STRATEGIC AND BEHAVIORAL DECOMPOSITION OF $2 \times 2 \times \cdots \times 2$ GAMES

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ABSTRACT. It is shown how to decompose all $2 \times 2 \times \cdots \times 2$ games into their strategic and behavioral parts. The strategic part of this unique decomposition contains all information needed to determine a strategic outcome, such as the Nash or Quantal Response Equilibria (QRE), while the behavioral portion captures the game's portion that can lead to other means of analysis, such as tit-for-tat, side-payments, etc. In this way, conditions are developed to determine when two games are indistinguishable from the perspective of Nash, or a QRE viewpoint. A measure of how this approach reduces the complexity of analysis is that it reduces the space of 2×2 games from $\mathbb{R}^8$ to points in a three-dimensional cube. Examples illustrating applications of this approach include showing what kinds of Nash (or QRE) structures can and cannot be captured when using zero-sum games.

1. INTRODUCTION

By being an important way to analyze cooperative and competitive interactions, game theory has become a valued tool for several disciplines. Within a given area, games can be used in various ways ranging from the classic single-shot setting, to repeated games, evolutionary games, quantum games, bargaining solutions, etc. While these varying approaches may seek different answers, what is constant and basic to all of them is the "initial game" that determines what is being analyzed. Thus, consequential to all of these methods is an understanding of a game's basic structure. This structure is developed here.

The central structure for games has not been previously fully developed because the space of games can be surprisingly complex. This is captured by the fact that even the simplest n-person game, where each player has only two strategies, requires $n2^n$ values. As such, each game can be identified with a point in the Euclidean space $\mathbb{R}^{n2^n}$, and each point in $\mathbb{R}^{n2^n}$ defines a game. Thus the space of the simplest 2×2 games is the eight-dimensional $\mathbb{R}^8$, while the space of three player $2 \times 2 \times 2$ games is $\mathbb{R}^{24}$.

This complexity is further captured by the partial characterization of the simplest 2×2 games as developed by Robinson and Goforth [7]. By identifying 144 central 2×2 ordinal games, they showed how to partition $\mathbb{R}^8$ into 144 regions. Recent work by Hopkins [3] captures the inherent complexity of this space by reducing portions of this analysis to five-dimensional manifolds, while another method leads to a torus with genus 37 (think of this as the number of holes).

Date: March 20, 2013.

The research of both authors was supported, in part, by NSF Grant NSF CMMI-1016785.

A natural goal is to find ways to reduce the dimension – and complexity – of a space of games. A clever approach that emphasized the “Nash strategic flow” in a game was developed in Candogan, Menache, Ozdaglar, and Parrilo [1]. Aspects of what they do is captured by our coordinate system. Another approach is to divide a game into cooperative and strategic portions. Kalai and Kalai [4] do this by uniquely decomposing a game into its competitive zero-sum and cooperative parts. By using appropriate terms from each component, they develop a thought-provoking “coco” (cooperative-competitive) bargaining solution. Our coordinate system provides a new way to analyze their contribution (Sect. 3.4).

We show how to divide any given $2\times 2\times\ldots\times 2$ game into the portion that affects strategic concerns from the part that affects other behavioral/cooperative actions. In this way, the space of 2×2 games is reduced from the eight dimensional $\mathbb{R}^8$ to a three-dimensional cube; key aspects of any game can be extracted from points in this cube. Similarly the decomposition reduces three-player games from 24 dimensions to a 17-dimensional cube. For n-player games, the analysis is reduced by $2n+1$ dimensions. Elsewhere we show how to decompose $k_1\times k_2$ games, $k_1, k_2 \geq 3$.

$$\mathcal{G}_1 = \begin{array}{|cc|cc|}\hline 0 & 0 & 4 & 2 \\ \hline 2 & 4 & 6 & 6 \\ \hline \end{array}, \quad \mathcal{G}_2 = \begin{array}{|cc|cc|}\hline 4 & 4 & -2 & 6 \\ \hline 6 & -2 & 0 & 0 \\ \hline \end{array}, \quad \mathcal{G}_3 = \begin{array}{|cc|cc|}\hline 8 & 0 & 0 & 2 \\ \hline 10 & 2 & 2 & 4 \\ \hline \end{array}. \tag{1}$$

To suggest what is achieved with our approach, notice how all three Eq. 1 games share the same BR (bottom-right) Nash equilibrium strategy. Stark differences among these games are reflected by how they elicit different reactions and forms of analysis. The $\mathcal{G}_1$ game is uncontroversial because the BR Nash strategy is a Pareto superior outcome. Game $\mathcal{G}_2$, however, is a Prisoner’s Dilemma where the BR Nash equilibrium is Pareto inferior to the TL (top-left) outcome offering each player 4 units. Game $\mathcal{G}_3$ is more ambiguous; if the entries represent money, to obtain the BL outcome with a superior total suggests using side payments to induce the column player to play L.

Aside from the Nash structure, then, it is reasonable to expect that a game can have other features that change the way in which it is analyzed. A natural question is whether these other aspects can be identified and separated. As we show, this can be done. Moreover, as proved, there is a unique way to decompose a game that separates the strategic from the other information.

To achieve this objective, any given n player $2\times 2\times\cdots\times 2$ game is divided into three parts: the first portion identifies strategic interests, the second part is what influences other behavioral reactions, and the third “kernel” portion merely adds the same value to each of a player’s entries. With the Prisoner’s Dilemma ($\mathcal{G}_2$), then, the behavioral portion is what motivates seeking ways to achieve the TL strategy. Indeed, the three Eq. 1 games will be shown to be Nash strategically equivalent, which means that all differences in analysis or behavior of these games are due to their behavioral terms.

The way in which a game is uniquely divided into these three components was discovered with the mathematics of symmetry groups and representation theory. But once the decomposition is known, no knowledge of these mathematical structures is needed; everything can be expressed in simpler terms. To introduce the central notions, 2×2 games are described first. With only minor modifications, this structure extends to $2 \times \cdots \times 2$ games.

One of several advantages of this decomposition is that it makes it possible to run experiments (which are being done) to determine whether, and in what ways, certain solution concepts are actually determined by a game's strategic part, or influenced by the behavioral portion. What makes this possible to do is that a wide class of games with an identical solution component can be created by just changing the behavioral terms.

Another advantage is that the decomposition identifies large classes of games with specified kinds of properties. A desired modeling, for instance, may require a certain Nash structure; perhaps a model needs two competing pure strategies. With our decomposition, rather than examining a specific game, *all possible games* with the desired features can be identified. The way in which this is done is developed in Sect. 3.3.

1.1. **Equivalent games.** To decompose these games, first adopt a solution concept; e.g., Nash equilibria. All games with an identical solution structure are collected into the same set by using the following definition (which is later refined).

Definition 1. *(Preliminary) With a given solution concept SC, games $\mathcal{G}_i$ and $\mathcal{G}_j$ satisfy the binary relationship $\sim_{SC}$ if they have an identical SC structure.*

To illustrate Def. 1 with Nash equilibria and its binary relationship $\sim_N$, both

$$\mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array}, \quad \mathcal{G}_5 = \begin{array}{|cc|cc|}\hline 2 & 11 & 8 & 9 \\ \hline 0 & 0 & 10 & 4 \\ \hline\end{array}. \tag{2}$$

have the same two pure strategies of TL, BR, and the mixed strategies of $(\frac{2}{3}, \frac{1}{3})$ for player one (the row player), and $(\frac{1}{2}, \frac{1}{2})$ for player two. As such, $\mathcal{G}_4 \sim_N \mathcal{G}_5$. This means that strictly from the perspective of Nash equilibria, these games are indistinguishable.

We also examine QRE (Quantal Response Equilibria) developed by McKelvey and Palrey [6]; this is a behavioral solution concept intended to capture the frequently observed feature where players deviate from Nash equilibrium play. In particular, the "logit" QRE systems analyzed here can be viewed as modeling player as being limitedly rational where rationality is measured by a single parameter $\lambda \in [0, \infty)$; larger λ values indicate more adept players. As such, when QRE defines the $\sim_{SC}$ relationship, it is reasonable to expect that λ plays a central role; for this reason, we use the notation $\sim_{QRE,\lambda}$. A natural question (answered below) is whether there exists a λ value where $\mathcal{G}_4 \sim_{QRE,\lambda} \mathcal{G}_5$, or, more dramatically by using Eq. 1 games with radically different behavioral structures, whether there exists a $\lambda > 0$ so that $\mathcal{G}_1 \sim_{QRE,\lambda} \mathcal{G}_2$. If this can happen, it would assert that, strictly from the perspective of QRE and the specified λ value, the uncontroversial $\mathcal{G}_1$ is indistinguishable from the Prisoner's Dilemma.

Clearly $\sim_{SC}$ is an equivalence relationship. (Namely, $\mathcal{G}_i \sim_{SC} \mathcal{G}_i$. If $\mathcal{G}_i \sim_{SC} \mathcal{G}_j$, then $\mathcal{G}_j \sim_{SC} \mathcal{G}_i$. Finally, if $\mathcal{G}_i \sim_{SC} \mathcal{G}_j$ and $\mathcal{G}_j \sim_{SC} \mathcal{G}_k$, then $\mathcal{G}_i \sim_{SC} \mathcal{G}_k$.) As such, the $\sim_{SC}$ relationship partitions the space of games into equivalence classes. Each $\sim_N$ equivalence class, for instance, consists of all games that share the same Nash equilibria strategies; e.g., $\mathcal{G}_4$ and $\mathcal{G}_5$ are but two of an infinite number of entries in their $\sim_N$ class.

To analyze the structure of the $\sim_{SC}$ equivalence classes, the goals are:

(1) Characterize each equivalence class by identifying an essential game theoretic aspect that defines all games in the class. This trait (which *cannot* be satisfied by a game from any other class) identifies a game's SC aspect.
(2) Determine how games within a class differ.
(3) Determine how and whether different solution concepts partition the space of games in different ways. Differences between $\sim_{QRE,\lambda}$ equivalences sets with different λ values, for instance, would identify structural variations of games that could explain differences in QRE predictions. All dissimilarities and similarities among $\sim_N$ and different $\sim_{QRE,\lambda}$ equivalence classes are fully identified.

1.2. **The second issue and a decomposition.** As shown next, $\mathcal{G}_1 \sim_N \mathcal{G}_2 \sim_N \mathcal{G}_3$ (with Def. 1 and a later refined definition). That these diverse games belong to the same Nash equivalence class underscores the importance of the second question, which is to understand how and why games within a class can differ. As the Eq. 1 choices suggest, answers involve a behavioral aspect.

As developed starting in Sect. 2, each game $\mathcal{G}$ has a *unique decomposition* into the $\mathcal{G}^N$ part that determines the Nash strategies, the $\mathcal{G}^B$ part that captures other behavioral aspects, and the $\mathcal{G}^K$ part that just scales the entries. The unique division of the Eq. 1 games, given in the $\mathcal{G} = \mathcal{G}^N + \mathcal{G}^B + \mathcal{G}^K$ order, follows:

$$\mathcal{G}_1 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline -2 & -2 & 2 & -2 \\ \hline -2 & 2 & 2 & 2 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline \end{array}, \tag{3}$$

$$\mathcal{G}_2 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline 3 & 3 & -3 & 3 \\ \hline 3 & -3 & -3 & -3 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array}. \tag{4}$$

$$\mathcal{G}_3 = \begin{array}{|cc|cc|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline 4 & -1 & -4 & -1 \\ \hline 4 & 1 & -4 & 1 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline 5 & 2 & 5 & 2 \\ \hline 5 & 2 & 5 & 2 \\ \hline \end{array}. \tag{5}$$

The three games share an identical $\mathcal{G}^N$ Nash strategic form (the first component) that determines the Nash equilibrium. The third $\mathcal{G}^K$ component merely changes each player's payoffs by adding the same value to each of the player's entries. That $\mathcal{G}^N$ fully extracts all Nash strategic elements of the game is indicated by the second behavioral component $\mathcal{G}^B$; there are no differences in the $\mathcal{G}^B$ rows for the row player, nor in the $\mathcal{G}^B$ columns for the column player. Thus each player's $\mathcal{G}^B$ outcome is determined by the other player's action: a preferred $\mathcal{G}^B$ result depends on behavioral aspects ranging from altruism, cooperation, tit-for-tat, to "You scratch my back, I'll scratch yours," etc.

A key $\mathcal{G}^B$ feature is its Pareto superior entry. To see its importance, while the $\mathcal{G}_1$ strategic and behavioral structures both select BR, the locations of these two features diametrically disagree for $\mathcal{G}_2$. Namely, the BR Nash location for $\mathcal{G}_2^N$ is the Pareto inferior position for $\mathcal{G}_2^B$; the Pareto superior TL position for $\mathcal{G}_2^B$ is the Nash inferior location (*both* players wish to move) for $\mathcal{G}_2^N$. This conflict between the two components is what generates the Prisoner's Dilemma. For $\mathcal{G}_3$, the BL position of the $\mathcal{G}_3^B$ Pareto superior term creates interest in the BL outcome.

As developed next, all 2×2 games uniquely decompose into a $\mathcal{G}^N$ strategic term that determines the Nash equilibria, a $\mathcal{G}^B$ behavioral term dominated by the Pareto structure, and a $\mathcal{G}^K$ kernel component. With only minor, technical modifications, this structure extends to n-person $2 \times \cdots \times 2$ games.

2. The Nash decomposition of two-by-two games

To motivate the decomposition by using $\mathcal{G}_4$, if the column player's L-R strategy is determined by probabilities $(q, 1-q)$, for $q \in [0,1]$, the row player's preferences between T and B are determined by their expected values; e.g., the row player's expected outcome by playing T is $6q + 0(1-q)$ and B is $4q + 2(1-q)$. Thus, the difference between these two expected values (which is then used in a "best response" analysis) is

$$EV(T) - EV(B) = [6q + 0(1-q)] - [4q + 2(1-q)] = [6-4]q + [0-2](1-q). \tag{6}$$

The critical terms are the bracket values in the last expression. Rather than the 6 and 4, it is their difference of 2 that determines the q coefficient; rather than the 0 and 2, it is their $[0-2]$ value that determines the $(1-q)$ coefficient. As such, each term in each pair can be replaced by how it differs from the pair's average; e.g., replacing 6 with $6 - \frac{6+4}{2} = 1$ and 4 with $4 - \frac{6+4}{2} = -1$ does not affect the q coefficient in Eq. 6. Similarly, in the second column, replacing 0 with $0 - \frac{0+2}{2} = -1$ and 2 with $2 - \frac{0+2}{2} = 1$ does not affect the $(1-q)$ coefficient. While it is not obvious, terms *must* be replaced by how they deviate from the pair's average in order to separate a game's strategic and behavioral aspects. As developed in Sect. 7, any other choice need not extract all strategic information into the first component as the second component could have a Nash equilibrium.

The same argument holds when comparing the differences between second player's expected values by playing L and R when the first player uses the mixed strategies of p and $(1-p)$ for T and B. This leads to the game $\mathcal{G}_4^N$ with an identical Nash structure as $\mathcal{G}_4$.

$$\mathcal{G}_4^N = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -2 & 1 & 2 \\ \hline\end{array} \tag{7}$$

What remains is the $\mathcal{G}_4 - \mathcal{G}_4^N$ difference; e.g., the first term in the upper-left corner of $\mathcal{G}_4 - \mathcal{G}_4^N$ must be $6 - 1 = 6 - [6 - \frac{6+4}{2}] = \frac{6+4}{2}$, or the average of the 6 and 4 entries. More generally, for each Nash pair of $\mathcal{G}_4$ entries, the $\mathcal{G}_4 - \mathcal{G}_4^N$ term is the pair's average. This leads to the decomposition

$$\mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -2 & 1 & 2 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 5 & 5 & 1 & 5 \\ \hline 5 & -2 & 1 & -2 \\ \hline\end{array}. \tag{8}$$

The second term can be further decomposed by continuing this approach of extracting deviations from the average. To do so, let κ_j be the j^{th} player's average of $\mathcal{G}_4$ entries; e.g., $\kappa_1 = (6+4+0+2)/4 = 3$, while $\kappa_2 = 1.5$. Because player one's column entries are 5 and

1 and $\kappa_1 = 3$, replace 5 with $5 - 3 = 2$ and 1 with $1 - 3 = -2$. Doing the same for player two with $\kappa_2 = 1.5$ leads to

(9)
$$\mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -2 & 1 & 2 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 2 & 3.5 & -2 & 3.5 \\ \hline 2 & -3.5 & -2 & -3.5 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 3 & 1.5 & 3 & 1.5 \\ \hline 3 & 1.5 & 3 & 1.5 \\ \hline\end{array}.$$

In general, replace each $\mathcal{G}$ entry in a pair of relevant numbers (from the strategic consideration) by how it differs from the pair's average to define the $\mathcal{G}^N$ Nash strategic part. The j^{th} player's $\mathcal{G}^K$ entries all are κ_j (the average of the player's four $\mathcal{G}$ entries). To compute $\mathcal{G}^B$, replace each of the j^{th} player's $\mathcal{G} - \mathcal{G}^N$ entries by how it differs from κ_j.

2.1. **Decomposition of the Nash $\sim_N$ sets.** To represent a 2×2 game as a vector, let $\mathbf{g} = (g_{1,1}, g_{2,1}, g_{3,1}, g_{4,1}; g_{1,2}, g_{2,2}, g_{3,2}, g_{4,2})$ (the second subscript identifies the player) where

$$\mathcal{G} = \begin{array}{|cc|cc|}\hline g_{1,1} & g_{1,2} & g_{2,1} & g_{2,2} \\ \hline g_{3,1} & g_{3,2} & g_{4,1} & g_{4,2} \\ \hline\end{array}. \tag{10}$$

Thus $\mathcal{G}_4$ has the $\mathbf{g}_4 = (6, 0, 4, -4; 6, 4, -4, 0) \in \mathbb{R}^8$ vector representation.

As shown in Sect. 7, an orthogonal coordinate system for $\mathbb{R}^8$ that decomposes a game into its Nash and behavioral components is given by

$$\begin{array}{ll} \mathbf{n}_{1,1} = (1, 0, -1, 0; 0, 0, 0, 0), & \mathbf{n}_{2,1} = (0, 1, 0, -1; 0, 0, 0, 0), \\ \mathbf{n}_{1,2} = (0, 0, 0, 0; 1, -1, 0, 0), & \mathbf{n}_{2,2} = (0, 0, 0, 0; 0, 0, 1, -1), \\ \mathbf{b}_1 = (1, -1, 1, -1; 0, 0, 0, 0), & \mathbf{k}_1 = (1, 1, 1, 1; 0, 0, 0, 0), \\ \mathbf{b}_2 = (0, 0, 0, 0; 1, 1, -1, -1), & \mathbf{k}_2 = (0, 0, 0, 0; 1, 1, 1, 1). \end{array} \tag{11}$$

where $\mathbf{n}_{i,j}$, $\mathbf{b}_j$, and $\mathbf{k}_j$ define directions in, respectively, the Nash, behavioral, and kernel directions for the j^{th} player where $i = 1, 2$ refers to top-bottom or left-right.

The $\mathbf{g} = (g_{1,1}, g_{2,1}, g_{3,1}, g_{4,1}; g_{1,2}, g_{2,2}, g_{3,2}, g_{4,2})$ component in the $\mathbf{n}_{1,1}$ direction is the scalar product of $\mathbf{g}$ with the unit vector $\frac{\mathbf{n}_{1,1}}{\sqrt{2}}$: This is $\frac{g_{1,1}-g_{3,1}}{\sqrt{2}}$ or $\frac{g_{1,1}-g_{3,1}}{\sqrt{2}} \frac{\mathbf{n}_{1,1}}{\sqrt{2}} = \frac{g_{1,1}-g_{3,1}}{2} \mathbf{n}_{1,1}$. As $\eta_{1,1} = \frac{g_{1,1}-g_{3,1}}{2}$ equals $g_{1,1} - \frac{g_{1,1}+g_{3,1}}{2}$, the two non-zero components of $\eta_{1,1}\mathbf{n}_{1,1}$ capture how $g_{1,1}$ and $g_{3,1}$ differ from their average. All four $\eta_{i,j}$ entries are similarly defined; i.e.,

$$\eta_{1,1} = \frac{g_{1,1} - g_{3,1}}{2},\ \eta_{2,1} = \frac{g_{2,1} - g_{4,1}}{2},\ \eta_{1,2} = \frac{g_{1,2} - g_{2,2}}{2},\ \eta_{2,2} = \frac{g_{3,2} - g_{4,2}}{2}. \tag{12}$$

While the second subscript identifies the player, subscript i of $\eta_{i,1}$ means that the relevant pair is player one's top and bottom row entries in column i. For the second player, $\eta_{i,2}$ refers to difference from the average of the left and right column entries in row i.

For the behavioral directions, the $\beta_1\mathbf{b}_1$ coefficient is $\beta_1 = \frac{1}{4}[g_{1,1} - g_{2,1} + g_{3,1} - g_{4,1}] = \frac{g_{1,1}+g_{3,1}}{2} - \frac{g_{1,1}+g_{2,1}+g_{3,1}+g_{4,1}}{4} = \frac{g_{1,1}+g_{3,1}}{2} - \kappa_1$, or the difference of the average of player one's entries in the first column and the overall average of player one's four entries. Similarly, $\beta_2 = \frac{1}{4}[g_{1,2} + g_{2,2} - g_{3,2} - g_{4,2}] = \frac{g_{1,2}+g_{2,2}}{2} - \kappa_2$.

The Nash strategic part of the decomposed game is the first bracket of

$$[\eta_{1,1}\mathbf{n}_{1,1} + \eta_{2,1}\mathbf{n}_{2,1} + \eta_{1,2}\mathbf{n}_{1,2} + \eta_{2,2}\mathbf{n}_{2,2}] + [\beta_1\mathbf{b}_1 + \beta_2\mathbf{b}_2] + [\kappa_1\mathbf{k}_1 + \kappa_2\mathbf{k}_2], \tag{13}$$

while the second bracket defines the behavioral portion, and the third captures the kernel. The game matrix form of Eq. 13 is $\mathcal{G} = \mathcal{G}^N + \mathcal{G}^B + \mathcal{G}^K$ where

$$\mathcal{G}^N = \begin{array}{|cc|cc|}\hline \eta_{1,1} & \eta_{1,2} & \eta_{2,1} & -\eta_{1,2} \\ \hline -\eta_{1,1} & \eta_{2,2} & -\eta_{2,1} & -\eta_{2,2} \\ \hline\end{array}, \tag{14}$$

$$\mathcal{G}^B = \begin{array}{|cc|cc|}\hline \beta_1 & \beta_2 & -\beta_1 & \beta_2 \\ \hline \beta_1 & -\beta_2 & -\beta_1 & -\beta_2 \\ \hline\end{array}, \tag{15}$$

$$\mathcal{G}^K = \begin{array}{|cc|cc|}\hline \kappa_1 & \kappa_2 & \kappa_1 & \kappa_2 \\ \hline \kappa_1 & \kappa_2 & \kappa_1 & \kappa_2 \\ \hline\end{array}. \tag{16}$$

Component $\mathcal{G}^B$ has no Nash strategic content. This is because each row is the same $(\beta_1, -\beta_1)$ for the row player; each column is the same $(\beta_2, -\beta_2)$ for the column player. Because $\mathcal{G}^B$ and $\mathcal{G}^K$ admit no strategic content, the $\mathcal{G}^N$ component has extracted all aspects of game $\mathcal{G}$ needed to determine its Nash strategy. A refined version of Def. 1 is that two games are equivalent if and only if they define the same $\mathcal{G}^N$. If $\eta_{1,j}\eta_{2,j} < 0$, $j = 1, 2$ (so they have different signs), the mixed strategy is given by

$$p = \frac{|\eta_{2,2}|}{|\eta_{1,2}| + |\eta_{2,2}|}, \quad q = \frac{|\eta_{2,1}|}{|\eta_{1,1}| + |\eta_{2,1}|}. \tag{17}$$

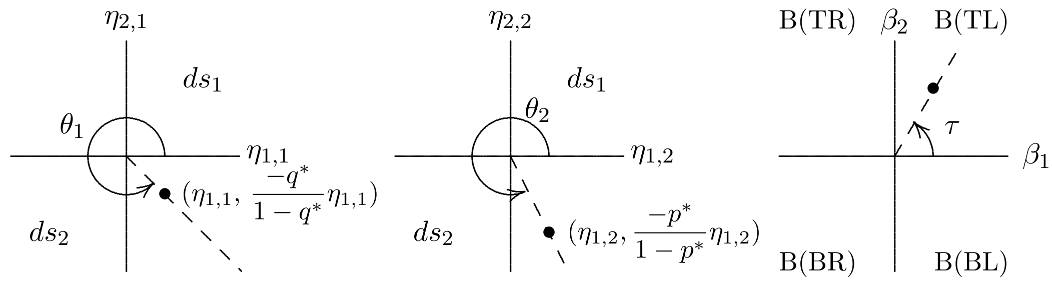
If $\beta_j \neq 0$ for $j = 1, 2$, then $\mathcal{G}^B$ always has a Pareto superior entry. This is because the larger of $\beta_1, -\beta_1$ (the one that is positive) identifies a particular column while the larger of $\beta_2, -\beta_2$ identifies a specific row; the intersection of this row and column identifies the Pareto superior entry. (This is the unique position where both entries are positive.) If, for example, $\beta_1 > 0, \beta_2 < 0$, then the $\mathcal{G}^B$ Pareto superior entry is at BL; this behavioral Pareto position is denoted by $B(BL)$. The Pareto inferior term in $\mathcal{G}^B$ always is diametrically opposite the Pareto dominate term. (This assertion does not extend to $n \geq 3$ player games.) As noted, the location of $\mathcal{G}^B$'s Pareto superior entry can change the game's complexion.

This construction is summarized with the following theorem.

Theorem 1. *The orthogonal basis for $\mathbb{R}^8$ given by Eq. 11 uniquely decomposes the space of 2×2 games into three subspaces:*

1. *The four dimensional $\mathcal{G}^N$ strategic subspace spanned by $\{\mathbf{n}_{i,j}\}_{i,j=1,2}$ that contains all information needed to determine a game's Nash structure.*
2. *The two dimensional $\mathcal{G}^B$ behavioral subspace spanned by $\{\mathbf{b}_j\}_{j=1}^2$ determines other features of the game such as the position of the Pareto superior outcome.*
3. *The two dimensional $\mathcal{G}^K$ kernel subspace spanned by $\{\mathbf{k}_j\}_{j=1}^2$ that has no impact on the Nash or Pareto structure of a game.*

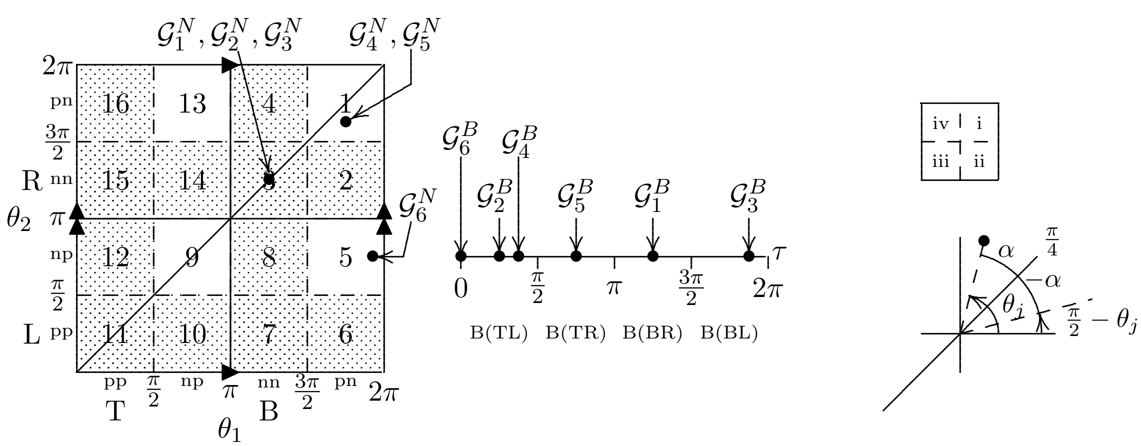
2.2. **A geometric representation.** To simplify the Thm. 1 description of the $\mathcal{G}^N, \mathcal{G}^B$ subspaces, first plot $\boldsymbol{\eta_j} = (\eta_{1,j}, \eta_{2,j})$, $j = 1, 2$, as points in $\mathbb{R}^2$. The Fig. 1 a, b choices of $\boldsymbol{\eta}_j = (\eta_{1,j}, \eta_{2,j})$ are from $\mathcal{G}_4^N$ with its Nash $p^* = \frac{2}{3}$ and $q^* = \frac{1}{2}$ mixed strategies. Next plot the $\boldsymbol{\beta} = (\beta_1, \beta_2)$ point; the $\boldsymbol{\beta} = (2, 3.5)$ choice in Fig. 1c is from $\mathcal{G}_4^B$.



a. Player 1; strategic **b.** Player 2; strategic **c.** Game $\mathcal{G}_4^B$

Figure 1. Strategic/behavior decomposition of the games $\mathcal{G}_4, \mathcal{G}_5$

An interesting feature about the $\mathcal{G}^N$ geometric representation, as illustrated by Figs. 1 a, b (and Eq. 17), is that all entries on a ray emanating from the origin and passing through a specified $\boldsymbol{\eta}_j$ have the same p and q values. (With Eq. 6, for instance, if the two bracket values are multiplied by the same positive scalar, the resulting equation would yield the same information about whether to play T or B.) This permits replacing the four-dimensional $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)$ representation for $\mathcal{G}^N$ with (θ_1, θ_2) points on the two-dimensional square $[0, 2\pi] \times [0, 2\pi]$ (Fig. 2a); the θ_j values identify the appropriate rays. (Angle 0 is identified with angle 2π, so rather than a square, Fig. 2a actually represents a torus T^2. The square is used to examine T^2 properties.) For strategic analysis, this square (torus), rather than the four-dimensional subspace, suffices. The entries along the Fig. 2a axes, such as pn, identify the quadrant signs of $(\eta_{1,j}, \eta_{2,j})$; e.g., pn on the horizontal axis means that $\eta_{1,1} > 0$ and $\eta_{2,1} < 0$.



a. Strategic $\mathcal{G}^N$ decomposition **b.** Behavioral $\mathcal{G}^B$ **c.** Rotating

Figure 2. Decomposition of games

If θ_j defines a ray in the first quadrant, then $\eta_{1,j}$ and $\eta_{2,j}$ are both positive, so the j^{th} agent has a dominant strategy. This region is denoted in Figs. 1a, b, with ds_1; in Fig. 2a, the dominant strategy of T or L is listed by the axis. Similarly, if θ_j is in the third quadrant, then $\eta_{1,j}$ and $\eta_{2,j}$ are negative so the $-\eta_{1,j}$ and $-\eta_{2,j}$ values in the other row (column) are positive; again, the j^{th} agent has a dominant strategy. (This second dominant strategy region is denoted by ds_2; in Fig. 2a, it is captured by B or R.)

A mixed strategy accompanies θ_j in the second or fourth quadrants (as with Fig. 1a). Thus a mixed strategy exists only where $\frac{\pi}{2} < \theta_j < \pi$ and $\frac{3\pi}{2} < \theta_j < 2\pi$, $j = 1, 2$. Similarly, $\theta_j = a\frac{\pi}{2}$ values for integer a (so θ_j is on either the vertical or horizontal axis) defines where p and/or q equals either zero or unity; it corresponds to a pure strategy transition between a dominated and mixed strategy.

For $\mathcal{G}^B$, all points along a ray passing through $\boldsymbol{\beta} = (\beta_1, \beta_2)$ have an identical Pareto structure for the players; differences just involve a positive scalar multiple. Thus these terms can be represented by angle τ as in Fig. 1c. Each quadrant represents a different Pareto superior position as indicated in Figs. 1c, 2b; e.g., $B(TR)$ means that the Pareto superior entry for the behavioral component is in position TR, so the Pareto inferior term of $\mathcal{G}^B$ is in BL.

2.3. **Using the decomposition.** Before describing which kinds of games are in each of the Fig. 2a,b regions, some immediate consequences are described.

2.3.1. *Strategic behavioral strategies.* The strategic portion of game, $\mathcal{G}^N$, can be used in several ways. As an example, the decomposition of an asymmetric matching penny game $\mathcal{G}_6$, from Goeree and Holt [2], is

(18)

$$\mathcal{G}_6 = \begin{array}{|cc|cc|}\hline 320 & 40 & 40 & 80 \\ \hline 40 & 80 & 80 & 40 \\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 140 & -20 & -20 & 20 \\ \hline -140 & 20 & 20 & -20 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 60 & 0 & -60 & 0 \\ \hline 60 & 0 & -60 & 0 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 120 & 60 & 120 & 60 \\ \hline 120 & 60 & 120 & 60 \\ \hline\end{array}$$

with the mixed Nash equilibrium of $p = 0.50, q = 0.125$. The puzzle Goeree and Holt raise is to explain why $p = 0.5$ differs so strongly from their experimental data value of $p = 0.96$; the $q = 0.16$ experimental value is somewhat compatible with the Nash value.

The $\mathcal{G}_6^N$ term captures player one's loss-gain strategic structure; this player can lose 140 by playing B, but at most 20, with a large possible reward, by playing T. As shown with empirical evidence by Luce [5], even if a lottery has two expressions with precisely the same outcome (which goes beyond the expected value comparisons central for Nash), where the first is expressed in terms of gains and the second in terms of losses from a given amount, people tend to avoid the choice expressed in terms of losses. This proclivity suggests a stronger behavioral tendency to play T than given by Nash predictions. If player two suspects this is the case, R is the appropriate strategy.

2.3.2. *Closeness of games.* The geometry provides an appropriate structure to determine whether two games are "close" to each other. But "closeness" must be specified in terms of an appropriate condition. For instance, whether the following seemingly dissimilar games $\mathcal{G}_7$ and $\mathcal{G}_8$ are close to each other depends on the desired criterion.

(19)
$$\mathcal{G}_7 = \begin{array}{|cc|cc|}\hline 13 & -8 & -12 & -16 \\ \hline 7 & 11 & -8 & 13 \\ \hline\end{array}, \quad \mathcal{G}_8 = \begin{array}{|cc|cc|}\hline 35 & 12 & -24 & 4 \\ \hline -27 & -9 & 16 & -7 \\ \hline\end{array}$$

If the criterion is in terms of Nash, the answer is determined by whether their Fig. 2a positions ((θ_1, θ_2) values) are near each other. Games that are close to each other have similar Nash strategies. To compute these values, the $\mathcal{G}_j^N$ components and θ_j values are:

$$\mathcal{G}_7^N = \begin{array}{|rr|rr|}\hline 3 & 4 & -2 & -4 \\ \hline -3 & -1 & 2 & 1 \\ \hline\end{array}, \quad \mathcal{G}_8^N = \begin{array}{|rr|rr|}\hline 31 & 4 & -20 & -4 \\ \hline -31 & -1 & 20 & 1 \\ \hline\end{array} \tag{20}$$

Because $\tan(\theta_j) = \frac{\eta_{2,j}}{\eta_{1,j}}$, the $\mathcal{G}_7^N$ Fig. 2a point is $(\arctan(\frac{-2}{3}), \arctan(4)) = (-0.5880, 1.3258)$ while that for $\mathcal{G}_8^N$ is the nearby $(\arctan(\frac{-20}{31}), \arctan(4)) = (-0.5730, 1.3258)$. Rather than computing these angles, their closeness becomes obvious by plotting rays passing through $(-\frac{2}{3}, 4)$ and $(-\frac{2}{3.1}, 4)$.

If the criterion involves behavioral aspects, then $\mathcal{G}_7$ and $\mathcal{G}_8$ are "far apart" as determined by $\mathcal{G}^B$ values and the location of points along the Fig. 2b line segment. Indeed, for $\mathcal{G}_7^B$, the $\beta_1 = 10, \beta_2 = -12$ values define $\tau = \arctan(-1.2)$ with a $B(BL)$ Pareto point. In contrast, for $\mathcal{G}_8^B$, the $\beta_1 = 4, \beta_2 = 8$ values define a significantly different $\tau = \arctan(2)$ with a $B(TL)$ Pareto point. (This metric sense of closeness can differ significantly from that developed in Robinson and Gogorth [7].)

2.3.3. *Tit-for-Tat.* The decomposition nicely captures the tension between the strategic and behavioral parts in a Prisoner's Dilemma. To demonstrate with $\mathcal{G}_2$, recall that its decomposition (Eq. 4) is as follows (with $\kappa_1 = \kappa_2 = 2$ defining $\mathcal{G}_2^K$).

$$\mathcal{G}_2 = \begin{array}{|rr|rr|}\hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline\end{array} + \begin{array}{|lr|rr|}\hline 3 & 3 & -3 & 3 \\ \hline 3 & -3 & -3 & -3 \\ \hline\end{array} + \mathcal{G}_2^K.$$

A standard analysis has player one using the tit-for-tat strategy: play T on step $k = 1$; on step k, play T if player two plays L on the previous step $k - 1$, otherwise play B. The goal is to determine player two's best response, where $\delta \in [0, 1)$ is his discount rate.

If player two cooperates by constantly playing L, the earnings, given by $E(L)$, are

$$E(L) = \sum_{k=1}^{\infty} -\delta^{k-1} + \sum_{k=1}^{\infty} 3\delta^{k-1} + \sum_{k=1}^{\infty} 2\delta^{k-1} = \frac{-1}{1-\delta} + [3 + \frac{3\delta}{1-\delta}] + \frac{2}{1-\delta}, \tag{21}$$

where the first and second functions on the right-hand side represent the tension of losses by not playing according to the strategic $\mathcal{G}_2^N$ portion, but compensated by cooperating to achieve Pareto superior values of $\mathcal{G}_2^B$. The third function is the $\mathcal{G}_2^K$ return.

In contrast, by always playing R, the $E(R)$ earnings are

$$E(R) = \frac{1}{1-\delta} + [3 - \frac{3\delta}{1-\delta}] + \frac{2}{1-\delta}. \tag{22}$$

Cooperation requires $E(L) - E(R) > 0$, which is if $\frac{3\delta}{1-\delta} > \frac{1}{1-\delta}$, or if $3\delta > 1$. For general PD games, this $\beta_2 > \frac{|\eta_{2,2}| + |\eta_{1,2}|}{2} + \frac{1-\delta}{\delta}|\eta_{1,2}|$ condition captures the relationship among the discount rate, Nash structure, and $\mathcal{G}^B$ Pareto structure needed for cooperation. In particular, the Nash term and discount rate define a lower bound for the behavioral $\mathcal{G}^B$'s strength for which tit-for-tat successfully establishes cooperation. As an illustration, if the $\eta_{i,j}$ and δ values are given, and β_2 does not satisfy this inequality, but β_1 has a large value, this suggests that a side payment to increase the β_2 value will allow tit-for-tat to obtain cooperation. (A complete analysis, which requires examining strategies involving a mixture of R and L, yields a similar relationship.)

2.3.4. *Symmetric and zero-sum games.* A reasonable question is to determine which $\mathcal{G}^N$ structures occur with widely used classes of games. The symmetric games, for instance, are located along the Fig. 2a diagonal. Namely, if a game is symmetric, its $\mathcal{G}^N$ point is on this diagonal; if a point is on this diagonal, it defines a two parameter family of symmetric games in its equivalence class given by the parameters $\beta_1 = \beta_2$, and $\kappa_1 = \kappa_2$. Each point on this diagonal defines points along the Fig. 1ab rays. As such, it follows that the symmetric games define a four-dimensional linear subspace of $\mathbb{R}^8$. As the diagonal misses many Fig. 2a regions, it follows that the associated Nash strategic structures for symmetric games are limited.

An example where the answer is not as immediate is the class of zero-sum games. These games are often used for convenience, so it is reasonable to determine their associated $\mathcal{G}^N$ structures. That is, the goal is to determine what kinds of Nash behavior are admitted, or omitted, by using zero-sum games. The answer is given by following theorem and Fig. 3b.

Theorem 2. *If $\mathcal{G}$ is a 2×2 zero-sum game, then entries in its $\mathcal{G}^N$ component satisfy*

$$\eta_{1,1} - \eta_{2,1} = \eta_{2,2} - \eta_{1,2}. \tag{23}$$

Conversely, if Eq. 23 is satisfied, there is a zero-sum game in its equivalence class, which is uniquely defined by

$$\beta_1 = -\frac{\eta_{1,2} + \eta_{2,2}}{2}, \quad \beta_2 = -\frac{\eta_{1,1} + \eta_{2,1}}{2} \tag{24}$$

Different from the symmetric games, the $\mathcal{G}^N$ component of a zero-sum game typically is not zero-sum. (This can be illustrated by using Eq. 23 and, for instance, $\eta_{1,1} = 9, \eta_{2,1} = 8, \eta_{2,2} = 2$, and $\eta_{1,2} = 1$.) Instead (Eq. 14), a necessary and sufficient condition for $\mathcal{G}^N$ to be zero-sum is $\eta_{1,1} = \eta_{2,2} = -\eta_{2,1} = -\eta_{1,2}$, which corresponds to a "matching pennies" game; these coordinates locate these games at the center points in squares 5 and 13. Here Eq. 24 forces $\beta_1 = \beta_2 = 0$, so the $\mathcal{G}^B$ component plays no role when $\mathcal{G}^N$ has a zero-sum structure. More generally and across $\sim_N$ classes, the zero-sum games form an interesting four-dimensional smooth manifold in a four-dimensional linear subspace of the $\mathbb{R}^8$ space of games; it is defined by the Eqs. 23, 24 combination of $\mathcal{G}^N$ and $\mathcal{G}^B$ with $\mathcal{G}^K$ parts.

Proof of Thm. 2: By writing out the zero sum condition for each game entry, four equations (in the order of the game matrix) are defined where $K = (\kappa_1 + \kappa_2)$:

$$\begin{array}{ll} \eta_{1,1} + \eta_{1,2} = -\beta_1 - \beta_2 - K, & \eta_{2,1} - \eta_{1,2} = \beta_1 - \beta_2 - K, \\ -\eta_{1,1} + \eta_{2,2} = \beta_2 - \beta_1 - K, & -\eta_{2,1} - \eta_{2,2} = \beta_1 + \beta_2 - K \end{array} \tag{25}$$

Adding the first and fourth equations (from TL and BR), and then the second and third gives two expressions for $-2K$. Setting them equal yields Eq. 23.

Adding the first and fourth equations and using Eq. 23 it follows that $2K = 0$, or that $\kappa_1 = -\kappa_2$. By using $K = 0$ and Eq. 23, it follows that the first and fourth equations and the second and third equations are equivalent. Thus, using the first and second (along with Eq. 23), it follows that the required β_j values are given by Eq. 24. □

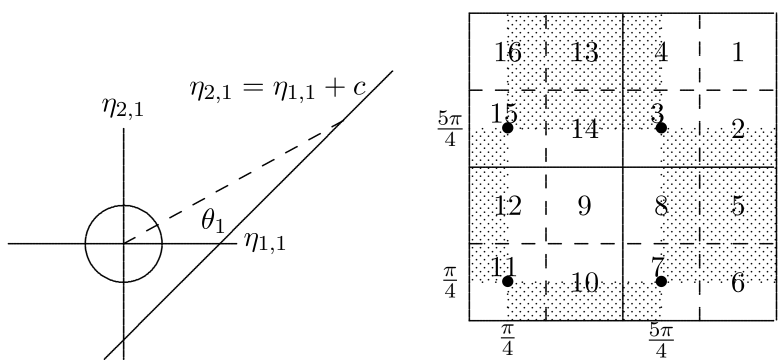
While the $\mathcal{G}^N$ symmetric games are only on the main diagonal of Fig. 2a, the admissible $\mathcal{G}^N$ zero-sum structures essentially fill half of the Fig. 2a square. They are given by the shaded regions of Fig. 3b; the only boundary points that are included are the four bullets.

Corollary 1. *Half of the Fig. 2a Nash structures can be achieved through zero-sum games, but half of them cannot.*

To derive Fig. 3b, set Eq. 23 equal to a constant; this defines the two linear equations

$$\eta_{2,1} = \eta_{1,1} + c, \quad \eta_{2,2} = \eta_{1,2} - c. \tag{26}$$

All values satisfying both equations are found in the same manner as illustrated in Fig. 3a. The strategy is to graph both Eq. 26 equations and then determine the corresponding and admissible θ_j values.



a. Player 1; strategic **b.** $\mathcal{G}^N$ Zero sum games

Figure 3. Strategic characteristics of zero sum games

If $c = 0$, both lines pass through the origin, so the θ_j values are combinations of $\theta_1, \theta_2 = \frac{\pi}{4}, \frac{5\pi}{4}$, which define the four Fig. 3b bullets. For any other c value, one Eq. 26 line passes above the origin, while the other passes below.

Compute each line's admissible θ_j values by determining which θ_j rays meet the line. As indicated in Fig. 3a, the line below the origin has θ_j ranging from $\frac{\pi}{4} - \pi$ to $\frac{\pi}{4}$, while the θ values for the companion line above the origin vary between $\frac{\pi}{4}$ and $\frac{\pi}{4} + \pi$; the end-points (which represent points at infinity) cannot be achieved. These constraints, where θ_1 is in one interval while θ_2 is in the other, define the shaded Fig. 3b regions.

It now follows that the symmetric and zero-sum games meet only at the two bullet points along the Fig. 3b diagonal. This intersection consists of all "zero-sum, symmetric games;" it defines a line in $\mathbb{R}^8$. To show this, the locations of the bullets require the $\mathcal{G}^N$ terms to be $\eta_{1,1} = \eta_{2,1} = \eta_{1,2} = \eta_{2,2} = x$. According to Eq. 24, $\beta_1 = \beta_2 = -x$. Symmetric games require $\kappa_1 = \kappa_2$ while the zero-sum condition is $\kappa_1 = -\kappa_2$, so $\kappa_1 = \kappa_2 = 0$. Thus the vector form of this subspace of games, $(0, 2x, -2x, 0; 0, -2x, 2x, 0)$, defines a line in $\mathbb{R}^8$, or

$$\begin{array}{|rr|rr|} \hline 0 & 0 & 2x & -2x \\ \hline -2x & 2x & 0 & 0 \\ \hline \end{array}, \quad x \in \mathbb{R},$$

where $x > 0$ corresponds to the square 11 bullet and $x < 0$ to the square 3 bullet.

3. Characterizing all 2×2 games

All games are characterized by points in the cube represented by the product of the $\mathcal{G}^N$ square (Fig. 2a) with the $\mathcal{G}^B$ line segment (Fig. 2b). (By identifying angles 0 and 2π, this cube actually represents a three-torus T^3.) Thus information about games reduces to examining the game structures in each of the $16 \times 4 = 64$ smaller cubes.

Fortunately, as shown in Sect. 3.1, natural symmetries decrease the number of relevant cubes from 64 to 16, with a further reduction based on Fig. 1c symmetries. The Nash structure of each Fig. 2a square is described (Thm. 3) in Sect. 3.2. In Sect. 3.3, features of the games in each cube are described.

3.1. **A reduction.** To reduce the number of Fig. 2a squares that need to be analyzed, notice how Fig. 2a has the structure of four copies of the region consisting of the {1, 2, 3, 4} squares. This similarity reflects how a game remains essentially the same after interchanging columns and/or rows. (The main difference is an exchange of p and $(1-p)$, and/or q and $(1-q)$.) As shown next, this symmetry allows attention to be focused on the squares {1, 2, 3, 5}, rather than all 16 of them. In turn, this reduces the need to examine what happens in only $4 \times 4 = 16$ small cubes, rather than all 64 of them.

The reduction is based on how a game's Fig. 2a position changes by interchanging its columns and/or rows. Interchanging the top and bottom rows of $\mathcal{G}^N$ (Eq. 14) has the effect of changing $\boldsymbol{\eta}_1 = (\eta_{1,1}, \eta_{2,1})$ to $(-\eta_{1,1}, -\eta_{2,1}) = -\boldsymbol{\eta}_1$. In angular terms, θ_1 is moved to $\theta_1 + \pi$. The $\boldsymbol{\eta}_2 = (\eta_{1,2}, \eta_{2,2})$ term is changed to $(\eta_{2,2}, \eta_{1,2})$, which interchanges the "x" and "y" coordinates. But a (x, y) to (y, x) change is a reflection about the $y = x$ line, which is the ray $\theta = \frac{\pi}{4}$. If (as in Fig. 2c) angle θ differs from $\frac{\pi}{4}$ by angle α (so $\alpha = \theta - \frac{\pi}{4}$), the reflected point is on the ray that differs from $y = x$ by angle $-\alpha$; i.e., the new angle is $\frac{\pi}{4} - \alpha = \frac{\pi}{4} - (\theta - \frac{\pi}{4}) = \frac{\pi}{2} - \theta$. This is formalized in the following proposition:

Proposition 1. *If the rows of a $\mathcal{G}^N$ matrix represented by (θ_1, θ_2) are interchanged, the new matrix is represented by $(\theta_1 + \pi, \frac{\pi}{2} - \theta_2)$. If the columns of a (θ_1, θ_2) matrix are interchanged, the new matrix is $(\frac{\pi}{2} - \theta_1, \theta_2 + \pi)$.*

To assist in the description of Prop. 1, the Fig. 2a squares are further subdivided as indicated in Fig. 2c. So, let the game matrix of a point be in square 4i; that is, (θ_1, θ_2) is in the upper-right corner of square 4. Interchanging the rows (Prop. 1) forms an $(\theta_1 + \pi, \frac{\pi}{2} - \theta_2)$ matrix. The $\theta_1 + \pi$ value positions the new matrix in the right-hand side of one of the {11, 12, 15, 16} squares (the first column of Fig. 2a). To determine which one, notice that the vertical distance from the horizontal $\frac{\pi}{2}$ line (Fig. 2a) to the top-right region in square 4 is over $2\frac{1}{2}$ squares; this is the $\theta_2 - \frac{\pi}{2}$ distance. To find the new point with value $\frac{\pi}{2} - \theta_2$, go down this "more than $2\frac{1}{2}$ squares" distance from the top horizontal 2π line (which is identified with the bottom horizontal line); it is in the lower right section of square 12. Thus, changing the rows of a matrix represented by a point in square 4i defines a unique point in 12ii.

Points (i.e., games) that are identified with each other can be determined with the dynamic of first exchanging columns, then rows, then columns, and then rows to return to the starting point. Illustrating with square 4, a point in $4i$ is mapped to point in $8iv$. The full sequence is $4i \to 8iv \to 16iii \to 12ii \to 4i$.

Proposition 2. *Starting with a matrix (θ_1, θ_2), perform the following operations in the same manner: Interchange columns, then rows, then columns, then rows. These operations define points in the following regions:*

$$4i \to 8iv \to 16iii \to 12ii \to 4i, \tag{27}$$

$$2i \to 10iv \to 6iii \to 14ii \to 2i, \quad 3i \to 7iv \to 11iii \to 15ii \to 3i \tag{28}$$

$$1i \to 9iv \to 1iii \to 9ii \to 1i, \quad 5i \to 13iv \to 5iii \to 13ii \to 5i. \tag{29}$$

Similar sequences arise by starting in any other sector of the starting square. According to Eq. 28, analyzing a game in square 2 is the same as doing so with a game in squares 6, 10, or 14. Similarly, analyzing all games in square 3 describes what happens in squares {3, 7, 11, 15}, and square 4 handles squares {4, 8, 12, 16}. Square 1, however, only handles squares {1, 9} while square 5 only handles squares {5, 13}. This dynamic shows that it suffices to examine the game structures only in squares {1, 2, 3, 4, 5}. This dynamic, for instance, identifies the four Fig. 3b bullets. It also shows that the two matching pennies games located at the center of squares 5 and 13 define the same game properties.

A second reduction uses the Fig. 2a diagonal line: Interchanging row and column players flips the Fig. 2a square about this diagonal. As an illustration, in square 4, player one has the dominant strategy of B, and player two reacts accordingly; in square 2, player two has the dominant strategy of R, and player one reacts accordingly. This symmetry reduces the $\mathcal{G}^N$ analysis to the squares {1, 2, 3, 5}, which is then combined with the $\mathcal{G}^B$ properties from the four Fig. 2b regions. (As the diagonal passes through squares 1 and 3, only the half of these squares below the diagonal needs to be examined.)

3.2. **The Nash structure of the square.** The Nash structure of each Eq. 2a square. follows immediately from the $\eta_{i,j}$ signs (specified along the Fig. 2a axes) and the $\mathcal{G}^N$ form (Eq. 14). According to Sect. 3.1, it suffices to describe the squares {1, 2, 3, 5}.

Theorem 3. *The Nash structures of games in the Fig. 2a torus are as follows:*

1. *Games in square 1 (where $\eta_{1,j} > 0$ and $\eta_{2,j} < 0$ for $j = 1, 2$) have the two pure strategies TL and BR, and a mixed strategy.*
2. *Games in square 2 (where $\eta_{1,1} > 0, \eta_{2,1} < 0$ and $\eta_{i,2} < 0$, $i = 1, 2$) have a dominant R strategy for player 2. Because $\eta_{2,1} < 0$, the first player plays B.*
3. *Games in square 3 (where $\eta_{i,j} < 0$) have BR as the dominant Nash strategy.*

(4) *Games in square 5 (where $\eta_{1,1} > 0, \eta_{2,1} < 0$ and $\eta_{1,2} < 0, \eta_{2,2} > 0$) have no pure strategies and one mixed strategy.*

By using ± 1 to define canonical examples, the center points of squares 1, 2, 3, and 5 are characterized, respectively, by $\mathcal{G}_9^N$, $\mathcal{G}_{10}^N$, $\mathcal{G}_2^N$, and $\mathcal{G}_{11}^N$ where:

$$\mathcal{G}_9^N = \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 \\ \hline\end{array},\ \mathcal{G}_{10}^N = \begin{array}{|cc|cc|}\hline 1 & -1 & -1 & 1 \\ \hline -1 & -1 & 1 & 1 \\ \hline\end{array},\ \mathcal{G}_{11}^N = \begin{array}{|cc|cc|}\hline 1 & -1 & -1 & 1 \\ \hline -1 & 1 & 1 & -1 \\ \hline\end{array} \tag{30}$$

To indicate how Thm. 3 captures what happens in all 16 squares, it follows from square 2's structure (Thm. 3 and $\mathcal{G}_{10}^N$) and Eq. 28 that player two's dominant strategy is R in squares 2 and 14, and L in 6 and 10. Reflecting about the diagonal, player one's dominant strategy is B in squares 4 and 8, and T in 12 and 16. A similar analysis holds for the other squares; e.g., only in squares 1 and 9 does each game have two pure and one mixed strategy; in 5 and 13, each game has a mixed strategy but no pure strategies.

3.3. **The general structure of each cube.** We now can design all possible games that have a variety of desired properties, This is done just by adding appropriate $\mathcal{G}^B$ terms (Fig. 2b) to a desired Nash structure (Thm. 3). What essentially happens is that the $\mathcal{G}^B$ Pareto term is either in conflict, or in agreement, with the Nash structure.

Indeed, for a specified $\mathcal{G}^N$, $\mathcal{G}^N + \mathcal{G}^B$ defines a two parameter (β_1, β_2) family of games with the same Nash structure. Illustrating with square 3, this family starts with the $\mathcal{G}_2^N$ game that has a dominant strategy and branches off in one direction to create a Prisoner's Dilemma structure, while in a different direction to suggest using side payments. Square 2 has surprisingly similar structures. Square 1 replaces these earlier dominant strategies with two pure and one mixed strategy; in one direction the parameterized family separates the equilibria, and another direction creates interest in non-equilibria outcomes. Finally, all pure strategies are dropped in square 5.

3.3.1. *Square 3.* All of these games have a dominant BR Nash strategy; a well known example in this region is the Prisoner's Dilemma.

Cube $(3, B(BR))$*:* The $\mathcal{G}^B$ Pareto superior point in this cube reinforces the Nash dominant strategy, so, as true with $\mathcal{G}_1$, the Nash outcome coincides with expected behavior.

Cube $(3, B(TL))$*:* Games in this cube have $\beta_1, \beta_2 \geq 0$ and the general form

$$\mathcal{G}^N + \mathcal{G}^B = \begin{array}{|cc|cc|}\hline -|\eta_{1,1}| + \beta_1 & -|\eta_{1,2}| + \beta_2 & -|\eta_{2,1}| - \beta_1 & |\eta_{1,2}| + \beta_2 \\ \hline |\eta_{1,1}| + \beta_1 & -|\eta_{2,2}| - \beta_2 & |\eta_{2,1}| - \beta_1 & |\eta_{2,2}| - \beta_2 \\ \hline\end{array}. \tag{31}$$

Thus Eq. 31 defines a two parameter family of games starting from its $\mathcal{G}^N$ $(\beta_1 = \beta_2 = 0)$ to a Prisoner's Dilemma. As the characterization involves which matrix entries dominate, set equal each player's TL and BR entries to find the transition values

$$\beta_1^* = \frac{|\eta_{1,1}| + |\eta_{2,1}|}{2}, \quad \beta_2^* = \frac{|\eta_{1,2}| + |\eta_{2,2}|}{2}. \tag{32}$$

Equation 34 illustrates two of the three types with $\mathcal{G}_2^N$, so $\beta_1^* = \beta_2^* = 1$.

(1) If $\beta_j < \beta_j^*$, $j = 1, 2$, the dominant BR Nash strategy also is a Pareto point. An example with $\beta_1 = \beta_2 = 0.5$ is the first Eq. 34 game.

(2) If $\beta_j > \beta_j^*$ for only one player, then an outcome different from that of BR becomes attractive to this player. Similar to $\mathcal{G}_3$, a sufficiently large β_j can suggest cooperative strategies with side payments. This is illustrated in the second Eq. 34 game with $\beta_1 = 4, \beta_2 = 0$. More generally, if

$$\beta_j > \frac{|\eta_{2,j}| - |\eta_{1,j}|}{2} + |\eta_{2,i}| \tag{33}$$

(where "i" is the other player), then the sum of terms in another matrix entry is larger than that of BR, which makes a cooperative solution more attractive.

(3) If $\beta_j > \beta_j^*$, $j = 1, 2$, the game is a Prisoner's Dilemma.

$$\begin{array}{|cc|cc|}\hline -0.5 & -0.5 & -1.5 & 1.5 \\ \hline 1.5 & -1.5 & 0.5 & 0.5 \\ \hline \end{array}, \quad \begin{array}{|cc|cc|}\hline 3 & -1 & -5 & 1 \\ \hline 5 & -1 & -3 & 1 \\ \hline \end{array} \tag{34}$$

Cubes $(3, B(TR))$, $(3, B(BL))$*:* As with $\mathcal{G}_3$, in these cubes the $\mathcal{G}^B$ component emphasizes components different from the Nash structure; this can make certain matrix entries more attractive than that of the Nash dominant strategy. The symmetry where both $B(TR)$ and $B(BL)$ emphasize off diagonal terms means that games in these two cubes have similar characteristics, so it suffices to describe only cube $(3, B(TR))$, which, with Eq. 31 has $\beta_1 < 0$ and $\beta_2 > 0$. A transition value $\beta_2 > |\eta_{2,1}| + \frac{|\eta_{2,2}|-|\eta_{1,2}|}{2}$ (which agrees with Eq. 33) is where the sum of TR entries exceeds that of BR, which suggests using a cooperative bargaining solution.

3.3.2. *Square 2.* All games in this square have player two with a dominant R strategy. Because $\eta_{2,1} < 0$, in $\mathcal{G}_N$, player one plays B leading to a BR outcome.

Cube $(2, B(BR))$*:* The structure is similar to that of cube $(3, B(BR))$ where the behavioral term adds support to the Nash outcome.

Cube $(2, B(TL))$*:* With only minor differences from Eq. 31, the general form of a game in this cube (where $\beta_1, \beta_2 > 0$) is

$$\mathcal{G}^N + \mathcal{G}^B = \begin{array}{|cc|cc|}\hline |\eta_{1,1}| + \beta_1 & -|\eta_{1,2}| + \beta_2 & -|\eta_{2,1}| - \beta_1 & |\eta_{1,2}| + \beta_2 \\ \hline -|\eta_{1,1}| + \beta_1 & -|\eta_{2,2}| - \beta_2 & |\eta_{2,1}| - \beta_1 & |\eta_{2,2}| - \beta_2 \\ \hline \end{array}. \tag{35}$$

While this Eq. 35 parameterized family (where the $\mathcal{G}^B$ terms add appeal to TL in conflict with the Nash BR) does *not* include the Prisoner's Dilemma, the features are similar. The only difference from Eq. 32 is the transition value $\beta_1^* = \frac{|\eta_{2,1}|-|\eta_{1,1}|}{2}$, which means that cooperation becomes attractive to player one at a smaller β_1 value. Once both $\beta_j > \beta_j^*$, strategies such as tit-for-tat become applicable.

Cubes $(2B(TR))$, $(2, B(BL))$*:* The description is essentially the same as for cubes $(3, B(TR))$ and $(3, B(BL))$.

3.3.3. *Square 1.* All 2×2 games with two pure and one mixed strategy belong to this square. Thanks to the $\mathcal{G}^N$ symmetry, where the pure strategies are located at TL and BR, there are essentially two classes of games; they are given by if the $\mathcal{G}^B$ component supports, or conflicts with, the Nash solutions. An example, which also illustrates Prop. 1, is the following game of chicken with $\kappa_1 = \kappa_2 = -2$.

$$\mathcal{G}_{chick} = \begin{array}{|cc|cc|}\hline -9 & -9 & 5 & -3 \\ \hline -3 & 5 & -1 & -1 \\ \hline \end{array} = \begin{array}{|cc|cc|}\hline -3 & -3 & 3 & 3 \\ \hline 3 & 3 & -3 & -3 \\ \hline \end{array} + \begin{array}{|cc|cc|}\hline -4 & -4 & 4 & -4 \\ \hline -4 & 4 & 4 & 4 \\ \hline \end{array} + \mathcal{G}^K_{chick}.$$

Here $\mathcal{G}^N_{chick}$ is at the center of square 9. By interchanging a row, or a column, this game is identified with a game at the center point of square 1 (Prop. 1). While both games are symmetric (both are on the diagonal), the difference caused by this transfer is that the strategies of, say, where both players decide to evade now can be BL rather than BR. The main point made by this example is how the $\mathcal{G}^B_{chick}$ Pareto superior $B(BR)$ location conflicts with the Nash pure strategy structure; the $\mathcal{G}^B_{chick}$ term creates a distinction between the two $\mathcal{G}^N_{chick}$ pure strategies (making BR more appealing than TL because of the behavioral component) and makes another strategy appealing (which is what makes the game interesting) by adding importance to survival where both players evade – and live. Another interesting point is how this decomposition proves how different the game of chicken is from the PD.

In the same manner, many other examples can be created. But the analysis of how the β_j values can focus attention on certain matrix entries is similar to that described above with other settings, so we change the emphasis to describe other features that can be extracted from the two-parameter family of games.

Cubes $(1, B(TL))$ *and* $(1, B(BR))$*:* This is an "agreement setting" where $\mathcal{G}^B$ adds support to one of the pure strategies. Doing so can make one Nash outcome Pareto superior to the other. The "Stag Hunt" characterizes this behavior; a $\kappa_1 = \kappa_2 = 2$ example is

$$\mathcal{G}_{SH} = \begin{array}{|cc|cc|}\hline 4 & 4 & 0 & 2 \\ \hline 2 & 0 & 2 & 2 \\ \hline \end{array} = \mathcal{G}^N_9 + \begin{array}{|cc|cc|}\hline 1 & 1 & -1 & 1 \\ \hline 1 & -1 & -1 & -1 \\ \hline \end{array} + \mathcal{G}^K \tag{36}$$

The $\mathcal{G}^N_9$ symmetry does not distinguish between TR and BL. Therefore, TL is the $\mathcal{G}_{SH}$ Pareto superior choice strictly because of the $B(TL)$ choice of $\mathcal{G}^B$. Indeed, this $\mathcal{G}^B$ component plays a crucial role in achieving the Stag Hunt's defining features, which are:

1. There is a Pareto dominant Nash outcome.
2. But, there is a risk-free option that provides an incentive to deviate from the Pareto dominant choice.

A definition for a "risk-free strategy" is that by using it, the outcome remains the same independent of what the other player decides to do. If a game is risk-free in this sense, when player one plays B and player two plays R, it must be (with Eq. 10) that $g_{3,1} = g_{4,1}$ and $g_{2,2} = g_{4,2}$. These conditions become, respectively, $-\eta_{1,1} + \beta_1 = -\eta_{2,1} - \beta_1$ and $-\eta_{1,2} + \beta_2 = -\eta_{2,2} - \beta_2$. Thus, the risk-free condition requires

$$\beta_j = \frac{\eta_{1,j} - \eta_{2,j}}{2}, \quad j = 1, 2. \tag{37}$$

Because $\mathcal{G}^N$ is in square 1, $\eta_{1,j} > 0, \eta_{2,j} < 0$, so both β_j's are positive, so B(TL). These positive β_j values ensure that TL is the Pareto superior outcome, which ensures the Stag-Hunt structure. By including the $\mathcal{G}^K$ variables, it follows from Eq. 37 that the stag-hunt structure holds for a portion of a six-dimensional subspace of $\mathbb{R}^8$.

"Pure coordination games" are in these cubes; one choice, with the $B(TL)$ values of $\beta_1 = \beta_2 = 0.5$, along with $\kappa_1 = \kappa_2 = 1.5$, is

$$\mathcal{G}_{PC} = \begin{array}{|cc|cc|}\hline 4 & 4 & 0 & 0 \\ \hline 0 & 0 & 2 & 2 \\ \hline \end{array} = \begin{array}{|rr|rr|}\hline 2 & 2 & -1 & -2 \\ \hline -2 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|rr|rr|}\hline 0.5 & 0.5 & -0.5 & 0.5 \\ \hline 0.5 & -0.5 & -0.5 & -0.5 \\ \hline \end{array} + \mathcal{G}^K_{PC}.$$

To have equal non-equilibrium payoffs in the general case, the $\mathcal{G}^B$ values must be determined by the $\mathcal{G}^N$ values. This conditions requires $-\eta_{1,j} + \beta_j = \eta_{2,j} - \beta_j$, or that β_j equals the average $\frac{1}{2}(\eta_{1,j} + \eta_{2,j})$. This β_j dependency of $\eta_{i,j}$ means that one equilibrium is made Pareto superior to the other by the choice of $\mathcal{G}^N$; $\mathcal{G}^N$ is in either 1i or 1iii. The 1i corner of square 1 has $\eta_{1,j} > |\eta_{2,j}|$, $j = 1, 2$, so TL is the Pareto superior outcome, while $\mathcal{G}^N$ in 1iii with $|\eta_{2,j}| > \eta_{1,j}$ makes BR the Pareto superior choice. Component $\mathcal{G}^K$ allows assigning zero values to the non-equilibrium terms. The following captures the $\mathcal{G}^B$ and $\mathcal{G}^K$ values as determined by these conditions and the $\mathcal{G}^N$ choice:

$$\beta_j = \frac{1}{2}(\eta_{1,j} + \eta_{2,j}), \quad \kappa_j = \eta_{1,j} - \beta_j = \frac{1}{2}(\eta_{1,j} - \eta_{2,j}), \quad j = 1, 2. \tag{38}$$

Cubes $(1, B(TR))$ *and* $(1, B(BL))$: The "Battle of the Sexes" game resides in these cubes and has a description similar to the coordination game. The main exception is that each player prefers a different Nash outcome as illustrated in the following ($\kappa_1, \kappa_2 = 1.5$)

$$\begin{array}{|cc|cc|}\hline 4 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 4 \\ \hline \end{array} = \begin{array}{|rr|rr|}\hline 2 & 1 & -1 & -1 \\ \hline -2 & -2 & 1 & 2 \\ \hline \end{array} + \begin{array}{|rr|rr|}\hline 0.5 & -0.5 & -0.5 & -0.5 \\ \hline 0.5 & 0.5 & -0.5 & 0.5 \\ \hline \end{array} + \mathcal{G}^K,$$

which has a $B(BL)$ behavioral component. Also, the positioning of $\mathcal{G}^N$ in square 1 played a role. An analysis similar to that of Eq. 38 determines the β_j values needed to achieve this kind of setting.

Another feature of games in this cube is how the $\mathcal{G}^B$ component can direct attention to non-equilibrium solutions. To indicate the differences, to create a Stag Hunt game, add a $\mathcal{G}^B$ with $\beta_1 = \beta_2 = 1$ to $\mathcal{G}^N_9$ (which emphasizes one Nash choice over the other); this is the first game of Eq. 39.) But by adding a sufficiently large $B(TR)$ component of $\beta_1 = -10, \beta_2 = 10$ to $\mathcal{G}^N_9$, the resulting game is the second one in Eq. 39; the two games are Nash indistinguishable, but the second game attracts attention to TR.

$$\begin{array}{|rr|rr|}\hline 2 & 2 & -2 & 0 \\ \hline 0 & -2 & 0 & 0 \\ \hline \end{array}, \quad \begin{array}{|rr|rr|}\hline -9 & 11 & 9 & 9 \\ \hline -11 & -11 & 11 & -9 \\ \hline \end{array} \tag{39}$$

3.3.4. *Square 5.* All 2×2 games with a single mixed strategy, which include variants of "Matching Pennies," belong to square 5. As true with the above, a sufficiently strong $\mathcal{G}^B$ component added to $\mathcal{G}_{11}$ can direct attention to a particular $\mathcal{G}$ entry.

3.4. **A bargaining outcome.** As described above, for any choice of $\mathcal{G}^N$, it is possible to select $\mathcal{G}^B$ terms that will emphasize a matrix entry that differs from the Nash structure. Such settings, as with $\mathcal{G}_3$, suggest exploring the use of a cooperative solution. One possibility is the "coco" solution developed by Kalai and Kalai [4]. To review their approach, let $\mathcal{A}$ and $\mathcal{B}$ be, respectively, the first and second player's game matrix,. Decompose $\mathcal{G}$ into a "cooperative" ($\mathcal{G}_{coop}$) plus a "competitive" ($\mathcal{G}_{ZS}$) part as

$$\mathcal{G} = (\mathcal{A},\mathcal{B}) = (\frac{\mathcal{A}+\mathcal{B}}{2},\frac{\mathcal{A}+\mathcal{B}}{2}) + (\frac{\mathcal{A}-\mathcal{B}}{2},\frac{\mathcal{B}-\mathcal{A}}{2}) = \mathcal{G}_{coop}+\mathcal{G}_{ZS}. \tag{40}$$

The $\mathcal{G}_{coop}$ cooperative (first) matrix takes the sum of each $\mathcal{G}$ matrix entry and equally divides it among the players, while the competitive $\mathcal{G}_{ZS}$ matrix is a zero sum game where each player's term in $\mathcal{G}$ is replaced by its difference from the average of the entries. Illustrating with $\mathcal{G}_4$ (Eq. 9),

$$\mathcal{G}_4 = \begin{array}{|cc|cc|}\hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline\end{array} = \begin{array}{|cc|cc|}\hline 6 & 6 & 2 & 2 \\ \hline 0 & 0 & 1 & 1 \\ \hline\end{array} + \begin{array}{|cc|cc|}\hline 0 & 0 & -2 & 2 \\ \hline 4 & -4 & 1 & -1 \\ \hline\end{array} \tag{41}$$

Presumably, $\mathcal{G}_{ZS}$ captures a sense of what the players could achieve in a Nash competitive setting. To capture this competitive aspect, the coco solution adds the minmax values from $\mathcal{G}_{ZS}$ to the largest value from $\mathcal{G}_{coop}$, call it $\frac{S}{2}$. With Eq. 41 where $\frac{S}{2}=6$, players one and two receive, respectively, $6+1=7$ and $6-1=5$.

Kalai and Kalai want to encourage a discussion of cooperative solutions. To do so in terms of our decomposition, a possible objection and a strength of the coco solution are described. A possible objection is that the competitive $\mathcal{G}_{ZS}$ structure does not match $\mathcal{G}$'s actual $\mathcal{G}^N$ competitive structure. After all, any game can be divided as in Eq. 40, but only half of the 2×2 Nash structures accompany zero-sum games (Cor. 1). This suggests that $\mathcal{G}_{ZS}$ distorts the $\mathcal{G}^N$ structure.

Indeed, $\mathcal{G}_4^N$ is represented by a point in square 1, but the Nash structure of $\mathcal{G}_{4,ZS}$ (or $\mathcal{G}_{4,ZS}^N$) is in square 4! Thus the actual $\mathcal{G}_4^N$ structure has two pure and one mixed strategy, while $\mathcal{G}_{4,ZS}^N$ endows player one with a dominant strategy. A reason for these differences is that each player's $\mathcal{G}_{ZS}^N$ terms are determined, in part, by the *other player's* $\mathcal{G}$ strategic and behavioral terms.

Theorem 4. *For game $\mathcal{G}$ and its Eqs. 14, 15, 16 decomposition, the $\mathcal{G}_{ZS}^N$ decomposition is defined by*

$$\begin{array}{ll} \eta^*_{1,1} = \frac{\eta_{1,1}}{2} - \frac{1}{4}\{\eta_{1,2}-\eta_{2,2}+2\beta_2\}, & \eta^*_{2,1} = \frac{\eta_{2,1}}{2} - \frac{1}{4}\{\eta_{2,2}-\eta_{1,2}+2\beta_2\}, \\ \eta^*_{1,2} = \frac{\eta_{1,2}}{2} - \frac{1}{4}\{\eta_{1,1}-\eta_{2,1}+2\beta_1\}, & \eta^*_{2,2} = \frac{\eta_{2,2}}{2} - \frac{1}{4}\{\eta_{2,1}-\eta_{1,1}+2\beta_1\} \end{array} \tag{42}$$

By setting $\eta^*_{i,j}=\eta_{i,j}$ and solving, it follows that for any $\mathcal{G}$, each agent's strategic structure as given by $\mathcal{G}_{ZS}$ depends on the other agent's strategic and behavioral terms from $\mathcal{G}$. A natural goal is to create a solution concept where each person's strategic structure does not involve the other agent's strategic terms. Surprisingly, in one direction, this includes coco.

This approach requires using information for each agent from $\mathcal{G}^N$ and $\mathcal{G}^B$. One approach is to adopt the $\mathcal{G}^N$ minmax entry. If it is $(\eta_{i,1}, \eta_{j,2})$, then the $\eta_{i,1} - \eta_{j,2}$ difference can determine a difference in the bargaining outcome; i.e., players one and two receive, respectively, $\frac{S}{2} + \frac{\eta_{i,1}-\eta_{j,2}}{2}$ and $\frac{S}{2} + \frac{\eta_{j,2}-\eta_{i,1}}{2}$. A related approach (that can yield different outcomes) is to start with these differences; replace each agent's $\mathcal{G}^N$ entry with how it differs from the average; e.g., the entry in the i^{th} row and j^{th} column is $(\frac{\eta_{i,1}-\eta_{j,2}}{2}, \frac{\eta_{j,2}-\eta_{i,1}}{2})$. The minmax entry of the resulting matrix determines what to add to $(\frac{S}{2}, \frac{S}{2})$. With $\mathcal{G}_4$, the first and second players receive, respectively, $6 - 0.5 = 5.5$ and $6 + 0.5 = 6.5$, which differs from the coco outcome. (With more players, replace each player's $\mathcal{G}^N$ entry by how it differs from the average.)

Further embellishments of either type are to involve $\mathcal{G}^B$ and/or $\mathcal{G}^K$ entries. Using the differences between all of these terms leads to a matrix where the first agent's TL entry is $\frac{1}{2}[(\eta_{1,1} - \eta_{1,2}) + (\beta_1 - \beta_2) + (\kappa_1 - \kappa_2)]$. Applying minmax to this matrix recovers the coco solution. Namely, rather than capturing $\mathcal{G}$'s competitive Nash structure, the purpose of $\mathcal{G}_{ZS}$ is to capture *differences* between the players' Nash, behavioral, and kernel entries.

4. Analyzing QRE structures

We now answer the Sect. 1.1 questions raised about QRE where we wondered whether a "rationality" parameter $\lambda > 0$ value exists so that $\mathcal{G}_4 \sim_{QRE,\lambda} \mathcal{G}_5$, or, more dramatically, so that $\mathcal{G}_1 \sim_{QRE,\lambda} \mathcal{G}_2$. But first, a brief review of QRE is required.

The QRE transforms the expected payoffs of a strategy choice into positive weights, where the probability that a strategy is chosen is designed to be proportional to its weight relative to the weights of all of the other strategic choices. If, for example, the second player's given mixed-strategy choice is q and the resulting expected payoff for the first player choosing strategy s_1 is $\pi_1(q)$, then the weight assigned to s_1 is $w_1 = \exp\{\lambda\pi_1(q)\}$; λ is the QRE parameter. The probability of choosing s_1 is $p_1(\lambda) = \frac{w_1}{w_1+\cdots+w_n}$.

In terms of our Nash decomposition, if the second player is playing the mixed-strategy q, then the expected payoffs for player one of choosing the top or bottom row are

$$
\begin{aligned}
EV(T) &= q(\eta_{1,1} + \beta_1) + (1-q)(\eta_{2,1} - \beta_1) = q\eta_{1,1} + (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1, \\
EV(B) &= q(-\eta_{1,1} + \beta_1) + (1-q)(-\eta_{2,1} - \beta_1) = -q\eta_{1,1} - (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1.
\end{aligned}
$$

The corresponding weights are

$$
\begin{aligned}
w_T &= \exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1]\}, \\
w_B &= \exp\{\lambda[-q\eta_{1,1} - (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1]\},
\end{aligned}
$$

so the corresponding choice probabilities are functions of λ. The choice for T is $p(\lambda) = \frac{w(T)}{w(T)+w(B)}$.

The λ dependency makes it reasonable to expect that λ will strongly influence the structure of the $\sim_{QRE,\lambda}$ equivalence classes. With $\lambda = 0$, for instance, $p(0) = 0.5$. This expectation is what makes the following theorem surprising from a game theoretic perspective.

Theorem 5. *For any $\lambda > 0$, two games $\mathcal{G}_i$ and $\mathcal{G}_j$ are Nash equivalent, $\mathcal{G}_i \sim_N \mathcal{G}_j$ if and only if they are QRE equivalent, $\mathcal{G}_i \sim_{QRE,\lambda} \mathcal{Q}_j$.*

In other words, for any $\lambda > 0$, the highly predictable $\mathcal{G}_1$ and the Prisoner's Dilemma $\mathcal{G}_2$ from Eq. 1 satisfy $\mathcal{G}_1 \sim_{QRE,\lambda} \mathcal{G}_2$! This means that from the perspective of QRE, these two games are indistinguishable! Theorem 5 means that only the $\mathcal{G}^N$ structure is used to determine QRE probabilities. Although the $\mathcal{G}^B$ and $\mathcal{G}^K$ terms can significantly alter how players react to a game (as true with $\mathcal{G}_1$ and $\mathcal{G}_2$), they play *no role in a QRE analysis.*

While Thm. 5 may be surprising from a game theoretic standpoint, it is an expected conclusion from an algebraic perspective. Similar to how addition of reals can be identified with multiplication through the isomorphism $y = e^{\lambda x}$ for a fixed $\lambda > 0$, the algebraic matrix structure of Nash and QRE remain essentially the same. Indeed, this algebraic equivalence is essentially the proof of the result.

5. The $2 \times 2 \times \cdots \times 2$ games

While it takes only minor modifications to extend the approach to all $2 \times 2 \times \cdots \times 2$ games, the analysis of the possible structures (as done for 2×2 games in Sects. 3.2, 3.3) is more complicated, so this will appear elsewhere. To have a manageable exposition, the space of three-player $2 \times 2 \times 2$ games is used to illustrate the construction.

With three players, there are eight possible strategies created by combinations of the first, second, and third players strategies of, respectively, (T,B), (L, R), and (F=front, Ba=Back). An example of a three player game is in Eq. 43. For $n \geq 2$ players, there are 2^n possible strategies. In the three player game "matrix," each entry specifies an outcome for each player, so three values are specified. This means that $3 \times 2^3 = 24$ values are specified. With n players, $n2^n$ values are required.

$$(43) \qquad \mathcal{G}_{12} \quad \text{Front } = \begin{array}{|ccc|ccc|}\hline 6 & 7 & -2 & 2 & 5 & 5 \\ \hline 4 & 3 & 9 & 4 & 5 & 2 \\ \hline \end{array}, \quad \text{Back } = \begin{array}{|ccc|ccc|}\hline 6 & 7 & 0 & -8 & 9 & 7 \\ \hline 8 & -7 & 7 & -6 & -5 & 4 \\ \hline \end{array}$$

5.1. **Space of $\mathcal{G}^N$ games.** If the second and third players use, respectively, $(q, 1-q)$ and $(r, 1-r)$ mixed strategies, the first player's expected payoff for playing T and B can be computed. The difference between these two expected values is a quadratic equation with variables $qr, q(1-r), (1-q)r, (1-q)(1-r)$. The coefficient for each quadratic term is determined by the choice made by each of the two other players. For instance, the qr variable is where player two plays L and player three plays F. This choice identifies the appropriate T and B cells in the first player's game matrix. The qr coefficient is the difference between player one's T and B entries in these cells. As true with Eq. 6, the actual values of player one's entries do not matter; the relevant value is their difference. So, replace each of the two entries by how it differs from the average of the two terms.

More generally, for player j in an n-player game $\mathcal{G}$, select a strategy for each of the other $n-1$ players; denote the combined strategies by σ_{-j}. This σ_{-j} identifies two matrix entries for the j^{th} player; $g_{(i,\sigma_{-j},j)}, i = 1, 2$. Replace each $g_{(i,\sigma_{-j},j)}$ with how it differs from the average $\frac{1}{2}[g_{(1,\sigma_{-j},j)} + g_{(2,\sigma_{-j},j)}]$ to define the $\eta_{(\sigma_{-j},j)} = \frac{1}{2}[g_{(1,\sigma_{-j},j)} - g_{(2,\sigma_{-j},j)}]$ entry that replaces $g_{(1,\sigma_{-j},j)}$ value; the $g_{(2,\sigma_{-j},j)}$ value is replaced by $-\eta_{(\sigma_{-j},j)}$. Doing so for each player and each possible choice of strategies for the other players defines the $\mathcal{G}^N$ component for game $\mathcal{G}$. The is illustrated with the $\mathcal{G}^N_{12}$ matrix given in Eq. 44 where the general form (Eq. 45) displays the three dimensions of the Nash terms. Namely, the η values for player one are in the vertical direction, for player two are in the horizontal direction, and for player three are in the front-back direction.

$$\mathcal{G}^N_{12} \quad \text{Front} = \begin{array}{|ccc|ccc|}\hline 1 & 1 & -1 & -1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 & 1 & -1 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|ccc|ccc|}\hline -1 & -1 & 1 & -1 & 1 & 1 \\ \hline 1 & -1 & -1 & 1 & 1 & 1 \\ \hline \end{array} \tag{44}$$

$$\begin{array}{ll} \mathcal{G}^N \quad \text{Front} = & \begin{array}{|ccc|ccc|}\hline \eta_{(LF,1)} & \eta_{(TF,2)} & \eta_{(TL,3)} & \eta_{(RF,1)} & -\eta_{(TF,2)} & \eta_{(TR,3)} \\ \hline -\eta_{(LF,1)} & \eta_{(BF,2)} & \eta_{(BL,3)} & -\eta_{(RF,1)} & -\eta_{(BF,2)} & \eta_{(BR,3)} \\ \hline \end{array}, \\ \text{Back} = & \begin{array}{|ccc|ccc|}\hline \eta_{(LBa,1)} & \eta_{(TBa,2)} & -\eta_{(TL,3)} & \eta_{(RBa,1)} & -\eta_{(TBa,2)} & -\eta_{(TR,3)} \\ \hline -\eta_{(LBa,1)} & \eta_{(BBa,2)} & -\eta_{(BL,3)} & -\eta_{(RBa,1)} & -\eta_{(BBa,2)} & -\eta_{(BR,3)} \\ \hline \end{array} \end{array} \tag{45}$$

The difference between the expected values for the two strategies for each of the three $\mathcal{G}_{12}$ players now follows from Eq. 44.

$$\begin{array}{ll} \text{Player 1:} & rq - r(1-q) - q(1-r) - (1-q)(1-r) \\ \text{Player 2:} & rp - r(1-p) - p(1-r) - (1-p)(1-r) \\ \text{Player 3:} & -pq - p(1-q) + q(1-p) - (1-p)(1-q). \end{array} \tag{46}$$

The Nash pure strategy is BRBa with the payoff of $-6, -5, 4$. Because the BLF payoff of $4, 3, 9$ from Eq. 43 is preferred by each of the three players, it is clear that this game has an influential $\mathcal{G}^B_{12}$ component, which is computed in Sect. 5.2.

With the 2^{n-1} pure strategies that the $(n-1)$ other players can adopt, the difference between the expected values of the j^{th} player's two strategies is determined by 2^{n-1} of the $\eta_{(\sigma_{-j},j)}$ coefficients; it defines a vector $\boldsymbol{\eta}_j \in \mathbb{R}^{2^{n-1}}$. Illustrating with Eq. 45, $\boldsymbol{\eta}_1 = (\eta_{(LF,1)}, \eta_{(RF,1)}, \eta_{(LBa,1)}, \eta_{(RBa,1)}) \in \mathbb{R}^4$. The best response analysis yields the same values even if each $\boldsymbol{\eta}_j$ is multiplied by a scalar. Thus $\boldsymbol{\eta}_j$ can be assumed to have length unity; e.g., it can be represented by a point in the sphere $\boldsymbol{\eta}_j \in S^{(2^{n-1}-1)}$. As this is true for each agent, it reduces the analysis by n dimensions; that is, $\mathcal{G}^N$ can be represented by a point in the compact $n(2^{n-1}-1)$ dimensional product space $\prod S^{(2^{n-1}-1)}$.

In the analysis for three player games, $\boldsymbol{\eta}_j$ can be in eight different octants, so there are $8^3 = 512$ different regions. (In general, $\boldsymbol{\eta}_j$ can be in 2^n different octants, so there are $(2^n)^n$ different Nash regions.) However, as true in Sect. 3.1, this number can be significantly reduced. Also, much of what happens can be determined from the above discussion of Fig. 2a. For instance, if a player in a three-player game has a dominant strategy, the other two assume one of the above described two-player structures. Illustrating with $\mathcal{G}_{12}$, while Eqs. 44 do not admit mixed strategy solutions, and while no player has a dominant strategy, if player three plays F, it follows from Eq. 44 that the first two players face a square-one game with two pure and one mixed strategies; whatever is done gives player three the -1

outcome. If player three plays Back, the first two players have a square-three game with the dominant strategy of BR; this choice gives the third player a "+1" outcome. But, as noted, $\mathcal{G}_{12}$ has outcomes that are Pareto preferred to this choice. The source of this different, the $\mathcal{G}^B_{12}$ is determined next.

5.2. **Spaces of $\mathcal{G}^B$ and $\mathcal{G}^K$ games.** The $\mathcal{G}^K$ matrix as in the 2×2 case; it is determined by the κ_j terms. Illustrating with $\mathcal{G}_{12}$, because $\kappa_1 = \frac{6+4+2+4+6-8-6}{8} = 2, \kappa_2 = 3$, and $\kappa_3 = 4$, the $\mathcal{G}^K$ matrix has the common $(2,3,4)$ entry for each of the 24 entries in its $2\times 2\times 2$ matrix.

To define $\mathcal{G}^B = \mathcal{G} - \mathcal{G}^N - \mathcal{G}^K$, replace each entry in each cell of the $\mathcal{G}-\mathcal{G}^N$ matrix by how it differs from κ_j. To illustrate with the Eq. 43 example, the $\mathcal{G}^B_{12}$ matrix is:

(47)

Front =

$\beta_{(LF,1)} = 3$	$\beta_{(TF,2)} = 3$	$\beta_{(TL,3)} = -5$	$\beta_{(RF,1)} = 1$	$\beta_{(TF,2)} = 3$	$\beta_{(TR,3)} = 2$
$\beta_{(LF,1)} = 3$	$\beta_{(BF,2)} = 1$	$\beta_{(BL,3)} = 4$	$\beta_{(RF,1)} = 1$	$\beta_{(BF,2)} = 1$	$\beta_{(BR,3)} = -1$

Back =

$\beta_{(LBa,1)} = 5$	$\beta_{(TBa,2)} = 5$	$\beta_{(TL,3)} = -5$	$\beta_{(RBa,1)} = -9$	$\beta_{(TBa,2)} = 5$	$\beta_{(TR,3)} = 2$
$\beta_{(LBa,1)} = 5$	$\beta_{(BBa,2)} = -9$	$\beta_{(BL,3)} = 4$	$\beta_{(RBa,1)} = -9$	$\beta_{(BBa,2)} = -9$	$\beta_{(BR,3)} = -1$

The extreme BRBa values in $\mathcal{G}^B_{12}$ are what direct attention away from the Nash solution.

Notice that $\beta_{(LF,1)}+\beta_{(LBa,1)}+\beta_{(RF,1)}+\beta_{(RBa,1)} = 0$. This is follows from the construction; the average of the j^{th} player's terms in $\mathcal{G}-\mathcal{G}^N$ equals κ_j, which is the average of the player's terms in $\mathcal{G}$. For this reason it always is true that

$$\sum \beta_{(i,\sigma_j,j)} = 0 \tag{48}$$

where the summation is over all choices of σ_j. Thus, the reduction of $\mathcal{G}$ to the $\mathcal{G}^N$ and $\mathcal{G}^B$ components drops $2n$ dimensions. An extra reduction, for a total of $2n+1$ dimensions, comes from scaling the collection of all β terms to have length unity.

An interesting feature is how the $\mathcal{G}^B$ structure becomes more complex once $n\geq 3$. With $n=2$ and non-zero β_j's, there is a unique $\mathcal{G}^B$ cell with a positive entry for each player. This Pareto dominant $\mathcal{G}^B$ term is diametrically opposite the Pareto inferior cell that has a negative entry for each player. But this convenient structure does not extend to $n\geq 3$. To see why, in Eq. 47, let $\beta_{(LF,1)} = 3$, and $\beta_{RF,1)} = \beta_{(LBa,1)} = \beta_{(R,Ba,1)} = -1$ to satisfy Eq. 48. Do the same for the second agent, except let $\beta_{(BBa,2)} = 3$, and set the other three terms equal to -1. Here, all eight cells have at least one negative entry. By assigning appropriate values for the third agent's β values, it can be that no cell has all positive, and no cell has all negative entries. These features suggest the added complexity that can be introduced by $\mathcal{G}^B$ with $n\geq 3$.

6. Summary

A purpose of this paper is to introduce a unique decomposition of any $2\times 2\times\cdots\times 2$ game. This decomposition simplifies the analysis by uniquely and completely separating a game's Nash equilibria information from the behavioral content that can influence or promote non-strategic kinds of analysis. As such, this decomposition not only reduces the complexity of the space of games (e.g., the $\mathbb{R}^8$ space of 2×2 games can be reduced to points in a three dimensional cube), but it also provides a tool that can be used in a variety of ways and settings. As illustrated by using symmetric and zero-sum games, it can be used to identify all possible Nash structures that can accompany a particular class of games. As demonstrated by considering two of the strategies that can be used against tit-for-tat, it provides insights how the behavioral terms influence the strategy and cooperation. This unified framework also can be used to identify, and then analyze algebraically equivalent games, which is illustrated here by showing that all QRE games have the same strategic structure of the Nash analysis.

Examining games in terms of this decomposition not only identifies the similarities between many different types of applications of game theory, but it also provides a new language with which to discuss games. Rather than focusing on a single game, the decomposition advances a simplified way to expand an analysis in terms of examining the entire space of games. This not only permits obtaining more general conclusions about classes of games, but it permits doing so in a way that simplifies the discussion and reveals essential features that may not have been previously known.

7. Proofs

Proof of Thm. 1: An advantage of knowing what will be the results in advance (by using representation theory) is that a much simpler proof can be constructed. To do so, start with the properties that a 2×2 game must have to ensure that it does *not* contain any Nash equilibria information. To explain what this means, by using the usual fixed point theorem approach to find Nash equilibria, this setting is defined by the degenerate case of games where all possible strategies are fixed points.

As it is easy to compute, this degenerate setting occurs if and only if each of the two rows for the row player, and each of the two columns for the column player, are identical. To illustrate, with this condition, the first player's expected values for playing T and B are identical no matter what strategy is adopted by the second player; for the second player, the expected values of playing R and L are identical no matter what strategy is adopted by the first player. If a game does not satisfy this degenerate condition, this assertion about the equality of expected values no longer holds.

Using the Sect. 2.1 notation, to satisfy this "degenerate game" condition for the first player, the game must be in the span of the two vectors $\mathbf{B}_{1,1} = (1,0,1,0;0,0,0,0)$ and $\mathbf{B}_{2,1} = (0,1,0,1;0,0,0,0)$; the first vector ensures that $g_{1,1} = g_{3,1}$ while the second ensures that $g_{2,1} = g_{4,1}$. Similarly, the space for the second player is spanned by $\mathbf{B}_{1,2} = (0,0,,0,0;1,1,0,0)$ and $\mathbf{B}_{2,2} = (0,0,0,0;0,0,1,1)$.

This means that all Nash information for the first player is in the orthogonal complement of the space spanned by $(\mathbf{B}_{1,1}, \mathbf{B}_{2,1})$. A direct computation proves that this two-dimensional space is spanned by $(\mathbf{n}_{1,1}, \mathbf{n}_{2,1})$. A similar description holds for the second player, which means that all Nash information for this player is in the space spanned by $(\mathbf{n}_{1,2}, \mathbf{n}_{2,2})$.

Finally, to provide a sharper discrimination of available information from the spaces spanned by $(\mathbf{B}_{1,j}, \mathbf{B}_{2,j})$, remove all common information. This common information is along the diagonal $\mathbf{k}_j = \mathbf{B}_{1,j} + \mathbf{B}_{2,j}$. As $\mathbf{b}_j$ defines the orthogonal direction to $\mathbf{k}_j$ in this space, the basis for this behavioral space becomes $(\mathbf{b}_j, \mathbf{k}_j)$. This completes the proof.

This proof clearly extends to all games with any finite number of players where each player can have any specified finite number of pure strategies. Of particular importance for Sect. 5, the proof extends to $2 \times 2 \times \ldots 2$ games. $\square$

Proof of Thm. 4: This is a direct computation. To carry it out for $\eta^*_{1,1}$, first find $\mathcal{G}_{ZS}$ for a given $\mathcal{G}$. The first player's TL entry is $\eta_{1,1}+\beta_1-\frac{1}{2}[\eta_{1,1}+\beta_1+\eta_{1,2}+\beta_2] = \frac{1}{2}[\eta_{1,1}+\beta_1-\eta_{1,2}-\beta_2]$. Similarly, this player's BL entry for $\mathcal{G}_{ZS}$ is $\frac{1}{2}[-\eta_{1,1}+\beta_1-\eta_{2,2}+\beta_2]$. According to Eq. 12, $\eta^*_{1,1}$ is half of the difference between these two terms, which is the $\eta^*_{1,1}$ value in Eq. 42. The other terms are found in the same manner. $\square$

Proof of Thm. 5: An abstract proof follows by showing that with a fixed $\lambda > 0$, there is an algebraic equivalence between the Nash and QRE settings. A simple computation, which does the same by showing that strategic algebraic structure of $\mathcal{G}$ is identical whether the solution concept is Nash or QRE with $\lambda > 0$, follows. The cancellation of the $\mathcal{G}^B$ terms becomes obvious from the following computation:

$$\begin{aligned}
p(\lambda) &= \frac{w_T}{w_T + w_B} \\
&= \frac{\exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1]\}}{\exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1]\} + \exp\{\lambda[-q\eta_{1,1} - (1-q)\eta_{2,1} + q\beta_1 - (1-q)\beta_1]\}} \\
&= \frac{\exp\{\lambda(q\beta_1 - (1-q)\beta_1)\}}{\exp\{\lambda(q\beta_1 - (1-q)\beta_1)\}} \left(\frac{\exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1}]\}}{\exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1}]\} + \exp\{\lambda[-q\eta_{1,1} - (1-q)\eta_{2,1}]\}} \right) \\
&= \frac{\exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1}]\}}{\exp\{\lambda[q\eta_{1,1} + (1-q)\eta_{2,1}]\} + \exp\{\lambda[-q\eta_{1,1} - (1-q)\eta_{2,1}]\}}
\end{aligned}$$

This expression shows that the relevant QRE game structure is the same as for the Nash equilibrium: any two games sharing $\eta_{i,j}$ values are the same for all $\lambda > 0$ values.

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