On a class of meaningful permutable laws

Jean-Claude Falmagne
University of California, Irvine

September 21, 2012

Abstract

The permutability equation $G(G(x,y),z) = G(G(x,z),y)$ is satisfied by many scientific and geometric laws. A few examples among many are: The Lorentz-FitzGerald Contraction, Beer's Law, the Pythagorean Theorem, and the formula for computing the volume of a cylinder. If we required that a permutable law be meaningful, the possible forms of a law are considerably restricted. The class of examples described here contains the Pythagorean theorem.

The mathematical expression of a scientific law typically does not depend on the units of measurement. The most important rationale for this convention is that measurement units do not appear in nature[1]. Thus, any mathematical model or law whose form would be fundamentally altered by a change of units would be a poor representation of the empirical world. As far as I know, however, there is no agreed upon formalization of this type of invariance of the form of scientific laws, even though there has been some proposals (see Falmagne and Narens, 1983; Narens, 2002; Falmagne, 2004).

Expanding on the just cited references, I propose here a general condition of 'meaningfulness' constraining *a priori* the form of any function describing a scientific or geometric law expressed in terms of ratio scales variables such as mass, length, or time[2]. We define this meaningfulness condition in the second section of this paper. In this definition, all the units of the variables are explicitly specified by the notation, as opposed to being implicitly embedded in the concepts of 'quantities' and 'dimensions' of dimensional analysis (c.f. for example Sedov, 1943, 1956).

The interest of such a meaningfulness condition from a philosophy of science standpoint is that, in its context, general abstract constraints on the function, formalizing 'gedanken experiments', may yield the exact possible forms of a law, possibly up to some real valued parameters.

An example of such a constraint is the condition below, which applies to a real, positive valued function G of two real positive variables. It is formalized by the equation

$$G(G(y,r),t) = G(G(y,t),r), \tag{1}$$

where G is strictly monotonic and continuous in both real variables. An interpretation of $G(y,r)$ in Equation (1) is that the second variable r in modifies the state of the first variable y, creating an effect evaluated by $G(y,r)$ in the same measurement variable as y. The left hand side of (1) represents a one-step iteration of this phenomenon, in that $G(y,r)$ is then modified by t, resulting in the effect $G(G(y,r),t)$. Equation (1),

[1]The only exception is the counting measure, as in the case of the Avogadro number.
[2]The results can be extended to other cases, in particular interval scales.

which is referred to as the 'permutability' condition by Aczél (1966), formalizes the concept that the order of the two modifiers r and t is irrelevant.

Many, and various, scientific laws are 'permutable' in the sense of Equation (1). Some examples of permutable laws are the Lorentz-FitzGerald Contraction, Beer's law, the formula for computing the volume of a cylinder, and the Pythagorean theorem. For the Lorentz-FitzGerald Contraction, for example, written in the form

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

in which c is the speed of light, we have

$$L(L(\ell, v), s) = L(\ell, v)\sqrt{1 - \left(\frac{s}{c}\right)^2} = \sqrt{1 - \left(\frac{v}{c}\right)^2}\sqrt{1 - \left(\frac{s}{c}\right)^2} = L(L(\ell, s), v)\,.$$

Not all scientific laws are permutable. Van der Walls Law, for instance is not: see the Counterexample 4(f).

Under fairly general conditions of continuity and solvability making empirical sense, the permutability condition (1) implies the existence of a general representation

$$G(y, r) = f^{-1}(f(y) + g(r)), \tag{2}$$

where f and g are real valued, strictly monotonic continuous functions. This is stated precisely in our Lemma 7, which is due Falmagne (2012), and generalizes results of Hosszú (1962a,b,c) (cf. also Aczél, 1966). It is easily shown that the representation (2) implies the permutability condition (1): we have

$$\begin{aligned}
G(G(y, r), t) &= f^{-1}(f(G(y, r)) + g(t)) && \text{(by (2))}\\
&= f^{-1}(f(f^{-1}(f(y) + g(r))) + g(t)) && \text{(by (2) again)}\\
&= f^{-1}(f(y) + g(r) + g(t)) && \text{(simplifying)}\\
&= f^{-1}(f(y) + g(t) + g(r)) && \text{(by commutativity)}\\
&= G(G(y, t), r) && \text{(by symmetry)}\,.
\end{aligned}$$

We will also use a more general condition, called 'quasi permutability', which is defined by the equation

$$M(G(y, r), t) = M(G(y, t), r) \tag{3}$$

and lead to the representation

$$M(y, r) = m((f(y) + g(r)) \tag{4}$$

(see also Falmagne, 2012, and our Theorem 7 here).

The combined consequences of permutability or quasi permutability and meaningfulness are powerful ones. For instance, suppose that meaningfulness holds, that the function G is symmetric, and that it also satisfies a certain quasi permutability condition. Our Theorem 9 states that, under reasonable solvability conditions, there are only two possible forms for G, which are:

1. $$G(y, x) = \theta y x \qquad (\theta > 0) \tag{5}$$

2. $$G(y, x) = \left(y^\theta + x^\theta\right)^{\frac{1}{\theta}} \qquad (\theta > 0)\,. \tag{6}$$

With $\theta = 1$, the first equation is the formula for the area of a rectangle. With $\theta = 2$, the second one is the Pythagorean Theorem.

In our first section, we state some basic definitions and we describe a few examples of laws, taken from physics and geometry, in which the permutability condition applies. We also give one example, van der Waals Equation, which is not permutable. The second section is devoted to meaningfulness and ancillary concepts. The third section contains preparatory lemmas. The last section contain the main results of the paper.

Basic Concepts and Examples

1 Definition. We write $\mathbb{R}_+$ and $\mathbb{R}_{++}$ for the nonnegative and the positive reals, respectively. For some positive integer $n \geq 2$, let $I_1, I_2, \ldots, I_{n+1}$ be $n+1$ non negative real intervals of positive length. A *n-dimensional (numerical) code*, or an *n-code* fort short, is a function

$$M : I_1 \times \ldots \times I_n \longrightarrow I_{n+1} \tag{7}$$

that is strictly monotonic and continuous in its n arguments, and strictly increasing in its first argument. As we mostly deal with 2-codes in this paper, we simplify our language and just write 'code' to mean 2-code.

A 2-code M is *solvable* if it satisfies the following two conditions.

[S1] If $M(x, t) < p \in H$, there exists $w \in J$ such that $M(w, t) = p$.

[S2] The function M is *1-point right solvable*, that is, there exists a point $x_0 \in J$ such that for every $p \in H$, there is $v \in J'$ satisfying $M(x_0, v) = p$. In such a case, we may say that M is *x_0-solvable*.

By the strict monotonicity of M, the points w and v of [S1] and [S2] are unique.

Two functions $M : J \times J' \to H$ and $G : J \times J' \to H'$ are *comonotonic* if

$$M(x, s) \leq M(y, t) \quad \iff \quad G(x, s) \leq G(y, t), \qquad (x, y \in J; s, t \in J'). \tag{8}$$

In such a case, the equation

$$F(M(x, s)) = G(x, s) \qquad (x \in J; s \in J') \tag{9}$$

defines a strictly increasing continuous function $F : H \overset{\text{onto}}{\longrightarrow} H'$. We may say then that G is *F-comonotonic* with M.

We turn to the key condition of this paper.

2 Definition. A function $M : J \times J' \longrightarrow H$ is *quasi permutable* if there exists a function $G : J \times J' \to J$ co-monotonic with M such that

$$M(G(x, s), t) = M(G(x, t), s) \qquad (x, y \in J; s, t \in J'). \tag{10}$$

We say in such a case that M is *permutable with respect to G*, or *G-permutable* for short. When M is permutable with respect to itself, we simply say that M is *permutable*, a terminology consistent with Aczél (1966, Chapter 6, p. 270).

3 Lemma. *A function $M : J \times J' \to H$ is G-permutable only if G is permutable.*

This is straightforward. Suppose that G is F-comonotonic with M. For any $x \in J$ and $s, t \in J'$, we get $G(G(x,s),t) = F(M(G(x,s),t)) = F(M(G(x,t),s)) = G(G(x,t),s)$.

Many scientific laws embody permutable or quasi permutable numerical 2-codes, and hence can be written in the form of Equation (2). We give four quite different examples below. In each case, we derive the forms of the functions f and g in the representation equation (2).

4 Four Examples and One Counterexample.

(a) The Lorentz-FitzGerald Contraction. This term denotes a phenomenon in special relativity, according to which the apparent length of a rod measured by an observer moving at the speed v with respect to that rod is a decreasing function of v, vanishing as v approaches the speed of light. This function is specified by the formula

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}, \tag{11}$$

in which $c > 0$ denotes the speed of light, ℓ is the actual length of the rod (for an observer at rest with respect to the rod), and $L : \mathbb{R}_+ \times [0, c[\overset{\text{onto}}{\longrightarrow} \mathbb{R}_+$ is the length of the rod measured by the moving observer.

The function L is a permutable code. Indeed, L satisfies the strict monotonicity and continuity requirements, and we have

$$L(L(p,v),w) = p\left(1 - \left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}}\left(1 - \left(\frac{w}{c}\right)^2\right)^{-\frac{1}{2}} = L(L(p,w),v). \tag{12}$$

Solving the functional equation

$$\ell\sqrt{1 - \left(\frac{v}{c}\right)^2} = f^{-1}(f(\ell) + g(v)) \tag{13}$$

leads to the Pexider equation (c.f. Aczél, 1966, pages 141-165)

$$f(\ell y) = f(\ell) + k(y) \tag{14}$$

with

$$y = \sqrt{1 - \left(\frac{v}{c}\right)^2} \qquad \text{and} \qquad k(y) = g\left(c\sqrt{1 - y^2}\right).$$

As the background conditions (monotonicity and domains of the functions[3]) are satisfied, the unique forms of f and k in (14) are determined. They are: with $\xi > 0$,

$$\begin{aligned} f(\ell) &= \xi \ln \ell + \theta \\ k(y) &= \xi \ln y. \end{aligned} \tag{15}$$

So, we get for the function g in (13):

[3]Note that the standard solutions for Pexider equations are valid when the domain of the equation is an open connected subset of $\mathbb{R}^2$ rather than $\mathbb{R}^2$ itself. Indeed, Aczél (1987, see also Aczél, 2005, Chudziak and Tabor, 2008, and Radó and Baker, 1987) has shown that, in such cases, this equation can be extended to the real plane.

$$g(v) = \xi \ln \left(\sqrt{1 - \left(\frac{v}{c} \right)^2} \right) . \tag{16}$$

(b) BEER'S LAW. This law applies in a class of empirical situations where an incident radiation traverses some absorbing medium, so that only a fraction of the radiation goes through. In our notation, the expression of the law is

$$I(x, y) = x\, e^{-\frac{y}{c}}, \qquad (x, y \in \mathbb{R}_+,\ c \in \mathbb{R}_{++} \text{ constant}) \tag{17}$$

in which x denotes the intensity of the incident light, y is the concentration of the absorbing medium, c is a reference level, and $I(x, y)$ is the intensity of the transmitted radiation. The form of this law is similar to that of the Lorentz-FitzGerald Contraction and the same arguments apply. Thus, the function $I : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$ is also a permutable code. The solution of the functional equation

$$x\, e^{-\frac{y}{c}} = f^{-1}(f(x) + g(y))$$

follows a pattern identical to that of Equation (13) for the Lorentz-FitzGerald Contraction. The only difference lies in the definition of the function g, which is here

$$g(y) = -\xi \frac{y}{c} .$$

The definition of f is the same, namely (15). So, we get

$$I(x, y) = f^{-1}(f(x) + g(y)) = \exp \left(\frac{1}{\xi} (\xi \ln x + \theta - \xi \frac{y}{c} - \theta \right) = x\, e^{-\frac{y}{c}} .$$

(c) THE VOLUME OF A CYLINDER. The permutability equation applies not only to many physical laws, but also to some fundamental formulas of geometry, such as the volume $C(\ell, r)$ of a cylinder of radius r and height ℓ, for example. In this case, we have

$$C(\ell, r) = \ell \pi r^2, \tag{18}$$

which is permutable. We have

$$C(C(\ell, r), v) = C(\ell \pi r^2, v) = \ell \pi r^2 \pi v^2 = C(C(\ell, v), r).$$

Solving the functional equation

$$\ell \pi r^2 = f^{-1}(f(\ell) + g(r))$$

yields the solution

$$f(\ell) = \xi \ln \ell + \theta$$

(again, the function f is the same as in the two preceding examples), and

$$g(r) = \xi \ln \left(\pi r^2 \right) ,$$

with

$$f^{-1}(f(\ell)+g(r))=\exp\left(\frac{1}{\xi}\left(\xi\ln\ell+\theta+\xi\ln\left(\pi r^2\right)-\theta\right)\right)=\ell\pi r^2\,.$$

We give another geometric example below, in which the form of f is different.

(d) The Pythagorean Theorem. The function

$$P(x,y)=\sqrt{x^2+y^2} \qquad\qquad (x,y\in\mathbb{R}_{++}), \tag{19}$$

representing the length of the hypothenuse of a right triangle in terms of the lengths of its sides, is a permutable code. We have indeed

$$P(P(x,y),z)=\sqrt{P(x,y)^2+z^2}=\sqrt{x^2+y^2+z^2}=P(P(x,z),y).$$

The other conditions are clearly satisfied, and so is Condition [S1]. Condition [S2] would be achieved by taking an appropriate restriction of the function P as in the case of Examples 4(a) and (b). Notice that the code P is a symmetric function: we have $P(x,y)=P(y,x)$ for all $x,y\in\mathbb{R}_{++}$

So, we must solve the equation

$$\sqrt{x^2+y^2}=f^{-1}\left(f(x)+f(y)\right)$$

or, equivalently,

$$f\left(\sqrt{x^2+y^2}\right)=f(x)+f(y)\,. \tag{20}$$

With $z=x^2$, $w=y^2$, and defining the function $h(z)=f\left(z^{\frac{1}{2}}\right)$, Equation (20) becomes

$$h(z+w)=h(z)+h(w)\,,$$

a Cauchy equation on the positive reals, with h strictly increasing. It has the unique solution $h(z)=\xi\,z$, for some positive real number ξ (c.f. Aczél, 1966, page 31). So, we get $f(x)=\xi x^2$ and

$$f^{-1}(f(x)+f(y))=\left(\frac{1}{\xi}\left(\xi x^2+\xi y^2\right)\right)^{\frac{1}{2}}=\sqrt{x^2+y^2}\,.$$

(f) The Counterexample: van der Waals Equation. One form of this equation is

$$T(p,v)=K\left(p+\frac{a}{v^2}\right)(v-b), \tag{21}$$

in which p is the pressure of a fluid, v is the volume of the container, T is the temperature, and a, b and K are constants; K is the reciprocal of the Boltzmann constant. It is easily shown that the function T in (21) is not permutable.

Meaningful Collection of Codes

One of our goals in this paper is to axiomatize a particular type of invariance that must hold for all scientific or geometric laws. The consequence of this axiomatization should be that the form of an expression representing a scientific law should not be altered by changing the units of the variables. The next definition, which generalizes that used by Falmagne (2004) (see also Falmagne and Narens, 1983; Narens, 2002) applies to codes regarded as functions of n real ratio scale variables.

We illustrate the definition by our Example 4(a) involving the Lorentz-FitzGerald Contraction, which we expressed by the equation

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}. \tag{22}$$

The trouble with this notation is its ambiguity: the units of ℓ, which denotes the length of the rod, and of v, for the speed of the observer, are not specified. Writing $L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to 70 meters and 3 kilometers per second, respectively. Such a parenthetical reference is standard in a scientific context, but is not instrumental for our purpose, which is to express, formally, an invariance with respect to any change in the units[4].

To rectify the ambiguity, we propose to interpret $L(\ell, v)$ as a shorthand notation for $L_{1,1}(\ell, v)$, in which ℓ and L on the one hand, and v on the other hand, are measured in terms of two particular initial or 'anchor' units fixed arbitrarily. Such units could be m (meter) and km/sec, if one wishes. The $_{1,1}$ index of $L_{1,1}$ signifies these initial units. Describing the phenomenon in terms of other units amounts to multiply ℓ and v in any pair (ℓ, v) by some positive constants α and β, respectively. At the same time, L also gets to be multiplied by α, and the speed of ligh c by β. Doing so defines a new function $L_{\alpha,\beta}$, which is different from $L = L_{1,1}$ if either $\alpha \neq 1$ or $\beta \neq 1$ (or both), but carries the same information from an empirical standpoint. For example, if our new units are *km* and *m/sec*, then the two expressions

$$L_{10^{-3},10^3}(.007, 3000) \quad \text{and} \quad L(70, 3) = L_{1,1,1}(70, 3),$$

while numerically not equal, should describe the same empirical situation. This points to the appropriate definition of $L_{\alpha,\beta}$. We should write:

$$L_{\alpha,\beta}(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{\beta c}\right)^2}. \tag{23}$$

The connection between L and $L_{\alpha,\beta}$ is thus

$$\begin{aligned} \frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) &= \left(\frac{1}{\alpha}\right)\alpha\ell\sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} \\ &= \ell\sqrt{1 - \left(\frac{v}{c}\right)^2} = L(\ell, v). \end{aligned} \tag{24}$$

[4]A relevant point is made by Suppes (2002, see "Why the Fundamental Equations of Physical Theories Are not Invariant", p. 120).

This implies, for any α, β, ν and μ in $\mathbb{R}_{++}$,

$$\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu} L_{\nu,\mu}(\nu\ell, \mu v), \tag{25}$$

which is the invariance equation we were looking for, in this case, and which is generalized as Equation (27) in the next definition. Note that the range of the second variable of the function $L_{\alpha,\beta}$ is now $[0, \beta c[$ instead of $]0, c[$. The range of the first variable of L is the non negative reals and so did not change.

It is clear from our discussion of this example and from Equation (25) that the definition of 'meaningfulness' must apply to a collection of codes, each of which corresponds to another choice of units, that is, the choice of (α, β) and (ν, μ) in the case of Equation (25). We formulate the definition in the general case of a family of n-codes.

5 Definition. Let $[a_1, a'_1[, \ldots, [a_{n+1}, a'_{n+1}[$ be $n+1$ half open intervals, and let

$$\mathcal{M} = \{M_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} = (\alpha_1 \ldots, \alpha_n) \in \mathbb{R}^n_{++}\} \tag{26}$$

be a collection of n-codes, with for the initial code M

$$M = M_{\underbrace{1,\ldots,1}_{n \text{ indices}}} : [a_1, a'_1[\times \ldots \times [a_n, a'_n[\longrightarrow [a_{n+1}, a'_{n+1}[\,.$$

Each of the terms $\alpha_1, \ldots, \alpha_n$ in $\boldsymbol{\alpha}$ represents a change of the unit of one of the measurement scale. We will specify the domain and range of any code $M_{\boldsymbol{\alpha}}$ in a moment.

Let $\delta_1, \ldots, \delta_n$ be a finite sequence of rational numbers. The collection of n-codes $\mathcal{M}$ is $(\delta_1, \ldots, \delta_n)$-*meaningful* if for any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n_{+}$ and any pair of vectors

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{++} \qquad \text{and} \qquad \boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n_{++},$$

the following equality holds:

$$\frac{1}{\alpha_1^{\delta_1} \cdot \ldots \cdot \alpha_n^{\delta_n}} M_{\boldsymbol{\alpha}}(\alpha_1 x_1, \ldots, \alpha_n x_n) = \frac{1}{\mu_1^{\delta_1} \cdot \ldots \cdot \mu_n^{\delta_n}} M_{\boldsymbol{\mu}}(\mu_1 x_1, \ldots, \mu_n x_n), \tag{27}$$

which implies

$$\frac{1}{\alpha_1^{\delta_1} \cdot \ldots \cdot \alpha_n^{\delta_n}} M_{\boldsymbol{\alpha}}(\alpha_1 x_1, \ldots, \alpha_n x_n) = M(x_1, \ldots, x_n)\,. \tag{28}$$

Accordingly, any code $M_{\boldsymbol{\alpha}}$ in a $(\delta_1, \ldots, \delta_n)$-meaningful family $\mathcal{M}$ satisfies

$$M_{\boldsymbol{\alpha}} : [\alpha_1 a_1, \alpha_1 a'_1[\times \ldots \times [\alpha_n a_n, \alpha_n a'_n[\longrightarrow \left[\left(\alpha_1^{\delta_1} \cdot \ldots \cdot \alpha_n^{\delta_n}\right) a_{n+1}, \left(\alpha_1^{\delta_1} \cdot \ldots \cdot \alpha_n^{\delta_n}\right) a'_{n+1} \right[\,.$$

The exponents δ_i's are called the *markers* of the family $\mathcal{M}$.

The next concept will also play an important role. A collection of n-codes $\mathcal{M}$ is *self-transforming*, or an *ST-collection*, if for all codes $M_{\boldsymbol{\alpha}}$ in the collection, the measurement unit of the output of the code $M_{\boldsymbol{\alpha}}$ is the same as the measurement unit of its first variable. In other words, if for every vector $\boldsymbol{\alpha} = (\alpha_1 \ldots, \alpha_n) \in \mathbb{R}^n_{++}$, we have

$$\alpha_1^{\delta_1} \cdot \ldots \cdot \alpha_n^{\delta_n} = \alpha_1. \tag{29}$$

In the rest of this paper, we apply these concepts to the case of a collection of 2-codes $\mathcal{M} = \{M_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{R}_{++}\}$. For $[a, a'[$, $[b, b'[$ and $]d, d'[$, three real non negative intervals, we have

$$M = M_{1,1} : [a, a'[\times [b, b'[\overset{\text{onto}}{\longrightarrow} [d, d'[,$$

and so

$$M_{\alpha,\beta} : [\alpha a, \alpha a'[\times [\beta b, \beta b'[\overset{\text{onto}}{\longrightarrow} [\alpha^{\delta_1}\beta^{\delta_2} d, \alpha^{\delta_1}\beta^{\delta_2} d' [, \qquad (\alpha, \beta \in \mathbb{R}_{++}).$$

Accordingly, the meaningfulness equation (27) specializes into

$$\frac{1}{\alpha^{\delta_1}\beta^{\delta_2}} M_{\alpha,\beta}(\alpha x, \beta r) = \frac{1}{\mu^{\delta_1}\nu^{\delta_2}} M_{\mu,\nu}(\mu x, \nu r), \qquad (x \in [a, a'[;\ r \in [b, b'[). \tag{30}$$

Let us exercise this definition in the case of our four examples. We will see that in one case—the Pythagorean Theorem—the exponents δ's in (30) are not integers.

6 Examples. (a) THE LORENTZ-FITZGERALD CONTRACTION. We have a collection $\mathcal{L} = \{L_{\alpha,\beta} \,|\, (\alpha, \beta) \in \mathbb{R}^2_{++}\}$ of codes. We require that the collection $\mathcal{L}$ be $(1, 0)$-meaningful. This implies that

$$\frac{1}{\alpha^1\beta^0} L_{\alpha,\beta}(\alpha\ell, \beta v) = \left(\frac{1}{\alpha}\right) \alpha\ell \sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2} = L(\ell, v)$$

and so

$$\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu} L_{\nu,\mu}(\nu\ell, \mu v),$$

which is our equation (25) and is a special case of (27) and (30). Clearly, the family $\mathcal{L}$ is self-transforming. So is the family of our next example.

(b) BEER'S LAW. The form of this law is similar to the preceding one. We have a collection $\mathcal{I} = \{I_{\alpha,\beta} \,|\, (\alpha, \beta) \in \mathbb{R}^2_{++}\}$ of codes, which is also $(1, 0)$-meaningful. This gives

$$\frac{1}{\alpha^1\beta^0} I_{\alpha,\beta}(\alpha x, \beta y) = \frac{1}{\alpha}(\alpha x)\, e^{-\frac{\beta y}{\beta c}} = x e^{\frac{y}{c}}$$

yielding

$$\frac{1}{\alpha} I(\alpha x, \beta y) = \frac{1}{\nu} I(\nu x, \mu u),$$

another special case of (27)-(30).

(c) THE VOLUME OF A CYLINDER. This example is quite different. We have a collection of codes $\mathcal{C} = \{C_{\alpha,\beta} \,|\, (\alpha, \beta) \in \mathbb{R}^2_{++}\}$, which must be $(1, 2)$-meaningful. We get

$$\frac{1}{\alpha^1\beta^2} C_{\alpha,\beta}(\alpha x, \beta y) = \frac{1}{\alpha^1\beta^2}(\alpha\ell)\, \pi\, (\beta r)^2 = \ell\pi r^2,$$

and so

$$\frac{1}{\alpha^1\beta^2} C_{\alpha,\beta}(\alpha x, \beta y) = \frac{1}{\nu^1\mu^2} C_{\nu,\mu}(\nu x, \mu y).$$

The collection $\mathcal{C}$ is not an ST-collection since $\alpha^1\beta^2 \neq \alpha$ if $\beta \neq 1$.

(d) THE PYTHAGOREAN THEOREM. Here, we have only one measurement scale, which is the same for the two input variables and for the output variable. We require the collection of codes $\mathcal{P} = \{P_{\alpha,\alpha} \,|\, \alpha \in \mathbb{R}_{++}\}$ to be $(\frac{1}{2}, \frac{1}{2})$-meaningful. We obtain

$$\frac{1}{\alpha^{\frac{1}{2}}\alpha^{\frac{1}{2}}} P_{\alpha,\alpha}(\alpha x, \alpha y) = \frac{1}{\alpha^{\frac{1}{2}}\alpha^{\frac{1}{2}}} \sqrt{(\alpha x)^2 + (\alpha y)^2} = \sqrt{x^2 + y^2}\,. \tag{31}$$

Preparatory Lemmas

We recall a recent result (see Falmagne, 2012), which generalizes Hosszú (1962a,b,c) (cf. also Aczél, 1966).

7 Lemma. (i) *A solvable code $M : J \times J' \to H$ is quasi permutable if and only if there exists three continuous functions $m : \{f(y) + g(r) \,|\, x \in J,\, r \in J'\} \to H$, $f : J \to \mathbb{R}$, and $g : J' \to \mathbb{R}$, with m and f strictly increasing and g strictly monotonic, such that*

$$M(y, r) = m(f(y) + g(r)). \tag{32}$$

(ii) *A solvable code $G : J \times J' \to J$ is a permutable code if and only if, with f and g as above, we have*

$$G(y, r) = f^{-1}(f(y) + g(r)). \tag{33}$$

(iii) *If a solvable code $G : J \times J \to J$ is a symmetric function—that is, $G(x, y) = G(y, x)$ for all $x, y \in J$— then G is permutable if and only if there exists a strictly increasing and continuous function $f : J \to J$ satisfying*

$$G(x, y) = f^{-1}(f(x) + f(y)). \tag{34}$$

The meaningfulness condition introduced in Definition 5 and Equation (30) is a powerful one. In particular, it enables some properties of one of the codes in $\mathcal{M}$ to extend to all the others codes in that collection. The next lemma illustrates this point.

8 Lemma. *Let $\mathcal{M}$ be a (δ_1, δ_2)-meaningful collection of codes—so all the codes in $\mathcal{M}$ are functions of two variables. Suppose that some code $M_{\alpha,\beta}$ in the collection $\mathcal{M}$-satisfies any of the following five properties:*

(i) *$M_{\alpha,\beta}$ is solvable;*

(ii) *$M_{\alpha,\beta}$ is differentiable in both variables;*

(iii) *$M_{\alpha,\beta}$ is quasi permutable;*

(iv) *$M_{\alpha,\beta}$ is a symmetric function, with $\alpha = \beta$;*

(v) *$\mathcal{M}$ is a self-transforming collection and $M_{\alpha,\beta}$ is permutable.*

Then all the codes in $\mathcal{M}$ satisfy the same property. Moreover, if $M_{\alpha,\beta} = M$ is solvable and permutable, so that $M(x,r) = f^{-1}(f(x)+g(r))$ by Lemma 7(ii), then for any code $M_{\mu,\eta}$ in the collection $\mathcal{M}$, we have

$$M_{\mu,\eta}(x,r) = \mu^{\delta_1}\nu^{\delta_2} f^{-1}\left(f\left(\frac{x}{\mu}\right) + g\left(\frac{r}{\eta}\right)\right). \tag{35}$$

Proof. Without loss of generality, we suppose that $\alpha = \beta = 1$, with $M = M_{1,1}$. As the family $\mathcal{M}$ is (δ_1, δ_2)-meaningful, we have, for all positive real numbers μ and ν and writing $\eta = \mu^{\delta_1}\nu^{\delta_2}$ for simplicity:

$$M_{\mu,\nu}(x,r) = \eta M\left(\frac{x}{\mu}, \frac{u}{\nu}\right) \qquad (x \in [\alpha a, \alpha a'[\,; r \in [\beta b, \beta b'[\,). \tag{36}$$

(i) Suppose that the code M is solvable. If $M_{\mu,\nu}(x,r) < p$, for some code $M_{\mu,\nu}$ in $\mathcal{M}$, then $M(\frac{x}{\mu}, \frac{u}{\nu}) < \frac{p}{\eta}$ follows from (36). As the code M satisfies [S1], there must be some $w \in [b, b'[$ such that $M(\frac{x}{\mu}, w) = \frac{p}{\eta}$. Defining $t = \nu w$, we get

$$M_{\mu,\nu}(x,t) = \eta M\left(\frac{x}{\mu}, \frac{t}{\nu}\right) = p.$$

Thus, the code $M_{\mu,\nu}$ also satisfies [S1]. Since M satisfies [S2], there exists some x_0 in $[a, a'[$ such that M is x_0-solvable. Define $y_0 = \mu x_0 \in [\mu a, \mu a'[$ and take any q in the range of the function $M_{\mu,\nu}$. This implies that $\frac{q}{\eta}$ is in the range of M, and by [S2] applied to M, there is some w such that $M(x_0, w) = \frac{q}{\eta}$ or, equivalently with $v = \beta w$,

$$q = \eta M\left(\frac{y_0}{\mu}, \frac{v}{\nu}\right) = M_{\mu,\nu}(y_0, v),$$

by the meaningfulness of the family $\mathcal{M}$. Thus, $M_{\mu,\nu}$ is y_0-solvable.

(ii) The differentiability of $M_{\mu,\nu,\eta}$ results from that of M via (36).

(iii) Suppose now that M is quasi permutable. (We do not assume here that $\mathcal{M}$ is a self-transforming family.) Thus, there exists a code $G : [a, a'[\times [b, b'[\to [a, a'[$ co-monotonic with M such that

$$M(G(x,s),t) = M(G(x,t),s) \qquad (x, y \in [a, a'[\,; s, t \in [b, b'[\,). \tag{37}$$

For any pair of parameters (μ, ν), define the function $G_{\mu,\nu} : [a, a'[\times [b, b'[\to [a, a'[$ by the equation

$$G_{\mu,\nu}(x,r) = \mu G\left(\frac{x}{\mu}, \frac{u}{\nu}\right). \tag{38}$$

Thus, $G_{\mu,\nu}$ is comonotonic with $M_{\mu,\nu}$ and we have successively

$$\begin{aligned}
M_{\mu,\nu}(G_{\mu,\nu}(x,r),v) &= \eta M\left(\frac{1}{\mu}G_{\mu,\nu}(x,r),\frac{v}{\nu}\right) && \text{(by (δ_1,δ_2)-meaningfulness)}\\
&= \eta M\left(G\left(\frac{x}{\mu},\frac{r}{\nu}\right),\frac{v}{\nu}\right) && \text{(by the definition of $G_{\mu,\nu}$)}\\
&= \eta M\left(G\left(\frac{x}{\mu},\frac{v}{\nu}\right),\frac{r}{\nu}\right) && \text{(by the permutability of G)}\\
&= M_{\mu,\nu}(G_{\mu,\nu}(x,v),r) && \text{(by symmetry).}
\end{aligned}$$

Consequently, any code $M_{\mu,\nu,\eta}$ is $G_{\mu,\nu}$-permutable.

(iv) This follows from the definition of the (δ_1,δ_2)-meaningfulness of the collection.

(v) Suppose that $\mathcal{M}$ is a self-transforming collection and that M is permutable. We have thus, for any $M_{\mu,\nu}$,

$$\begin{aligned}
\frac{1}{\mu}M_{\mu,\nu}\left(M_{\mu,\nu}(x,r),v\right) &= M\left(\frac{1}{\mu}M_{\mu,\nu}(x,r),\frac{v}{\nu}\right) && \text{(by (δ_1,δ_2)-meaningfulness)}\\
&= M\left(M\left(\frac{x}{\mu},\frac{r}{\nu}\right),\frac{v}{\nu}\right) && \text{(by (δ_1,δ_2)-meaningfulness)}\\
&= M\left(M\left(\frac{x}{\mu},\frac{v}{\nu}\right),\frac{r}{\nu}\right) && \text{(by the permutability of M)}\\
&= \frac{1}{\mu}M_{\mu,\nu}\left(M_{\mu,\nu}(x,v),u\right) && \text{(by symmetry).}
\end{aligned}$$

We have thus $M_{\mu,\nu}\left(M_{\mu,\nu}(x,r),v\right) = M_{\mu,\nu}\left(M_{\mu,\nu}(x,v),r\right)$ and so $M_{\mu,\nu}$ is permutable.

Equation (35) results from Equation (33) of Theorem 7(ii) and Equation (36). □

Main Result

9 Theorem. *Let $\mathcal{G} = \{G_{\alpha,\beta} \mid \alpha,\beta \in \mathbb{R}_{++}\}$ be a 2-meaningful collection of codes, with $G_{\alpha,\beta} : \mathbb{R}_{++} \times \mathbb{R}_{++} \overset{onto}{\longrightarrow} \mathbb{R}_{++}$ for all $\alpha,\beta \in \mathbb{R}_{++}$. Moreover, suppose that one of these codes, say the code $G_{\alpha,\beta}$, is solvable, strictly increasing in both variables, and permutable with respect to the initial code G.*

(i) *Then the initial code $G = G_{1,1}$ must have the following form*

$$G(y,r) = y\,r^{\theta} \tag{39}$$

for some positive constant θ. This implies that, for all $G_{\alpha,\beta} \in \mathcal{G}$:

$$G_{\alpha,\beta}(y,r) = \frac{1}{\alpha\,\beta^{\theta}}\,y\,r^{\theta}\,. \tag{40}$$

(ii) *If, in addition, the code $G_{\alpha,\beta}$ is a symmetric function, with $\alpha = \beta$ and $G_\alpha = G_{\alpha,\alpha}$, then we have two possible cases.*

1. CASE 1.
$$G(y,x) = \theta y x\,. \qquad (\theta > 0) \tag{41}$$
with for all $G_\alpha \in \mathcal{G}$
$$G_\alpha(y,x) = \frac{\theta}{\alpha^2} y x\,. \tag{42}$$
2. CASE 2.
$$G(y,x) = \left(y^\theta + x^\theta\right)^{\frac{1}{\theta}} \qquad (\theta > 0) \tag{43}$$
with for all $G_\alpha \in \mathcal{G}$
$$G_\alpha(y,x) = \frac{1}{\alpha}\left(y^\theta + x^\theta\right)^{\frac{1}{\theta}}\,. \tag{44}$$

PROOF. (i) By Lemma 8, all the codes in $\mathcal{G}$ are solvable, permutable, and strictly increasing in both variables, and we get for all $G_{\alpha,\beta}$ in $\mathcal{G}$:

$$G_{\alpha,\beta}(y,r) = f^{-1}\left(f\left(\frac{y}{\alpha}\right) + g\left(\frac{r}{\beta}\right)\right)\,, \tag{45}$$

with in particular

$$G(y,r) = f^{-1}(f(y) + f(x))\,. \tag{46}$$

We get successively

$$\begin{aligned}
&G_{\alpha,\beta}(G(y,r),s)\\
&\quad = G_{\alpha,\beta}(f^{-1}(f(y)+g(r)),s) && \text{(by Lemma 7(iii))} && (47)\\
&\quad = G\left(\frac{1}{\alpha}f^{-1}(f(y)+g(r)),\frac{s}{\beta}\right) && \text{(by 2-meaningfulness)} && (48)\\
&\quad = f^{-1}\left(f\left(\frac{1}{\alpha}f^{-1}\left(f(y)+g(r)\right)\right)+g\left(\frac{s}{\beta}\right)\right) && \text{(by Lemma 7(iii))} && (49)\\
&\quad = f^{-1}\left(f\left(\frac{1}{\alpha}f^{-1}\left(f(y)+g(s)\right)\right)+g\left(\frac{r}{\beta}\right)\right) && \text{(by quasi permutability).} && (50)
\end{aligned}$$

Equating the last two r.h. sides and applying the function f on both sides, we get

$$f\left(\frac{1}{\alpha}f^{-1}\left(f(y)+g(r)\right)\right)+g\left(\frac{s}{\beta}\right) = f\left(\frac{1}{\alpha}f^{-1}\left(f(y)+g(s)\right)\right)+g\left(\frac{r}{\beta}\right)\,. \tag{51}$$

Setting $s = f(y)$, $t = g(r)$, fixing $s = 1$, and fixing also temporarily $\nu = \frac{1}{\alpha}$ and $\eta = \frac{1}{\beta}$, transform Equation (51) into

$$f\left(\nu f^{-1}\left(s+t\right)\right) + g\left(\eta\right) = f\left(\nu f^{-1}\left(s+g(1)\right)\right) + g\left(\eta g^{-1}(t)\right)\,. \tag{52}$$

Defining the functions

$$\begin{aligned}
h_\nu &= f \circ \nu f^{-1},\\
k_\nu(s) &= h_\nu(s + f(1)),\\
p_\eta(t) &= g\left(\eta g^{-1}(t)\right) - g\left(\eta\right)\,,
\end{aligned}$$

(62) becomes

$$h_\nu(s+t) = k_\nu(s) + p_\eta(t)\,,$$

a Pexider equation. Its solution is

$$h_\nu(s) = a\,s + b(\nu) + c\,, \tag{53}$$
$$k_\nu(s) = a\,s + b(\nu)\,, \tag{54}$$
$$p_\eta(s) = a\,s + c\,, \tag{55}$$

for some constants a and c and some function b which vary with ν but not η. Indeed, the function k_ν does not depend upon η and the function p_η does not depend upon ν, so a and c must be constants.

We first rewrite Equation (55) to get the form of the function g. We get

$$p_\eta(t) = g\left(\eta g^{-1}(t)\right) - g\left(\eta\right) = a\,t + c$$

As $g^{-1}(t) = r$, the second equation gives

$$g(\eta r) = ag(r) + g(\eta) + c \tag{56}$$

With $\eta = 1$, we get

$$g(r) = ag(r) + g(1) + c, \tag{57}$$

yielding

$$g(r)(1-a) = g(1) + c\,.$$

as g is strictly monotonic, we must have

$$a = 1 \qquad \text{and} \qquad g(1) + c = 0\,.$$

Equation (56) simplifies into

$$g(\eta r) = g(r) + g(\eta) + c\,,$$

the only solution of which is, for some constant ϕ_1 and with $c = -\psi_1$,

$$g(r) = \phi_1 \ln r + \psi_1\,.$$

We now rewrite Equation (53) in terms of the functions f. As $s = f(y)$, we get

$$h_\nu(s) = (f \circ \nu f^{-1})(s) = f(\nu y).$$

Equation (53) becomes—with $c = -\psi_1$ and $a = 1$,

$$f(\nu y) = f(y) + b(\nu) - \psi_1$$

again, a Pexider equation. Its unique solutions for the functions f and m are

$$f(y) = \phi_2 \ln y + \psi_2$$
$$b(y) = \phi_2 \ln y + \psi_1\,.$$

We finally obtain

$$\begin{aligned}G(y,r) &= f^{-1}(f(y)+g(r))\\ &= e^{\frac{1}{\phi_2}(f(y)+g(r)-\psi_2)}\\ &= e^{\frac{1}{\phi_2}(\phi_2 \ln y+\psi_2+\phi_1 \ln r+\psi_1-\psi_2)}\\ &= e^{\frac{1}{\phi_2}\left(\ln y^{\phi_2}+\ln r^{\phi_1}+\psi_1\right)}\\ &= e^{\ln y+\ln r^{\frac{\phi_1}{\phi_2}}+\frac{\psi_1}{\phi_2}}\,.\end{aligned}$$

So, with $\theta = e^{\frac{\psi_1}{\phi_2}}$ and $\phi = \frac{\phi_1}{\phi_2}$, we finally get

$$G(y,r) = \theta\, y\, r^{\phi}\,. \tag{58}$$

Equation (40) follows from (58) and 2-meaningfulness: $G_{\alpha,\beta}(y,r) = G\left(\frac{y}{\alpha},\frac{r}{\beta}\right)$.

Proof of (ii). Suppose now that $G_{\alpha,\beta} = G_\alpha$ is a symmetric function, with $\alpha = \beta$. By Lemma 8 (iii), all the codes in $\mathcal{G}$ are symmetric, and we have

$$G_{\alpha,\beta}(y,x) = f^{-1}\left(f\left(\frac{y}{\alpha}\right)+f\left(\frac{x}{\alpha}\right)\right) \tag{59}$$

replacing Equation (45), with in particular

$$G(y,r) = f^{-1}(f(y)+f(x))\,. \tag{60}$$

Applying the same derivation as in asymmetric case, namely Eqs. (47)-(50), we get

$$f\left(\frac{1}{\alpha}f^{-1}\left(f(y)+f(x)\right)\right)+f\left(\frac{z}{\alpha}\right) = f\left(\frac{1}{\alpha}f^{-1}\left(f(y)+f(z)\right)\right)+f\left(\frac{x}{\alpha}\right)\,, \tag{61}$$

instead of Equation (51). We then proceed as in our proof of (i). (We will, however, end up with a different functional equation, which will give us two possible solutions.) Setting $s = f(y)$, $t = f(x)$, and fixing $z = 1$, and fixing also temporarily $\beta = \frac{1}{\alpha}$, Equation (51) becomes

$$f\left(\beta f^{-1}\left(s+t\right)\right)+f\left(\beta\right) = f\left(\beta f^{-1}\left(s+f(1)\right)\right)+f\left(\beta f^{-1}(t)\right)\,. \tag{62}$$

Defining the functions $h_\beta = f\circ\beta f^{-1}$, and $k_\beta : s \mapsto h_\beta(s+f(1))-f(\beta)$, (62) yields

$$h_\beta(s+t) = k_\beta(s)+h_\beta(t)\,,$$

a Pexider equation. Because the functions h_β and κ_β are defined on the reals and are strictly monotonic, its solution is

$$h_\beta(s) = w(\beta)s+v(\beta) \tag{63}$$

$$k_\beta(s) = w(\beta)s, \tag{64}$$

for some constants $w(\beta)$ and $v(\beta)$ which may, hovever, depends on β. Rewriting now (63) in terms of the function f, we get

$$f(\beta y) = w(\beta) f(y) + v(\beta)\,, \tag{65}$$

another standard functional equation (Aczél, 1966). We thus have two possible solutions for the function f.

Case 1. With w a constant function in (65):

$$f(y) = \phi \ln y + \psi \qquad\qquad (\phi > 0)\,. \tag{66}$$

Replacing f in the representation equation (60) by its form in (66) leads for the code G to the equation

$$G(y, x) = \theta y x\,, \tag{67}$$

with $\theta = e^{\frac{\psi}{\phi}}$. With $G_\alpha(y, x) = G\left(\frac{y}{\alpha}, \frac{x}{\alpha}\right)$, we get

$$G_\alpha(y, x) = \frac{\theta}{\alpha^2} y x\,.$$

Case 2. With w not constant in (65):

$$f(y) = \psi y^\theta \qquad\qquad (\psi\theta > 0)\,. \tag{68}$$

Replacing f in (60) by its form in (68) leads to

$$G(y, x) = \left(y^\theta + x^\theta\right)^{\frac{1}{\theta}} \qquad\qquad (\theta > 0)\,. \tag{69}$$

This implies

$$G_\alpha(y, x) = G\left(\frac{y}{\alpha}, \frac{x}{\alpha}\right) = \frac{1}{\alpha}\left(y^\theta + x^\theta\right)^{\frac{1}{\theta}}\,.$$

□

The Pythagorean Theorem

With $\theta = 2$, Case 2 of Theorem 9 is the formula for the Pythagorean Theorem. In fact, Theorem 9 can provide us with still another proof of the Pythagorean theorem, to be added to the several hundreds that already exists.

We suppose that the length $P(x, y)$ of the hypotenuse of a right triangle with leg lengths $x > x_0$ and $y > x_0$ (for some $x_0 > 0$) is a symmetric solvable code[5]; thus $P : [x_0, \infty[\, \times \mathbb{R}_+ \to [x_0, \infty[$. We take the function P to be the initial code of a family of codes $\{P_\alpha\}$. We establish the permutability and the quasi permutability of the code P with respect to P_α, for any $\alpha > 0$, by an elementary geometric argument.

[5]Cf. our discussion of Condition [S2] in the context of Example 4(e).

10 The Permutability of P. A right triangle $\triangle ABC$ with leg lengths x and y and hypothenuse of length $P(x, y)$ is represented in Figure 1A. Thus $AB = x$, $BC = y$ and $P(x, y) = AC$. Another right triangle $\triangle ACD$ is defined by the segment $\overline{CD}$ of length z, which is perpendicular to the plane of $\triangle ABC$. The length of the hypothenuse $\overline{AD}$ of $\triangle ACD$ is thus $P(P(x, y), z) = AD$. Still another right triangle $\triangle EAB$ is defined by the perpendicular $\overline{AE}$ to the plane of $\triangle ABC$. We choose E such that $AE = z = CD$; we have thus $EB = P(x, z)$. Since $\overline{AE}$ is perpendicular to the plane of $\triangle ABC$ and $\triangle ABC$ is a right triangle, $\overline{EB}$ is perpendicular to $\overline{BC}$. The lines $\overline{BC}$ and $\overline{BE}$ are perpendicular. (Indeed, the perpendicular L at the point B to the plane of triangle $\triangle ABC$ is coplanar with $\overline{AE}$. So, as $\overline{BC}$ is perpendicular to both $\overline{AE}$ and L, it must be perpendicular to to the plane of $\triangle EAB$, and so it must be perpendicular to $\overline{EB}$.) Accordingly, $EC = P(P(x, z), y)$ is the length of the hypothenuse of the right triangle $\triangle EBC$. It is clear that, by construction, the four points A, C, D and E are coplanar. They define a rectangle whose diagonals $\overline{AD}$ and $\overline{EC}$ must be equal. So, we must have $P(P(x, y), z) = P(P(x, z), y)$, establishing the permutability of the code P.

11 The Quasi Permutability of P_α. For any positive real number α, the triangle $\triangle A'B'C'$ pictured in Figure 1B, with $C' = c$, A collinear with $A'C'$, B collinear with $B'C'$, and $A'B' = \frac{x}{\alpha}$, $B'C' = \frac{y}{\alpha}$ and $A'C' = \frac{P(x,y)}{\alpha}$, is similar to the triangle $\triangle ABC$ also represented in Figure 1B. So, we have

$$P\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) = \frac{P(x, y)}{\alpha}. \tag{70}$$

The function P is the initial code of the meaningful family of codes $\{P_\alpha\}$. For the code P_α in that family, we get

$$\begin{aligned}
P_\alpha\left(P(x, y), z\right) &= \alpha P\left(\frac{P(x, y)}{\alpha}, \frac{z}{\alpha}\right) && \text{(by } \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\text{-meaningfulness)}\\
&= \alpha P\left(P\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right), \frac{z}{\alpha}\right) && \text{(by Equation (70))}\\
&= \alpha P\left(P\left(\frac{x}{\alpha}, \frac{z}{\alpha}\right), \frac{y}{\alpha}\right) && \text{(by the permutability of } P\text{)}\\
&= \alpha P\left(\frac{P(x, z)}{\alpha}, \frac{y}{\alpha}\right) && \text{(by Equation (70))}\\
&= P_\alpha\left(P(x, z), y\right) && \text{(by } \left(\tfrac{1}{2}, \tfrac{1}{2}\right)\text{-meaningfulness).}
\end{aligned}$$

We conclude that any code P_α in the family $\{P_\alpha\}$ is quasi permutable with respect to the initial code P.

References

J. Aczél. Utility of extension of functional equations—when possible. *Journal of Mathematical Psychology*, 49:445–449, 2005.

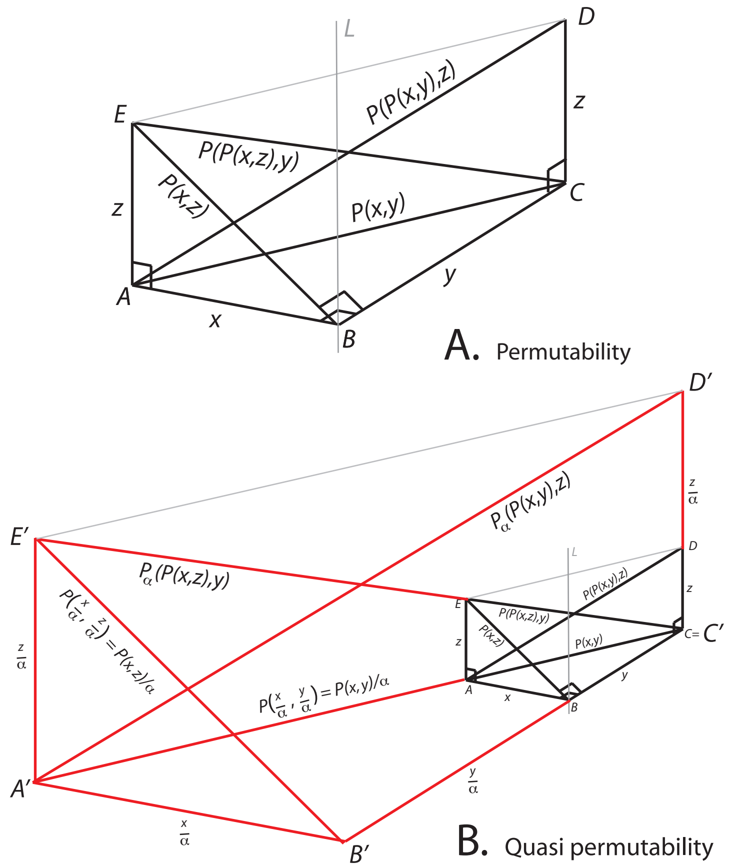

Figure 1: The upper graph A illustrates the permutability condition formalized by the equation $P(P(x,y),z) = P(P(x,z),y)$. The lower graph B shows that the quasi permutability condition formalized by the equation $P_\alpha(P(x,y),z) = P_\alpha(P(x,z),y)$ only involves a rescaling of all the variables pictured in Figure 1A, resulting in a similar figure, with the rectangle $A'B'C'D'$ similar to the rectangle $ABCD$. The measures of the two diagonals of the rectangle $A'B'C'D'$ are $P_\alpha(P(x,y),z)$ and $P_\alpha(P(x,z),y)$.

J. Aczél. *Lectures on Functional Equations and their Applications*. Academic Press, New York and San Diego, 1966.

J. Aczél. *A short course on functional equations based on recent applications to the social and behavioral sciences*. Reidel/Kluwer, Dordrecht and Boston, 1987.

J.-Cl. Falmagne. Meaningfulness and order invariance: two fundamental principles for scientific laws. *Foundations of Physics*, 9:1341–1384, 2004.

J.-Cl. Falmagne. A set of independent axioms for positive Hölder systems. *Philosophy of Science*, 42(2):137–151, 1975.

J.-Cl. Falmagne. On a bounded version of Holder's Theorem and an application to the permutability equation. Submitted to a volume honoring Patrick Suppes on the occasion of his 90th birthday. arXiv:1207.3848, 2012.

J.-Cl. Falmagne and L. Narens. Scales and meaningfulness of quantitative laws. *Synthese*, 55:287–325, 1983.

O. Hölder. Die axiome der quantitat und die lehre von mass. *Berichte uber die Verhandlungen der Koniglichen, Sachsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematische-Physysische Classe* , 53:1–64, 1901.

M. Hosszú. Note on commutable mappings. *Publ. Math. Debrecen*, 9:105–106, 1962a.

M. Hosszú. Néhány lineáris függvényegyenletröl. *Mat. Lapok*, 13:202, 1962b.

M. Hosszú. Algebrai rendszereken értelmezett függvényegyenletek, i. algebrai módszerek a függvényegyenletek elméletében.
Magyar Tud. Acad. Mat. Fiz. Oszt. K ozl, 12:303–315, 1962c.

L. Narens. *Theories of Meaningfulness*. Lawrence Erlbaum Associates, New Jersey and London, 2002.

L.I. Sedov. *Similarity and dimensional methods in mechanics*. English translation of the 1956 edition by M. Holt and M. Friedman, New York, Academic Press, 1959 Moscow, 1943, 1956.

P. Suppes. *Representation and Invariance of Scientific Structures*. Stanford, CA: CSLI Publications, 2002.