models; and others emphasize properties that are valid only in infinite ones. For example, in finite, ordered domains, maximal and minimal elements necessarily exist. Further, a desirable generalization may exclude such elements, as for example: "For each American middle class individual, there is a slightly richer American middle class individual." Or a useful generalization may use the concept of a homogeneous domain (as explained below) – a condition that is a basis of many scientific laws and one that usually requires infinite domains.

The crux of the matter is that science – at least as currently practiced – relies on special kinds of generalizations rather than exact descriptions, and such generalizations necessarily need infinite domains, and hence actual scientific domains need to be idealized to infinite ones. There are many ways to carry out such idealizations, and unfortunately at the present time there are, in our view, no acceptable formal theories of idealization. While this is obviously a very important problem in the foundations of science, it appears to be a very difficult one. Various strategies have been suggested – e.g., idealize through some use of potential infinity or through some recursive process – but so far these attempts have failed to capture the full power of the current practices of mathematical science.

Intuitively, the idealization of a finite domain should be to some denumerably infinite one. However, mathematical science routinely employs nondenumerable domains (e.g., continua, finite-dimensional Euclidean space, etc.) for idealizations, because these have special, desirable modeling properties that are not possible for denumerable domains. (E.g., for continua X the proposition, "Each continuous function from a closed interval of X into X takes on maximum and minimum values," is true, whereas for densely ordered, denumerable X it is false.) While it is an interesting exercise to try and replace assumptions that imply nondenumerability of particular classes of scientific models with others that are consistent with denumerability, we will not go into this topic here, since for this paper it is somewhat peripheral.

Instead, we will simply take as a starting point that science employs idealizations of actual (non-mathematical) domains, and that these qualitative idealizations, which are usually infinite (and often nondenumerable) are used qualitatively to express generalizations about the original domain. But we emphasize that this is a major lacuna in the philosophy of science, one about which we do not, at present, have anything useful to say. In our opinion, understanding exactly when such idealizations are helpful is one major gap in understanding the effectiveness of mathematics in science and is a part of its perceived "unreasonableness".

With such idealizations accepted, the next problem is then to explain *why mathematics is so effective in drawing useful inferences about the qualitative, idealized domain.* To investigate this, the properties of qualitative structures that permit their quantification will be looked at first; that is, we will first look at how numbers enter in science through processes of measurement.

3. Empirical Bases for Number Systems

The number systems most used in science are algebraic subsystems of the ordered field of real numbers,[b] $\langle \mathrm{Re}, \geq, +, \bullet \rangle$. There are many such subsystems, and for the purposes of this paper, the most important ones are the subsystems of positive integers, positive rationals, and positive reals, respectively $\langle I^+, \geq, +, \bullet \rangle$, $\langle Ra^+, \geq, +, \bullet \rangle$, and $\langle Re^+, \geq, +, \bullet \rangle$. The subsystem of positive integers has two important empirical realizations: a system based on ordered counting (ordinal numbers) and a system based upon matching finite sets in terms of their being in one-to-one correspondence (cardinal numbers). Although these two realizations are *empirically different*, they are *algebraically identical*, since each can be shown to be isomorphic to the algebraic system of positive integers.

The system of positive rationals has a variety of empirical realizations, some in terms of the realizations of the system of positive integers. These latter will not be discussed in this paper for lack of space. Instead, we will focus on those that result from extensive measurement (discussed below), which is basic to much of mathematical science and which can also produce empirical realizations of the system of positive real numbers.

[b] It has become conventional in part of the measurement literature to use *Re*, *Ra*, and *I* for the real numbers, the rationals, and the integers, respectively, and to superscript them with + to denote the positive restrictions.

In understanding the role of numbers in science, it is important to describe the *empirical correlates* to the number domain, to the numerical ordering relation $\geq$ on it, and to the operations + and $\bullet$ on it. In our view, this is not accomplished successfully for continuous domains by the well-known classical mathematical construction of the nineteenth century due to Peano and Dedekind, which constructs the system of the positive rationals out of the positive integers, and then the system of positive real numbers out of the positive rational numbers. The reason for its failure is that the *empirical correlates* that one wants for addition and multiplication on the reals are *not related in any empirical way via the construction* to the empirical realization(s) of the positive integers.

The way we shall proceed in this paper is to start with algebraic structures in which the domain, ordering relation, and addition operation are empirically realizable, and then investigate various empirical ways to realize multiplication.

3.1. Extensive Measurement

Extensive structures are algebraic structures of the form $\mathcal{X} = \langle X, \geq', \bigcirc \rangle$, where X is a non-empty set of objects; $\geq'$ is a binary relation on X; and $\bigcirc$ is binary operation on X, called the *concatenation operation*. It is assumed that:

(i) $\geq'$ is a total ordering;
(ii) $\bigcirc$ is associative and commutative;
(iii) $\bigcirc$ is monotonic in the sense that it is strictly increasing relative to $\geq'$ in each variable;
(iv) $\bigcirc$ is positive ($x \bigcirc y >' x$), where $>'$ is the strict part of $\geq'$; and
(v) the structure $\mathcal{X}$ is such that all elements are "commensurable" in the sense that no two distinct elements of X are infinitely far apart nor infinitesimally close together in terms of the ordering and concatenation operation. (Elements x and y, where $x >' y$, are said to be "infinitely far apart" if and only if for all finite positive integers n, n concatenations of y remains strictly less than x, where for example "three concatenations" of y is defined by $(y \bigcirc y) \bigcirc y$. "Infinitesimally close" has a similar but slightly more complicated formulation.)

The important theorem about extensive structures is that they are isomorphically imbeddable in the numerical structure $\langle Re^+, \geq, + \rangle$, and from this it follows immediately that empirical realizations of extensive structures are empirical realizations of subsystems of the positive additive reals.

Extensive structures have many empirical realizations, including many of the basic physical dimensions such as length, charge, mass, etc. For example, for physical length, concatenation is accomplished by placing two perfectly straight measuring rods end-to-end in a line and forming a third rod by abutting them; and $\geq'$ is determined by putting two rods side-by-side in the same direction with left endpoints corresponding and observing which spans the other.

In empirical realizations, the extensive ordering is the empirical correlate to the usual numerical ordering, and the extensive concatenation operation is the empirical correlate to numerical addition. What is missing is an empirical correlate to numerical multiplication. In measurement theory, the empirical correlate to multiplication rarely is a directly observed additional concatenation operation; it usually appears in much more subtle and indirect ways.

Throughout the rest of the paper, we will assume, unless explicitly stated otherwise, that the extensive structures discussed are either mappable onto $\langle Ra^+, \geq, + \rangle$ or $\langle Re^+, \geq, + \rangle$. These are the two most important situations and ones that can easily be described in terms of the defining relations, $\geq'$ and $\bigcirc$, of extensive structures.

We now consider three different ways in which multiplication can be empirically introduced.

3.2. One Empirical Base for Multiplication: Multiplicative Representations

The first way considers multiplication as transformed addition. Although this idea, in itself, will not suffice to explain multiplication in the system of positive reals, in particular its distributivity over addition, it is nevertheless instructive to examine it.

Extensive structures $\langle X, \geq', \bigcirc \rangle$ have isomorphisms into $\langle Re^+, \geq, + \rangle$ and thus have isomorphisms into $\langle (1, \infty), \geq, \bullet \rangle$. By deleting positivity from the assumptions of an extensive structure, a generalization results that has isomorphisms into $\langle Re^+, \geq, \bullet \rangle$, and thus may have an identity element (i.e., an element mapping into 1) and negative elements (those mapping onto (0,1)). In science, such generalized extensive structures often appear in indirect ways, especially in those situations where an attribute is affected by two (or more) factors that can be manipulated independently. Examples abound. In physics, varying either the volume and/or the substance filling the volume affects the resulting ordering by mass. In psychology and economics, varying the amount of a reward and the delay in receiving it each affects its ordering according to value. For a detailed discussion, see Chap. 6 of Krantz *et al.* (1971).

Abstractly, there are two disjoint sets X and P and a relation $\succsim$ on $X \times P$, where $(x,p)\succsim(y,q)$ is interpreted to mean that (x,p) exhibits at least as much of the attribute in question as does (y,q). Four empirically testable properties are invoked:

(i) $\succsim$ is a *weak order*, i.e., $\succsim$ is *transitive* – if $(x,p)\succsim(y,q)$ and $(y,q)\succsim(z,r)$, then $(x,p)\succsim(z,r)$ – and *connected* – either $(x,p)\succsim(y,q)$ or $(y,q)\succsim(x,p)$.

(ii) *Independence*, which amounts to monotonicity in each factor separately, i.e., if $(x,p)\succsim(y,p)$ for some p, then $(x,q)\succsim(y,q)$ for all $q \in P$, and a similar statement for the P-component.

From these two properties, it follows that there is a unique weak order induced on each factor, which is denoted $\succsim_i, i = X, P$, namely $x \succsim_X y$ if $(x,p)\succsim(y,p)$ for some p, and a similar statement for $\succsim_P$.

(iii) Each $\succsim_i$ is a total order.

The final property is a cancellation one that involves two equivalences implying a third.

(iv) *Thomsen Condition*: if $(x,r) \sim (z,q)$ and $(z,p) \sim (y,r)$, then $(x,p) \sim (y,q)$.

As in the case of an operation, two less testable conditions are also invoked. One is a form of solvability which in its strongest version asserts that given any three of $x,y \in X, p,q \in P$, the fourth exists such that $(x,y) \sim (p,q)$. The other is an Archimedean property that guarantees that all elements are commensurable. From these, one then shows that an operation can be induced on a component, say $\bigcirc_X$ on X, that captures all of the information about the conjoint structure, and that $\langle X, \succsim_X, \bigcirc_X\rangle$ is a generalized extensive structure with an isomorphism into $\langle Re, \geq, +\rangle$. This representation can then be reflected back to give a representation of $\mathcal{C} = \langle X \times P, \succsim\rangle$, namely, there are two positive real functions $\psi_i, i = X, P$, such that their sum is order preserving, i.e.,

$$(x,p)\succsim(y,q) \text{ iff } \psi_X(x) + \psi_P(p) \geq \psi_X(y) + \psi_P(q)\,.$$

Such structures are called *additive conjoint* ones. By taking exponentials, the representation is transformed into a *multiplicative representation* (θ_X, θ_P), i.e.,

$$(x,p)\succsim(y,q) \text{ iff } \theta_X(x)\theta_P(p) \geq \theta_X(y)\theta_P(q)\,,$$

where $\theta_i = \exp \psi_i, i = X, P$. There is, so far, no reason to favor one kind of representation over the other since we do not have distinct notions of addition and multiplication that should be related by the usual distribution property, $r(s+t) = rs + rt$. We turn to that approach next.

3.3. A Second Empirical Base for Multiplication: Automorphisms

Let $\mathcal{X} = \langle X, \geq', \bigcirc\rangle$ be an extensive sturcture. Consider the class of transformations of X *onto* itself that leave $\geq'$ and $\bigcirc$ invariant – the *automorphisms* of the structure. These, of course, capture symmetries of the structure in the sense that everything "looks the same" before and after the transformation. Although the concept of "automorphism" is highly abstract, individual automorphisms and sets of automorphisms are often realized empirically. For example, in the structure $\mathcal{X}$ it is easy to verify empirically that functions like $g(x) = x \bigcirc x$ and $h(x) = (x \bigcirc x) \bigcirc x$ are automorphisms.

A very important fact about the usual axiomatization discussed above is that the structure is *homogeneous* in the sense that each element "looks like" each other element: given any two elements, there is an automorphism that takes the one into the other. Another important fact is that it is *1-point unique* in the sense that if two automorphisms agree at one point, then they agree at all points. Both homogeneity and 1-point uniqueness, despite their abstractness, play an important role in the *empirical* aspects of measurement theory. This is because they are often implied by purely empirical considerations.

Let m be an isomorphism of $\mathcal{X}$ onto $\mathcal{R} = \langle R, \geq, +\rangle$, where R is either Ra^+ or Re^+. Let α denote an automorphism of $\mathcal{X}$, and for $x \in X$, let $m(x)$ denote the number assigned to it. Then a function f_α is defined on R by:

$$f_\alpha[m(x)] = m[\alpha(x)]\,.$$

Since the automorphisms form a group with many nice properties (see below), these functions combines nicely and define the operation we know as multiplication. Among other things, it is related to addition by the familiar law of distribution, which reflects nothing more than the fact that it arises from automorphisms: For suppose $x, y \in X$. Then

$$\begin{aligned} f_\alpha[m(x) + m(y)] = f_\alpha[m(x \bigcirc y)] &= m[\alpha(x \bigcirc y)] \\ &= m[\alpha(x) \bigcirc \alpha(y)] \\ &= m[\alpha(x)] + m[\alpha(y)] \\ &= f_\alpha[m(x)] + f_\alpha[m(y)]\,. \end{aligned}$$

It is well-known that this equation defines multiplication in the usual sense, i.e., we may write for some positive *real* r_α,

$$f_\alpha[m(x)] = r_\alpha m(x) .$$

In this notation, the automorphism property becomes simply

$$r(s+t) = rs + st ,$$

which is referred to as the *distribution* of multiplication over addition.

It should also be noted that one can show that changing the element to which the numeral 1 is assigned induces an automorphism of the structure, and so a change of unit is reflected in the numerical representation as multiplication by a constant.

3.4. A Third Empirical Basis for Multiplication: Distributive Operations

Consider an additive conjoint structure $\mathcal{C}$ that has an independently given operation $\bigcirc$ on the first component such that $\mathcal{X} = \langle X, \succsim_X, \bigcirc \rangle$ satisfies the properties of an extensive structure. We then say that the operation $\bigcirc$ *distributes in* $\mathcal{C}$ if and only if for each $x, y, u, v \in X, p, q \in P$, if $(x,p) \sim (y,q)$ and $(u,p) \sim (v,q)$, then $(x \bigcirc u, p) \sim (y \bigcirc v, q)$. This is just the sort of property that holds when X is a domain of volumes, P of substances, and $\succsim$ is an ordering by mass. Let ϕ_X be an additive representation of $\mathcal{X}$. Then one is able to show that we may take $\theta_X = \phi_X$ in a multiplicative representation (θ_X, θ_P) of $\mathcal{C}$. Thus, the main consequence is that relative to the operation $\bigcirc$, the impact of the second component P is that of an automorphism of $\mathcal{X}$, since multiplications are the automorphisms of the isomorphic image $\langle Re^+, \geq, + \rangle$ of $\mathcal{X}$. So by what we showed earlier, ordinary distribution of multiplication over addition again obtains. For details, see Luce and Narens (1985) and Narens (1985).

3.5. What Else is Measurable?

For many sciences, there do not appear to be many, if any, observable extensive structures on which to base measurement. To some degree the situation is alleviated by the representations of additive conjoint structures, but even that continues to place a very high premium on additivity – either associativity of an induced operation or equivalently, at the observational level, the Thomsen condition. Many concatenation and conjoint situations simply do not admit additive or multiplicative representations. Does this mean that measurement of such variables is impossible? Some, starting with Campbell (1920, 1928), have held that one must either have an extensive structure at the empirical level or, as some now accept, be able to show that an extensive structure is implicit, as in additive conjoint measurement. The question is whether more is possible at the implicit level, and the claim of modern work is, "Yes, a lot more."

The key new idea for this was developed in a series of papers (Alper, 1987; Cohen and Narens, 1979; Luce, 1986, 1987; Narens, 1981a,b) and is summarized in Chapter 20 of Luce, Krantz, Suppes, and Tversky (1990). The major mathematical result, which was completed by Alper (1987), is that for any ordered structure *on* the positive real numbers that is both homogeneous (see the definition given in the section on automorphisms) and finitely unique in the sense that for some integer N, any two automorphisms that agree at N distinct points are identical, there exists an isomorphic structure on the positive real numbers for which the automorphism group lies between the similarity group $(x \rightarrow rx, r > 0)$ and the power group $(x \rightarrow rs^s, r > 0, s > 0)$. An important consequence is that each member of this widely diverse class of structures has a subgroup of the automorphisms – called the *translations* – that is isomorphic to $\langle Re^+, \geq, + \rangle$. This consequence is useful because we can characterize readily what this means qualitatively.

In general, for an arbitrary, totally-ordered structure $\mathcal{X} = \langle X, \geq', R_j \rangle_{j \in J}$, where each R_j is a relation of finite order on X, define the *translations* to consist of the identity map together with all automorphisms α of $\mathcal{X}$ that do not have a fixed point (i.e., for all $x \in X, \alpha(x) \neq x$). The set T of translations can be ordered as follows: For $\sigma, \tau \in T, \sigma \geq'' \tau$ if $\forall x \in X, \sigma(x) \geq' \tau(x)$. The critical assumption is that $\mathcal{T} = \langle T, \geq'', * \rangle$ is an Archimedean, totally-ordered group, where $*$ denotes function composition. If this is so, then $\mathcal{T}$ is isomorphic to a subgroup of $\langle Re^+, \geq, + \rangle$. This is essentially the same construction as for extensive structures. We will refer to $\mathcal{T}$ as the *implicit extensive structure* of $\mathcal{X}$.

If, further, $\mathcal{T}$ is homogeneous, then it is not difficult to show that $\mathcal{X}$ can be mapped isomorphically into the translation group, so that the translations of the isomorphic image of $\mathcal{X}$ are left multiplications of the translation group of $\mathcal{X}$. By the above, $\mathcal{X}$ is isomorphic to a numerical structure that has multiplication by positive real numbers (i.e., similarities) as its translations. Thus, measurement is effected for any ordered relational structure whose translations form a homogeneous, Archimedean ordered group. This class is far from vacuous: For example, consider any operation $\bigcirc$ on the positive reals, Re^+, for which there is a function $f : Re^+$ onto Re^+ such that f is strictly increasing, $f(x)/x$ is strictly decreasing, and $x \bigcirc y = yf(x/y)$. Then the structure $\langle Re^+, \geq, \bigcirc \rangle$ has as its translations the similarity group. The details can be found in Cohen and Narens (1979) and Luce and Narens (1985).

If this sort of implicit extensive measurement is deemed acceptable – and there is a growing body of literature which suggests that it is – then the study of measurement reduces to uncovering the properties of the set of translations. Doing so is not necessarily easy. An important problem is to give empirical conditions that for a broad class of situations will insure that the translations form a homogeneous group and reasonable conditions that will insure it is Archimedean. Some progress along these lines has been reported.

A question can be raised about these general structures that closely parallels the issue of the distribution of an extensive structure in a conjoint one.

3.6. Distribution of General Relational Structures in Conjoint Ones

Given our more general notion of measurement based upon an implicit extensive structure for the translations, one must inquire about the extent to which it behaves in familiar ways. In particular, it is important to understand the extent to which such general measures can be incorporated into the structure of physical quantities, which classically has involved just extensive structures distributing in conjoint ones. The first issue is to arrive at a suitable concept of distribution. This is done in two steps. First, given a conjoint structure $\mathcal{C} = \langle X \times P, \succsim \rangle$, two ordered n-tuples (x_i) and (y_i) from X are said to be *similar* if and only if there are $p, q \in P$ such that for $i = 1, \ldots, n$, $(x_i, p) \sim (y_i, q)$. Now, consider a relational structure on the first component of $\mathcal{C}$, namely, $\mathcal{X} = \langle X, \succsim_X, R_j \rangle_{j \in J}$, where each R_j is a relation of finite order on X. We say that $\mathcal{X}$ *distributes over* $\mathcal{C}$ if and only if for each $j \in J$, if (x_i) and (y_i) are similar ordered $n(j)$-tuples and (x_i) is in R_j, then (y_i) is also in R_j. A careful examination of the earlier definition of distribution for an extensive structure shows it to be a special case of the more general concept.

The key result (Luce, 1987) is that if the translations of a relational structure are homogeneous, form a group under function composition (or equivalently, are 1-point unique), and are Archimedean under the order

defined earlier, then that structure is isomorphic to a relational structure on the first component of an additive conjoint structure and it distributes over the conjoint structure. Conversely, if an additive conjoint structure has a relational structure on the first component that distributes over the conjoint structure, then the translations of the relational structure form a homogeneous, Archimedean-ordered group.

The significance of this result is that to the extent we are interested in measurement that relates to conjoint structures via the distribution property, then the condition that the translations form a homogeneous, Archimedean-ordered group is exactly what is needed because it provides for the existence of the proper numerical setting in which addition and multiplication arise simultaneously with natural emprical correlates.

4. The Use of Non-Empirical Mathematics

We take the following generalization of the above remarks as one of our basis theses: *In many empirical situations considered in science – particularly in classical physics – there is a good deal of mathematical structure* already present *in the empirical situation. Measurement produces numerical correlates of that structure.*

We have seen several ways in which the algebraic system of positive real numbers can arise in science as numerical correlates of empirical concepts. By extending the methods, the system can be generalized to include powers and logarithms. More inventive methods could probably produce additional empirical correlates to simple concepts of integration and differentiation. Thus, it is reasonable to expect that important parts of elementary mathematics concerned with analysis have empirical realizations. If science only used such "elementary" means, then the scientific effectiveness of mathematics would be easily understood. However, science also uses additional sophisticated mathematical concepts and methods to get results, and many – if not most – of these methods have no empirical correlates. Symbolically, this situation can be expressed as follows:

An empirical situation E has an empirical mathematics E_M associated with it, that through a measurement representation, m, is isomorphic to a fragment M_E of mathematics, M. A scientist uses M, sometimes including portions of $M - M_E$, to get a result r about E. For this result to be 'about E', it must somehow be translatable into E. With what has so far been given, this can only be done if r is a result about M_E and is translatable into E via m^{-1}. In other words, M is used so that $m^{-1}(r)$

can be concluded about E_M, which in turn is used to draw the empirical conclusion r about E. If this use of M is "unreasonably effective", then its "unreasonableness" consists in using some part of $M - M_E$ to draw conclusions about M_E, a matter formulated purely in terms of mathematics, and one whose explanation we believe is likely to be found within mathematics itself.

Put another way, it is commonplace in science that non-interpretable mathematics – that is, mathematics with no empirical correlates – is used to draw mathematical conclusions that are empirically interpretable. In our opinion, such practices have not been adequately justified.

5. Conclusion

Our analysis has had three parts:

First, actual empirical situations are usually conceptualized as structures on large finite sets. Because of the complexity of such structures and the irregularities and non-homogeneity often necessarily inherent in them, the actual empirical situation is often idealized to an infinite "empirical" situation, where the irregularities and non-homogeneities disappear; that is, the actual situation is idealized to a more mathematically tractable structure. A well reasoned account of the conditions under which such idealizations are acceptable is a major unresolved problem in the philosophy of science.

Second, in many important areas of science, such idealized empirical structures contain – often in non-obvious ways – a good deal of mathematical structure, which through the process of measurement is realized as familiar mathematical structures on numerical domains. So it is not unreasonable to expect the kinds of mathematics inherent in such numerical realizations to be effective in producing conclusions that can be translated back to the idealized empirical situations (and thereby to the actual empirical situations) by inverting the measurement process. This aspect of the problem is relatively well developed and understood.

Third, and what remains a mystery, is why mathematics outside of the numerical realizations should be so (unreasonably) useful in determining correct results about those realizations. This appears to be more of a problem in the philosophy of mathematics than in the philosophy of science.

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