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"On the Possible Psychophysical Laws" Revisited: Remarks on Cross-Modal Matching

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A nonrelational theory of cross-modal matching, including magnitude estimation, is proposed for the general class of 1-dimensional measurement structures that have real-unit (ratio and interval scale) representations. A key feature of these structures is that each point of the structure can be mapped into each other point by a translation, which is the structural analogue of ratio scale transformations of the representation. Let M denote a 1:1 matching relation between 2 (not necessarily distinct) unit structures. The major assumption is that for each translation τ of one structure, there is a translation σ_τ of the other such that if xMs, then $\tau(x)M\sigma_\tau(s)$. This property is shown to be equivalent to a power law holding between the unit representations. A concept of similar relations is taken from dimensional analysis, and 2 matching relations are shown to be similar if and only if their power laws differ only in the unit (modulus), not the exponent. A relation R between pairs in each system is said to be a ratio relation relative to a matching relation M satisfying the above condition provided that $(x, y)R(s, t)$ obtains if and only if for some translation τ both $\tau(x)Ms$ and $\tau(y)Mt$. In that case, R is unique and is represented by the ratio of the values assigned to x and y in their unit representation being equal to a power of the ratio of the values assigned to s and t in their unit representation; the exponent is that of M. This theory is related to an earlier article by the author and is compared with the relational theory of Shepard (1978, 1981) and Krantz (1972).

The following remarks arise from my reflections on four articles and commentaries on them. Historically, the first was my 1959 article "On the Possible Psychophysical Laws," which attempted to account for why two ratio-scaled variables, such as those encountered in the simplest version of cross-modal matching, should be related by power functions. In it, I postulated that if x and y are two ratio-scaled variables that are related by some law $y = f(x)$, where f is a strictly increasing function, and if the units of x are changed by a ratio transformation r, then there is a corresponding ratio change, $s(r)$ of y such that for all positive x and r,

$$s(r)f(x) = s(r)y = f(rx). \quad (1)$$

As is easily demonstrated, this functional equation for f implies that f is a power function.

The key issue surrounding the article, which was first critically discussed by Rozeboom (1962a, 1962b), is, Why should one assume Equation 1? I had spoken of it as a "principle of theory construction," thinking it to be on a par with the dimensional invariance of physical laws postulated in the method of dimensional analysis (for a detailed discussion of that method, see chapter 10 of Krantz, Luce, Suppes, & Tversky, 1971, and chapter 22 of Luce, Krantz, Suppes, & Tversky, in press). In the face of Rozeboom's criticism, I (1962) retreated from that position. Later, I (1964); Aczél, Roberts, and Rosenbaum (1986); Roberts and Rosenbaum (1986); and Osborne (1970) studied various generalizations of Equation 1, and Falmagne and Narens (1983) gave a detailed analysis of a collection of closely related principles, showing how they interrelate and what they imply about the forms of laws. Despite these interesting developments, I have never felt that we have gained a full understanding of what is really involved empirically in assuming Equation 1, except I have come to recognize that it is not really the specialization of dimensional invariance that I had thought it was in 1959.

The next two articles were by Krantz (1972) and Shepard (1978, 1981)[1] on modeling magnitude estimation and cross-modal matching.[2] Krantz, who saw a manuscript version of

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[1] The Shepard work was first written in 1966, circulated in draft, revised and circulated in 1968, published in 1978, and reprinted in 1981.

[2] In an earlier article, Krantz (1968) developed an abstract theory of matching when the stimuli to be matched are from the same modality but perception of them is affected by their surrounds. This was motivated primarily by work on color matches with different, colored surrounds. Because this is inherently more than one dimensional, it is outside the main thrust of the present work. However, because there are

Shepard's work, attempted to give a deeper, formal axiomatization of what Shepard suggested may be involved. Both articles rest on the idea that such direct psychophysical judgments always entail a comparison of stimulus ratios, possibly with an implicit standard. The fact that some cross-modal matching and magnitude estimation experiments make no reference to a standard is not accepted at face value; Krantz and Shepard contended that some form of implicit standard plays a role in all judgments of this character.

Shepard (1981, p. 36) made an additional argument to the following effect: Absolute judgments are highly variable, whereas discriminations between stimulus ratios are much less so, and subjects should therefore favor ratio judgments when they have the opportunity because that will lead to higher quality performance. Discriminations between two stimuli certainly provide evidence for a less variable internal representation of signals than is found with the absolute identification. And in certain domains, such as auditory pitch, frequency ratios probably can be absolutely identified more accurately than can absolute frequencies, at least if an octave is not spanned. However, in regard to the intensity of signals, neither interpretation seems relevant when considering procedures such as magnitude estimation and cross-modal matching, both of which are far more similar to absolute identification than to discrimination. For such modalities, the relevant studies are of variability in magnitude estimates, ratio estimates, and cross-modal matches. These show a variability comparable with and often greater than that found in absolute judgments (Green & Luce, 1974; Green, Luce, & Duncan, 1977; and Luce, Green, & Weber, 1976). Matching judgments of signal intensity behave more like absolute judgments and less like ratio discriminations, so I do not believe that Shepard's (1978, 1981) argument has force for signal intensity. The theory described herein does not appear to be applicable to the more qualitative modalities, such as pitch.

Finally, Krantz's (1972) theory, which is based on such ratio judgments, is not completely satisfactory because it results in matching relations that are determined only up to a free, strictly increasing function. He removed this lack of specificity only by taking one continuum—he selected length—as being matched veridically. I discuss this issue more fully in the Ratio Relations and Conclusions sections.

One goal of this article is to develop a matching theory that is based entirely on qualitative considerations and that is free from such an indeterminacy. Within that context, the discussion in the Ratio Relations section makes clear exactly why an indeterminacy arises in Krantz's (1972) formulation but does not in my theory.

The fourth article stimulating my work is an article of mine published in 1987, which is an outgrowth and generalization of aspects of five articles and a book: Alper (1987), Cohen and Narens (1979), Luce (1986), and Narens (1981a, 1981b, 1985). This body of work has shown, at a general level, what it means for an ordered structure to have a numerical representation with ratio scale properties (this is described more fully in the next section, and the exact definition is in Appendix A). One important insight is that the concept of scale type, such as ratio scalability, which of course inheres to the structure itself, does not require a numerical representation to be studied. The relevant aspect of the structure is its symmetries, to use the physicist's term, or its automorphisms, to use the mathematician's term. These are the transformations of the structure onto itself that leave the structure unchanged. For example, for a square, the automorphisms are rotations through 90°, reflections about the diagonals, and reflections about midlines parallel to an edge. In other words, automorphisms are self-isomorphisms, that is, isomorphisms of the structure with itself.

Within measurement, a numerical structure defined on all of the real numbers is said to be an *interval scale structure* if the affine transformations $x \rightarrow rx + s$, where $r > 0$ and s is real, take the structure into itself. If only the translations $x \rightarrow x + s$ preserve the structure, then it is a ratio scale structure.[3] Basically, the two cases differ in that the group of admissible transformations (automorphisms) have 2 and 1 degrees of freedom, respectively. An interval scale structure is necessarily far more symmetric than a ratio scale structure because it has every symmetry of the ratio scale case (i.e., those with $r = 1$), as well as many others (all with $r \neq 1$). Indeed, one of the major scientific drawbacks to interval scale structures is their great symmetry, which does not admit many structural possibilities.

The number of degrees of freedom of the group of automorphisms can be studied for any structure, not just numerical structures or ones that are known to be isomorphic to numerical structures. In this research, the subset of automorphisms corresponding to the ratio case—the translations in the real structure—are found to be especially important in understanding the ratio change. These are defined, independent of any representation, to consist of the identity map, in which every point remains fixed, together with the automorphisms that have no fixed point. This is true of the real translations because $s = 0$ yields the identity map and for $s \neq 0$, $x + s \neq x$. Therefore, the terminology is generalized, and the identity and any automorphism without a fixed point are referred to as *translations*.

Three important properties of the translations imply the existence of a numerical representation with at least as much symmetry as the ratio scale property. To state them exactly requires some notation, which is given in Appendix A, but the gist is as follows. The action of composing one automorphism with another is like a combining or concatenation operation, such as is used in the measurement of mass and length. A partial ordering of the automorphisms can be defined in terms of the order of the structure. Under an appropriate restriction, basically that the degrees of freedom of the group of automorphisms is finite, this is a *total order.* That is, it is transitive and holds between each pair of automorphisms. Thus, the set of automorphisms is both ordered and has an operation of combining. The translations are different from all other automorphisms in that they

many points of similarity, it is discussed more fully at the end of the Ratio Relations section.

[3] Usually, one thinks of ratio scale transformations as of the form $x \rightarrow tx$, which is correct when the representation is on the positive real numbers, but when it is on the reals, a log transformation must be taken and the difference transformations therefore arise.

act like infinitesimals; they are not comparable. However, the translations can be compared with one another, and when they are, they have a simple numerical representation as the multiplication of positive numbers. The exact conditions that must be satisfied for this to be so are the conditions underlying what is known as extensive measurement (see Krantz et al., 1971, chap. 3). The most important are that translations acting on other translations yield translations, and the ordering satisfies the condition of commensurability known as Archimedeanness. The third condition, which is discussed later, assumes that there are many translations, one corresponding to each real number.

The important fact in this context is that a translation, which is an action on the qualitative structure, is more basic and interesting than is its numerical embodiment as a change of units. True, the class of possible representations mapping onto the positive real numbers can be characterized in terms of the translations, but quite independent of that, the translations provide a way of designating systematic changes within the qualitative domain. I use that fact to illuminate the meaning of Equation 1 and to develop a nonrelational theory of matching.

Ordered Relational and Unit Structures

The study of the foundations of scientific measurement has grown from a focus on relatively specific structures to very general ones. Among the specific types of structures that have been closely studied, three stand out. The earliest were extensive structures of the sort just mentioned. These were used to model several basic attributes of physics. They are ratio scale structures. The next was additive (or multiplicative) conjoint structures in which the ordering is over stimuli or entities that can be thought of as having n distinct components. If the effect of the components on the attribute underlying the order are suitably independent, then the numerical representation can be taken to be either additive on the real numbers or multiplicative on the positive real numbers; it does not matter which. Physicists have used the latter representation for such conjoint quantities as energy, work, density, and so forth. Behavioral scientists frequently study trade-offs exhibited by conjoint structures—speed and accuracy, equal loudness, amount versus delay of reward, and so forth—but rarely do these structures exhibit truly independent effects of the factors. When they do or are assumed to, then behavioral scientists typically use the additive rather than the multiplicative representation. The third type of structure is like the first in that both an order and an operation of combining exist, but they differ in what happens when two objects are combined. In the extensive case, the combined object exceeds either of its constituents. In the third type, it lies between them, and these are known as intensive or intern structures. The prototypic representation is as a weighted average. This representation also arises in physics (e.g., temperature), as well as in the behavioral sciences, where the best known example is subjective expected utility theory. Additive conjoint and intensive structures have interval scale representations.

In a natural but sweeping manner, these three measurement structures can be generalized to what is known as an *ordered relational structure.* Three kinds of primitives are involved. The first is a set X, which is intended to be identified with the objects or entities that one is trying to measure. The second is a (qualitative) ordering $\succsim$ of pairs of objects according to the attribute being measured. That is, for all x, y in X (abbreviated $\forall x \;\; y \in$ X), either x exhibits at least as much of the attribute as y, in which case, $x \succsim y$, or the other way around, $y \succsim x$. If both statements are true, then they exhibit the attribute in the same degree: $x \sim y$. The third type of primitive is any remaining structure, such as the operation of combining found in both extensive and intensive structures. The added structure is postulated to take the form of a finite number of additional structural relations, each of finite order, $X_1, X_2, \ldots, X_J$, where J is used to denote both an integer and the set of integers up to and including J. This structure is usually summarized by a name, $\mathscr{X}$, and by a listing of its terms: $\mathscr{X} = \langle X, \succsim, X_1, X_2, \ldots, X_J \rangle$ or, more compactly, $\langle X, \succsim, X_j \rangle_{j \in J}$.

A fairly general type of well-understood structure, encompassing extensive and intensive structures, is the class of concatenation structures, in which aside from $\succsim$, there is just one additional relation, and it is, in fact, an operation—a mapping of pairs from X into X. The operation is often denoted O, and it is assumed to be monotonically[4] related to the ordering $\succsim$, in the sense that

$$(\forall x, y, z)\; x \succsim y \quad \text{if and only if} \quad xOz \succsim yOz \qquad 2$$
$$\text{if and only if} \quad zOx \succsim zOy.$$

In words, concatenating the same (or equivalent) objects to two others does not alter the order between them. In this case, the structure is simply $\langle X, \succsim, O \rangle$ Any extensive structure is an example of a concatenation structure in which the operation exhibits certain special properties; specifically, it is associative,

$$(\forall x, y, z \in \mathrm{X}), \qquad (xOy)Oz = xO(yOz), \qquad 3$$

commutative,

$$(\forall x, y \in \mathrm{X}), \qquad xOy = yOx, \qquad 4$$

and positive,

$$(\forall x, y \in \mathrm{X}), \qquad xOy > \max(x, y). \qquad 5$$

In the intensive case, rather different assumptions are made because weighted averages fail all three of these properties.

As I mentioned earlier, a major avenue for studying these general structures is to inquire about their symmetries. Structures that are highly symmetric are the ones that have simple numerical representations and that have traditionally played a role in at least physical measurement. Formally, a symmetry or, as I shall refer to it, an *automorphism,* α, of any ordered relational structure $\mathscr{X}$ is a 1:1 mapping of the domain X of the structure onto itself that also "preserves" the structure. That is, it is an isomorphism of the structure with itself. Preserving the struc-

[4] The term *monotonic* is used because if the operation is treated as a function, that is, $xOy = f(x, y)$, then the property is the same as saying that the function is strictly monotonic, increasing in each of its variables.

ture simply means that for each relation X_j of order $n(j)$, if $(x_1, x_2, \ldots, x_{n(j)})$ is in the relation X_j, so then is $[\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_{n(j)})]$. Special cases of this condition are that $x \gtrsim y$ holds if and only if $\alpha(x) \gtrsim \alpha(y)$ and, in the case of a concatenation structure, $\alpha(x\text{O}y) = \alpha(x)\text{O}\alpha(y)$. The concept of an automorphism seems, at first, quite abstract, but it is an essential concept in understanding certain aspects about the numerical representation of structure and, in particular, to providing a clear meaning to Equation 1.

As I said earlier, any automorphism τ that does not fix any point—that is, for each $x \in X$, it is not the case that $\tau(x) \sim x$—is called a translation. It is also convenient to treat the automorphism that fixes every point, the identity, as a translation. As also mentioned earlier, for ordinary interval scale representations, the set of all affine transformations $x \to rx + s$, $r > 0$ forms the group of automorphisms of the representation, and $x \to x + s$ forms the subgroup of translations.

I now focus on a special class of ordered relational structures, the *unit structures*. These may be described in either of two equivalent ways. They are structures that are isomorphic to, or have exactly the same structure as, a real relational structure with the usual ordering of greater than or equal to ($\geq$) and in which the translations are represented by multiplication by positive constants. This includes structures with a ratio scale representation. The numerical representing structure is called a *real-unit structure*. The other way is to say that the translations are isomorphic to a subgroup of the additive real numbers and that they are *homogeneous* in the sense that any element is transformed into any other element under some translation:

$$(\forall x, y \in X) \quad \text{there exists a translation } \tau \text{ such that } \tau(x) \sim y.$$

For a precise definition of what is meant by a real-unit structure and for a qualitative formulation in terms of properties of the translations of the qualitative structure, see Appendix A.

In most of the following, the domain of the real-unit structures is assumed to be the positive real numbers, Re^+; that is, the qualitative structure is assumed to map onto the positive reals.

For example, consider a real concatenation structure $\langle \text{Re}^+, \geq, \otimes \rangle$. The subclass of real-unit concatenation structures is quite simple to describe (Luce & Narens, 1985). For each real-unit concatenation structure, there exists a function f from Re^+ mapping onto Re^+ such that for all $x, y \in \text{Re}^+$:

(i) f is a strictly increasing function: if $x > y$, then $f(x) > f(y)$.

(ii) $f(x)/x$ is a strictly decreasing function of x: if $x > y$, then $f(x)/x < f(y)/y$.

(iii) $x \otimes y = yf(x/y)$.

Properties (i) and (ii) correspond, under (iii), to the two sides of the monotonicity property of the operation $\otimes$.

Observe that the transformation $\tau(x) = tx$, where $t > 0$, is a translation. It either has no fixed point or all points are fixed ($t = 1$), it is order preserving, and $\otimes$ is invariant under τ because

$$\tau(x) \otimes \tau(y) = tx \otimes ty = tyf(tx/ty)$$
$$= tyf(x/y) = t(x \otimes y) = \tau(x \otimes y).$$

In what follows, I shall discuss matching in the framework of unit structures, their translations, and their real-unit representations. The reason for this level of generality is that it encompasses a broad class of one-dimensional physical stimuli and admits the possibility of matching other one-dimensional stimuli that are not structured in the additive fashion of many physical stimuli. Thus, the theory has the potential of being applicable to the matching of nonphysical attributes.

One-Dimensional Matching Relations

Intuitively, a matching relation simply associates in a systematic manner an element in one (possibly qualitative) relational structure to an element in a different (possibly qualitative) relational structure. For example, in magnitude estimation, a numerical value is matched to a physical stimulus, such as a light or sound of some intensity. In cross-modal matching, stimuli from one modality, such as sounds of varying intensity, are matched to stimuli from a different modality, such as lights also varying in intensity. The term *matching* is broad, and I shall only investigate the important special case of matching one-dimensional stimulus sets. So, for example, this article does not address metameric matches, which involve color stimuli that physically are of infinite dimension, nor does it deal with context effects of the type treated by Krantz (1968). The one-dimensional case is, of course, an idealization, not unlike the example of objects falling in a vacuum or of frictionless surfaces, which can only be partially achieved experimentally. For example, if one is matching tone intensity to brightness, the tones should be presented at one frequency in an acoustically quiet environment, and the lights should be monochromatic against a dark field.

To formulate formally this hypothesis of matching one-dimensional structures, I use the notation for ordered, relational structures.

Definition 1. Suppose that $\mathcal{X} = \langle X, \gtrsim_X, X_j \rangle_{j \in J}$ and $\mathcal{S} = \langle S, \gtrsim_S, S_k \rangle_{k \in K}$ are both ordered relational structures. A funetion M from X onto S is called a *matching relation* if and only if it is strictly increasing:

$$(\forall x, y \in X) \qquad x \gtrsim_X y \quad \text{if and only if} \quad M(x) \gtrsim_S M(y).$$

The following notations are commonly used to say that s is the stimulus that is matched to x under M: $s = M(x)$ and xMs.

Without any restrictions on the possible matching functions, there is little to say. There are at least two methods of imposing restrictions. One way is to provide an explicit perceptual theory that in one way or another characterizes mechanisms whereby a stimulus from $\mathcal{S}$ is selected to match one from $\mathcal{X}$. To carry out such an explicit perceptual theory requires that something more specific be imposed on the two structures involved than mere one dimensionality. The other, which I shall pursue, is to investigate a condition analogous to dimensional invariance in physics. In Luce (1978), I showed that the somewhat illusive property of dimensional invariance is really nothing more nor less than the assertion that a physical law remains invariant under automorphic transformations of the underlying physical dimensions, which is in essence the same as the meaningfulness property postulated by Stevens (1946, 1951) for statistical assertions. The condition investigated in this article is that not only do the elements correspond under the matching relation, but so do their automorphisms. This requirement is captured in the next definition.

Definition 2. A matching relation M is said to be *translation consistent* if and only if for each translation τ of $\mathcal{X}$ there is a translation σ_τ of $\mathcal{S}$ such that $(\forall x \in X, s \in S)$,

$$xMs \quad \text{if and only if} \quad \tau(x)M\sigma_\tau(s).$$

Obviously, any matching relation between two structures that each have the identity map as the only translation are, trivially, translation consistent. However, the cases to be considered are those in which the group of translations is homogeneous in that for each ordered pair of elements there is a translation that maps the first onto the second (see Appendix A), and so has many members. In the case of homogeneous translations, the apparent asymmetry of Definition 2 disappears, as is proved in Lemma 1 of Appendix B.

Note that the concept of translation consistency is closely related to the concept embodied in Equation 1, but it is different. It says nothing about representations of the structures or about how changes in the representations relate to one another. Rather, it asserts a way that transformations within the two structures describing the two modalities might systematically relate to one another—if one pair of stimuli match, then to each translation of one structure there is coordinated translation of the second structure that preserves the match. The concept says in the case of cross-modal matches of physical continua that the psychological substance embodied in the matching relation M is consistent with (or invariant under) certain pairs of translations from the two modalities that it relates. Assuming that the modalities are physical, as is often the case, the hypothesis that the matching relation is translation consistent amounts to the postulate: The psychology is conceptually consistent with the physics.

Unlike Equation 1, for which it is difficult to know what constitutes a suitable test, the form of consistency embodied in Definition 2 can be tested empirically provided that the translations are known. Of course, the test may be empirically falsified. By contrast, statements about representations and the numerical law relating them are not so clearly empirical.

The proviso that the translations be known is important. To test for consistency, one must know which 1:1 mappings of the structure are translations of the structure, and in general, there is no method for explicitly identifying translations in terms of the given structure. This may seem almost contradictory, but it is not. If one is given a 1:1 mapping of the structure onto itself, one can verify whether or not it is a translation. That is not the same as saying that any such function can be constructed. Fortunately, for one (important) case, there is an explicit way to construct many translations. This case includes the class of unit concatenation structures for which the operation is positive in that

$$(\forall x, y \in X) \qquad x \bigcirc y > \max(x, y).$$

Consider the function $\tau_2(x) = x \bigcirc x$. It is easy to verify, with the unit representation described earlier, that τ_2 is a translation. The key part of the argument is to note that if ϕ is the automorphism to the real unit representation, then

$$\phi(x) \otimes \phi(x) = \phi(x)f[\phi(x)/\phi(x)] = \phi(x)f(1).$$

By positivity, $f(1) > 1$, therefore, this translation is nontrivial. This observation generalizes to what are known as the "n-copy operators", which can be defined inductively as follows:

$$\tau_n(x) = \tau_{n-1}(x) \bigcirc x.$$

These are all translations (Cohen & Narens, 1979). Or, when the real-unit representations are available, as they are for the usual physical structures, then stimulus pairs in a constant ratio form the translations (Appendix B, Theorem 1). For additional remarks about the issue of isolating translations, see the section, A Method of Constructing Translations.

The first result asserts that for unit structures of any type (not just positive concatenation structures) the empirical content of translation invariance is exactly that the numerical representations of the unit structures are related by a power law. More specifically, suppose $\mathcal{X}$ and $\mathcal{S}$ are unit structures with real-unit representations ϕ and ψ onto the positive real numbers, and suppose that M is a matching relation between them (see Definition 1). Then it can be shown (Appendix B, Theorem 1) that M is translation consistent (see Definition 2) if and only if there exist positive constants K and ρ such that for all $x \in X$, $s \in S$,

$$xMs \quad \text{if and only if} \quad \psi(s) = K\phi(x)^{\rho}. \tag{2}$$

In this case, it is not difficult to show that the mapping from τ to σ_τ is an isomorphism of the translation groups of $\mathcal{X}$ and $\mathcal{S}$. Indeed, under the conditions of the theorem, if in the unit representations T and S_T are the translations corresponding to τ and σ_τ, then $S_T = T^{\rho}$.

A generalization of Definition 1 and Theorem 1 to stimuli with several independent components is given in Appendix C.

For the special case in which $\mathcal{X}$ is a unit concatenation structure, a matching relation from $\mathcal{X}$ to $\mathcal{S}$ can be established so that a concatenation operation $*$ can be defined on $\mathcal{S}$. This somewhat technical matter is taken up in Theorem 2 of Appendix B. The major finding is that if the matching relation is translation consistent, then the induced concatenation operation is invariant under the translations of $\mathcal{S}$; in other words, if σ is a translation of $\mathcal{S}$ and $*$ is the induced operation, then $\sigma(r*s) = \sigma(r)*\sigma(s)$. This fact can be interpreted as saying that $*$ is meaningful within the structure of $\mathcal{S}$.

As a psychological example of a translation-consistent matching relation that relates a concatenation structure to itself, consider intensity discrimination of pure tones. If I denotes the intensity of a pure tone of fixed frequency and $\Delta(I)$ the corresponding intensity jnd, it has been found, to a good approximation, that $\Delta(I) = cI^{1-\beta}$, where β is comparatively small relative to 1, positive, and independent of signal frequency (Jesteadt, Wier, & Green, 1977; McGill & Goldberg, 1968). McGill and Goldberg have referred to this empirical generalization as the "near-miss to Weber's law" because when $\beta = 0$, the relation is referred to as Weber's law and β is moderately small (approximately 0.075) relative to 1. Because sound intensity is a physical concatenation structure $\mathcal{X} = \langle X, \succsim, \bigcirc \rangle$, the discrimination data establish a matching relation M between $\mathcal{X}$ and itself defined as follows. For each $y \in X$, let $x \in X$ be such that xMy holds if and only if y and $y\bigcirc x$ are just noticeably different. By Theorem 1, M is translation consistent if and only if a power relation obtains, and the near-miss law is such a power relation. Thus, at the qualitative level, if τ is a translation, then there is another translation σ_τ such that if xMy, then $\tau(x)M\sigma_\tau(y)$, where $\tau(x)$ satisfies $\tau(y)\bigcirc\tau(x) = \tau(y\bigcirc x)$.

This example makes clear that there are important matching relations in psychology aside from the ones usually referred to as matching.

Similar Matching Relations

There are also important physical examples of matching relations. Two of the standard ones mentioned in discussions of dimensional analysis are Hooke's law, which states the force applied to a spring is proportional to its length, and Einstein's $E = mc^2$, which matches energy to mass. In all such cases, the matching law involves a third quantity, called a dimensional constant: the so-called spring constant in the case of Hooke's law and the square of the velocity of light in Einstein's law. The latter, involving a unique constant, is known as a universal dimensional constant. The former, which is identified as a characteristic of the spring, is not unique, and this results in a family of similar matching relations. These familes are addressed next.

As a psychological motive for looking at similar matching relations, consider the matching relations obtained when a subject is asked to shift the modulus he or she is using in making magnitude estimates or cross-modal matches, which is something subjects can easily do for at least magnitude estimation. Indeed, if power laws describe these matching relations at the level of unit representations, they are expected to differ only by a constant factor. The problem is to capture the concept of similarity qualitatively.

Fortunately, a general definition of similar sets has already arisen in the theory of dimensional analysis (Krantz et al., 1971, p. 508). The following definition is simply the specialization of the general one to the context of matching relations.[5]

Definition 3. Suppose $\mathcal{X}$ and $\mathcal{S}$ are unit structures and M and N are two matching relations between the structures. M and N are said to be *similar* if and only if there exist translations τ of $\mathcal{X}$ and σ of $\mathcal{S}$ such that for all $x \in X$ and $s \in S$,

$$\text{if } xMs, \text{ then } \tau(x)N\sigma(s).$$

This statement simply says that by an appropriate shift in point of view, via a translation, in each structure, the matching relation M is changed to that of N. Moreover, one can show that any two such changes in point of view in the unit structures generate a similar relation. That is, if the structures are unit structures, M is a matching relation between them, and τ and σ are translations of the two structures, then N, defined by xNs if and only if $\tau^{-1}(x)M\sigma^{-1}(s)$, is a matching relation that is similar to M. This is proved as Part (i) of Theorem 3 (Appendix B). Part (ii) shows that if M and N are translation-consistent matching relations, then their being similar is exactly the same as saying that they have the same exponent in the power function representation of Equation 2. In other words, for translation-consistent matching relations, similarity corresponds simply to a change in the modulus of their power function representations, but the exponent stays fixed.

The empirical evidence suggests that subjects are able to generate a wide range of similar matching relations. Far more problematic are matching relations that have different exponents. The data suggest, although Teghtsoonian and Teghtsoonian (1971) have argued that these data are being misinterpreted, that the exponents differ from subject to subject, and King and Lockhead (1981) have reported manipulating the exponent in magnitude estimation for individual subjects. To my knowledge this has not been replicated, and comparable results have not been reported for cross-modal matches.

A Method of Constructing Translations

As noted earlier, the only unit structures for which there is an explicit method of constructing translations are those with a positive, binary operation. This is not a serious limitation for matching of modalities of physical intensity or energy because a positive concatenation operation exists; however, for any other cases that may arise, explicitly finding translations can be troublesome. Therefore, it is natural to consider whether the matching elations themselves can be used to generate translations. The next result, which is stated formally as Theorem 4 in Appendix B, shows how to do this provided there is independent knowledge of when two matching relations are similar.

Suppose $\mathcal{X}$ and $\mathcal{S}$ are unit structures having representations defined onto the positive real numbers. Furthermore, suppose M and N are two matching relations between them. Define the functions $\tau = NM^{-1}$ and $\sigma = M^{-1}N$ on X and S, respectively, where M^{-1} is simply the inverse of M; that is, if xMs, then $sM^{-1}x$. Stated more explicitly, $y = \tau(x)$ provided that there exists some $u \in S$ such that xNu and yMu, and $\sigma(s) = t$ provided there exists $z \in X$ such that zMs and zNt. Given these definitions, then, according to Theorem 4, if M and N are each translation consistent and they are similar, then τ is a translation of $\mathcal{X}$ and σ is a translation of $\mathcal{S}$.

The difficulty in using this result is to convince oneself that two matching relations are similar. Presumably, this is a matter of finding in a context of positive concatenation structures, where some of the translations are known, that certain experimental manipulations generate similar matching relations and then extrapolating the impact of these manipulations to other unit structures for which the translations are not known.

Ratio Relations

Next, consider the idea of ratio matching between two unit structures, first in terms of a matching relation and then in isolation. This is important partly because the Shepard–Krantz theory places so much weight on ratios and attempts to explain matching in terms of them.

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[5] Note that the apparent asymmetry of Definition 3 is just that, apparent. For, suppose it holds and xNs; then the claim is that $\tau^{-1}(x)M\sigma^{-1}(s)$. To show this, let $r = M[\tau^{-1}(x)]$, and by Definition 3, $xN\sigma(r)$. However, N is a function, and by assumption xNs, so $s = \sigma(r)$.

Definition 4. Suppose M is a matching relation between ordered relational structures $\mathscr{X}$ and $\mathscr{S}$. A function R from $X \times X$ onto $S \times S$ is said to be a *ratio relation relative to* M provided that R is strictly increasing in each component and ($\forall x, y \in X$ and $r, s \in S$),

$$(x, y)R(r, s)$$ 15

if and only if for some translation θ of $\mathscr{X}$,

$$\theta(x)Mr \quad \text{and} \quad \theta(y)Ms.$$ 16

The intuition behind this definition entails two simple ideas: The first is that any translation preserves stimulus ratios; the second is that two pairs of matching stimuli have the same ratio. The major result is to establish that this intuition is really correct; the formal statement and proof constitute Theorem 5 in Appendix B.

Suppose M is a translation-consistent matching relation between unit structures $\mathscr{X}$ and $\mathscr{S}$ with real-unit representations ϕ and ψ onto Re^+. A unique ratio relation R may be defined relative to M in either of two ways, one qualitative and the other numerical. First, for each translation τ let σ_τ be defined as in Definition 2, and let R be defined by

$$[x, \tau(x)]R[s, \sigma_\tau(s)] \quad \text{if and only if} \quad xMs. \tag{3}$$ 17

Second, let R be defined by

$$(x, y)R(r, s) \quad \text{if and only if} \quad \psi(r)/\psi(s) = [\phi(x)/\phi(y)]^\rho, \tag{4}$$ 18

where ρ is the exponent of the power representation of M (Equation 2). This latter definition explicitly shows how ratios are being preserved by the relation R. Theorem 5, Part (i) establishes that Equations 3 and 4 are equivalent.

Two additional results are demonstrated in Theorem 5. One is that two translation-consistent matching relations that are similar generate exactly the same ratio relation. The other is that if Equations 2 and 4 both hold with the same exponent ρ, then the relation R defined by Equation 4 is ratio relative to the relation M defined by Equation 2.

The gist of these results is that ratio relations that satisfy Equation 4 are substantially equivalent to the class of similar, translation-consistent matching relations satisfying Equation 2. However, the way to characterize Equation 4 purely qualitatively in the absence of M is not clear. It is easy to verify, for example, that Equation 4 implies that R is dimensionally invariant in the following sense: If $\mathscr{X}$ and $\mathscr{S}$ are unit structures, $R: X \times X \twoheadrightarrow S \times S$ is strictly monotonic in each component, and R satisfies Equation 4, then for any translations τ of $\mathscr{X}$ and σ of $\mathscr{S}$,

$$(x, y)R(r, s) \quad \text{if and only if} \quad [\tau(x), \tau(y)]R[\sigma(r), \sigma(s)]. \tag{5}$$

However, Equation 5 does not imply Equation 4, only that there is a strictly increasing function F such that

$$(x, y)R(r, s) \quad \text{if and only if} \quad \psi(r)/\psi(s) = F[\phi(x)/\phi(y)]. \tag{6}$$

As Krantz (1972) showed, if ratio consistency is demanded among ratio relations between more than two ordered systems, then except for a single strictly increasing function, everything is related as powers of ratios of scale values. The only way the unknown function can be eliminated is to introduce additional information. For example, Krantz suggested that empirically, length ratios have the special status of being objectively correct, and so the function F is determined.

An argument can be made to the effect that having function F is an advantage because it affords an opportunity to fit data for which strict power laws are empirically violated. Such discrepancies are common for individual subjects and even for group data if the background stimulation is not negligible. The problem with this argument is that the function affects the entire interrelated family of matching relations, not just an isolated one, and so it must be interpreted as a feature of the person doing the matching, not of the experimental situation. It is hard to imagine that the function needed to cope with background noise in magnitude estimates of pure tones will also be needed for brightness estimates against a dark background.

One could, in principle, effect a mathematical generalization of the idea of a ratio relation relative to M for the case of more than one independent variable. For example, this is exactly what is needed when there are stimuli and context, such as lights with a surround. Krantz (1968) developed such a theory, and it has some similarities to my theory—and some sharp differences. Two major differences are (a) that the matching relation is restricted to the case of relating a domain to itself and (b) that the theory is inherently two dimensional in that the stimulus situation is assumed to have two independently manipulable factors, as in conjoint measurement. Krantz's theory is similar in that it focuses on a class of transformations of the first factor. Unlike my theory, which uses translations that form a mathematical group, his assumed a semigroup (i.e., inverses need not exist). This level of generality was motivated by considerations of color matching in which certain transformations are experimentally feasible, but inverses may not be. Within that limitation, he assumed that for any two elements of the first factor (the dimension along which the match is to be achieved) there are transformations of each element to a common value. If the semigroup were a group, this would amount to assuming that the group of transformations is homogeneous on the first component. Additional assumptions impose various patterns of consistency on these transformations and the matching relation. These assumptions include properties that are similar to a form of monotonicity in the conjoint structure, that there is solvability in that structure, and that the transformations from the semigroup have some properties not unlike translations. His major assumption, called *context invariance*, was strongly motivated by empirical properties of color vision.

Within the framework of this article, a class of two-dimensional matching theories could be developed in which one class of translations acts on the stimulus being matched and another acts on its surround; with such a class, one could investigate various possible interactions of such translations. I have not done this.

Conclusion

Krantz (1972, p. 171) listed five empirical generalizations that need to be accounted for by a matching theory:

(i) *Magnitude consistency.* Magnitude estimation functions with different moduli differ only by similarity transformations. . . .

(ii) *Pair consistency.* Pair estimates [= ratio judgments] behave like ratios. . . .

(iii) *Magnitude-pair consistency.* The pair estimate is equal to the ratio of magnitude estimates for the members of the pair. . . .

(iv) *Consistency of magnitude estimates and cross-modality matching.* If y_j is matched with y_i, in the cross-modality matching function with modulus (x_j, x_i), then the ratio of magnitude estimates of y_j to x_j equals the ratio of the magnitude estimates of y_i to x_i. . . .

(v) *Power law.* Magnitude estimates are approximately power functions of stimulus energy. . . .

Krantz concluded that "the principal reason for favoring relation theory over the others is that it gives a satisfactory account of generalization (iv): that magnitude estimates predict cross-modality matches, *independent of the choice of the moduli in the cross-modality matching*" (1972, p. 174).

In my theory of translation-consistent matching relations, the first result embodied in the power relation of Equation 2 covers Statement (v). The second result (Appendix B, Theorem 3) about similar, translation-consistent matching relations having power function representations with a common exponent and differing only in their modulus covers Statement (i). Together, the first two results yield Statement (iv). The final results about ratio relations (Theorem 5) yields Statement (iii), and so Statement (ii) because, as Krantz noted, (iii) implies (ii). What differs considerably between my theory and the Shepard–Krantz theory is the relatively incidental role of numerical representations.

In contrast to the views expressed by both Krantz and Shepard, I do not find it at all clear why a theory based on ratios is to be preferred to one based on matches, in which the empirical substance of the theory is entirely captured qualitatively by translation consistency (Definition 1). Of course, I do not contend that translation consistency is known to be exactly valid; indeed, the consistent departures from the power law of Equation 2, found for individual subjects making many magnitude estimates, certainly suggests that either the ways of eliciting matches are flawed or that the hypothesis of translation consistency is, at best, an idealization. Furthermore, by introducing contexts, such as background noise in loudness judgments, the shape of the magnitude estimation functions certainly deviates sharply from a power function. The present theory is simply inapplicable in this case because the stimulus domain cannot be idealized to be one dimensional.

To summarize the theory described in this article: If the only psychology embodied in the matching relation is consistent with the physics embodied in the assumption that the variables are described as unit structures, and so are one dimensional, then all of the empirical generalizations outlined by Krantz (1968, 1972) must hold. *Consistent* means that the psychological law can be formulated without exceeding the usual conventions for formulating a physical law; it can be seen as an empirical generalization formulated within the framework of the physics of the variables involved.

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Appendix A

Definition of Unit Structures

If the set of translations satisfies three important additional conditions, then the structure has a *real-unit* numerical representation. Moreover, three conditions are implied by that representation:

(i) The set $\mathcal{T}$ of translations actually forms a mathematical group under function composition in the following sense: The group operation is function composition, which is associative; the identity map is the group identity; and the function inverse is the group inverse. (The only substantive assumption involved is that a translation acting on a translation is itself a translation.)

To state the second assumption, an *asymptotic ordering* needs to be induced from the ordering of the structure on the group $\mathcal{T}$. This is done in the following way (Narens, 1981b): Suppose $\tau, \sigma \in \mathcal{T}$; then, $\tau \succsim' \sigma$ if and only if for some fixed $y \in X$, $\tau(x) \succsim \sigma(x)$ for all $x \in X$ with $x \succsim y$.

(ii) The asymptotic ordering of $\mathcal{T}$ is a total ordering, and it is assumed to exhibit the essential property of being Archimedean in the following group-theoretic sense: If $\tau, \sigma \in \mathcal{T}$ are such that $\sigma >' \iota$, where ι denotes the identity map, then for some integer n, $\sigma^n >' \tau$, where σ^n means n successive applications of σ.

By Hölder's theorem (Krantz et al., 1971, p. 53), these two assumptions imply that the translations are isomorphic to a subgroup of the positive real numbers under multiplication, and so the translations commute; that is, if $\tau, \sigma \in \mathcal{T}$, for each $x \in X$, $\tau[\sigma(x)] = \sigma[\tau(x)]$.

The third assumption is that the set of translations is very rich or, in other words, the underlying structure is highly symmetric.

(iii) The translations are *homogeneous* in the sense that given any two points in the structure, there is some translation that takes one into the other; that is, if $x, y \in X$, there is some $\tau \in \mathcal{T}$ such that $\tau(x) = y$.

When all three assumptions are true, the relational structure has a numerical representation that is a real-unit structure, in which the translations map into multiplication by positive constants. The qualitative structure in that case is called a unit structure. The formal definition (Luce, 1986, 1987) is as follows.

Definition 5. A real relational structure $\mathcal{R} = \langle R, \geq, R_j \rangle_{j \in J}$ is said to be a *real-unit structure* if and only if R is a subset of Re^+ and there is some subset U of Re^+ such that

1. U is a group under multiplication,
2. U maps R into R; that is, for each $r \in R$ and $u \in U$, then $ur \in R$,
3. U restricted to R is the set of translations of $\mathcal{R}$.

A qualitative, ordered relational structure is said to be a *unit structure* if and only if it is isomorphic to a real-unit structure. The isomorphism is referred to as a *real-unit representation.*

The use of the symbol R in this definition is not to be confused with its use in Definition 3 for a ratio relation.

Appendix B

Theorems and Proofs

Lemma 1. Suppose $\mathcal{X}$ and $\mathcal{S}$ are ordered relational structures, the translations of which form homogeneous groups. Let M be a translation-consistent matching relation. Then for each translation σ' of $\mathcal{S}$, there is a translation τ' of $\mathcal{X}$ such that for all $x \in X$, $s \in S$, sMs if and only if $\tau'(x)M\sigma'(s)$. In other words, for each pair τ, σ_τ, the map $\tau^{-1}M^{-1}\sigma_\tau M$ is the identity map of $\mathcal{X}$, and $\sigma_\tau^{-1}M\tau M^{-1}$ is the identity map of $\mathcal{S}$.

Proof of Lemma 1. Suppose yMr. Because M is onto S, there exists some z such that $zM\sigma'(r)$. By the homogeneity of $\mathcal{X}$, there is some τ' such that $\tau'(y) = z$. However, given τ', by hypothesis there is some σ'' such that $\tau'(y)M\sigma''(r)$. Because M is a function, $\sigma''(r) = \sigma'(r)$. By Theorem 2.1 of Luce (1986), the assumption that the translations form a group implies that $\sigma'' \equiv \sigma'$. Therefore, for any $x \in X$ and $s \in S$, xMs if and only if $\tau'(x)M\sigma''(s)$ if and only if $\tau'(x)M\sigma'(s)$.

Theorem 1. Suppose $\mathcal{X}$ and $\mathcal{S}$ are unit structures with real-unit representations ϕ and ψ onto Re^+ and with a matching relation M between them. Then M is translation consistent if and only if there exist positive constants K and ρ such that for all $x \in X$, $s \in S$,

$$xMs \quad \text{if and only if} \quad \psi(s) = K\phi(x)^\rho. \tag{B1}$$

In this case, the mapping from τ to σ_τ is an isomorphism of the translation groups of $\mathcal{X}$ and $\mathcal{S}$.

Proof of Theorem 1. Suppose Equation B1 holds. Let τ be a translation of $\mathcal{X}$ and let $t > 0$ correspond to it in the real-unit representation; that is, $\phi[\tau(x)] = t\phi(x)$. Then, letting σ be the translation of $\mathcal{S}$ corresponding to t^ρ,

$$\begin{aligned} xMs \quad &\text{if and only if} \quad \psi(s) = K\phi(x)^\rho \\ &\text{if and only if} \quad t^\rho\psi(s) = K[t\phi(x)]^\rho \\ &\text{if and only if} \quad \psi[\sigma(s)] = K\phi[\tau(x)]^\rho \\ &\text{if and only if} \quad \tau(x)M\sigma(s). \end{aligned}$$

Conversely, suppose M is translation consistent. For fixed real-unit representations ϕ and ψ, the real function G is defined by

$$G = \psi M\phi^{-1}. \tag{B2}$$

Because M is strictly increasing, G is well defined and strictly increasing. Because the real-unit structures are on Re^+, G is defined on Re^+. Let τ be a translation of $\mathcal{X}$ and let σ_τ be the corresponding one under translation consistency. Let positive t and u correspond to τ and σ_τ under the unit representations. Suppose xMs and therefore $\tau(x)M\sigma_\tau(s)$. By definition,

$$\psi(s) = G[\phi(x)] \quad \text{and} \quad u\psi(s) = \psi[\sigma_\tau(s)] = G(\phi[\tau(x)]) = G[t\phi(x)].$$

Setting $x' = \phi(x)$, G satisfies the functional equation

$$uG(x') = G(tx').$$

Setting $x' = 1$, $G(t) = uG(1)$, and so G satisfies

$$G(t)G(x') = G(tx')G(1).$$

Because G is strictly increasing, the solution is Equation B1, with $K = g(1)$.

The mapping $\tau \twoheadrightarrow \sigma_\tau$ is an isomorphism: Because M is a 1:1 map, σ_τ is clearly onto and 1:1. Therefore, it preserves function composition. Suppose τ, θ are translations of $\mathcal{X}$ and $s \in S$. There exists some $x \in X$ so that xMs. By translation consistency, xMs implies $\theta(x)M\sigma_\theta(s)$ and $\tau\theta(x)M\sigma_{\tau\theta}(s)$. Applying τ to the former yields $\tau\theta(x)M\sigma_\tau\sigma_\theta(s)$. Because M is 1:1, $\sigma_{\tau\theta}(s) = \sigma_\tau\sigma_\theta(s)$. Because s is arbitrary, $\sigma_{\tau\theta} \equiv \sigma_\tau\sigma_\theta$.

Corollary 1. Under the conditions of the theorem, suppose that within the unit representations T and S_T are the translations corresponding to τ and σ_τ; then, $S_T = T^\rho$.

Proof of Corollary 1. By Equation B1,

$$\begin{aligned} \psi(s) = K\phi(x)^\rho &\Leftrightarrow xMs \\ &\Leftrightarrow \tau(x)M\sigma_\tau(s) \\ &\Leftrightarrow \psi[\sigma_\tau(s)] = K\phi[\tau(x)]^\rho \\ &\Leftrightarrow S_T\psi(s) = KT^\rho\phi(x)^\rho \\ &\Leftrightarrow S_T = T^\rho. \end{aligned}$$

Theorem 2. Suppose $\mathcal{X} = \langle X, \succsim_X, \bigcirc \rangle$ and $\mathcal{S} = \langle S, \succsim_S, S_j \rangle_{j \in J}$ are unit structures with real-unit representations ϕ and ψ mapping onto Re^+ and that M is a matching relation from $\mathcal{X}$ to $\mathcal{S}$. Define the operation $*$ on S by

$$(\forall r, s \in S)\ r * s = M[M^{-1}(r) \bigcirc M^{-1}(s)]. \tag{B3}$$

Then,

(i) For all x, $y \in X$, r, $s \in S$, if xMr and yMs, then $(x \bigcirc y)M(r * s)$.

(ii) $*$ is a monotonic operation.

(iii) If M is translation consistent, then $*$ is invariant under the translations of $\mathcal{S}$; that is, $\sigma(r * s) = \sigma(r) * \sigma(s)$.

(iv) If M is translation consistent and Equation B1 holds, then $\langle \mathrm{Re}^+, \geq, \otimes \rangle$ is a real-unit representation of $\langle S, \succsim_S, * \rangle$, where $x \otimes y = yg(x/y)$ for $g(u) = f(u^{1/\rho})^\rho$.

Proof of Theorem 2.

(i) Suppose xMr and yMs. Then,

$$\begin{aligned} x \bigcirc y &= M^{-1}(r) \bigcirc M^{-1}(s) \\ &= M^{-1}(r * s), \end{aligned}$$

from which $(x \bigcirc y)M(r * s)$.

(ii) Suppose xMr, yMs, and zMq. By Part (i) of the theorem, $(x \bigcirc z)M(r * q)$ and $(y \bigcirc z)M(s * q)$, and therefore

$$\begin{aligned} r \succsim_s s \quad &\text{if and only if} \quad x \succsim_X y && \text{(strict monotonicity of } M) \\ &\text{if and only if} \quad x \bigcirc y \succsim_X y \bigcirc z && \text{(monotonicity of } \bigcirc) \\ &\text{if and only if} \quad (r * q) \succsim_S (s * q) && \text{(strict monotonicity of } M). \end{aligned}$$

The other side is similar.

(iii) Let σ be a translation of $\mathcal{S}$ and, by Lemma 1, let τ be the translation of $\mathcal{X}$ such that $(\forall r \in S)\ \tau M^{-1}(r) = M^{-1}[\sigma(r)]$. Then,

$$\begin{aligned} M^{-1}[\sigma(r * s)] &= \tau M^{-1}(r * s) && \text{(Lemma 1)} \\ &= \tau[M^{-1}(r) \bigcirc M^{-1}(s)] && \text{(Equation B3)} \\ &= \tau M^{-1}(r) \bigcirc \tau M^{-1}(s) && (\tau \text{ is a translation}) \\ &= M^{-1}[\sigma(r)] \bigcirc M^{-1}[\sigma(s)] && \text{(Lemma 1)} \\ &= M^{-1}[\sigma(r) * \sigma(s)], && \text{(Equation B3)} \end{aligned}$$

from which $\sigma(r * s) = \sigma(r) * \sigma(s)$, which establishes the invariance of $*$ under translations of $\mathcal{S}$.

(iv) By Part (i) of this theorem and Theorem 1,

$$\begin{aligned} \psi(r * s) &= K\phi(x \bigcirc y)^\rho \\ &= K\phi(x)^\rho f[\phi(x)/\phi(y)]^\rho \end{aligned}$$

$$= \psi(r)f([\psi(r)/\psi(s)]^{1/\rho})^{\rho}$$
$$= \psi(r)g[\psi(r)/\psi(s)].$$

Theorem 3. Suppose that $\mathcal{X}$ and $\mathcal{S}$ are unit structures with real-unit representations onto Re^+.

(i) If τ is any translation of $\mathcal{X}$, σ is one of $\mathcal{S}$, and M is a matching relation, then N is defined by xNs if and only if $\tau^{-1}(x)M\sigma^{-1}(s)$ is a matching relation that is similar to M.

(ii) Suppose that M and N are any two matching relations that are translation consistent between $\mathcal{X}$ and $\mathcal{S}$. Then the following statements are equivalent:

(a) M and N are similar.
(b) The power function representations (Equation B1) of M and N have the same exponent.
(c) For each translation τ, let σ_τ and η_τ be the isomorphisms generated by M and N, respectively; then, $\sigma_\tau \equiv \eta_\tau$.

Proof of Theorem 3.

(i) Suppose xMs, then $\tau^{-1}\tau(x)M\sigma^{-1}\sigma(s)$, and by definition, $\tau(x)N\sigma(s)$.

(ii) (a) $\Rightarrow$ (b). Suppose M and N are similar under the translation pair (τ, σ). Because M is translation consistent, there exists η such that if xMs, then $\tau^{-1}(x)M\eta(s)$. Let $\xi = \eta\sigma$. M and N are similar under (ι, ξ), where ι denotes the identity map. Suppose yNr. By hypothesis, there exists x, s such that xMs, $\tau(x) = y$, and $\sigma(s) = r$. Observe that because translations commute,

$$\begin{aligned}(\iota, \xi)[\tau(x), \eta^{-1}(s)] &= (\tau^{-1}, \eta)(\tau, \sigma)[\tau(x), \eta^{-1}(s)]\\ &= (\tau, \sigma)(\tau^{-1}, \eta)[\tau(x), \eta^{-1}(s)]\\ &= (\tau, \sigma)(x, s)\\ &= (y, r).\end{aligned}$$

Furthermore, with the definition of η and the monotonocity of M, xMs is equivalent to $\tau(x)M\eta^{-1}(s)$, proving (ι, ξ) renders M and N similar.

Let $u > 0$ be the multiplicative equivalent of ξ. Then xMs is equivalent to $\psi(s) = K\phi(x)^{\rho}$, and $xN\xi(s)$ is equivalent to $\psi[\xi(s)] = u\psi(s) = K'\phi(x)^{\rho'}$. Thus, $uK\phi(x)^{\rho} = K'\phi(x)^{\rho'}$, which is possible for all x only if $K' = uK$ and $\rho' = \rho$.

(b) $\Rightarrow$ (c). For any translation τ of $\mathcal{X}$, let σ_τ and η_τ be the translations corresponding to the translation consistency of M and N, respectively. Let T, S_T, and U_T be the corresponding real-unit translations. Then, from the corollary to Theorem 1 and the fact that the exponents are the same, $S_T = T^{\rho} = U_T$, from which $\sigma_\tau = \eta_\tau$.

(c) $\Rightarrow$ (a). Suppose M and N both have the same isomorphism between translations. Choose some x', r', x' such that $x'Ms'$ and $x'Nr'$, and by homogeneity, let η be the translation such that $\eta(s') = r'$. I now show that if xMs, then $xN\eta(s)$, and so M and N are similar. Select τ so that $\tau(x') = x$. Then,

$$x'Ms' \Rightarrow \tau(x')M\sigma_\tau(s') \Rightarrow xM\sigma_\tau(s').$$

By the fact that M is a function, $s = \sigma_\tau(s')$. From the commutativity of translations and this last fact,

$$x'Nr' \Rightarrow xN\sigma_\tau(r') \Rightarrow xN\sigma_\tau\eta(s') \Rightarrow xN\eta\sigma_\tau(s') \Rightarrow xN\eta(s).$$

Theorem 4. Suppose $\mathcal{X}$ and $\mathcal{S}$ are unit structures with representations onto Re^+ and M and N are matching relations between them.

(i) Define the functions $\tau: X \to X$ and $\sigma: S \to S$ by

$$\tau(x) = y$$

if and only if for some $u \in S$, xNu and yMu; that is, $\tau = NM^{-1}$.

$$\sigma(s) = t$$

if and only if for some $z \in X$, zMs and zNt; that is, $\sigma = M^{-1}N$. If M and N are translation consistent and similar, then τ is a translation of $\mathcal{X}$ and σ is one of $\mathcal{S}$.

(ii) If M is translation consistent, τ is a translation of $\mathcal{X}$, and σ is a translation of $\mathcal{S}$, then there exist matching relations N and N' that are similar to M, for which xMs implies $\tau(x)Ns$ and $xN'\sigma(s)$.

Proof of Theorem 4.

(i) The proof is given only for τ. Observe that τ is defined for all $x \in X$ because N is, and it is onto S because M is. By Theorems 1 and 3, there exist K, K', and ρ such that if xNu and yMu, then $\psi(u) = K\phi(x)^{\rho} = K'\phi(y)^{\rho}$. The ratio $(K/K')^{1/\rho}$ corresponds to some translation τ of $\mathcal{X}$, and $\tau(x) = y$.

(ii) Define N by xNs if and only if $\tau^{-1}(x)Ms$. By Theorem 1,

$$\psi(s) = K\phi[\tau^{-1}(x)]^{\rho} = (K/C^{\rho})\phi(x)^{\rho},$$

which proves that N is similar to M, and clearly, by definition, if xMs holds, then $\tau(x)Ns$ holds. The case for N' is similar.

Theorem 5. Suppose M is a translation-consistent matching relation between unit structures $\mathcal{X}$ and $\mathcal{S}$ with real-unit representations ϕ and ψ onto Re^+.

(i) Then there exists a unique ratio relation R relative to M, and $(\forall x, y \in X, s, t \in S)$ R is defined by either:

(a) For each translation τ with σ_τ defined in Definition 2,

$$[x, \tau(x)]R[s, \sigma_\tau(s)] \quad \text{if and only if} \quad xMs, \tag{B4}$$

or

$$\text{(b)}\ (x, y)R(r, s) \quad \text{if and only if} \quad \psi(r)/\psi(s) = [\phi(x)/\phi(y)]^{\rho}, \tag{B5}$$

where ρ is the exponent of the power representation of M (Equation B1).

(ii) If N is a translation-consistent matching relation, N is similar to M, and R' is the ratio relation relative to N, then $R' = R$.

(iii) If R is any relation for which Equation B5 holds for some exponent ρ and M is any relation from $\mathcal{X}$ to $\mathcal{S}$ that satisfies Equation B1 with the same exponent ρ, then R is a ratio relation relative to M.

Proof of Theorem 5.

(i-a) I show that R satisfies Definition 3 with $\theta = \iota$. Suppose $(x, y)R(s, t)$. By Equation B4, this is possible only if xMs, $y = \tau(x)$, and $t = \sigma_\tau(s)$, where σ_τ is defined in Definition 2. By translation consistency, $\tau(x)M\sigma_\tau(\sigma)$. Therefore, yMt, and so by Definition 3, R is ratio relative to M.

(i-b) By Theorem 1, Equation B1 holds for M, and therefore R can be defined by Equation B4. I prove that R is a ratio relation relative to M. Suppose $(x, y)R(r, s)$. Let $q \in S$ be such that xMq. By homogeneity, there exists a translation σ such that $\sigma(q) = r$. By Lemma 1, there exists a translation τ such that xMq implies $\tau(x)M\sigma(q)$; that is, $\tau(x)Mr$. Let t be the unit transformation corresponding to τ. Therefore, $\psi(r) = K\phi[\tau(x)]^{\rho} = Kt^{\rho}\phi(x)^{\rho}$. By Equation B5,

$$\begin{aligned}\psi(s) &= \psi(r)[\phi(y)/\phi(x)]^{\rho}\\ &= Kt^{\rho}\phi(y)^{\rho}\\ &= K\phi[\tau(y)]^{\rho},\end{aligned}$$

establishing that $\tau(y)Ms$, and therefore that R is a ratio relation relative to M.

To establish uniqueness, suppose R is a relation that is ratio relative to M. Therefore, for some translation τ, $(x, y)R(r, s)$ implies $\tau(x)Mr$ and $\tau(y)Ms$. By Theorem 1, $\psi(r) = Kt^{\rho}\phi(x)^{\rho}$ and $\psi(s) = Kt^{\rho}\phi(y)^{\rho}$, where t is the numerical translation corresponding to τ. Thus, Equation B5 holds, which proves R is unique:

(ii) By Theorem 3 and Part (i), the result is immediate.

(iii) By Equations B1 and B5, there exists some C such that $\psi(r) = CK\phi(x)^{\rho} = K\phi[\tau(x)]^{\rho}$, where τ is the translation of $\mathcal{X}$ corresponding to $C^{1/\rho}$, and $\psi(s) = K\phi[\tau(y)]^{\rho}$. Thus, $\tau(x)Mr$ and $\tau(y)Ms$.

Appendix C

Generalization to Several Independent Variables

In formulating the natural generalization of Definition 1 to two (or more) independent variables, it is convenient to use the vector notation $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$, etc.

Definition 6. Suppose that $\mathfrak{X}_i = \langle X_i, \succsim_i, X_{ij} \rangle_{j \in J}$, $i = 1, \ldots, n$, and $\mathcal{S} = \langle S, \succsim_S, S_k \rangle_{k \in K}$ are all ordered relational structures. Any strictly increasing function M from $X = \bigtimes_{i=1}^{n} X_i$ onto $\mathcal{S}$ is called a *matching relation* (on n independent variables).

1. A matching relation is said to be *translation consistent* if and only if for each vector $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$, $\tau_i \in \mathfrak{X}_i$, $i = 1, 2, \ldots, n$, there is a translation $\sigma = \sigma(\tau)$ of $\mathcal{S}$ such that $(\forall x_i \in X_i, s \in S)$,

$$\mathbf{x}Ms \quad \text{if and only if} \quad \tau(\mathbf{x})M\sigma(s). \qquad 42$$

2. Let M_i be a matching relation from X_i onto S. Then, M_i is said to be induced by M provided that for fixed choices $z_j, j \neq i$, from X_j,

$$x_i M_i s \quad \text{if and only if} \quad (z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n)Ms. \qquad 43$$

The following generalizes Theorem 1 to n independent variables.

Theorem 6. Suppose that $\mathfrak{X}_i = \langle X_i, \succsim_i, X_{ij} \rangle_{j \in J}$, $i = 1, \ldots, n$, and $\mathcal{S} = \langle S, \succsim_S, S_k \rangle_{k \in K}$ are ordered structures with real-unit representations ϕ_i, $i = 1, \ldots, n$ and ψ on Re^+, and M is a matching relation on the n independent variables.

(i) Then M is translation consistent if and only if there exist positive constants ρ_i and K such that for all $x_i \in S_i$ and $s \in S$,

$$\mathbf{x}Ms \quad \text{if and only if} \quad \psi(s) = K\prod_{i=1}^{n} \phi_i(x_i)^{\rho_i}. \qquad 44$$

If M is translation consistent,

(ii) any induced M_i is translation consistent, and

(iii) any two induced matching relations from X_i to S are similar.

Proof of Theorem 6.

(i) This is an easy modification of the proof of Theorem 1.

(ii) This follows immediately from Definition 5 by setting $\tau_j = \iota$ for $j \neq i$.

(iii) Let M_i and M_i' be two induced relations on X_i, and suppose the fixed coordinates are z_j and z_i', $j \neq i$. By the homogeneity of the unit representations, there exist τ_i such that $\tau(z_i') = z_i$. Let σ be the translation of $\mathcal{S}$ from Part 1 of Definition 5 that depends on $\tau_j, j \neq i$, and ι on X_i. Then

$$\begin{aligned} xM_i's &\Rightarrow (z_1', \ldots, x, \ldots, z_n')Ms \\ &\Rightarrow (z_1, \ldots, x, \ldots, z_n)M\sigma(s) \qquad 45 \\ &\Rightarrow xM_i\sigma(s), \end{aligned}$$

proving that M_i and M_i' are similar.

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