COOPERATION IN n-PLAYER REPEATED GAMES

DANIEL T. JESSIE AND DONALD G. SAARI

Abstract. After introducing the basics of a recent decomposition of single-shot games (Jessie and Saari [3]) into their strategic and behavioral parts, it is shown how this decomposition provides new results and insights into the nature of repeated interactions. For any two-strategy $n \geq 2$ player game, it is shown how to obtain a complete characterization of when a specified outcome is sustainable in an infinitely repeated setting, given a standard choice of punishment. This characterization is given by a simple relationship between strategic and non-strategic components. Furthermore, it is shown how this analysis can extend issues raised by simplified 2-player scenarios to a $n \geq 3$ player setting; specifically, results obtained for a 2-player analysis need not hold for the n-player case even under generous assumptions.

1. Introduction

When modeling events from the social sciences, it could be that no player wants to suffer the consequences of a specified Nash equilibrium. An illustration would be competing countries where the strategic structure leads to the dominant strategy outcome of war. Fortunately, and as it is well-known, there are equilibria in a repeated setting that differ from the single-shot Nash equilibria. This fact underscores an advantage of repeated games – that is, games involving repeated interactions – where an appropriate strategy can lead to cooperation.

A natural illustration is the Prisoner's Dilemma $\mathcal{G}_1$,

$$\mathcal{G}_1 = \begin{array}{|cc|cc|} \hline 4 & 4 & -2 & 6 \\ \hline 6 & -2 & 0 & 0 \\ \hline \end{array}. \qquad (1)$$

Although the single-shot Nash strategy is Bottom-Right (BR), the Pareto superior outcome in the Top-Left (TL) corner could be the equilibrium should the game be repeated infinitely often. But TL not a Nash equilibrium of a single-shot play, so a theme initiated here is to use a recently developed decomposition of games (Jessie and Saari [3]) to provide new insights into the cooperative structure that can accompany repeated interactions. As an example, it turns out that the Nash structure of a game assumes different roles as opportunities for cooperation change.

A second theme involves using a two-person model where a more realistic modeling would involve games with three or more players. As this is being written, a contemporary international difficulty involves the countries of Israel and Iran, where a natural objective is to determine whether a repeated game could create a cooperative solution other than an uncomfortable Nash outcome. But for the game to capture reality, another major player, the United States, must be included.

Two-player repeated games already can be complicated to analyze, so three-player games become daunting. So, to circumvent the added complexity if two of the players (here, the US and Israel) have fairly common objectives, it is reasonable to expect that a two-player analysis suffices. The theoretical issue is to determine whether this is true; to do so, relationships between two and three player games are examined. As we prove, a two-player game need not capture the realities of a

The research of both authors was supported, in part, by NSF Grant NSF CMMI-1016785.

three-player game. As an illustration, in an interesting analysis Brams and Kilgour show that by giving Israel the ability to detect Iran's strategy before choosing its own, a cooperative outcome can be reached in several different 2×2 games. We show, however, that once the United States is added as a third player, these results need not extend: the ability to detect Iran's strategy can lead to *less* cooperation.

A third theme also is motivated by common and continuing international events; it is where countries use their resources to change the game's structure. A game's strategic structure can be significantly altered, for instance, should a country develop a new bomb or aircraft. (To avoid serious shifts in strategic structures explains the reason for efforts such as nuclear non-proliferation treaties.) Our interest centers on other kinds of modifications, such as the possibility of an adversary's admission into a trade organization or scientific/cultural exchanges. While these rearrangements need not affect the game's strategic aspects, they could make cooperation more attractive in a repeated setting. To examine this issue, we analyze how to combine the strategic behavior in a repeated game with modifications in the game structure (i.e., the level of inducement) to attain and sustain cooperation. Answers, again, follow from our decomposition of games.

A way to describe other themes is to indicate how our approach differs from other methods. A current approach adopted to handle some of these issues is to use an interesting tool developed by Robinson & Goforth [4], and extended by Hopkins [2], to locate the different types of 2×2 interactions. This approach, however, is local in nature, computationally intensive, does not extend beyond the 2×2 case, and can discard relevant information about payoffs.

An advantage our decomposition has over a point-wise analysis is to emphasize the structure of $\mathbb{G}$, the space of games. In this way, classes of games with desired properties can be identified, rather than relying on individual choices. This global approach also highlights the sensitivity of results to small perturbations in a game. For instance, it is shown that results derived from a particular choice of payoffs need not hold for other games even if they are "close."

Another advantage of the decomposition is to demonstrate how non-strategic information can affect conclusions in a repeated setting and to examine differences between the repeated and single-shot equilibria. This relationship between strategic and non-strategic information underscores the role of the discount factor δ for a given strategy choice; it demonstrates how the δ discount factor interacts with changes in what we call the behavioral component of a game. Finally, this decomposition allows for a comparison of strategies, and it shows all games for which a particular strategy will "out-perform" another.

1.1. **Decomposition.** While details, proofs, and the motivation for our decomposition are in [3], a brief introduction is given here in terms of the Eq. 1 game $\mathcal{G}_1$. The goal is to divide the game into the component $\mathcal{G}_1^N$ that strictly determines all possible Nash strategic behavior, component $\mathcal{G}_1^B$ that, from a behavioral perspective, contains no Nash information but can change the dynamics of the games in other ways, and a kernel term $\mathcal{G}_1^K$ that merely modifies entries. The important fact is that this decomposition is unique and can be used with all games.

To find $\mathcal{G}^N$ from a given game $\mathcal{G}$, consider the two matrix entries left to a player after a pure strategy is specified for each of the other players. With $\mathcal{G}_1$, for example, if L is specified for Player 2, then the two entries for Player 1 are "4" by playing T and "6" by playing B. Replace each entry by how it differs from their average of $(4+6)/2 = 5$. That is, replace the 4 with $4 - 5 = -1$ and 6 with $6 - 5 = 1$. Doing this for all players and options leads to

$$\mathcal{G}_1^N = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array}.$$

What remains is a listing of the averages, or

$$\mathcal{G}_1^A = \mathcal{G}_1 - \mathcal{G}_1^N = \begin{array}{|cc|cc|}\hline 5 & 5 & -1 & 5 \\ \hline 5 & -1 & -1 & -1 \\ \hline \end{array}.$$

The kernel component $\mathcal{G}_1^K$ replaces each $\mathcal{G}_1$ entry for a player by the average of all of the player's $\mathcal{G}_1$ entries. For $\mathcal{G}_1$, each player's average is $(4-2+0+6)/4 = 2$. The behavioral component is $\mathcal{G}_1^B = \mathcal{G}^A - \mathcal{G}_1^K$. Thus, $\mathcal{G}_1 = \mathcal{G}_1^N + \mathcal{G}_1^B + \mathcal{G}_1^K$ where

$$\mathcal{G}_1^B = \begin{array}{|cc|cc|}\hline 3 & 3 & -3 & 3 \\ \hline 3 & -3 & -3 & -3 \\ \hline \end{array}, \quad \mathcal{G}_1^K = \begin{array}{|cc|cc|}\hline 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array}$$

Matrices $\mathcal{G}_1^B$ and $\mathcal{G}_1^K$ contain no Nash information of any kind. This is because the rows in each matrix are the same for the row player, the columns are identical for the column player, so *individual* actions taken by a player cannot affect the player's $\mathcal{G}^B$ payoff. This means that $\mathcal{G}_1^N$ contains all of $\mathcal{G}_1$'s Nash information and that $\mathcal{G}_1^B$ contains all other aspects about $\mathcal{G}_1$ that affect behavior.

For instance, the Prisoner's Dilemma reflects the conflict where the TL location of the $\mathcal{G}_1^B$ Pareto superior entry coincides with the location of the $\mathcal{G}_1^N$ Pareto inferior term. *Individual actions* cannot affect the $\mathcal{G}_1^B$ outcome, so a cooperative action (i.e., TL) is required to extract the larger $\mathcal{G}_1^B$ outcome that defines the larger $\mathcal{G}_1$ outcome. More generally and for any game $\mathcal{G}$, all collective and cooperative actions that are allowed by $\mathcal{G}$ are based on the $\mathcal{G}^B$ structure and how it compares with $\mathcal{G}^N$, which is why $\mathcal{G}^B$ plays a central role in addressing the described themes.

The decomposition for a three player game, where each has two strategies, is the same. To illustrate, consider

$$(2) \quad \mathcal{G}: \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 0 & 3 & 5 & 4 & 1 & 2 \\ \hline 2 & 2 & 1 & 0 & 4 & 3 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 3 & 2 & 1 & 4 & -2 & 2 \\ \hline 1 & 4 & 5 & 8 & 6 & 7 \\ \hline \end{array}$$

The "Front" and "Back" labels correspond to Player 3's strategy choices. Although not labeled, "Top/Bottom" and "Left/Right" refer to Player 1's and Player 2's strategies, respectively. So the payoffs for the three players should they play Bottom-Left-Back, or BLBa, are 1, 4, 5, respectively.

To find $\mathcal{G}^N$, select a strategy for each of two players, and then replace the third player's identified entries by how they differ from their average. If T for Player 1 and L for Player 2 are specified, for instance, then Player 3's two entries are 5 from "Front" and 1 from "Back." Thus replace each term with how it differs from the average of 3. Doing so for all strategies and players, it follows that

$$\mathcal{G}^N: \text{Front} = \begin{array}{|ccc|ccc|}\hline -1 & 1 & 2 & 2 & -1 & 0 \\ \hline 1 & -1 & -2 & -2 & 1 & -2 \\ \hline \end{array}, \text{Back} = \begin{array}{|ccc|ccc|}\hline 1 & 2 & -2 & -2 & -2 & 0 \\ \hline -1 & -1 & 2 & 2 & 1 & 2 \\ \hline \end{array}$$

An important $\mathcal{G}^N$ feature is that it contains all information needed to determine everything about $\mathcal{G}$'s Nash strategic structure; nothing else is needed. To see that $\mathcal{G}^N$ has extracted all of the strategic information, notice that there is no difference in T or B for Player 1, L or R for Player 2, or F or Ba for Player 3 in

$$\mathcal{G}^A = \mathcal{G} - \mathcal{G}^N: \text{Front} = \begin{array}{|ccc|ccc|}\hline 1 & 2 & 3 & 2 & 2 & 2 \\ \hline 1 & 3 & 3 & 2 & 3 & 5 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 2 & 0 & 3 & 6 & 0 & 2 \\ \hline 2 & 5 & 3 & 6 & 5 & 5 \\ \hline \end{array}.$$

The average entry for players 1, 2, and 3, is, respectively, $\kappa_1 = \frac{11}{4}, \kappa_2 = \frac{5}{2}, \kappa_3 = \frac{13}{4}$; each matrix entry in $\mathcal{G}^K$ has these three terms. The $\mathcal{G}^B$ matrix is obtained by replacing the j^{th} player's $\mathcal{G}^A$ entry by how it differs from κ_j. So the Front matrix for $\mathcal{G}^B$ is

$$\text{Front} = \begin{array}{|ccc|ccc|}\hline 1-\frac{11}{4} & 2-\frac{5}{2} & 3-\frac{13}{4} & 2-\frac{11}{4} & 2-\frac{5}{2} & 2-\frac{13}{4} \\ \hline 1-\frac{11}{4} & 3-\frac{5}{2} & 3-\frac{13}{4} & 2-\frac{11}{4} & 3-\frac{5}{2} & 5-\frac{13}{4} \\ \hline \end{array}$$

In general and by construction,

$$\mathcal{G} = \mathcal{G}^N + \mathcal{G}^B + \mathcal{G}^K. \tag{3}$$

The general form of each of these matrices is given next where the second subscript refers to the player. Also, $\sum_i \beta_{i,j} = 0$ for $j = 1, 2, 3$.

$$\begin{aligned} \mathcal{G}^N : \text{Front} &= \begin{array}{|ccc|ccc|}\hline \eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \eta_{2,1} & -\eta_{1,2} & \eta_{2,3} \\ \hline -\eta_{1,1} & \eta_{2,2} & \eta_{3,3} & -\eta_{2,1} & -\eta_{2,2} & \eta_{4,3} \\ \hline \end{array}, \\ \text{Back} &= \begin{array}{|ccc|ccc|}\hline \eta_{3,1} & \eta_{3,2} & -\eta_{1,3} & \eta_{4,1} & -\eta_{3,2} & -\eta_{2,3} \\ \hline -\eta_{3,1} & \eta_{4,2} & -\eta_{3,,3} & -\eta_{4,1} & -\eta_{4,2} & -\eta_{4,3} \\ \hline \end{array}, \\ \mathcal{G}^B : \text{Front} &= \begin{array}{|ccc|ccc|}\hline \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & \beta_{2,1} & \beta_{1,2} & \beta_{2,3} \\ \hline \beta_{1,1} & \beta_{2,2} & \beta_{3,3} & \beta_{2,1} & \beta_{2,2} & \beta_{4,3} \\ \hline \end{array}, \\ \text{Back} &= \begin{array}{|ccc|ccc|}\hline \beta_{3,1} & \beta_{3,2} & \beta_{1,3} & \beta_{4,1} & \beta_{3,2} & \beta_{2,3} \\ \hline \beta_{3,1} & \beta_{4,2} & \beta_{3,3} & \beta_{4,1} & \beta_{4,2} & \beta_{4,3} \\ \hline \end{array}, \end{aligned} \tag{4}$$

where each matrix entry for the front and the back matrices of $\mathcal{G}^K$ is $(\kappa_1\, \kappa_2\, \kappa_3)$ and κ_j is the average of the $\mathcal{G}$ entries for the j^{th} player. For a 2×2 game, only use the "Front" matrix and drop the third player's entries.

1.2. **Advantages.** Advantages of the decomposition are apparent. Computing the Nash $\mathcal{G}^N$ component can be quickly done using only elementary arithmetic; this $\mathcal{G}^N$ component identifies all possible aspects of the Nash structure of the game. As an illustration, a normal Nash analysis for 2×2 games involves eight variables, but with $\mathcal{G}^N$, only the four defining $\eta_{1,1}, \eta_{2,1}, \eta_{1,2}, \eta_{2,2}$ variables are needed. More generally, the space of games for n agents, where each player has two strategies, is $\mathbb{G} = \mathbb{R}^{n2^n}$, so a Nash analysis typically involves all $n2^n$ terms. But with the decomposition, only half as many variables, the $n2^{n-1}$ variables $\{\eta_{i,j}\}$ from $\mathcal{G}^N$, are needed. Of importance for what is developed here, the $n(2^{n-1} - 1)$ $\{\beta_{i,j}\}$ variables that define $\mathcal{G}^B$ determine all other aspects of the game. What simplifies an analysis is the separation of the β variables from the η choices.

The computational simplicity of computing $\mathcal{G}^N$ provides an easy way to identify all pure Nash equilibria. This follows from the Eq. 4 representation for $\mathcal{G}^N$ where, when all other players select their strategies, the remaining player, say the j^{th}, has two options with payoffs $-|\eta_{i,j}|$ and $|\eta_{i,j}|$. To be a Nash equilibrium, this player must select the larger $|\eta_{i,j}|$ choice. This leads to the first Thm. 1.1 statement, where the "strict" modifier just eliminates $\eta_{i,j} = 0$ terms.

Theorem 1.1. *For games $\mathcal{G}$ involving $n \geq 2$ players where each has two strategies, the following are true:*

1. *For game $\mathcal{G}$, a strict pure Nash equilibrium occurs if and only if all of the entries in the identified $\mathcal{G}^N$ cell are positive.*
2. *A game $\mathcal{G}$ can be constructed to have k pure Nash equilibria where k is any integer satisfying $0 \leq k \leq 2^{n-1}$.*
3. *Mixed strategies are completely determined by the $n2^{n-1}$ variables $\{\eta_{i,j}\}$ that define $\mathcal{G}^N$.*

The second statement is proved with an obvious induction argument. To capture its flavor, notice that this decomposition makes it trivial to create a wide class of games with the same Nash structure, but different cooperative behaviors in repeated settings. To do so, use Eq. 3 and select appropriate $\mathcal{G}^N$ and $\mathcal{G}^B$ components. (This is how the Eq. 5 games in the next section were created.)

To illustrate, it can be difficult to create a class of four-player games where each game has the same $2^{4-1} = 8$ pure Nash equilibria, but different games suggest different cooperative behavior. With the decomposition and Eq. 3, the construction becomes immediate. To start by constructing $\mathcal{G}^N$, such equilibria must have a diametric positioning in $\mathcal{G}^N$. To explain, if TL is a Nash equilibria for a two player game, then the $\eta_{i,j}$ structure of $\mathcal{G}^N$ requires a negative entry in TR and BL. Thus the only remaining possibility for another Nash equilibria is in BR.

Similarly to build a three player game structure, start with the TL and BR structure of the two person game in the Front set of matrices. In these $\mathcal{G}^N$ cells, put a positive entry for Player 3 to create strict Nash equilibria at TLF and BRF. The $\eta_{i,3}$ properties require Player 3 to have a negative entry in TLBa and BRBa. As the entries for Players 1 and 2 for this Back choice are free to be selected, the only way to obtain two more equilibria is to place positive entries in the remaining two $\mathcal{G}^N$ Back cells. Thus the four equilibria must be diagonally located at TLF, BRF, BLBa, and TRBa.

A four player game is given by two sets, I and II, of "Front-Back" matrices. Namely, there are two sets of matrices with the Eq. 2 form except that each cell lists the payoffs for each of the four players. To have four equilibria in set I, put a positive value for Player 4 in each of TLFI, BRFI, BLBaI, and TRBaI. This forces a negative value for Player 4 in each of the TLFII, BRFII, BLBaII, and TRBaII cells. So, select positive $\eta_{i,j}$ values for the remaining four set II cells. By selecting all positive $\eta_{i,j}$ values to equal unity, a four-person game is constructed. More generally, moving from $(n-1)$ to n players doubles the maximum number of pure Nash equilibria. The mixed Nash equilibria for this four-person game are found in the standard way with these 32 $\eta_{i,j}$ variables. To construct a wide class of games with this same Nash structure but potentially different equilibria for repeated games, add appropriately selected $\mathcal{G}^B$ terms to the constructed $\mathcal{G}^N$.

1.3. **Three examples.** To introduce the types of games, strategies, and unusual conclusions that are analyzed in this paper, consider the three Eq. 5 games.

$$
\begin{array}{lll}
\mathcal{G}_2 \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 0 & 0 & 0 & 4 & 4 & 4 \\ \hline 4 & 4 & 5 & 8 & 8 & 7 \\ \hline \end{array}, & \text{Back} = \begin{array}{|ccc|ccc|}\hline 5 & 5 & 4 & 7 & 9 & 8 \\ \hline 9 & 7 & 9 & 11 & 11 & 11 \\ \hline \end{array} \\
\mathcal{G}_3 \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 7 & 7 & 7 & 5 & 11 & 5 \\ \hline 11 & 5 & 4 & 9 & 9 & 0 \\ \hline \end{array}, & \text{Back} = \begin{array}{|ccc|ccc|}\hline 4 & 4 & 11 & 0 & 8 & 9 \\ \hline 8 & 0 & 8 & 4 & 4 & 4 \\ \hline \end{array} \\
\mathcal{G}_4 \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 7 & 4 & 5 & 5 & 8 & 7 \\ \hline 11 & 7 & 4 & 9 & 11 & 0 \\ \hline \end{array}, & \text{Back} = \begin{array}{|ccc|ccc|}\hline 4 & 7 & 9 & 0 & 11 & 11 \\ \hline 8 & 0 & 8 & 4 & 4 & 4 \\ \hline \end{array}
\end{array} \tag{5}
$$

A first objective is to determine which states can be sustained in a repeated setting. For $\mathcal{G}_2$, the answer is immediate: Each player has a dominant strategy given by BRBa. As BRBa yields the game's Pareto superior outcome, by playing the dominant strategy, each player receives the games's highest possible payoff.

Results for $\mathcal{G}_3$ are not as obvious. Each player has a dominant BRBa strategy, but its outcome of 3 for each player is Pareto inferior to the TLF, TRF, TLBa, and BLF outcomes. It is reasonable to wonder whether any of these four other choices can be sustained should each player adopt a grim-trigger strategy. As we show, if the discount factor satisfies $\delta > \frac{4}{7}$, then TLF can be supported. In fact, with $\frac{4}{7} < \delta \leq \frac{4}{5}$, TLF is the only outcome that grim trigger can sustain. If $\delta > \frac{4}{5}$, then TRF can be sustained as well.

The next question is whether larger δ values could support BLF or TLBa: They cannot; no other state is inducible under grim trigger. Notice how this conclusion identifies an advantage for Player 2 over the other players. This is because the two sustainable grim trigger options for $\delta > \frac{4}{5}$ are TLF and TRF, and only Player 2 can select between them. Consequently, Player 2 can receive the personally higher TRF payoff at the expense of the other two. Of interest is how classes of games with this feature can be identified by using the decomposition and analysis developed below.

While the dominant strategy outcome for $\mathcal{G}_4$ is BRBa and TLF, BLF, TRF, and TLBa have Pareto preferred outcomes, $\mathcal{G}_4$ appears to be the most complicated game by not having a clear optimal state. Indeed, this complexity is reflected by the number of calculations that typically need to be performed in a standard analysis. However, by using the method outlined below, it can be quickly shown that, for any $\delta \in (0,1)$, it is impossible to induce a cooperative state with a grim-trigger strategy.

These three games illustrate interesting features that are due to behavioral $\mathcal{G}^B$ terms. This must be the case because games $\mathcal{G}_2, \mathcal{G}_3$ and $\mathcal{G}_4$ are strategically equivalent; that is $\mathcal{G}_2^N = \mathcal{G}_3^N = \mathcal{G}_4^N$. This equality means that all possible differences among these games in achieving cooperation are strictly due to differences in the games' behavioral $\mathcal{G}^B$ components. Namely, $\mathcal{G}_2^B$ features contribute to $\mathcal{G}_2$ having a dominant strategy that is the Pareto superior outcome, $\mathcal{G}_3^B$ features allow only two sustainable grim trigger cooperative outcomes, and $\mathcal{G}_4^B$ features prohibit any grim trigger sustainable cooperative outcome.

Of particular interest is the similarity of games $\mathcal{G}_3$ and $\mathcal{G}_4$: Player 1's entries in both games are identical, and there are only minor differences in the entries for the other two players. (The two games could have been made much closer in appearance, but we opted to use choices where all terms are integers.) While the game matrices are similar, the cooperative structures differ significantly in that $\mathcal{G}_3$ can ensure cooperation, but $\mathcal{G}_4$ cannot. This example makes it clear that changes in $\mathcal{G}^B$ can alter the kinds of cooperation that are possible.

It is reasonable to wonder whether these conclusions would change by replacing the grim-trigger with the tit-for-tat strategy. As shown for the Eq. 5 games, tit-for-tat strategies produce identical results.

1.4. **Assumptions.** Our goal is to show how a game's $\mathcal{G}^B$ behavioral structure can change the game's cooperative properties. Because of this emphasis, we consider only the commonly used grim-trigger and tit-for-tat strategies with the objective of determining *which* non-Nash outcomes are sustainable in a repeated setting. By considering all possible non-Nash outcomes, a special case is the typical choice of exploring whether a Pareto optimal outcome can be sustained.

The Eq. 5 games illustrate a feature of our more general perspective: While $\mathcal{G}_2$ and $\mathcal{G}_3$ have clear candidates for the cooperation state, BRBa for $\mathcal{G}_2$ and TLF for $\mathcal{G}_3$, it is not clear what should be selected for $\mathcal{G}_4$. With $\mathcal{G}_4$, for instance, the three states of TRBa, TRF, and BLF are Pareto equivalent. But by characterizing all sustainable outcomes in any game (independent of Pareto properties), this selection problem can be ignored as it becomes a special case of the analysis. That is, this more general approach sheds light on differing ways to define an optimal outcome.

By characterizing which states are inducible in a repeated setting, we also identify the role played by $\mathcal{G}^B$ in seeking cooperative outcomes and how $\mathcal{G}^B$ changes the function of $\mathcal{G}^N$ in the analysis. To simplify the calculations, the punishment state is given by a pure-strategy outcome. An interesting feature is that the punishment state need not be an equilibrium of the single-shot game. This allows for a minmax strategy against Player i to be i's punishment.

2. Sustaining Cooperation

It is instructive how the $\mathcal{G}^B$ structure changes the role played by $\mathcal{G}^N$ for a given game $\mathcal{G}$. Should $\mathcal{G}^B$ have a minimal impact on $\mathcal{G}^N$, because the magnitudes of the $\mathcal{G}^B$ entries are small in size (relative to $\mathcal{G}^N$) or $\mathcal{G}^B$ supports the $\mathcal{G}^N$ structure (by enhancing a Nash equilibrium as true with $\mathcal{G}_2^B$), then $\mathcal{G}^B$ plays a minimal role in the $\mathcal{G}$ game analysis. Most of what is interesting in the game can be attributed to $\mathcal{G}^N$ where what individuals obtain (the Nash equilibria) is determined by their own strategic actions. But a stronger $\mathcal{G}^B$ term, such as with $\mathcal{G}_3^B$, provides an inducement for the players to try to obtain a Pareto preferred outcome. For this to happen, ways must be found to extract the benefits that are provided by the $\mathcal{G}^B$ component: Doing so requires a level of cooperation among the players.

Our interest is to determine what it takes to ensure the cooperation that will, for *any* reason, support *any* specified non-Nash outcome for *any* game. For the desired cooperation to be feasible, each player receives a better outcome with the targeted outcome than what the player could get from at least one other setting. No assumption is made about the structure of this comparison setting, so this analysis includes, as a special case, the Prisoner's Dilemma (where the comparison entry for all players is determined by the dominant strategy). In particular, the targeted outcome need not be Pareto superior to the various Nash equilibria (Thm. 1.1), it just must be better than some other outcome, so this discussion applies to a wide selection of games.

A feature described in Sect. 2.1 is that the effort needed to support a non-Nash outcome *must always* provide a temptation for certain players to renege from cooperating. (This feature holds the Prisoner's Dilemma, but it is not obvious that it is true in general.) As a reneging player could destroy the designated cooperative outcome, if the targeted outcome is desired by the other players, they must adopt some form of collective action to enforce cooperation. A natural approach is to make it expensive to renege. Doing so may not be possible in a one-shot game, but the added opportunities offered by a repeated game may make it an option. Strategies such as the grim trigger and tit-for-tat share this objective; they differ in their efficiency and effectiveness.

To make "reneging" costly, a game must have a sufficiently distasteful aspect that can be converted into an enforcement tool where the non-cooperative player must endure this distasteful punishment. The question is whether a game always provides such enforcement tools. Surprisingly, it always does; this enforcing tool comes from the structure of $\mathcal{G}^N$!

In summary, for any given game $\mathcal{G}$, the role played by $\mathcal{G}^N$ is influenced by the $\mathcal{G}^B$ structure. When $\mathcal{G}^N$ is the dominating component of the game, it enjoys a positive, rewarding image of determining what should happen. But with changes in a game due to $\mathcal{G}^B$ inducements, the image of $\mathcal{G}^N$ now changes from describing rewards to that of providing a means for punishment for lack of cooperation.

2.1. **Changing roles of $\mathcal{G}^N$ due to $\mathcal{G}^B$.** Once each of $(n-1)$ players adopt a strategy, the remaining player, say Player 1, is left with two choices centered about their $\beta_{i,1}$ average. The choice with the $|\eta_{i,1}| + \beta_{i,1}$ payoff strategically dominates the $-|\eta_{i,1}| + \beta_{i,1}$ choice.

Now consider what is required to support a specified non-Nash outcome by enlisting cooperation. By being non-Nash, it must be that for some player to cooperate, say Player 1, the player must select the strategy offering $-|\eta_{i,1}| + \beta_{i,1}$ rather than the more tempting $|\eta_{i,1}| + \beta_{i,1}$; this is the above described guaranteed temptation. This comment applies to any non-Nash outcome, so it generalizes discussions associated with the Prisoner's Dilemma.

For Player 1 to consider cooperation, the poorer $-|\eta_{i,1}| + \beta_{i,1}$ payoff must be preferred to some other outcome where Player 1 receives $|\eta_{j,1}| + \beta_{j,1}$. That is,

$$|\eta_{i,1}| + \beta_{i,1} > -|\eta_{i,1}| + \beta_{i,1} > |\eta_{j,1}| + \beta_{j,1}. \tag{6}$$

There are no opportunities to ensure cooperation in a one shot game. But in a repeated game, the $(n-1)$ other players can select penalty strategies to force Player 1 to choose between $|\eta_{j,1}| + \beta_{j,1}$ and $-|\eta_{j,1}| + \beta_{j,1}$. The main differences between grim trigger and tit-for-tat are the frequency and reasons to penalize; our interest is to identify when cooperation can be achieved. The answer depends on the effect of the future penalties on Player 1 as determined by the player's discount factor $\delta \in (0, 1)$.

Theorem 2.1. *In an $n \geq 2$ player game, suppose with a targeted, non-Nash cooperative outcome, Player 1, with discount rate $\delta \in (0, 1)$, must select a strategy yielding the smaller of $\{-|\eta_{i,1}| + \beta_{i,1}, |\eta_{i,1}| + \beta_{i,1}\}$ as identified in Eq. 6. To enforce this cooperative action, the other $(n-1)$ players use the grim trigger option that would force Player 1 to select between $|\eta_{j,1}| + \beta_{j,1}$ and $-|\eta_{j,1}| + \beta_{j,1}$. It is in Player 1's interest to cooperate if*

$$\left(1 - \frac{2}{\delta}\right) |\eta_{i,1}| + \beta_{i,1} > |\eta_{j,1}| + \beta_{j,1} \tag{7}$$

If the other players adopt a tit-for-tat strategy, then it is in Player 1's interest to cooperate if

$$\left(-1 - \frac{2}{\delta}\right) |\eta_{i,1}| + \beta_{i,1} > -|\eta_{j,1}| + \beta_{j,1} \tag{8}$$

Proof. Suppose Player 1 is facing opponents who have a implemented a grim trigger in which he can play Top (Cooperate) for a payoff of $-|\eta_{i,1}| + \beta_{i,1}$ ad infinitum, or play Bottom (Defect) for a one-time payoff of $|\eta_{i,1}| + \beta_{i,1}$. But, our player did not cooperate, so the grim trigger ensures that the following payoffs are $|\eta_{j,1}| + \beta_{j,1}$ ad infinitum. This means that Player 1 will cooperate if

$$\begin{aligned}
\sum_{t=1}^{\infty} \delta^{t-1}(-|\eta_{i,1}| + \beta_{i,1}) &> |\eta_{i,1}| + \beta_{i,1} + \sum_{t=2}^{\infty} \delta^{t-1}(|\eta_{j,1}| + \beta_{j,1}) \\
\frac{-|\eta_{i,1}| + \beta_{i,1}}{1-\delta} &> |\eta_{i,1}| + \beta_{i,1} + \delta\left(\frac{|\eta_{j,1}| + \beta_{j,1}}{1-\delta}\right) \\
-|\eta_{i,1}| + \beta_{i,1} - (1-\delta)(|\eta_{i,1}| + \beta_{i,1}) &> \delta(|\eta_{j,1}| + \beta_{j,1}) \\
(\delta - 2)|\eta_{i,1}| + \delta\beta_{i,1} &> \delta(|\eta_{j,1}| + \beta_{j,1})
\end{aligned}$$

After some algebraic computations, Eq. 7 is obtained.

If Player 1 is faced with opponents who are playing tit-for-tat, then the grim-trigger in Eq. 7 must hold or Player 1 will never cooperate, as he retains the ability to defect ad infinitum against. However, there is also the possibility of first cooperation, then defection, then cooperation, etc. In order for cooperation to hold in this case, it is also needed that

$$\begin{aligned}
\sum_{t=1}^{\infty}\delta^{t-1}(-|\eta_{i,1}|+\beta_{i,1}) &> |\eta_{i,1}|+\beta_{i,1}+\delta\,(-|\eta_{j,1}|+\beta_{j,1}) \\
&\quad +\delta^2(|\eta_{i,1}|+\beta_{i,1})+\delta^3\,(-|\eta_{j,1}|+\beta_{j,1})+\cdots \\
\sum_{t=1}^{\infty}\delta^{t-1}(-|\eta_{i,1}|+\beta_{i,1}) &> \sum_{t=1}^{\infty}\delta^{2(t-1)}(|\eta_{i,1}|+\beta_{i,1})+\delta\sum_{t=1}^{\infty}\delta^{2(t-1)}(-|\eta_{j,1}|+\beta_{j,1}) \\
\frac{-|\eta_{i,1}|+\beta_{i,1}}{1-\delta} &> \frac{|\eta_{i,1}|+\beta_{i,1}}{1-\delta^2}+\delta\left(\frac{-|\eta_{j,1}|+\beta_{j,1}}{1-\delta^2}\right) \\
\left(\frac{1+\delta}{1+\delta}\right)\frac{-|\eta_{i,1}|+\beta_{i,1}}{1-\delta} &> \frac{|\eta_{i,1}|+\beta_{i,1}}{1-\delta^2}+\delta\left(\frac{-|\eta_{j,1}|+\beta_{j,1}}{1-\delta^2}\right) \\
(1+\delta)(-|\eta_{i,1}|+\beta_{i,1}) &> |\eta_{i,1}|+\beta_{i,1}+\delta\,(-|\eta_{j,1}|+\beta_{j,1}) \\
(-\delta-2)|\eta_{i,1}|+\delta\beta_{i,1} &> \delta\,(-|\eta_{j,1}|+\beta_{j,1})
\end{aligned}$$

This inequality is equivalent to Eq. 8. □

Equations 7, 8 completely characterize all pairs of states (the targeted and the penalty states) in which cooperation is inducible against strategic interests with either a grim trigger or a tit-for-tat strategy, respectively. Also, for $\delta \in (0,1)$, the left hand side of these equations is unbounded below whenever $\eta_{i,1} \neq 0$, and so all statements must be qualified with suitable conditions on the discount factor.

Equations 7 and 8 show that equilibrium outcomes in repeated game are affected by information ignored in the single-shot case. This is because the $\beta_{i,j}$ value has a large impact on whether or not cooperation can be induced, but it is irrelevant as far as the Nash equilibrium is concerned. Also note that the phrase "for δ large enough" could be substituted with either "for non-strategic interests large enough," or "for strategic factors small enough," when describing results on cooperative outcomes.

In this manner, the analysis captures one of our goals, which is to indicate how modifications in the game structure – the level of inducement given by $\mathcal{G}^B$ – can be introduced to attain and sustain cooperation. Furthermore, notice that these equations also identify the type of information that parameter δ measures; as δ is a coefficient only for the $\eta_{i,j}$ terms, non-strategic interests (i.e., $\mathcal{G}^B$ terms) can be altered without affecting the standard requirements of a "suitable δ."

2.2. **An application.** To explain the comments about the three Eq. 5 games with Thm. 2.1, first notice that when $|\eta_{i,1}| = |\eta_{j,1}|$, the inequalities in Eqs. 7 and 8 are identical. It is only when $|\eta_{i,1}| > |\eta_{j,1}|$ that the tit-for-tat inequality imposes a greater restriction. If $|\eta_{i,1}| < |\eta_{j,1}|$, then the grim trigger inequality is stricter; it must also hold for the tit-for-tat strategy to sustain cooperation. This shows that the strength of the inducement strategy is a function of the $\mathcal{G}^N$ strategic structure of the game; in some games, a tit-for-tat strategy is not stronger than a grim-trigger strategy. For the games in Eq. 5, the two are equivalent, which is clear when the decompositions are given.

(9)

$\mathcal{G}_i^N$ Front =

−1	−1	−1	−1	1	−1
1	−1	−1	1	1	−1

, Back =

−1	−1	1	−1	1	1
1	−1	1	1	1	1

(10)

$$\mathcal{G}_2^B \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline -4 & -4 & -4 & 0 & -4 & 0 \\ \hline -4 & 0 & 1 & 0 & 0 & 3 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 1 & 1 & -4 & 3 & 1 & 0 \\ \hline 1 & 3 & 1 & 3 & 3 & 3 \\ \hline \end{array}$$

$$\mathcal{G}_3^B \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 3 & 3 & 3 & 1 & 3 & 1 \\ \hline 3 & 1 & 0 & 1 & 1 & -4 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 0 & 0 & 3 & -4 & 0 & 1 \\ \hline 0 & -4 & 0 & -4 & -4 & -4 \\ \hline \end{array}$$

$$\mathcal{G}_4^B \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 3 & 0 & 1 & 1 & 0 & 3 \\ \hline 3 & 2 & 0 & 1 & 2 & -4 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 0 & 2 & 1 & -4 & 2 & 3 \\ \hline 0 & -4 & 0 & -4 & -4 & -4 \\ \hline \end{array}$$

(11)
$$\mathcal{G}_i^K \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline \end{array}$$

To see what states are equilibria in a repeated setting for games $\mathcal{G}_3$ and $\mathcal{G}_4$, first note that the punishment for defection can be assumed to be BRFr as this is both the minmax against each player and the Nash equilibrium with the lowest payoff. Because $\eta_{i,j} = -2$ for all i, j, Eq. 7 gives

$$2 - \frac{4}{\delta} + \beta_{i,j} > 2 - 4 \Rightarrow \beta_{i,j} > -4 + \frac{4}{\delta}$$

To determine which states are equilibria reduces to checking a condition on the behavioral component. In particular, if $\delta > \frac{4}{7}$, then a state is sustainable if $\beta_{i,j} \geq 3$ for every Player j. This is only true of the TLFr outcome in $\mathcal{G}_3$. As δ increases, it becomes easier to sustain cooperation, and for $\delta > \frac{4}{5}$, a state is sustainable if $\beta_{i,j} \geq 1$. This means that TRFr in $\mathcal{G}_3$ is also an equilibrium. The limiting case as $\delta \to 1$ is $\beta_{i,j} > 1$. But there are no other outcomes in either $\mathcal{G}_3$ or $\mathcal{G}_4$ in which each player has $\beta_{i,j} > 1$, so there are no more equilibria. Notice the extent to which the calculations were reduced in finding *all* repeated equilibria; by developing a characterization of sustainable states in terms of the structure of the game, the amount of calculating can be substantially reduced.

These examples also highlight the importance of the $\beta_{i,j}$ values in determining the repeated game equilibria. This relationship is captured by the following theorem where $\mathcal{G}_2 \sim \mathcal{G}_3$ means that $\mathcal{G}_2^N = \mathcal{G}_3^N$.

Theorem 2.2. *For any $\mathcal{G} \in \mathbb{G}$, any $\delta \in (0, 1)$, any choice of cooperation outcome, and any different choice of punishment outcome, there exists a game $\mathcal{G}' \in \mathbb{G}$ such that $\mathcal{G}' \sim_N \mathcal{G}$ and the cooperation outcome can be sustained in $\mathcal{G}'$ with either a tit-for-tat or grim-trigger strategy. Furthermore, the payoffs in $\mathcal{G}'$ can be restricted to any interval in $\mathbb{R}$.*

Proof. For any given $\mathcal{G}$, the $\beta_{i,j}$ values can be adjusted freely. Eqs. 7 and 8, as well as the Nash component $\mathcal{G}^N$, are invariant under both positive scalar multiplication and addition by a constant to all of Player i's payoffs. □

The significance of Thm. 2.2 is to demonstrate the importance of the actual magnitude of values that are in repeated games. Often, inspiration for a repeated game comes from a story about the Nash equilibrium structure of the single-shot game. However, this theorem shows that this information is insufficient for extension to a repeated analysis. Furthermore, Thm. 2.2 shows that there can be hidden difficulties in picking a "representative game" from an equivalence class of games, such as a reduction of a game to its ordinal information. An alternative approach is to use the Eq. 7, 8 inequalities to identify *classes* of relevant games in which a desired outcome holds.

3. Cooperation Inducement

Aside from payoff specification, another difficulty in analyzing repeated games comes from additional players. The results obtained from simplifying a situation to two players do not necessarily hold when additional players are added.

In [1], Brams and Kilgour provide an interesting analysis of cooperation inducement in a 2-player repeated setting. By giving one player the common knowledge ability to receive a noisy signal of the opponent's strategy before play, it is shown that a probabilistic tit-for-tat strategy can induce a cooperative outcome in a significant proportion of games. An interesting question is whether or not this extends to 3 players as well.

In a the 2-player setting, giving Player 1 the common knowledge ability to observe the opponent's strategy decreases, but does not eliminate, the incentive for defection: if Player 2 defects, the likelihood of receiving a higher one-time payoff is no longer certain, but the punishment in the following round remains. Consider the following simplified 2-player game consisting only of Player 2's payoffs

$$\mathcal{G}_4 = \begin{array}{|c|c|}\hline -|\eta_{1,2}| & |\eta_{1,2}| \\ \hline -|\eta_{2,2}| & |\eta_{2,2}| \\ \hline \end{array} + \begin{array}{|c|c|}\hline \beta_{1,2} & \beta_{1,2} \\ \hline \beta_{2,2} & \beta_{2,2} \\ \hline \end{array}. \tag{12}$$

We assume that Player 2 has the dominant strategy "Right," with the cooperative strategy being "Left." Without a detection mechanism in place, Player 2 will cooperate against a tit-for-tat strategy whenever Eq. 8 holds. Now suppose there is a detection mechanism which reduces the probability of reaching the state (T,R), which is the state in which Player 1 cooperates and Player 2 defects. Brams and Kilgour provide a more detailed discussion about the nature of detection mechanisms and the different types of errors, but here it is assumed that $Pr(\text{signal Left}|\text{played Left}) = 1$ and $Pr(\text{signal Right}|\text{played Right}) = p$. Examining this simple best-case[1] detector shows how the mechanism can change the incentives for a player. Under such a mechanism, cooperation is preferred to defection ad infinitum if

$$\begin{aligned}\sum_{t=1}^{\infty} \delta^{t-1}\left(-|\eta_{1,2}| + \beta_{1,2}\right) >& (1-p)(|\eta_{1,2}| + \beta_{1,2}) + p(|\eta_{2,2}| + \beta_{2,2}) \\ &+ \sum_{t=2}^{\infty} \delta^{t-1}(|\eta_{2,2}| + \beta_{2,2})\end{aligned} \tag{13}$$

This difference between this inequality and that in Eq. 7 is the first term on the right hand side: the benefit of defection has been replaced by a probability $1 - p \leq 1$ of obtaining the benefit. If there is no detector and $p = 0$, Eqs. 7 & 13 are equivalent. As the detector increases in accuracy, the expected benefit from defection decreases.

Note that the detection mechanism here makes cooperation more likely simply by lowering the benefit of defection, and there is no qualitative difference in the analysis when compared to the non-detection mechanism case. The calculations using the decomposition provide not only a simple way of analyzing extensions to the standard setting, but also a way of characterizing differences between mechanisms. Quantitatively, the detection mechanism works by the single substitution in Eq. 13 above; other mechanisms can be contrasted easily by seeing how they differ in terms of this modification.

3.1. **Iran, Israel, and the United States.** In light of the ability of a third player to expand the possibilities of repeated interactions, consider the interaction between Iran and Israel discussed in [1], but with the addition of the United States.

[1]This is a best-case detector because having $Pr(\text{signal Left}|\text{played Left}) < 1$ would decrease the incentive for cooperation by including the possibility of Player 1 defecting against Player 2's cooperation.

In this game, Iran has a dominant strategy to develop nuclear capacity (defect), with Israel and the U.S. seeking to sustain the cooperative outcome in which Iran does not develop nuclear weapons. As Brams and Kilgour demonstrate, a strategy detection mechanism for Israel in the two-player game induces Iran's cooperation against Iran's credible tit-for-tat strategy.

The question here is what new possibilities Iran might have with the inclusion of the United States, given the detection mechanism possessed by Iran's opponents. Naturally, what situation actually obtains is entirely dependent upon not just the strategic structure, i.e., that Iran has a dominant strategy, but of the particular payoffs chosen to represent the game. Because of this, only qualitative differences are discussed to highlight problems in generalizing results based on simplified scenarios.

One characteristic of dominant strategy games with more than two players is that there can be more than a single "cooperative" outcome. That is, even if (C, C, C) is possible, it might also be possible to sustain (C, C, D) even though the analogous two-player situation (C, D) is unrealizable. In terms of the case study, this possibility depends on the relationship of payoffs between Israel and the U.S., which is ignored in the two-player case. The (C, C, D) outcome can still be sustainable even if both Israel and the U.S. have strategic incentives to play D. For example, in $\mathcal{G}_3$ above, each player has a dominant strategy, but there are multiple sustainable states.

However, if Iran's strategy could, with common knowledge, be observed before the U.S. and Israel act, the possibility of (C, C, D) could be reduced, if not eliminated all together. In the two player case, the advantage of the detection mechanism was that the dominant strategy of defection became less profitable than without the detector, and the advantage lies with Israel. However, with multiple sustainable states, the detection mechanism can have another use, which in this case, benefits Iran. Namely, a signaling device: Iran is able to signal in advance which equilibrium it prefers, perhaps (C, C, D). If we take the commitment to develop nuclear capacity even in the face of international sanctions as a pre-commitment to defection ad infinitum, which is not an unreasonable interpretation, Iran is playing the game of equilibrium selection, acting on the belief that (C, C, D) is sustainable.

Note that this signaling feature is not limited necessarily to games with $n \geq 3$ players, but to games with multiple cooperation states. This was recognized and side-stepped in [1] by assuming one player had a dominant strategy. However, with more players, this strategic assumption no longer remains a valid way to avoid the problem, so the results cannot be generalized beyond the two-player case. The reason for limiting the analysis to two players was to avoid the increasing number of possible games in the combinatorial approach of Robinson & Goforth [4]. Instead of this method, the foundation for analysis can be simplified to Eq. 8 or 13 by using the strategic decomposition.

4. Conclusion

Although the concerns raised here about the typical approach to analyzing repeated interactions are known, there was no efficient method of addressing them. By applying the decomposition to repeated games, analysis can be extended from isolated cases to classes of games sharing a property of interest. Furthermore, the decomposition can be used not only to analyze payoff structures, but also the different strategies and inducement mechanisms used in a repeated setting.

References

[1] Brams, S., and Kilgour, D., Inducible Games: Using Tit-for-Tat to Stabilize Outcomes, Working Paper

[2] Hopkins, B., "Exploring and expanding the Robinson Goforth system for 2 by 2 games," Presentation, Special Session on Mathematics of Decisions, Elections, and Games, annual conference, American Mathematical Society, Jan. 11, 2013, San Diego, CA.

[3] Jessie, D., and Saari, D., Strategic and behavioral decomposition of $2 \times 2 \times \cdots \times 2$ Games, Technical Report, April 2013, Institute for Mathematical Behavioral Sciences, University of California, Irvine.

[4] Robinson, D., and D. Goforth, *The Topology of the* 2×2 *Games: A new Periodic Table*, Routledge, New York, 2005.

Institute for Mathematical Behavioral Sciences, University of California, Irvine, CA 92697-5100