CONNECTING PAIRWISE AND POSITIONAL ELECTION OUTCOMES

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Abstract. General conclusions relating pairwise tallies with positional (e.g., plurality, antiplurality ("vote-for-two")) election outcomes were previously known only for the Borda Count. While it has been known since the eighteenth century that the Borda and Condorcet winners need not agree, it had not been known, for instance, in which settings the Condorcet and plurality winners can disagree, or must agree. Results of this type are developed here for all three-alternative positional rules. These relationships are based on an easily used method that connects pairwise tallies with admissible positional outcomes; e.g., a special case provides the first necessary and sufficient conditions ensuring that the Condorcet winner is the plurality winner; another case identifies when a profile must exist whereby *each candidate* is the "winner" with a specific positional rule.

1. Introduction

After a quarter of a millennium of study, it is clear that the objective of determining which voting method most accurately reflects the views of the voters is a surprisingly subtle, major challenge. The complexity of this issue has forced researchers to adopt secondary measures, such as seeking properties of specific rules or probability estimates of paradoxical events. While providing useful information, these approaches remain surrogates for the true intent of identifying which profiles cause different kinds of election outcomes. Rather than determining the likelihood of particular paradoxical outcomes, for instance, a preferred outcome would be to identify all profiles that cause these difficulties.

To advance our understanding of which profiles create various conclusions, the approach introduced here identifies all three-alternative profiles that support specified paired majority vote tallies. An advantage of knowing all possible supporting profiles is that it now is possible to determine all of the associated positional outcomes.

To illustrate the variety of new questions that can be answered, suppose all we know about a profile is that its majority vote pairwise comparisons are

$$A \text{ beats } B \text{ by } 70{:}30,\ A \text{ beats } C \text{ by } 60{:}40, \text{ and } B \text{ beats } C \text{ by } 55{:}45.$$

Here A is the Condorcet winner (she beats all other candidates) and C is the Condorcet loser (she loses to everyone). Just from these tallies, where the two involving the Condorcet winner A are of "landslide proportions" (winning 60% or more of the vote), the goal

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is to determine all admissible plurality (vote-for-one), antiplurality (a "vote-for-two" is equivalent to a "vote against one"), Borda (assign two and one points, respectively, to a ballot's first and second positioned candidate) and other positional outcomes. Could middle-ranked B win a plurality election? Even though Condorcet winner A badly defeats the Condorcet loser C, are there profiles with these majority votes where C, rather than A, is the plurality winner? Could C be an antiplurality winner? Do the Borda and Condorcet winners agree or differ? (Complete answers are in Sect. 4.2.)

The easily used approach developed here connects majority votes with positional outcomes, so this method becomes a central tool to answer all such questions. As our intent to develop relationships between positional methods and pairwise votes, only sincere voting is considered. (Proofs are in Sect. 5, but the basic ideas are developed in Sects. 3.1, 3.2.)

1.1. **Basic definitions.** A profile lists each voter's ranking of the alternatives where it is assumed that each voter has complete, strict (no ties), transitive preferences. As for outcomes, assume in an $\{A, B\}$ majority vote that A always has at least as many votes as B, and, in a $\{B, C\}$ majority vote, B has at least as many as C. With this assumption and by denoting a strict preference by "$\succ$," a strict transitive outcome of these paired comparisons is $A \succ B \succ C$. So, if there is a Condorcet winner in what follows, it always is A, and C always is the Condorcet loser. If the rankings define a cycle, it has the $A \succ B, B \succ C, C \succ A$ form. As a "name change" converts any other situation into our setting, this assumption does not affect the generality of our conclusions.

Rather than using the actual tallies, the differences between majority vote tallies turns out to be a more useful way to analyze these issues.

Definition 1. *For a $\{X, Y\}$ majority vote election with n voters, let*

$$P(X, Y) = \{X's \text{ majority vote}\} - \{Y's \text{ majority vote}\}. \tag{1}$$

Illustrating with the introductory example, $P(A, B) = 70 - 30 = 40$, $P(A, C) = 20$, and $P(B, C) = 10$; e.g., the larger the $P(X, Y)$ value, the better X does against Y. Also, $P(X, Y) = -P(Y, X)$; e.g., in the introductory example, $P(B, A) = 30 - 70 = -40$. This notation converts our $A \succeq B, B \succeq C$ assumption about the paired elections (where "$\succeq$" has the obvious "preferred or indifferent to" meaning) into the equivalent $P(A, B) \geq 0$, $P(B, C) \geq 0$ condition.

With n voters, because $n = \{X's \text{ vote}\} + \{Y's \text{ vote}\}$, it follows that

$$\{X's \text{ vote}\} = \frac{1}{2}[n + P(X, Y)]. \tag{2}$$

So, with $n = 60$ voters, a $P(A, B) = 10$ outcome means that A received $\frac{1}{2}[60 + P(A, B)] = 35$ votes while B received $\frac{1}{2}[60 + P(B, A)] = \frac{1}{2}[60 - 10] = 25$ votes. Because X's tally is an integer, it follows from Eq. 2 that n and all three $P(X, Y)$'s must have the same parity; i.e., either all are odd integers, or all are even integers. This parity agreement is used throughout the paper.

Different profiles can yield the same pairwise tallies, so the following definition is introduced to collect all of them into one class.

Definition 2. *Two profiles* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *are "binary equivalent" (*$\mathbf{p}_1 \sim_{BE} \mathbf{p}_2$*) if they have identical* $P(A,B), P(B,C), P(A,C)$ *values.*

The $\sim_{BE}$ connection is an equivalence relationship. (That is, for each $\mathbf{p}_i$, $\mathbf{p}_i \sim_{BE} \mathbf{p}_i$; if $\mathbf{p}_1 \sim_{BE} \mathbf{p}_2$, then $\mathbf{p}_2 \sim_{BE} \mathbf{p}_1$; and, finally, if $\mathbf{p}_1 \sim_{BE} \mathbf{p}_2$ and $\mathbf{p}_2 \sim_{BE} \mathbf{p}_3$, then it must be that $\mathbf{p}_1 \sim_{BE} \mathbf{p}_3$.) Thus $\sim_{BE}$ partitions the space of profiles into equivalence classes; each class consists of all profiles with the same $P(X,Y)$ values, $X, Y = A, B, C$. To be useful, a trait must be found to identify which profiles belong to a particular $\sim_{BE}$ class. As described in Thm. 5, this is the profile's "essential part" – the unique portion of a profile that determines the $P(X,Y)$ values.

Answers for the above questions are found by characterizing how profiles in a $\sim_{BE}$ class differ. As a preview of what will be discovered, although profiles from the same class have identical $P(X,Y)$ values, their plurality and other positional rankings can differ.

Definition 3. *A* positional voting rule *tallies a ballot by assigning specific points to the three alternatives according to how they are positioned on a ballot. Here,* w_j *points are assigned to the* j^{th} *positioned choice. The conditions are that not all* w_j *values are equal, and* $w_1 \geq w_2 \geq w_3 = 0$*. The* normalized positional rule *is obtained by dividing all* w_j *values by* w_1 *to create* $\mathbf{w}_s = (1, s, 0)$ *where* $s = \frac{w_2}{w_1}$ *represents the number of second ranked points.*

The plurality vote is $\mathbf{w}_0 = (1,0,0)$, the anti-plurality vote is $\mathbf{w}_1 = (1,1,0)$, and the Borda Count (normally defined by $(2,1,0)$) has the $\mathbf{w}_{\frac{1}{2}} = (1, \frac{1}{2}, 0)$ normalized form. Although we discuss only positional methods, all results extend immediately to other rules, such as Approval Voting and Cumulative Voting, by using the techniques developed in (Saari [7]).

A technical assumption, which is needed to separate the different cases, follows:

Definition 4. *A profile satisfies the* strongly non-cyclic *condition if*

$$P(A,C) \geq \min(P(A,B), P(B,C)). \tag{3}$$

To explain Def. 4, if the pairwise rankings define a transitive $A \succ B \succ C$ outcome, the strongly non-cyclic condition (Eq. 3) just means that Condorcet winner A's victory over the Condorcet loser C is at least as decisive as A's victory over B, or B's victory over C. With the $P(A,B), P(B,C) \geq 0$ assumption, Eq. 3 requires $P(A,C) \geq 0$, so it precludes cycles; this leads to its "non-cyclic" name. The "strongly" modifier refers to the fact that even if weaker $P(A,C)$ values fail Eq. 3, they do not define cycles if $P(A,C) \geq 0$. A $n = 100$ example is where $P(A,B) = 20$, $P(B,C) = 14$ and $P(A,C) = 10$. (To create all profiles with these pairwise values, see Thm. 5 and Sect. 3.)

1.2. **Sample of outcomes.** Our approach decomposes a profile into the part that determines the $P(X,Y)$ paired comparison values – the portion essential for a profile to be in the $\sim_{BE}$ equivalence class – and the part that affects only positional outcomes. This decomposition simplifies discovering and proving new conclusions. Samples of the kinds of results that can be found are given in the following four theorems. (Proofs are in Sect. 5, but intuition and partial proofs are given in Sects. 3.1, 3.2.)

To introduce the first theorem, it is reasonable to wonder whether the plurality and Condorcet winners agree. The first two parts of Thm. 1 ensure that this always is true for at least one profile supporting the pairwise tallies. But the next two parts show that the pairwise tallies must satisfy exacting conditions for the Condorcet winner A to be the plurality winner for *all supporting profiles.*

Theorem 1. *With the strongly non-cyclic condition (Eq. 3), the following are true:*

(1) *A profile supporting the paired outcomes always exists where the Condorcet winner A is a plurality winner. She may, however, be tied with another candidate.*
(2) *If $P(A,B), P(A,C) > 0$, there is at least one profile supporting the paired tallies where the Condorcet winner A is the sole plurality winner.*
(3) *If with alternative Y, $P(A,Y) > 0$ is the largest pairwise victory, then a necessary and sufficient condition for all supporting profiles to have the Condorcet winner A as the sole plurality winner (where X is the third alternative) is*

$$2P(A,X) + P(A,Y) > n. \tag{4}$$

If Eq. 4 is an equality, there are profiles where A is tied with X. If the Eq. 4 inequality is reversed, then some profiles have X as the sole plurality winner.
(4) *For $P(B,C) > \max(P(A,B), P(A,C))$, a necessary and sufficient condition for all profiles to have A as the sole plurality winner is*

$$2P(A,B) + P(B,C) > n. \tag{5}$$

Statement 1 is required for completeness. The ten-voter profile where five voters prefer $A \succ B \succ C$ and five prefer $C \succ A \succ B$, for instance, has $P(A,B) = 10$ and $P(A,C) = P(B,C) = 0$, so it satisfies Def. 4 but not the part 2 condition. It will turn out (Thm. 5) that this is the only ten-voter profile supporting these $P(X,Y)$ values; its plurality $A \sim C \succ B$ outcome (where "$\sim$" means a tie) is as stated in Thm. 1, part 1.

Part (2) asserts that *some supporting profiles* must elect A, but not necessarily all of them. For instance, the fifteen voter profile where eight prefer $A \succ B \succ C$, five prefer $C \succ A \succ B$ and two prefer $C \succ B \succ A$ has the $P(A,B) = 11, P(B,C) = P(A,C) = 1$ values with the plurality $A \succ C \succ B$ outcome. In contrast, the profile where six prefer $A \succ B \succ C$, seven prefer $C \succ A \succ B$ and two prefer $B \succ A \succ C$ has identical $P(X,Y)$ values but a different plurality winner with the $C \succ A \succ B$ plurality ranking.

This example motivates Eq. 4, which identifies all possible settings where the plurality winner always is the Condorcet winner. With $P(A,C) = 10$ and $P(A,B) = 8$ (so $Y = C, X = B$), for instance, the Condorcet winner A must be the plurality winner if and only if the number of voters satisfies $2(8) + 10 = 26 > n$. Thus, A is both the plurality and Condorcet winner for all supporting profiles if and only if there are no more than 24 voters. With 26 voters, a profile exists with these $P(A,B), P(A,C)$ values where A and B are tied in a plurality election; with 28 voters, a supporting profile has B as the plurality winner. (Equations 4 and 5 differ slightly because the pairwise middle-ranked B, rather than the Condorcet winner A, defines the largest pairwise victory.)

The lower bounds for Eqs. 4, 5 involve n, which can require huge $P(X,Y)$ victory margins to ensure agreement between the plurality and Condorcet winners. To satisfy Eq. 4, for instance, it must be that $P(A,Y) > \frac{1}{3}n$, which means that the Condorcet winner

A receives more than two-thirds of the vote when compared with Y.[1] In other words, unless the Condorcet winner A exhibits exceptional dominance over the other alternatives in paired comparisons, expect the Condorcet and plurality winners to differ.

This required level of dominance exceeds even the landslide proportions of the introductory example! To use Thm. 1 to analyze this case, because $P(A, B) = 40$ is the strongest pairwise victory, $Y = B$, $X = C$, and $n = 100$. These pairwise tallies define a strict transitive ranking, so (Thm. 1, part 2) some supporting profiles have A as the sole plurality winner. But $2P(A, C) + P(A, B) = 80 < 100$ reverses the Eq. 4 inequality, so other supporting profiles have the Condorcet loser C as the sole plurality winner. (As developed later, there are no supporting profiles that elect B.)

If the strongly non-cyclic condition is not satisfied, the requirements for A to always be the plurality winner are slightly more complicated, but similar in form.

Theorem 2. *Suppose Eq. 3 is not satisfied (so $P(A, C) < \min(P(A, B), P(B, C))$). Whether there is, or is not, a cycle, a necessary and sufficient conditions for A to always beat B, and for A to always beat C, in a plurality vote are, respectively,*

$$2P(A, B) + P(B, C) > n, \quad P(A, B) + 2P(B, C) > n. \tag{6}$$

Both inequalities must be satisfied for A to always be the plurality winner. A benefit of Thm. 2 is that it also applies to cycles, such as $P(A, B) = 6, P(B, C) = 14, P(A, C) = -4$. Although C beats A in their pairwise vote, it follows from Eq. 6 that A always is plurality ranked over C if and only if $6 + 2(14) = 34 > n$, or if there are no more than 32 voters. It follows from the first Eq. 6 inequality that for A to always beat B (and to be the sole plurality winner for all supporting profiles), there can be no more than 24 voters.

These stringent conditions make it difficult for the Condorcet and plurality winners to always agree. This suggests exploring whether other positional rules enjoy more relaxed requirements. But as asserted next (Thm. 3), this is not the case for the antiplurality rule; instead, conditions ensuring that the Condorcet and antiplurality winners always agree impose even more demanding constraints on the paired victories.

Theorem 3. *With the strongly non-cyclic condition (Eq. 3), the following are true:*

(1) *If with alternative Y, $P(A, Y)$ is the largest paired victory, a necessary and sufficient condition for A to be the only antiplurality winner for all supporting profiles is*

$$2P(A, Y) > n + P(B, C). \tag{7}$$

(2) *If $P(B, C) > \max(P(A, B), P(A, C))$, A cannot be the only antiplurality winner. If $2P(B, C) > n + P(A, C)$, then A never can be an antiplurality winner.*

Because the $P(B, C) \geq 0$ value is on the right-hand side, Eq. 7 requires a much stronger pairwise victory for A over some alternative than needed for the plurality vote (Eq. 4). Illustrating with the introductory example, where $2P(A, B) = 80 < 100 + P(B, C)$ violates

[1] The best case for Y is if $P(A, X) = P(A, Y)$ where $P(A, Y) > \frac{n}{3}$. According to Eq. 2, A receives $\frac{1}{2}[n + P(A, Y)] > \frac{1}{2}[n + \frac{n}{3}] = \frac{2}{3}n$ votes.

Eq. 7, it follows that some supporting profiles elect someone other than A as the antiplurality winner. The Eq. 7 condition also demonstrates that it is difficult for A to be the sole antiplurality winner. Even in the extreme case where B and C tie (so $P(B,C)=0$, which means from Eq. 7 that $P(A,Y)>\frac{n}{2}$), it follows from Eqs. 2, 7 that A's victory over some candidate must give her more than 75% of the vote!

The last assertion shows that a strong pairwise victory of middle-ranked B over the Condorcet loser C jeopardizes A's antiplurality standing. With $n=60$ voters, if $P(B,C)=40$ and $P(A,B)=P(A,C)=16$, then the Condorcet winner A can never be an antiplurality winner! In contrast, all supporting profiles have A as the sole plurality winner (Eq. 5).

A last illustration of the kind of results that can be derived from our approach compares the Condorcet and Borda winners.

Theorem 4. *In all cases (that is, independent of whether Eq. 3 is satisfied), if A is a Condorcet winner, then a necessary and sufficient condition for the Borda and Condorcet winners to agree is the more relaxed*

$$2P(A,B)+P(A,C)>P(B,C). \tag{8}$$

Illustrating with the introductory example, as $2P(A,B)+P(A,C)=100>P(B,C)=10$, it follows from Eq. 8 that A is the Condorcet and Borda winner.

Equation 8 is a significantly more relaxed condition than required for the plurality and antiplurality rules, which makes it easier and far more likely for the Condorcet and Borda winners to agree than, say, the Condorcet and plurality winners. Even more, the form of Eq. 8 suggests that it can be difficult to find actual elections where the Condorcet and Borda winners differ. This is because to violate Eq. 8 (which makes B the Borda winner), the Condorcet winner A must experience narrow victories over both the Condorcet loser C and the middle-ranked B, while the middle-ranked B must have a substantial victory over the Condorcet loser C. (Also see Saari [4].)

1.3. **Contributions to the literature.** It is often stated that the Borda and Condorcet winners need not agree. To appreciate whether this comment has any significance, it is necessary to identify the settings in which the two winners disagree. Theorem 4 precisely specifies where agreement can, and cannot, happen. It follows that to have different winners, the profile must be of a special, perhaps unusual type.

Beyond settling this question for the Borda and Condorcet winners, similar relationships are needed for other positional rules. With the wide use of the plurality vote, for instance, a valuable result is to determine when the plurality and the Condorcet winners must agree, and when they can disagree. Prior to this paper and Thm. 1, general conditions in terms of the paired comparison tallies were not known.

The plurality and antiplurality rules fare so poorly with respect to the Condorcet winner that the next step is to explore whether this burden extends to other positional voting rules. Theorem 4 proves that the requirements are significantly lifted for the Borda Count. But what conditions are needed, for example, to ensure that the $(3,1,0)$ winner (i.e., $\mathbf{w}_{\frac{1}{3}}$) always agrees with the Condorcet winner? Necessary and sufficient conditions of this kind are developed (Thms. 6, 7) for all three-alternative positional voting rules. They prove that as $s \to \frac{1}{2}$ (i.e., as the normalized form of the positional rule approaches that of the Borda Count), requirements ensuring that the $\mathbf{w}_s$ and Condorcet winners agree become more relaxed. Thus consistency is more likely to occur with those positional methods that more closely resemble the Borda Count.

Another contribution of this paper is to introduce an easily used method to construct profiles. To illustrate, while various $P(X, Y)$ values have been stated in the above discussion, it need not be obvious whether profiles exist to support them. This is particularly true with lopsided values such as $P(A, B) = 20, P(B, C) = 40, P(A, C) = -30$. But should the $P(X, Y)$ values satisfy a minimal requirement (Cor. 1), profiles always exist. A way to construct them is developed in Thm. 5 and Sect. 3.1. This result makes it simple to find, say, all possible 100 voter profiles with specified $P(X, Y)$ values. This new ability to identify all possible supporting profiles is our central tool.

As this method also makes it possible to find all possible profiles that generate specified conflicting outcomes, it allows new concerns to be addressed. For instance, while it has been known since the eighteenth century that the Borda and Condorcet winners can disagree, to the best of our knowledge the likelihood of this disagreement has not been computed. Similarly, it is worth determining the likelihood that the Condorcet and plurality winners disagree (Eq. 4). By using results and the approach developed here, issues of this kind are addressed in a companion paper.

1.4. **Finding all outcomes.** The above theorems, and the following extensions, are derived by identifying all profiles that support specified paired comparison tallies. That is, the approach finds all profiles in a $\sim_{BE}$ equivalence class, which means that all associated positional outcomes can be determined. This emphasis on which positional election outcomes can accompany specified paired majority vote tallies makes our results a converse to the Sieberg and McDonald [9] contribution of identifying which majority vote outcomes can accompany specified plurality tallies; e.g., they examined when a plurality tally ensures whether a Condorcet winner, or cycle, can arise.

Our emphasis on tallies also distinguishes our results from the literature; e.g., for any number of candidates, it is known which majority vote and positional *rankings* can accompany each other (Saari [2, 5, 6]). With three candidates, for instance, anything can happen with a non-Borda method (i.e., for $\mathbf{w}_s$, $s \neq \frac{1}{2}$). Namely, select any ranking for each pair of candidates and any ranking for the triplet; there exists a profile with these majority and $\mathbf{w}_s$ vote outcomes. (To construct illustrating profiles, see [4] or [6, Chap. 4].) The missing refinement (which is developed here) is to connect majority vote tallies with all associated positional outcomes. Our approach depends upon a profile decomposition (Saari [4, 6]) that identifies the precise profile portions that cause all possible differences between, say, the plurality and antiplurality election outcomes, or with paired comparisons.

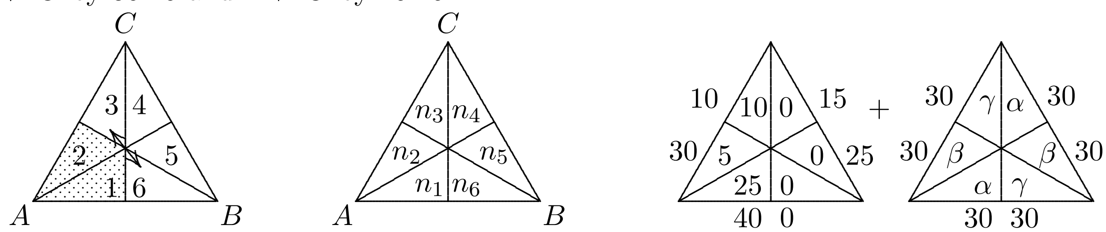
2. Paired comparisons

Central to our approach is a geometric way to tally ballots (developed in Saari [3, 8]; for applications to actual elections, see Nurmi [1]). Start by assigning each alternative, A, B, C, to a vertex of an equilateral triangle. The ranking assigned to a point in the triangle is determined by its distance to each vertex where "closer is better." Thus, points in the Fig. 1a labeled regions have the rankings:

(9)

No.	Ranking	No.	Ranking	No.	Ranking
1	$A \succ B \succ C$	2	$A \succ C \succ B$	3	$C \succ A \succ B$
4	$C \succ B \succ A$	5	$B \succ C \succ A$	6	$B \succ A \succ C$

For a given profile, let n_j be the number of voters with the j^{th} preference ranking; place n_j in the j^{th} ranking region as indicated in Fig. 1b. The geometry conveniently separates these values in a manner to simplify the tallying of ballots. For example, to compute pairwise votes, just sum the numbers on each side of an edge's perpendicular bisector as indicated in the first Fig. 1c triangle; the paired comparison tallies are listed below the appropriate edge. For instance, the vertical line separates preferences where $A \succ B$ (on the left) from $B \succ A$ (on the right) leading to the 40:0 tally supporting $A \succ B$. Similarly, $A \succ C$ by 30:10 and $B \succ C$ by 25:15.



a. Profiles and tallying **b.** Example **c.** Initial example

Figure 1. Computing tallies

2.1. **Computing positional outcomes.** A candidate's plurality tally is the sum of entries in the two regions sharing her vertex. For A, this tally is the sum of entries in the two Fig. 1a shaded regions. A candidate's $\mathbf{w}_s$ tally is

{her plurality tally} plus {s times the number of voters who have her second ranked}.

For A with Fig. 1b values, add to her plurality tally s times the sum of entries in the two Fig. 1a regions with arrows in them; e.g., the $\mathbf{w}_s$ positional tallies for A and B are, respectively, $(n_1+n_2)+s(n_3+n_6)$ and $(n_5+n_6)+s(n_1+n_4)$. For the first Fig. 1c triangle, the positional tallies for

$$A : B : C \text{ are, respectively, } 30 + 10s : 25s : 10 + 5s. \tag{10}$$

With these tallies, the Condorcet winner A wins with any $\mathbf{w}_s$, the Condorcet loser C is second ranked for $\mathbf{w}_s$, $0 \leq s < \frac{1}{2}$, but B advances to second position for $\frac{1}{2} < s \leq 1$.

To list the tallies as a point in $\mathbb{R}^3$, let $V_s(\mathbf{p})$ be profile $\mathbf{p}$'s $\mathbf{w}_s$ tallies listed in the (A, B, C) order. So with Eq. 10 ($\mathbf{p}$ from the first Fig. 1c triangle), $V_s(\mathbf{p}) = (30 + 10s, 25s, 10 + 5s)$. Notice that $V_s(\mathbf{p}) = (1-s)(30, 0, 10) + s(40, 25, 15) = (1-s)V_0(\mathbf{p}) + sV_1(\mathbf{p})$ defines a straight line connecting the profile's plurality and antiplurality tallies. This line (which is used in Sect. 4) represents a general behavior: For any profile $\mathbf{p}$,

$$V_s(\mathbf{p}) = (1-s)V_0(\mathbf{p}) + sV_1(\mathbf{p}), \quad 0 \leq s \leq 1. \tag{11}$$

The Eq. 11 line segment in $\mathbb{R}^3$ is called the *procedure line* (Saari [3]); the point s^{th} of the way along this line (from the plurality to the antiplurality tally) is the $\mathbf{w}_s$ tally for $\mathbf{p}$.

2.2. **The "essential profile".** Because the $P(X,Y)$ values in the first Fig. 1c triangle agree with those of the initial example, any profile supporting the initial example and this Fig. 1c profile belong to the same $\sim_{BE}$ equivalence class. Even stronger, it will follow from the next theorem that the Fig. 1c profile is the *essential profile*, denoted by $\mathbf{p}_{ess}$, for the original choice. What makes this profile "essential" is that, as developed below, all possible profiles that support these $P(X,Y)$ values build upon $\mathbf{p}_{ess}$; i.e., the essential profile characterizes all profiles in its $\sim_{BE}$ equivalence class.

Definition 5. *For specified $P(A,B), P(A,C), P(B,C)$ values, the* essential profile *is the profile with the smallest number of voters that have the specified $P(X,Y)$ values.*

As it will become clear, each profile in a $\sim_{BE}$ equivalence class includes the class's essential profile as a component. For strongly non-cyclic settings, a way to derive $\mathbf{p}_{ess}$ is to take the largest $P(X,Y) > 0$ value, and subtract Y's tally from each candidate's tally in each pairwise comparison. Illustrating with the initial example, while the original A:B tallies were 70:30, the reduced tallies are A:B by 40:0, B:C by 25:15, and A:C by 30:10. It follows from Thm. 5 (below) that these reduced tallies uniquely define the essential profile.

Theorem 5 identifies all essential profiles (four of them) and the corresponding number of $\mathbf{p}_{ess}$ voters. In defining an essential profile, the n_j value is given by e_j. But as an essential profile has the smallest number of voters, certain n_j terms must be expected to equal zero. This is the case; for each $\mathbf{p}_{ess}$, Thm. 5 specifies which n_j values equal zero. If an $n_j = e_j$ need not equal zero, the e_j value is determined by its Fig. 1b ranking; i.e. if the j^{th} ranking is $X \succ Y \succ Z$, then

$$e_j = \frac{1}{2}[P(X,Y) + P(Y,Z)]. \tag{12}$$

Illustrating with e_1 that represents $A \succ B \succ C$, it follows from Eq. 12 that $e_1 = \frac{1}{2}[P(A,B) + P(B,C)]$. Similarly, $e_5 = \frac{1}{2}[P(B,C) + P(C,A)] = \frac{1}{2}[P(B,C) - P(A,C)]$. A useful relationship when computing outcomes with Eq. 12 is that $e_{j+3} = -e_j$, $j = 1,2,3$. (So if a computation involves $e_1 - e_6$, it could be computed as $e_1 + e_3$ with Eq. 12 values.)

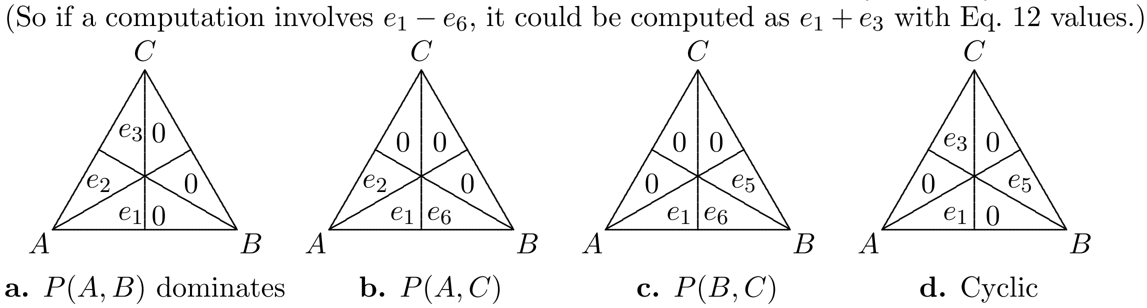


Figure 2. The four essential profiles

Figure 2 represents the four essential profiles; the first three correspond to strongly non-cyclic settings. The last one, Fig. 2d, represents where the strongly non-cyclic condition is violated. Recall from the comments following Def. 4 that Fig. 2d includes cyclic and those non-cyclic outcomes that fail to qualify as being "strongly non-cyclic."

Theorem 5. *A specified set of $P(X,Y)$ values defines a unique essential profile: The values of $n_j = e_j$ terms that need* not *equal zero are given by Eq. 12. With the standard assumptions that $P(A,B) \geq 0$, $P(B,C) \geq 0$, there are four essential profiles.*

(1) *With the strongly non-cyclic condition, the essential profile has one of the Fig. 2 a, b, c forms where the choice is defined by the largest $P(X,Y) > 0$ value; call it the XY essential profile. The XY $\mathbf{p}_{ess}$ has $P(X,Y)$ voters. The three essential profiles are:*
   (a) *If $P(A,B)$ is the largest value (Fig. 2a), then the AB essential profile has $n_4 = n_5 = n_6 = 0$.*
   (b) *If $P(A,C)$ is the largest value (Fig. 2b), then the AC essential profile has $n_3 = n_4 = n_5 = 0$.*
   (c) *if $P(B,C)$ is the largest value (Fig. 2c), then the BC essential profile has $n_2 = n_3 = n_4 = 0$.*

(2) *If the $P(X,Y)$ values do not satisfy the strongly non-cyclic condition, the "cyclic essential profile" is given by Fig. 2d where $n_2 = n_4 = n_6 = 0$. The number of $\mathbf{p}_{ess}$ voters is $\{P(A,B) + P(B,C) - P(A,C)\}$.*

A direct computation using Thm. 5 with $P(X,Y)$ values from the introductory example proves that the first Fig. 1c triangle is its essential profile. To illustrate a different aspect of Thm. 5, if three arbitrarily selected integer values with the same parity are specified as tentative $P(X,Y)$ values, it need not be clear whether there is a supporting profile. But an immediate corollary of Thm. 5 is that such a profile always exists.

Corollary 1. *For any three integers with the same parity, I_1, I_2, I_3, there exists a profile supporting the values $P(A,B) = I_1$, $P(B,C) = I_2$, $P(A,C) = I_3$*

If I_1 and/or I_2 are negative, changing the candidates' names converts everything to our setting. (For instance, $P(A,B) = -7, P(B,C) = -11$ is equivalent to $P(C,B) = 11$ and $P(B,A) = 7$, so rename C as A^*, B as B^* and A as C^*; the A^*, B^*, C^* names satisfy the required $P(A^*,B^*), P(B^*,C^*) \geq 0$.) Thus, assume that $I_1, I_2 \geq 0$. The I_k values both identify the appropriate essential profile and define the e_j's (Eq. 12).

If, for instance, $P(A,B) = 10$, $P(B,C) = 6$, and $P(A,C) = 8$, the AB essential profile is given by $e_1 = \frac{1}{2}[P(A,B) + P(B,C)] = 8$, $e_2 = \frac{1}{2}[P(A,C) + P(C,B)] = \frac{1}{2}[8 - 6] = 1$, $e_3 = \frac{1}{2}[P(C,A) + P(A,B)] = \frac{1}{2}[-8 + 10] = 1$, and $n_4 = n_5 = n_6 = 0$. This $\mathbf{p}_{ess}$ has ten voters, so all other supporting profiles have an even number of voters where $n > 10$.

3. Positional voting outcomes

All possible supporting profiles for given $P(X,Y)$ values are found by adding to $\mathbf{p}_{ess}$ profile components that never affect $P(X,Y)$ values; all ways this can be done identify all profiles in a given $\sim_{BE}$ equivalence class. With the original $n = 100$ voter example, for instance, its AB essential profile has 40 voters, so 60 voters must be added in appropriate ways. The different choices of doing this is what creates the different positional outcomes.

One approach to add terms without affecting $P(X,Y)$ values is to use "reversal pairs;" e.g., a voter preferring $A \succ B \succ C$ has a companion who prefers the reversed $C \succ B \succ A$. As a reversal pair defines a tied majority vote for all pairs of candidates, it does not change $P(X,Y)$ values. This is illustrated in the second Fig. 1c triangle where α voters have $A \succ B \succ C$ preferences and another α voters have the reversed $C \succ B \succ A$ preferences. The other reversal pairs are indicated by the β and γ terms in this triangle. Notice, each candidate in each paired comparison receives $\alpha+\beta+\gamma$ votes to create complete ties.

What is not obvious is that, as proved in [4, 6], *adding reversal pairs to a* $\mathbf{p}_{ess}$ *is the only way to preserve all* $P(X,Y)$ *values.*[2] This result is what makes it possible to identify all possible supporting profiles; just add reversal pairs to $\mathbf{p}_{ess}$. Conversely, to compute the $\mathbf{p}_{ess}$ for a specified profile, *remove* as many reversal pairs as possible. Illustrating with the 40 voter profile $(10,8,2,6,11,3)$, removing pairs $(6,0,0,6,0,0)$, $(0,8,0,0,8,0)$, and $(0,0,2,0,0,2)$ creates the eight-voter BC essential profile $(4,0,0,0,3,1)$.

The reason "pairs" can be added to $\mathbf{p}_{ess}$ to create n-voter profiles is that Eq. 2 ensures that n and each $P(X,Y)$ have the same parity. Thus (Thm. 5) for any admissible n, the number of voters not in $\mathbf{p}_{ess}$ is an even integer $2q$, so q pairs can be added. In particular, it follows from Thm. 5 that with the strongly non-cyclic condition and if $P(X,Y)$ is the maximum pairwise victory, then

$$q = \frac{1}{2}[n - P(X,Y)] \tag{13}$$

pairs are added. If the strongly non-cyclic condition is not satisfied, the number becomes

$$q = \frac{1}{2}[n - (P(A,B) + P(B,C) + P(C,A))]. \tag{14}$$

In all cases, the reversal pairs (see the second Fig. 1c triangle) must satisfy the equality

$$\alpha + \beta + \gamma = q. \tag{15}$$

3.1. **Finding new results.** To use Thm. 5 to find and prove new results, first determine which Fig. 2 choice applies. Next, add the α, β, γ terms (as in the second Fig. 1c triangle) and then just compute and compare tallies. To illustrate by developing an explanation why B can never be a plurality winner for the initial example, notice that answers depend on the value of q (the number of reversal pairs to be added to the essential profile). A large q value provides a rich assortment of positional outcomes. But large q values correspond to small $P(X,Y)$ values, which require more competitive, closer pairwise election outcomes. So, expect a wealth of positional outcomes to accompany competitive paired comparison elections. A listing of what can happen with plurality winners follows:

Corollary 2. *With the strongly non-cyclic condition, if* $P(X,Y)$ *is the largest paired victory, a necessary and sufficient condition for there to be at most two different plurality outcomes is*

$$3P(X,Y) > n - 4. \tag{16}$$

[2] This statement is not true for four or more alternatives, which means that our results do not extend in a simple way to settings with more than three alternatives.

If this largest $P(X,Y)$ satisfies $3P(X,Y) \leq n-4$, then, for each of the three alternatives, a supporting profile exists where that alternative is the plurality winner.

If the strongly non-cyclic condition is not satisfied, then a necessary and sufficient condition that A is the sole plurality winner is

$$\min(2P(A,B)+P(B,C),\ 2P(B,C)+P(A,B)) > n. \tag{17}$$

Furthermore, a necessary and sufficient condition for there to be at least one profile where alternative X is the sole plurality winner is

$$n-4 \geq 3P(Y,X) \tag{18}$$

where Y is the alternative immediately preceding X in the listing A, B, C, A.

Notice, Eqs. 17, 18 hold whether there is, or is not, a cycle. The reason B is not a plurality winner with the initial example follows from Eq. 16; its left-hand side is 120 while the right is 96. These pairwise outcomes, then, allow at most two different plurality winners. But it already has been established that the Condorcet winner and loser, A and C, can be plurality winners, so B cannot.

Corollary 2 identifies interesting properties about settings with small number of voters. With $n=8$, for instance, Eq. 16 is satisfied should any candidate win a majority election. (As n is an even integer, a victory requires $P(X,Y) \geq 2$.) So, with the strongly non-cyclic condition and no more than eight voters, some alternative never is the plurality winner.

With more voters, the situation changes. To ensure for $n=100$ that, for each candidate, a profile can be constructed where that candidate is the plurality winner, it follows from Eq. 16 that this happens if each $P(X,Y) \leq 32$. In other words, each candidate is the plurality winner with some supporting profile if no candidate receives more than 66 of the 100 votes in paired comparisons. Thus even landslide outcomes, such as where A beats B by 62:38, A beats C by 66:34, and B beats C by 64:36, admit enormous flexibility in the associated plurality winners; with this example, for each candidate there is a supporting profile where she is the plurality winner.

3.1.1. *Intuition and proofs of parts of Thm. 1 and Cor. 2.* To illustrate how to use Thm. 5, Cor. 2 is proved in the strongly non-cyclic case where $P(A,B)$ has the largest victory (as with the initial example). The structure builds upon the AB essential profile, so the plurality tallies for A, B, and C are, respectively,

$$e_1+e_2+\alpha+\beta, \quad \beta+\gamma, \quad e_3+\alpha+\gamma. \tag{19}$$

For only A to be the plurality winner, A must always have the largest tally; that is, it must always be that $e_1+e_2+\alpha+\beta > \beta+\gamma$ (for A to always beat B) and $e_1+e_2+\alpha+\beta > e_3+\alpha+\gamma$ (for A to always beat C). These inequalities reduce, respectively, to $e_1+e_2 > \gamma-\alpha$ and $P(A,C) = e_1+e_2-e_3 > \gamma-\beta$. The worse case scenario threatening A's status is where $\alpha=\beta=0$ and $\gamma=q$, these inequalities are satisfied if $P(A,C) > \gamma = q = \frac{1}{2}[n-P(A,B)]$. Collecting terms leads to $P(A,B)+2P(A,C) > n$, which is Eq. 4 in Thm. 1. With equality, this $\gamma=q$ value creates a profile with a $A \sim C$ tie; if the inequality is reversed, then a profile exists where C is the plurality winner.

Again with the AB essential profile, to determine what it takes for *each candidate* to be a plurality winner with some supporting profile, because (as just shown) it is easier with a AB essential profile for C to be a plurality winner, just find conditions where B can be the plurality winner. Using Eq. 19, this requires finding a profile where in a plurality election B beats both A and C. Computing these tallies leads, respectively, to the inequalities

$$\beta+\gamma > e_1+e_2+\alpha+\beta, \quad \beta+\gamma > e_3+\alpha+\gamma,$$

or $\gamma > e_1+e_2+\alpha$ and $\beta > e_3+\alpha$. Thus, minimal conditions for such a profile are where $\alpha = 0, \gamma \geq \gamma_{min} = e_1+e_2+1$, and $\beta \geq \beta_{min} = e_3+1$. Such values exist if and only if Eq. 15 is satisfied, or if $\beta_{min}+\gamma_{min} = (e_3+1)+(e_1+e_2+1) = P(A,B)+2 \leq q = \frac{1}{2}[n-P(A,B)]$. Collecting terms leads to the $3P(A,B) \leq n-4$ condition. This inequality establishes necessary and sufficient conditions for each of the three candidates to be the plurality winner with some supporting profile, so it is equivalent to Eq. 16.

Illustrating with $P(A,B) = 6$, $P(A,C) = P(B,C) = 4$ (so $\mathbf{p}_{ess}$ has the Fig. 2a form), it follows that each candidate can be a plurality winner with some profile as long as $3P(A,B) = 18 \leq n-4$. Thus, with any even number of voters where $n \geq 22$, these pairwise outcomes allow such profiles to be constructed. For large n values, then, expect this behavior to occur if the winner of each pairwise election receives less than two-thirds of the vote. Stated in a different manner, to avoid this behavior, some candidate must have an exceptionally strong pairwise victory. All remaining assertions in Thms. 1 - 4 and Cor. 2 are proved in this same manner by applying elementary algebra to the admissible tallies.

3.2. **Results for other positional rules.** With this approach, Thms. 1 - 4 can be extended to all $\mathbf{w}_s$ rules. To see how to do this, the strongly non-cyclic assumption makes $\mathbf{p}_{ess}$ one of Fig. 2 a, b, or c; e.g., if $P(A,B)$ has the largest value, the AB $\mathbf{p}_{ess}$ is given by Fig. 2a. To Fig. 2a, add the α, β, γ values and compute the $\mathbf{w}_s$ tallies. To ensure it always is true that $A \succ C$ with $\mathbf{w}_s$, for instance, A's tally must always be larger so

$$(e_1+e_2+\alpha+\beta)+s(e_3+2\gamma) > (e_3+\alpha+\gamma)+s(e_2+2\beta).$$

Collecting terms leads to

$$e_1+(1-s)(e_2-e_3) > (1-2s)\gamma+(2s-1)\beta. \tag{20}$$

The $(1-2s)$ term (on the right-hand side with reversal pairs) differs in sign depending on whether $s > \frac{1}{2}$ or $s < \frac{1}{2}$. To make it difficult for A by enhancing C's tally, let $\beta = 0$ and $\gamma = q$ for $s < \frac{1}{2}$, and let $\beta = q$ and $\gamma = 0$ for $s > \frac{1}{2}$. This means that the right-hand side of Eq. 20 becomes $|1-2s|q$. Using the Eqs. 12 and 13 values leads to the assertion that, in this setting, A always beats C in a $\mathbf{w}_s$ election if and only if

$$(s+|1-2s|)P(A,B)+sP(B,C)+2(1-s)P(A,C) > |1-2s|n. \tag{21}$$

Similarly, by writing down the tallies and collecting terms, it follows that A always beats B in a $\mathbf{w}_s$ election if and only if $(1-s)e_1+e_2+se_3 > |1-2s|q$, or

$$(1+|1-2s|)P(A,B)+(1-s)P(A,C) > |1-2s|n+sP(B,C). \tag{22}$$

For A to always be the sole winner, both inequalities must be satisfied. The two extremes of $s = 0, 1$, capture and prove Thms. 1 and 3 statements.

It is interesting how these two equations capture the transition from Eq. 4 for $s = 0$ to Eq. 7 for $s = 1$. Here, Eq. 21 is the more demanding for $s = 0$, while Eq. 22 is the more demanding for $s = 1$. It also follows from this computation that if either Eq. 21 or 22 is an equality, the above choices ($\beta = q$ or $\gamma = q$) create a profile with A being tied with the appropriate candidate; if the inequality is reversed, the construction shows how to select reversal terms to create profiles where the appropriate candidate is the $\mathbf{w}_s$ winner.

An interesting Borda Count feature is how the $s = \frac{1}{2}$ value forces Eq. 20 to drop the right-hand side, which consists of reversal terms. (This also happens with Eq. 22.) This demonstrates the known fact (e.g., [4, 5] and [6, Chap. 4]) that the Borda Count is the only positional method never affected by reversal terms. These equations also illustrate why conditions for the Borda Count are more relaxed and do not involve the n value. The Eq. 20 condition for A to be the Borda winner is $P(A, B) + P(B, C) + 2P(A, C) > 0$, which, unless there is a complete tie, must always be satisfied. This statement is a special case of the following (also see Thm. 4 and [4]):

Corollary 3. *With the strongly non-cyclic condition and an alternative Y where the largest pairwise victory is A over Y, then A is both the Borda and Condorcet winner.*

The following theorem, which generalizes Thms. 1 - 4 to all $\mathbf{w}_s$ rules, is proved in the same manner as Eqs. 21, 22. Each strongly non-cyclic essential profile defines a pair of conditions. The minor differences in the expressions for the pairs reflect subtle differences in the essential profiles.

Theorem 6. *With the assumption that the strongly non-cyclic condition holds:*

(1) *If $P(A, B)$ is the strongest pairwise victory, then*

(a) *A always beats B in a $\mathbf{w}_s$ election if and only if*

$$(1 + |1 - 2s|)P(A, B) + (1 - s)P(A, C) > |1 - 2s|n + sP(B, C) \tag{23}$$

(b) *and A always beats C in a $\mathbf{w}_s$ election if and only if*

$$(s + |1 - 2s|)P(A, B) + sP(B, C) + 2(1 - s)P(A, C) > |1 - 2s|n. \tag{24}$$

(2) *If $P(A, C)$ is the strongest pairwise victory, then*

(a) *A always beats B in in a $\mathbf{w}_s$ election if and only if*

$$2(1 - s)P(A, B) + (s + |1 - 2s|)P(A, C) > |1 - 2s|n + sP(B, C), \tag{25}$$

(b) *and A always beats C in a $\mathbf{w}_s$ election if and only if*

$$(1 - s)P(A, B) + sP(B, C) + (1 + |1 - 2s|)P(A, C) > |1 - 2s|n. \tag{26}$$

(3) *If $P(B, C)$ is the unique strongest pairwise victory, then*

(a) *A always beats B in a $\mathbf{w}_s$ election if and only if*

$$2(1 - s)P(A, B) + sP(A, C) > |1 - 2s|n + (s - |1 - 2s|)P(B, C), \tag{27}$$

(b) *and A always beats C if and only if*

$$(1-s)P(A,B)+(1-s+|1-2s|)P(B,C)+2sP(A,C)>|1-2s|n. \tag{28}$$

For the Condorcet winner A to always be the $\mathbf{w}_s$ winner, both (a,b) conditions in a pair must be satisfied. Notice that, with the possible exception of Eq. 27, the Borda Count ($s=\frac{1}{2}$) satisfies all of these conditions, so these favorable conclusions always hold for Borda's method. For $s=\frac{1}{2}$, Eq. 27 reduces to Eq. 8, which guarantees that A is the Borda winner. Incidentally, the "uniqueness condition" prior to Eq. 27 is needed only to ensure that the Fig. 2c essential profile, and only this $\mathbf{p}_{ess}$, is the relevant one. Without uniqueness, one of the other Thm. 6 conditions would apply.

As an example with $P(A,B)=30, P(B,C)=10, P(A,C)=20$, and $n=100$, Eq. 23 is not satisfied for the plurality vote ($s=0$) because $2P(A,B)+P(A,C)=80$ is smaller than $n=100$, nor for the antiplurality vote ($s=1$) because 60 is smaller than 110. Thus these $P(X,Y)$ values admit supporting profiles where B is plurality ranked over A and supporting profiles where B is antiplurality ranked over A. But as noted, Eq. 23 is satisfied by the Borda Count ($s=\frac{1}{2}$), so all supporting profiles have A Borda ranked above B. The fact Eq. 23 is satisfied for one positional rule (the Borda Count) motivates the goal of finding all s values for which A always beats B. With these $P(X,Y)$ values and $0\le s\le\frac{1}{2}$, Eq. 23 becomes $80(1-s)>(1-2s)100+10s$, so Eq. 23 is satisfied if $\frac{2}{11}<s\le\frac{1}{2}$. With $\frac{1}{2}\le s\le 1$, Eq. 23 becomes $60s+20(1-s)>100(2s-1)+10s$, or $\frac{12}{17}>s$. Thus, for $\frac{2}{11}<s<\frac{12}{17}$ and these $P(X,Y)$ values, A always is $\mathbf{w}_s$ ranked above B.

This example suggests that the positional $\mathbf{w}_s$ rules that satisfy certain Thm. 6 conditions can cluster around the Borda Count. The following corollary asserts that this clustering effect holds for most of the Thm. 6 conditions.

Corollary 4. *With the possible exception of Eq. 27, if the $P(X,Y)$ values satisfy a particular inequality in Thm. 6 for s_1 in $0\le s_1\le\frac{1}{2}$ and for s_2 in $\frac{1}{2}\le s_2\le 1$, then they satisfies the inequality for all s in $s_1\le s\le s_2$. So, if a condition holds for the plurality vote, it must hold for at least all s in $0\le s\le\frac{1}{2}$. If it holds for the plurality and antiplurality rules, it holds for all positional rules. These statements extend to Eq. 27 if Eq. 8 holds.*

Incidentally, Eq. 27 (which extends Eq. 8) proves that if $P(B,C)$ has the largest value in a strongly non-cyclic setting, then A cannot be the sole antiplurality winner (which proves the last statement of Thm. 3). This is because Eq. 27 would require the impossible $P(A,C)>n$; i.e., A would need to receive more than all of the votes in an $\{A,C\}$ election.

The next result specifies what happens without the strongly non-cyclic condition.

Theorem 7. *If $P(A,C)<\min(P(A,B),P(B,C))$ (the outcome is not strongly non-cyclic), then*

(1) *A always is $\mathbf{w}_s$ ranked above B if and only if*

$$(1+|1-2s|)P(A,B)+(-s+|1-2s|)P(B,C)+(s-1+|1-2s|)P(C,A)>|1-2s|n. \tag{29}$$

(2) *and A always is $\mathbf{w}_s$ ranked above C if and only if*

$$(s+|1-2s|)P(A,B)+(1-s+|1-2s|)P(B,C)+(-1+|1-2s|)P(C,A)>|1-2s|n. \tag{30}$$

Notice the added burden (from Eqs. 23, 25, 27 and Eq. 29 when there is a Condorcet winner) for the Condorcet winner A to always $\mathbf{w}_s$ beat the middle pairwise ranked B; A's pairwise victories must be sufficiently dominant to overcome the $P(B,C)$ terms that make the inequalities more stringent. But, the Eq. 29 condition for A beating B need not ensure that with a Condorcet winner (remember, if Eq. 3 is not satisfied, the paired outcomes could define a cycle, or a Condorcet winner), she is the $\mathbf{w}_s$ winner; e.g., for smaller values of s, there may be profiles where C beats A. This reflects the fact that with each pair of equations from Thms. 6, 7, different s values make one inequality more difficult to meet than the other. Illustrating with the AB setting of Eqs. 23, 24, Eq. 24 is the more difficult to meet for $s = 0$ while Eq. 23 is more demanding for $s = 1$.

Indeed, the first of two variables in play is the choice of the positional $\mathbf{w}_s$ voting rule. It follows immediately from Thms. 6, 7 that the larger the $|s - \frac{1}{2}|$ value, the more difficult it becomes to satisfy the conditions. Stated in another manner, the closer a $\mathbf{w}_s$ rule resembles the Borda Count, the easier it is for the $\mathbf{w}_s$ winner to be the Condorcet winner (Cor. 4); conversely, the more removed a positional rule is from the Borda Count, the more freedom there is to admit profiles forcing the $\mathbf{w}_s$ and Condorcet winners to differ.

This added freedom to create conflicts is captured by the second variable; this is the q term (from Eqs. 13, 14) that determines the number of reversal pairs (Eq. 15) to add to an essential profile: The smaller the number of reversal pairs that can be added, the smaller the variance in the possible positional outcomes. But small q values require large $P(X,Y)$ values, so unless the pairwise victories are very decisive, expect differences between the positional and Condorcet winners.

3.3. **Smaller victories.** If sizable pairwise victories are required to have consistency between the $\mathbf{w}_s$ and Condorcet winners, it is reasonable to wonder what happens with smaller, more common $P(X,Y)$ values. (Many other new results can be discovered in the following manner, so our emphasis is to show how to find them.) The question we explore whether there exists *a single profile* that allows *each candidate* to win with an appropriate $\mathbf{w}_s$ rule.

To illustrate with $n = 100$ where Condorcet winner A beats B by 55:45; B beats C by 58:42, and A beats C by 60:40, as $P(A,B) = 10, P(A,C) = 20, P(B,C) = 16$, all supporting profiles are created by adding $q = 40$ reversal pairs to the AC essential profile. The objective is to determine whether there is a profile where *each candidate* is the winner with an appropriate $\mathbf{w}_s$ rule. As computed next, it is *not* possible, but only barely.

The AC essential profile's positive entries are $e_1 = \frac{1}{2}[10 + 16] = 13$, $e_2 = \frac{1}{2}[20 - 16] = 2$, $e_6 = \frac{1}{2}[-10+20] = 5$. As A is the Borda winner (Cor. 3), our objective requires some non-Borda positional method to elect B while another elects C. It follows from the procedure line (Eq. 11) that if this can be done, the easiest way to do so is with the plurality and antiplurality rules. The most undemanding approach is to have A as the Borda winner, B the plurality winner (Thm. 1, part 3), and C the antiplurality winner.

By substituting the e_j and α, β, γ values into Fig. 2b and computing the plurality tallies, it follows that a necessary and sufficient condition for B to be the plurality winner is $\gamma > 10 + \alpha$. Setting the antiplurality tallies so that C beats A, it follows after collecting terms that this occurs if and only if $\beta > 18 + \gamma$. The minimal values satisfying these expressions are $\alpha_{min} = 0, \gamma_{min} = 11$, and $\beta_{min} = 30$, which must satisfy (Eq. 15) $\alpha_{min} + \beta_{min} + \gamma_{min} \leq q = 40$. But the sum is 41, which barely violates Eq. 15, so no profile has this property. However, a slightly tighter $\{A, C\}$ election, say 59:41 instead of 60:40, has the larger $q = 41$ value, so these computations show how to construct such a profile.

Using this kind of analysis, the following result is proved. This theorem indicates how easy it is to have a variety of positional election outcomes.

Theorem 8. *With the strongly non-cyclic conditions and an alternative Y where $P(A, Y)$ is the maximum paired victory, a necessary and sufficient condition for a profile to exist where the plurality, Borda, and antiplurality winners all differ is*

$$2P(A, Y) + 4P(A, X) + P(X, Y) \leq n - 6. \tag{31}$$

Examples, then, never exist for $n \leq 7$. To explore what Thm. 8 allows, Eq. 31 is satisfied with $n = 100$ voters even if each paired election winner receives around 56 votes. (For instance, if $P(A, C) = 14$ (so A receives 57 votes) and $P(A, B) = P(B, C) = 12$, then $Y = C$, $X = B$ and $2(14) + 4(12) + 12 = 88 < 100 - 6 = 94$.) As such, rather than being an unusual event, Eq. 31, which allows almost anything to happen, is satisfied with even decisive paired election outcomes. Also notice how if the strongest pairwise outcome is $P(A, B)$, then Eq. 31 is easier to satisfy because $P(X, Y) = P(C, B) = -P(B, C) \leq 0$. Related results using the other XY essential profiles are proved in the same manner.

3.4. **General statement.** The above results connect paired tallies to positional election outcomes. The following more general conclusions (Thm. 9) completely specify all possible $\mathbf{w}_s$ rankings that can accompany specified $P(X, Y)$ values.

As reversal pairs are central to what follows, let $\mathbf{r}_{X,Y} = \{X \succ Z \succ Y, Y \succ Z \succ X\}$; i.e., the subscript names the pair's two top-ranked candidates. For example, $\mathbf{r}_{B,C} = \{B \succ A \succ C,\ C \succ A \succ B\}$. The number of pairs of voters with $\mathbf{r}_{A,C}, \mathbf{r}_{A,B}, \mathbf{r}_{B,C}$ preferences are given, respectively, by α, β, γ where $\alpha + \beta + \gamma = q$.

Theorem 9. *a. The $\mathbf{w}_s$ tally for $\mathbf{r}_{X,Y}$ assigns one point to X and to Y, and $2s$ points to Z. Thus*

$$V_s(\mathbf{r}_{A,C}) = (1, 2s, 1), \quad V_s(\mathbf{r}_{A,B}) = (1, 1, 2s), \quad V_s(\mathbf{r}_{B,C}) = (2s, 1, 1). \tag{32}$$

b. For n voters, all possible supporting profiles for specified $P(X, Y)$ values are given by

$$\{\mathbf{p}_{ess} + \alpha\mathbf{r}_{A,C} + \beta\mathbf{r}_{A,B} + \gamma\mathbf{r}_{B,C} \mid \alpha, \beta, \gamma \text{ are non-negative integers; } \alpha + \beta + \gamma = q\}. \tag{33}$$

c. The set of all associated $\mathbf{w}_s$ tallies is

$$V_s(\mathbf{p}_{ess}) + \alpha(1, 2s, 1) + \beta(1, 1, 2s) + \gamma(2s, 1, 1), \quad \alpha + \beta + \gamma = q. \tag{34}$$

According to Thm. 9, all possible $\mathbf{w}_s$ tallies for the introductory example are given by

$$(30 + 10s, 25s, 10 + 5s) + (\alpha + \beta + 2s\gamma, \beta + \gamma + 2s\alpha, \alpha + \gamma + 2s\beta);\ \alpha + \beta + \gamma = 30. \tag{35}$$

All corresponding election rankings are computed in Sect. 4.2 and Eq. 37.

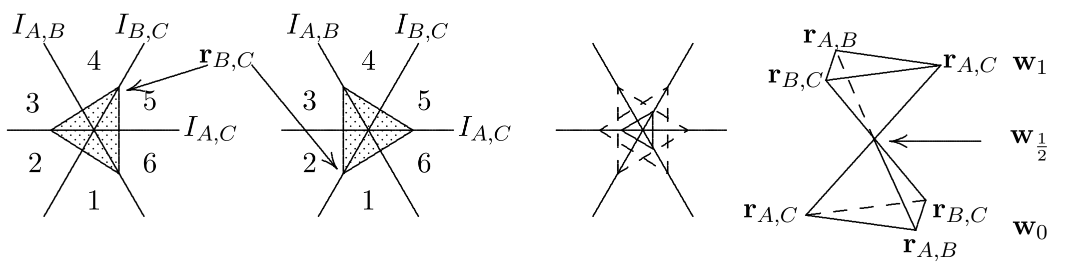
For an $n = 20$ voter cyclic outcome example where $P(A, B) = 4, P(B, C) = 10, P(A, C) = -2$, the essential profile (Thm. 5) is $(7, 0, 3, 0, 6, 0)$. Because $n = 20$ and the essential profile has 16 voters, so $2q = 4$ and only two $\mathbf{r}_{X,Y}$ pairs can be added. Thus all six supporting profiles are $(9,0,3,2,6,0)$, $(7,2,3,0,8,0)$, $(7,0,5,0,6,2)$, $(8,1,3,1,7,0)$, $(8,0,4,1,6,1)$, and $(7,1,4,0,7,1)$. As $V((7,0,3,0,6,0)) = (7+3s, 6+7s, 3+6s)$, the $\alpha = 2$ tally is $(7+3s, 6+7s, 3+6s) + 2(1, 2s, 1) = (9+3s, 6+11s, 5+6s)$.

4. The associated positional rankings

Although Thm. 9 completes the problem by identifying all profiles and $\mathbf{w}_s$ tallies associated with specified $P(X,Y)$ values, the algebraic computations required to find new results and determine positional rankings can be messy. In this section, a geometric tool is developed to simplify the analysis of the $\mathbf{w}_s$ rankings. This tool plays a central role when computing probabilities of paradoxical behavior.

4.1. Geometry.

To simplify the geometry of the $V_s(\mathbf{p})$ points in $\mathbb{R}^3$, let $I_{X,Y}$ be the indifference plane of points where X and Y are tied; e.g., $I_{A,B} = \{(x,y,z) \mid x = y\}$. Figure 3 represents how the three indifferences planes intersect a plane given by $x + y + z = c$, where constant c is described below. The numbers (from Fig. 1a) between indifference lines identify the ranking of a tally that lands in the region. The three indifference planes intersect along the main diagonal $t(1,1,1)$ of $\mathbb{R}^3$, where all alternatives are tied. This diagonal (a point in each of the above triangles) is the line of *complete indifference.*



a. $\mathbf{w}_0$ outcomes **b.** $\mathbf{w}_1$ outcomes **c.** $\mathbf{w}_s$ **d.** Pyramidal cone, $0 \le s \le 1$

Figure 3. Positional outcomes

For a specified q, the set of all $\mathbf{w}_s$ tallies, $V_s(\alpha\mathbf{r}_{A,C} + \beta\mathbf{r}_{A,B} + \gamma\mathbf{r}_{B,C})$, is in convex hull defined by the three extreme tallies $\{V_s(q\mathbf{r}_{A,C}), V_s(q\mathbf{r}_{A,B}), V_s(q\mathbf{r}_{B,C})\}$. According to Thm. 9 and Eq. 32, these $V_s(q\mathbf{r}_{X,Y})$ points are the vertices of an equilateral triangle in the plane defined by $c = (2+2s)q$; the triangle's center point is on the complete indifference line. Denote this triangle by $T_s(q)$.

Figure 3a depicts the $T_0(q)$ triangle defined by the vertices $(q,0,q)$, $(q,q,0)$, $(0,q,q)$, which are the $V_0(q\mathbf{r}_{X,Y})$ plurality tallies. Similarly, Fig. 3b represents the antiplurality $T_1(q)$ triangle in the $c = 4q$ plane; the three vertices are $(q,2q,q)$, $(q,q,2q)$, $(2q,q,q)$. When applied to reversal terms, the plurality and antiplurality rankings reverse each other; e.g., the plurality ranking for $q\mathbf{r}_{X,Y}$ is $X \sim Y \succ Z$, while the antiplurality ranking is the reversed $Z \succ X \sim Y$. This ranking reversal is indicated by the arrows between Figs. 3a and b for the $\mathbf{r}_{B,C}$ ranking.

The diagram depicting all V_s tallies is Fig. 3d; the vertical direction depicts different $c = (2+2s)q$ values, $0 \leq s \leq 1$. (This direction is along the complete indifference diagonal.) Think of this figure as placing Fig. 3b (on the $c = 4q$ plane) above Fig. 3a (on the $c = 2q$ plane). Next, connect with straight lines the $V_0(q\mathbf{r}_{X,Y})$ and $V_1(q\mathbf{r}_{X,Y})$ vertices (in the plurality and antiplurality planes); that is, with the procedure lines (Eq. 11) defined by the reversal pairs. The pyramid is the region defined by these lines. All three procedure lines (actually, all procedure lines of reversal terms) cross at a common point on the $c = 3q$ plane, which is the $\mathbf{w}_{\frac{1}{2}}$ complete tie outcome. (This condition reflects the requirement (Eq. 32) that the Borda outcome ($s = \frac{1}{2}$) of reversal terms must be a complete tie.)

The $T_s(q)$ triangle is the intersection of the pyramid with the $c = q(2+2s)$ plane; the vertices are $(q, 2qs, q)$, $(q, q, 2qs)$, $(2qs, q, q)$. Notice how $T_{\frac{1}{2}+t}(q)$ and $T_{\frac{1}{2}-t}(q)$, $0 < t \leq \frac{1}{2}$, have the same size; they differ only in orientation where one is the reversal of the other. This geometry reflects how the $\mathbf{w}_{\frac{1}{2}+t}$ and $\mathbf{w}_{\frac{1}{2}-t}$ rankings of a $\mathbf{r}_{X,Y}$ reverse each other. Also, the $T_s(q)$ triangles are embedded in the manner indicated by Fig. 3s with centers along the complete indifference line; the dashed lines represent the $T_0(q)$ and $T_1(q)$ triangles. As the Fig. 3c shaded $T_s(q)$ has the same orientation as $T_0(q)$, it represents an s value where $0 < s < \frac{1}{2}$. For $\frac{1}{2} < s < 1$ s values, the $T_s(q)$ orientation is that of $T_1(q)$.

The above figure and descriptions, Thm. 9, and Eq. 34 provide all tools needed to describe what $\mathbf{w}_s$ rankings could possibly accompany specified $P(X, Y)$ values. In particular, no matter what reversal pairs are added to $\mathbf{p}_{ess}$, the Borda ranking always agree with the $V_{\frac{1}{2}}(\mathbf{p}_{ess})$ ranking. All V_s rankings now are given by rankings in the set

$$V_s(\mathbf{p}_{ess}) + T_s(q).$$

4.2. **The initial example.** To conclude, these methods are illustrated by finding all $\mathbf{w}_s$ rankings for the initial example. They are determined by

$$V_s(\mathbf{p}) = (30 + 10s, 25s, 10 + 5s) + T_s(30), \quad 0 \leq s \leq 1. \tag{36}$$

So the Borda ranking ($s = \frac{1}{2}$) is $A \succ B \sim C$.

To find the admissible plurality rankings, if the translated $T_0(30)$ crosses any Fig. 3 ranking region, it intersects the triangle's boundary. Thus, to identify all admissible plurality rankings, just analyze what happens with the $s = 0$ extreme points $(30, 0, 10) + (30, 0, 30) = (60, 0, 40), (60, 30, 10)$, and $(30, 30, 40)$; the boundary lines between these vertices give transition rankings. To illustrate, the vertices define, respectively, the rankings $A \succ C \succ B, A \succ B \succ C$, and $C \succ A \sim B$. The line between the first two vertices, $(1-t)(60, 0, 40) + t(60, 30, 10) = (60, 30t, 40 - 30t)$, defines one of the three boundaries for the triangle. This boundary admits a tie when $30t = 40 - 30t$, or $t = \frac{2}{3}$. Thus the only new ranking is the transition ranking $A \succ B \sim C$ that lies between the first two strict rankings. The line between the first and third vertex, $(60 - 30t, 30t, 40)$, defines a second boundary line; it admits a tie only at the $t = 1$ endpoint, so no new rankings are created. The final boundary line, $(60 - 30t, 30, 10 + 30t)$ admits a $60 - 30t = 10 + 30t$, or $t = \frac{5}{6}$ point with an $A \sim C \succ B$ ranking transitioning from the first ranking to a $C \succ A \succ B$ plurality outcome. Thus, the only admissible plurality rankings by profiles supporting the initial $P(X,Y)$ values are $A \succ C \succ B, A \succ B \succ C, C \succ A \sim B$ and the obvious need for the transition rankings $A \succ B \sim C$ and $A \sim C \succ B$.

To analyze this example for any s, the vertices are $(30+10s, 25s, 10+5s)+(30, 60s, 30) = (60+10s, 85s, 40+5s), (60+10s, 30+25s, 10+65s), (30+70s, 30+25s, 40+5s)$. Again, the analysis reduces to elementary algebra. For instance, with the first vertex, $\{A,B\}, \{A,C\}$, $\{B,C\}$ ties require, respectively, $60+10s = 85s, 60+10s = 40+5s, 85s = 40+5s$), which have respective solutions $s = \frac{4}{5}, -4, \frac{1}{2}$. As only the first and third lie in the $0 \le s \le 1$ range, the vertex has the $A \succ C \succ B$ ranking for $0 \le s < \frac{1}{2}$, the $A \succ B \succ C$ ranking for $\frac{1}{2} < s < \frac{4}{5}$, and the $B \succ A \succ C$ ranking for $\frac{4}{5} < s \le 1$, with two obvious transition rankings involving a tie. The second vertex defines $A \succ B \succ C$ for $0 \le s < \frac{1}{2}$, $A \succ C \succ B$ for $\frac{1}{2} < s < \frac{10}{11}$, and $C \succ A \succ B$ for $\frac{10}{11} < s \le 1$ with with transition rankings with ties at $s = \frac{1}{2}, \frac{10}{11}$, while the third vertex has the plurality $C \succ A \sim B$ that becomes $C \succ A \succ B$ for $0 < s < \frac{2}{13}$, continues as $A \succ C \succ B$ for $\frac{2}{13} < s < \frac{1}{2}$, and ends as $A \succ B \succ C$ for $\frac{1}{2} < s \le 1$.

A complete analysis follows from the above with the transition values of $s = 0, \frac{2}{13}, \frac{1}{2}, \frac{4}{5}, \frac{10}{11}$. Leaving out the obvious transition rankings for each s, a complete analysis of the initial example is given by the following list of vertex rankings:

(37)

s	Rankings	s	Rankings
$s = 0,$	$A \succ C \succ B, A \succ B \succ C, C \succ A \sim B$	$0 < s < \frac{2}{13},$	$A \succ C \succ B, A \succ B \succ C, C \succ A \succ B$
$s = \frac{2}{13},$	$A \succ C \succ B, A \succ B \sim C, C \sim A \succ B$	$\frac{2}{13} < s < \frac{1}{2},$	$A \succ C \succ B, A \succ B \succ C$
$s = \frac{1}{2},$	$A \succ B \sim C$	$\frac{1}{2} < s < \frac{4}{5},$	$A \succ B \succ C, A \succ C \succ B,$
$s = \frac{4}{5},$	$A \sim B \succ C, A \succ C \succ B, A \succ B \succ C$	$\frac{4}{5} < s < \frac{10}{11},$	$B \succ A \succ C, A \succ C \succ B, A \succ B \succ C$
$s = \frac{10}{11},$	$B \succ A \succ C, A \sim C \succ B, A \succ B \succ C$	$\frac{10}{11} < s \le 1,$	$B \succ A \succ C, C \succ A \succ B, A \succ B \succ C$

Thus, for $0 \le s < \frac{2}{13}$, either A or C could be the $\mathbf{w}_s$ winner, for $\frac{2}{13} < s < \frac{4}{5}$, A must be both the Condorcet and $\mathbf{w}_s$ winner, for $\frac{4}{5} < s < \frac{10}{11}$, either A or B could be the $\mathbf{w}_s$ winner, and for $\frac{10}{11} < s \le 1$, anyone could be the positional winner.

5. Proofs

Proof of Thms. 1 – 4: Some assertions are proved in Sect. 3, others are special cases ($s = 0, 1$) of Thms. 6, 7. The exceptions are proved here. Assume Thm. 5 is correct.

To prove Thm. 1, parts 1 and 2, add α, β, γ values to each of the first three Fig. 2 essential profiles and determine what it takes to ensure the assertion. With Fig. 2a, because $P(A,C) \ge 0$, we have that $e_1 + e_2 \ge e_3$, or that the plurality tally for the essential profile has A at least tied for first. If the ranking is strict (i.e., $P(A,C) > 0$), then A is the sole plurality winner. Adding β terms improves A's tally at the expense of C. (B cannot gain enough advantage.) The same argument holds for Fig. 2b by interchanging B and C. A similar argument holds for Fig. 2c, where the $P(A,B) \ge 0$ assumption ensures A can be a plurality winner. If $P(A,B) > 0$, A is the sole plurality winner for the essential profile. Adding α terms helps A at B's expense.

The argument for Thm. 3 and the antiplurality rule involves minor differences. To prove Eq. 7, start with the AB essential profile where the A, B, C antiplurality tallies are, respectively,

$$e_1 + e_2 + e_3 + \alpha + \beta + 2\gamma, \quad e_1 + 2\alpha + \beta + \gamma, \quad e_2 + e_3 + +\alpha + 2\beta + \gamma. \tag{38}$$

By comparing tallies, for A to always beat B, it must always be that $e_2 + e_3 > \alpha - \gamma$, for A to always beat C, it must always be that $e_1 > \beta - \gamma$. So, for A to beat B for all supporting profiles, it must be that $e_2 + e_3 = \frac{1}{2}[P(A,C) + P(C,B)] + \frac{1}{2}[P(C,A) + P(A,B)] = \frac{1}{2}[P(A,B) - P(B,C)]$ is greater than $q = \frac{1}{2}[n - P(A,B)]$. This requires $2P(A,B) > n + P(B,C)$, which is Eq. 7. The other constraint of $e_1 = \frac{1}{2}[P(A,B) + P(B,C)] > q = \frac{1}{2}[n - P(A,B)]$ provides the more relaxed $2P(A,B) + P(B,C) > n$.

The analysis for the AC essential profile is the essentially the same; the A, B, C antiplurality tallies are, respectively, $e_1+e_2+e_6+\alpha+\beta+2\gamma$, $e_1+e_6+2\alpha+\beta+\gamma$, and $e_2+\alpha+2\beta+\gamma$. The sharpest constraint again involves ensuring that A always beats B; this requires $e_2 = \frac{1}{2}[P(A,C)+P(C,B)] > q = \frac{1}{2}[n - P(A,C)]$, which becomes the Eq. 7 condition $2P(A,C) > n + P(B,C)$. The condition ensuring that A beats C is the more relaxed $2P(A,C) + P(B,C) > n$.

To prove the Thm. 3 assertion that a BC essential profile makes it impossible for A to be the sole antiplurality winner, the A, B, C antiplurality tallies with the BC essential profile are, respectively, $e_1 + e_6 + \alpha + \beta + 2\gamma$, $e_1 + e_5 + e_6 + 2\alpha + \beta + \gamma$, and $e_5 + \alpha + 2\beta + \gamma$. By comparing the A and B tallies, for A to always be the sole antiplurality winner, it must always be that $\gamma > e_5 + \alpha$. Because $e_5 \geq 0$ and γ can be set equal to zero, this inequality cannot always be satisfied. But if $\gamma < e_5 + \alpha$ always is true, then (as asserted in Thm. 3), A *never* is an antiplurality winner. This holds if and only if $e_5 > q \geq \gamma$, or $\frac{1}{2}[P(B,C) + P(C,A)] > q = \frac{1}{2}[n - P(B,C)]$, or $2P(B,C) > n + P(A,C)$.

To prove Thm. 4 and Eq. 8, it is well known that a Condorcet winner (A) always is Borda ranked over a Condorcet loser (C). Thus, the only three strict Borda rankings are $A \succ B \succ C$, $A \succ C \succ B$, and $B \succ A \succ C$. To show when the last choice cannot occur, compute A's and B's Borda tallies. Because a candidate's Borda tally is the sum of her tallies from all paired comparison elections, they are, respectively, $(P(A,B)+\frac{n}{2})+(P(A,C)+\frac{n}{2})$ and $(P(B,A)+\frac{n}{2})+(P(B,C)+\frac{n}{2})$. By setting A's tally over B's, canceling terms and using $P(B,A) = -P(A,B)$, Eq. 8 follows. □

Proof of Cor. 2: All settings satisfying Def. 4 are proved as described in Sect. 3.1.1. Only the case not satisfying Def. 4 remains. From Fig. 2d, the plurality tallies for A, B, and C are, respectively,

$$e_1 + \alpha + \beta, \quad e_5 + \beta + \gamma, \quad e_3 + \alpha + \gamma.$$

The proof reduces to simple algebra; e.g., for A to always beat B, it must be for all α, β, γ that $e_1 + \alpha + \beta > e_5 + \beta + \gamma$, or $e_1 - e_5 > \gamma$. As the largest choice of γ is q, the condition becomes

$$e_1 - e_5 = \frac{1}{2}[P(A,B) + P(B,C) - P(B,C) - P(C,A)] > q = \frac{1}{2}[n - P(A,B) - P(B,C) - P(C,A)],$$

which is $2P(A,B) + P(B,C) > n$. All other assertions are similarly proved. □

Proof of Thm. 5: Results in [4, 6] prove that only reversal pairs have no effect upon $P(X,Y)$ values. Start with any profile in the Fig. 1b form and remove reversal pairs; these are pairs where one entry is in region j, $j = 1, 2, 3$, and the other is in the diametrically opposite region $j + 3$. Removing entries from each $\{j, j+3\}$ pair until one entry is zero has no effect on $P(X,Y)$ values; as the remaining entry is non-negative there are, at most, three positive entries.

This reduction defines eight possible arrangements; either all entries on one side of one of the perpendicular dividers are zero, or an arrangement arises where the zeros occur in an alternating fashion to define either $(n_1, 0, n_3, 0, n_5, 0)$ or $(0, n_2, 0, n_4, 0, n_6)$. All eight settings can be catalogued with Fig. 2; each case has a parallel arrangement where the e_j's in the figure are replaced by zeros and the zeros with e_j terms. That three of the sought after arrangements are given by Figs. 2 abc follows from the $P(A,B), P(B,C), P(A,C) \geq 0$ assumptions; a direct computation proves that the parallel choices violate these conditions.

For the last two settings, the $(n_1, 0, n_3, 0, n_5, 0)$ choice is represented by Fig. 2d with appropriate n_j values; Fig. 2d clearly is compatible with the $P(A,B), P(B,C) \geq 0$ conditions. To show that the remaining $(0, n_2, 0, n_4, 0, n_6)$ choice is not compatible, the $P(A,B) \geq 0$ constraint requires $n_2 \geq n_4 + n_6$. But unless $n_4 = 0$ and $n_2 = n_6$ (which defines a Fig. 2b setting), this is incompatible with the $P(B,C) \geq 0$ condition that requires $n_6 \geq n_2 + n_4$.

To find the e_j values, each setting defines a linear algebra problem; e.g., Fig. 2a defines the set of three equations and three unknowns

$$n_1 + n_2 + n_3 = P(A,B), \quad n_1 - n_2 - n_3 = P(B,C), \quad n_1 + n_2 - n_3 = P(A,C), \tag{39}$$

with similar equations for each of the three remaining settings. As each set of equations is linearly independent, each set admits a unique solution. Thus, to show that these solutions are giving by Eq. 12, it suffices to show that the e_j values satisfy the equations. With the first of Eq. 39, for instance, this becomes

$$e_1 + e_2 + e_3 = \frac{1}{2}[P(A,B) + P(B,C)] + \frac{1}{2}[P(A,C) + P(C,B)] + \frac{1}{2}[P(C,A) + P(A,B)].$$

Using the relationships $P(C,B) = -P(B,C)$, $P(C,A) = -P(A,C)$, it follows that the sum is $P(A,B)$ as required. All other sets of equations are verified in the same manner.

All remaining assertions in the theorem are immediate. □

Proof of Thms. 6, 7: This elementary algebra exercise follows the outline leading to Eq. 20. □

Proof of Cor. 4: These conditions are equivalent to determining the s values for which a linear equation is positive. To illustrate with $0 \leq s < \frac{1}{2}$, Eq. 23 can be expressed as determining where the linear equation $(1-s)[2P(A,B) + P(A,C)] - (1-2s)n - sP(B,C)$ is positive. But if a linear equation is positive at two points, it is positive for all values between these points. The conclusion follows because, with the possible exception of Eq. 27, all conditions are satisfied for the Borda Count $s = \frac{1}{2}$. Thus, if the conditions hold for $s_1 < \frac{1}{2}$ and $s_2 > \frac{1}{2}$, they hold for $s_1 \leq s \leq \frac{1}{2}$ and $\frac{1}{2} \leq s \leq s_2$, or for all s in $s_1 \leq s \leq s_2$. When Eq. 8 is satisfied (so the Borda and Condorcet winners agree), the assertion extends to Eq. 27. (If Eq. 8 is not satisfied, then a similar "clustering" assertion holds for s values that violate, rather than agree with, Eq. 27.) □

Proof of Thm. 8: Again, this is a direct algebraic computation following the lead of the example developed before the statement of the theorem. □

Proof of Thm. 9: This is a direct computation. □

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