UNIFYING VOTING THEORY FROM NAKAMURA'S TO GREENBERG'S THEOREMS

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Abstract. Cycles, empty cores, intransitivities, and other complexities affect group decision and voting rules. Approaches developed to prevent these difficulties include the Nakamura number, Greenberg's theorem, and single peaked preferences. The results derived here significantly extend and subsume these assertions while providing a common, surprisingly simple explanation for these seemingly dissimilar conclusions.

1. Introduction

Decision and economic issues involving paired comparison decision and voting methods can be plagued by cyclic outcomes, empty cores, and other difficulties. The extensive nature and practical consequences of these problems are captured by Arrow's Impossibility Theorem (Arrow, 1951), path dependency problems where inferior choices can be selected (McKelvey, 1979), questions raised about court decisions (Cohen, 2010, 2011), and even perversities in our laws (Katz, 2012).

In response to these difficulties, a variety of seemingly dissimilar results (described below), such as the Nakamura number, Greenberg's theorem, and single peaked preferences, describe ways to avoid these problems. But for applications, where conditions may differ from what is specified in theorems, it is necessary to appreciate why the results are true so that, if necessary, they can be modified or extended. Unfortunately, an understanding why these approaches are successful and how they may be improved is obscured by their technical mathematical proofs.

Thus it is surprising that these results have the same simple explanation. This is because the culprits causing N-alternative paired comparison mysteries are profiles dominated by what are called "ranking wheel configurations" (denoted by $\mathcal{RWC}_N$ and introduced in Sect. 2). By identifying the single source for these difficulties, simpler, stronger arguments emerge to extend major conclusions, identify new strengths and weaknesses, and relate seemingly disparate results. The general theorems developed here (Thms. 5 and 6) have a significantly wider range of applicability that unify the above approaches by including them as special cases.

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As for notation, the N alternatives $\{a_1, \dots, a_N\}$ are represented by capital letters $A, B, \dots$ with specific examples. There are n voters, each is assumed to have complete, transitive preferences over the specified alternatives where "$\succ$" represents "strictly preferred to;" e.g., $A \succ B$ means "A is strictly preferred to B." The symbol "$\sim$" refers to a "tied" ranking; e.g., $A \sim B$ means that in an $\{A, B\}$ election, neither A beats B nor B beats A. A profile is a list of the preferences for the n voters; in what follows, the $\mathcal{RWC}_N$ terms usually are components of a profile. (Proofs are in the appendix.)

1.1. **Supermajority voting rules.** In supermajority voting, a winning proposition must receive at least a quota of q votes. In the US Senate, for instance, a vote must receive at least 60 of the 100 possible votes to avoid a filibuster.

Definition 1. *A q-rule with n voters, denoted by (q, n), is where $q > \frac{n}{2}$ and a winning proposition receives at least q of the n votes.*

Majority vote cycles require just three alternatives, but it is reasonable to expect that exacting q-rules make it difficult to have q-rule cycles. They can occur; Nakamura determined the needed number of alternatives to construct them.

Theorem 1. *(Nakamura, 1978) Nakamura's number for a (q, n)-rule is $\nu(q, n) = \lceil \frac{n}{n-q} \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x. A q-rule cycle never occurs with $N < \nu(q, n)$ alternatives. With $N \geq \nu(q, n)$ alternatives, there are profiles with q-rule cycles.*

With the US Senate's q-rule, Thm. 1 asserts that "60-vote" cycles can occur with $\nu(60, 100) = \lceil \frac{100}{100-60} \rceil = 3$ alternatives. A three-fourths rule requiring 75 votes is spared three-alternative cycles, but it admits $\nu(75, 100) = \lceil \frac{100}{25} \rceil = 4$ alternative cycles. In developing an interpretation for $\nu(q, n)$ (by using $\mathcal{RWC}_N$'s), new results are obtained. For instance, when using Thm. 1 to anticipate voting problems, it is crucial to know whether the allowed cycles are likely, or sufficiently rare to be safely ignored. Can we, for example, safely ignore the possibility of a 60-vote cycle in the US Senate? To the best of my knowledge, this kind of issue has not been investigated; results directed toward these concerns are developed here.

Another way to avoid cycles is by restricting profiles. A popular choice is Black's single peaked condition (Black, 1958) where, for three alternatives, nobody ranks the same alternative at the bottom. Sen (1966) extended Black's result to a best possible majority vote conclusion for any number of alternatives. Although several authors used "simple games" to find extensions for (q, n)-rules (e.g., Dummett and Farquharson, 1961; Nakamura, 1975; Pattanaik, 1971; Salles and Wendell, 1977), general restrictions holding for all (q, n)-rules had not been developed. This elusive objective finally is resolved here. This best possible result (Thm. 5) for (q, n)-rules includes Nakamura's and Sen's results as special cases.

1.2. **Spatial voting.** Similar difficulties arise in spatial voting where each axis of an Euclidean space represents a different issue; e.g., a k-issue setting is modeled with points in $\mathbb{R}^k$. The coordinates of an agent's *ideal point* in $\mathbb{R}^k$ reflect his ideal combination of the level of each issue. Preferences often are modeled in terms of the Euclidean distance a proposal is from an agent's ideal point where closer is better.

A *core point* is a proposal that cannot be defeated with a specified voting rule. The *core* is the set of all core points; denote the core for the (q, n)-rule by $\mathbb{C}(q, n)$. An example of a majority vote core is the well-known "median voter theorem" where, with a single issue and an odd number of voters, $\mathbb{C}(\lceil \frac{n}{2} \rceil, n)$ coincides with the median voter's ideal point; a proposal not in $\mathbb{C}(\lceil \frac{n}{2} \rceil, n)$ can be defeated in a majority vote.

McKelvey (1979) demonstrated the importance of the core by proving that an empty majority vote core must be accompanied by cycles and even worse behavior. As he showed, it is possible to start with any initial proposal and, with majority vote victories between counterproposals (or amendments) specified by an appropriately designed agenda, to end up with a conclusion so inferior that nobody prefers it to the initial choice. Tataru (1999) extended the "McKelvey chaos theorem" to (q, n)-rules.

These wild behaviors cannot occur if $\mathbb{C}(q, n) \neq \emptyset$, so it is important to know when the core is non-empty. This fact underscores the significance of Greenberg's result, which specifies conditions for which $\mathbb{C}(q, n) \neq \emptyset$ no matter where the voters' ideal points are located.

Theorem 2. *(Greenberg, 1979) For k-issue spatial voting, let domain $\mathcal{D} \subset \mathbb{R}^k$ be a compact (i.e. closed and bounded), convex subset. A necessary and sufficient condition for $\mathbb{C}(q, n) \neq \emptyset$ for any positioning of the n ideal points in $\mathcal{D}$ is if $k \leq \nu(q, n) - 2$.*

With q-rules, the common "single issue" assumption used with voting models (to avoid cycles) can be relaxed to allow up to $\nu(q, n) - 2$ issues.

The significance of Thm. 2 makes it important to determine whether its conclusion can be extended to more general settings. It is doubtful, for instance, whether Thm. 2 holds for models examining consequences of different proposals or laws. My generalization (Thm. 6) provides the strongest possible extension, so it subsumes Thms. 2 and 5 (hence Namamura's and Sen's conclusions). In this way, these major results are extended and unified.

2. Ranking Wheel

What simplifies the analysis is that all "trouble creating profiles" are known; they are strictly caused by $\mathcal{RWC}_N$ terms (introduced below). While it has been known since Condorcet (1785) that these kind of terms cause cycles and paired comparison problems, it was not known whether other aspects of a profile could also cause difficulties. This gap limited progress; e.g., the need to circumvent this missing information is why the proofs of the above results can be mathematically technical and involve restrictive assumptions (e.g., Thm. 2).

Results showing that all three-alternative paired comparison difficulties are caused by $\mathcal{RWC}_3$ terms, and only these terms, (e.g., Zwicker, 1991; Saari, 1999) are not sufficient to analyze Nakamura's and the other conclusions because they involve more than three alternatives. What is needed is to know for any $N \geq 3$ that $\mathcal{RWC}_N$ terms, and *only* $\mathcal{RWC}_N$ terms, cause these difficulties: This result is proved in (Saari, 2000) as part of a decomposition of N-candidate profiles.

Because nothing else can cause pairwise difficulties, all problems and negative conclusions about paired comparisons are caused by, and reflect properties of, $\mathcal{RWC}_N$ terms. As shown in Chap. 2 of (Saari, 2008), for instance, these $\mathcal{RWC}_N$ terms explain Arrow's Impossibility Theorem (1951), Sen's Minimal Liberalism result (1970), and other negative paired comparisons assertions. What was not known is whether this approach could explain, unify, and significantly extend the above tools into a form that can be applied to more general models. This is done here.

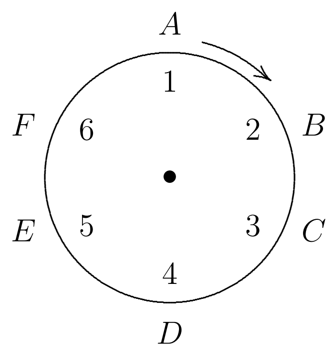


Figure 1. Creating a $\mathcal{RWC}_6$ profile.

2.1. **Ranking wheel.** To define a N-alternative *"ranking wheel configuration"* ($\mathcal{RWC}_N$) (Saari, 2008), attach a freely rotating wheel to a surface. Place in an equally spaced manner the numbers 1, 2, ..., N along the wheel's edge. As illustrated in Fig. 1, list the candidates' names on the surface in the order determined by a desired initial ranking; e.g., the j^{th} ranked candidate's name is positioned next to number j.

This initial position defines the first ranking; in Fig. 1 it is $A \succ B \succ C \succ D \succ E \succ F$. Rotate the wheel to place "1" by the next candidate and read off the second ranking; in Fig. 1, moving "1" next to B defines the second ranking of $B \succ C \succ D \succ E \succ F \succ A$. Create rankings in this manner until each candidate is in first place precisely once to define N rankings. The $\mathcal{RWC}_6$ for Fig. 1 consists of the six rankings

$$
\begin{array}{ll}
A \succ B \succ C \succ D \succ E \succ F, & B \succ C \succ D \succ E \succ F \succ A, \\
C \succ D \succ E \succ F \succ A \succ B, & D \succ E \succ F \succ A \succ B \succ C, \\
E \succ F \succ A \succ B \succ C \succ D, & F \succ A \succ B \succ C \succ D \succ E
\end{array} \tag{1}
$$

Similarly, the $\mathcal{RWC}_3$ defined by $A \succ B \succ C$ is the Condorcet triplet

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B, \tag{2}$$

while the $\mathcal{RWC}_3$ defined by the reversed $C \succ B \succ A$ is

$$C \succ B \succ A, \quad B \succ A \succ C, \quad A \succ C \succ B. \tag{3}$$

A $\mathcal{RWC}_N$ (e.g., Eq. 1) has each candidate in first, second, ..., last place precisely once, which makes it arguable that no candidate is favored; this completely tied outcome is satisfied by all positional rules. (A "positional rule" (Riker, 1982) tallies ballots by assigning specified points to candidates based on their ballot position; e.g., the plurality vote assigns a point only to a voter's first place candidate.) But this completely tied outcome *fails to hold* with majority votes over pairs!

To see this with Eq. 1, start with $\{E, F\}$. The first five rankings have $E \succ F$; only the last ranking has $F \succ E$, so E beats F by a 5:1 vote. This same phenomenon happens with each adjacent pair of candidates, so Eq. 1 defines the cycle

$$F \succ A, \quad A \succ B, \quad B \succ C, \quad C \succ D, \quad D \succ E, \quad E \succ F$$

where each outcome has the decisive 5:1 tally. More generally (as stated in Prop. 1), $\mathcal{RWC}_N$ defines a pairwise voting cycle over the N alternatives (given by adjacent entries) where each has a (N-1):1 tally. (So the three-alternative cycles defined by Eqs. 2, 3 have 2:1 tallies.) These cyclic outcomes reflect both the cyclic way in which a $\mathcal{RWC}_N$ is constructed and the rule's myopic limitations where, by using only small portions of available information from $\mathcal{RWC}_N$, the rule cannot recognize global symmetries that may demonstrate the outcome should be a tie.

The tallies for non-adjacent alternatives are not as decisive. With Eq. 1, rather than 5:1 tallies, the $A \succ C$ tally is 4:2, while the $A \sim D$ tally is the 3:3 tie. The general situation is described in Prop. 1.

Proposition 1. *With a $\mathcal{RWC}_N$ generated by $a_1 \succ a_2 \succ \cdots \succ a_N$, the tallies are*

$$N - s : s \text{ for } a_j \succ a_{j+s},\ 1 \leq s < \frac{N}{2} \text{ and } a_j \sim a_{j+s} \text{ if } s = \frac{N}{2}. \tag{4}$$

Here, a_{N+k} is identified with a_k. Because the tallies tighten for $s \geq 2$ (Eq. 4), non-adjacently ranked alternatives in $\mathcal{RWC}_N$ play no role in what follows.

2.2. Worst case scenarios. For a restriction (e.g., Nakamura's number, Greenberg's Theorem) to prevent cycles, it must be able to exclude "worst case scenarios"; i.e., the first profiles to encounter problems. This reality makes it easier to explain and extend conclusions because worst case scenarios must be created with $\mathcal{RWC}_N$ profiles, and only $\mathcal{RWC}_N$ profiles.

As shown in Prop. 1, the most dramatic pairwise tallies involve adjacently ranked $\mathcal{RWC}_N$ alternatives. So if a profile consists of two different $\mathcal{RWC}_N$'s, some pair is not adjacent in both of the $\mathcal{RWC}_N$'s. As this lowers the relative tally of this pair in a cycle, it does not constitute a worst case setting.

In other words, profiles creating worst case scenarios consist strictly of multiples of the same $\mathcal{RWC}_N$. Therefore, to analyze various paired comparison assertions, just examine what happens with these special profiles. As shown starting with the Nakamura number, this approach significantly simplifies the analysis.

3. Supermajority votes: Nakamura's number

Nakamura sought the maximum number of alternatives, N, with which it is impossible to create a (q, n)-rule cycle. Because paired comparison cycles are caused by $\mathcal{RWC}_N$ profile components, his concern now is easy to answer; i.e., find the critical N value so that appropriate multiples of a $\mathcal{RWC}_N$ never admit (q, n)-rule cycles.

To illustrate by finding this N for the $(90, 100)$-rule, it follows from the above that an extreme 11-alternative cycle is created by using a $\mathcal{RWC}_{11}$. This choice yields a cycle with 10:1 tallies, so a q-cycle with $q = 90$ requires using $\frac{90}{10} = 9$ copies of the $\mathcal{RWC}_{11}$. As a $\mathcal{RWC}_{11}$ has eleven voters, such a profile can be created: The nine copies define the preferences for 99 of the 100 voters, and the last voter's preferences can be anything. Similarly with $N = 10$; as the $\mathcal{RWC}_{10}$ cycle has 9:1 tallies, $\frac{90}{9} = 10$ copies are needed. Each $\mathcal{RWC}_{10}$ has ten voters, so the preferences of all 100 voters are determined to define a profile with a cycle. With $N = 9$, the $\mathcal{RWC}_9$ cycle has 8:1 tallies, so $\lceil \frac{90}{8} \rceil = 12$ copies are needed. But as twelve copies require $12 \times 9 = 108$ voters when there are only 100, it is impossible to create such a profile. Thus, to avoid (90, 100)-rule cycles, use no more than $N = 9$ alternatives.

To summarize, a $\mathcal{RWC}_N$ profile defines a cycle where $N - 1$ points are assigned to the winning alternative of each pair. To create a (q, n)-rule cycle, $\lceil \frac{q}{N-1} \rceil$ copies of the N-voter $\mathcal{RWC}_N$ are needed. The full profile cannot have more than n-voters, so it must be that $\lceil \frac{q}{N-1} \rceil N \leq n$. By solving for N, it follows that *a (q, n) cycle can be created if and only if*

$$N \geq \lceil \frac{n}{n-q} \rceil = \nu(q, n). \tag{5}$$

These results explain and prove Thm. 1.

This computation introduces a new interpretation for $\nu(q,n)$: As summarized in Thm. 3, the purpose of $\nu(q,n)$ is to identify which $\mathcal{RWC}_N$ (i.e., $N=\nu(q,n)$) and how many copies (i.e., $\lceil\frac{q}{\nu(q,n)-1}\rceil$) are needed to create worst case scenarios. For this reason, expect all (q,n) difficulties to involve $\mathcal{RWC}_{\nu(q,n)}$ terms.

Theorem 3. *For a (q,n)-rule with $\nu(q,n)$ alternatives, a q-rule cycle is created if the profile has $\lceil\frac{q}{\nu(q,n)-1}\rceil$ copies of a $\mathcal{RWC}_{\nu(q,n)}$. To create a profile with a q-rule cycle where a maximum number of voters can be assigned arbitrary preferences, the profile must include $\lceil\frac{q}{\nu(q,n)-1}\rceil$ copies of a $\mathcal{RWC}_{\nu(q,n)}$; the preferences for the remaining $n-\lceil\frac{q}{\nu(q,n)-1}\rceil\nu(q,n)$ voters can be selected in an arbitrary manner. A q-rule cycle cannot occur with $N<\nu(q,n)$ alternatives.*

3.1. **Likelihood.** While cycles can occur with $\nu(q,n)$ alternatives, a realistic worry is whether they are robust or unlikely to arise. The answers suggest exercising caution when using Nakamura's number to anticipate voting difficulties.

Definition 2. *If $\frac{n}{n-q}$ is an integer for a (q,n)-rule, call it a $\nu(q,n)$ bifurcation value.*

To motivate what follows, a $(150,200)$-rule has a $\nu(150,200)=4$ bifurcation value. According to Thm. 3, a four-alternative cycle with this rule must use $\lceil\frac{q}{\nu(q.n)-1}\rceil=\frac{150}{3}=50$ multiples of a $\mathcal{RWC}_4$. So, let each of the $\mathcal{RWC}_4$ rankings

$$A\succ B\succ C\succ D,\ B\succ C\succ D\succ A,\ C\succ D\succ A\succ B,\ D\succ A\succ B\succ C$$

be supported by 50 voters. Here, A beats B, B beats C, C beats D, and D beats A where the winning candidate in each election receives *precisely* 150 of the 200 votes.

"Precisely" is emphasized because should even *one* of the two hundred voters change her preference ranking in any manner, the cycle disappears. If, for instance, a voter preferring $A\succ B\succ C\succ D$ just interchanges how she ranks her bottom two candidates to create the $A\succ B\succ D\succ C$ preferences, C receives 149 votes in the $\{C,D\}$ election while D receives 51. By failing to reach the $q=150$ threshold, the societal ranking is the "tied" $C\sim D$ outcome leading to the $D\succ A, A\succ B, B\succ C$ and $C\sim D$ conclusion that (barely) avoids the cycle and identifies D as the sole candidate who cannot be beaten; i.e., $\mathbb{C}(300,400)=\{D\}$.

As this example illustrates, $\nu(q,n)$ can represent highly delicate, unrealistic settings rather than a robust description of what to expect. In particular, with rare exceptions (there are six $\mathcal{RWC}_4$'s, so for only six profiles out of the more than a trillion trillions of possibilities), there will not be q-cycles and $\mathbb{C}(300,400)\neq\emptyset$ with four alternatives. In general, *at a $\nu(q,n)$ bifurcation value, cycles with $\nu(q,n)$ alternatives are highly unlikely to arise.*

Quirks of this instability are captured by the $(301,400)$-rule where the slightly larger q creates the larger $\nu(301,400)=5$, so four-alternative cycles are avoided. It follows that extreme $(301, 400)$ scenarios (Thm. 3) involve $\mathcal{RWC}_5$ profiles such as:

(6)

Number	Ranking	Number	Ranking
80	$A \succ B \succ C \succ D \succ E$	80	$B \succ C \succ D \succ E \succ A$
80	$C \succ D \succ E \succ A \succ B$	80	$D \succ E \succ A \succ B \succ C$
80	$E \succ A \succ B \succ C \succ D$		

Each pair's tally in the promised $A \succ B$, $B \succ C$, $C \succ D$, $D \succ E$, $E \succ A$ cycle is 320 to 80, which significantly exceeds $q = 301$. Rather than a highly delicate setting, Thm. 3 asserts that the cycle can be created by using $\lceil \frac{301}{5-1} \rceil = 76$, rather than 80, copies of this $\mathcal{RWC}_5$. It further follows from Thm. 3 that the preference rankings for the remaining $n - \lceil \frac{q}{\nu(q,n)-1} \rceil \nu(q, n)$ voters, or 20 in this example, can be selected in an arbitrary fashion, and the altered profile still supports the cycle. With this much larger number of supporting profiles, a (301, 400)-rule five alternative cycle is much more likely to occur than a (300, 400)-rule four alternative cycle.

More generally, a larger $\nu(q, n) - \frac{n}{n-q} > 0$ gap forces a larger $n - \lceil \frac{q}{\nu(q,n)-1} \rceil \nu(q, n)$ difference, which (Thm. 3) admits a larger set of profiles with q-rule cycles. To illustrate with $\nu(300 + x, 400) = 5$ for $1 \leq x \leq 20$, supporting examples involve $\mathcal{RWC}_5$ components. But as x approaches the next bifurcation value ($x = 20$ or $q = 320$), the $400 - \lceil \frac{300+x}{4} \rceil 5$ difference shrinks, which reduces the likelihood of cycles. Even stronger, at a bifurcation value a "cycle" is not a *generic property*, where "generic" means that the property is "expected" in that it holds even after the preferences of any agent are slightly changed.

Theorem 4. *At a $\nu(q, n)$ bifurcation value, $\frac{q}{\nu(q,n)-1} = n - q$. A profile defining a q-rule cycle must consist of $n-q$ copies of a $\mathcal{RWC}_{\nu(q,n)}$. With any standard probability assumption over profiles, cycles are unlikely; the generic property is that $\mathbb{C}(q, n) \neq \emptyset$.*

4. Supermajority votes: Profile restrictions

A way to avoid cycles without restricting the number of alternatives is to regulate which profiles are admissible. According to the above, for a profile restriction to succeed, it must be able to handle worst case scenarios: Profiles must be appropriately restricted to prevent certain $\mathcal{RWC}_N$ terms from creating cycles.

Interestingly, profile restrictions and Nakamura avoid cycles in the same way: Both temper the strong $\mathcal{RWC}_N$ tallies that create cycles to a more benign $\mathcal{RWC}_{N-s}$ level. Nakamura does this by restricting the number of alternatives. As Prop. 2 shows, dropping s rankings from a $\mathcal{RWC}_N$ reduces certain tallies to a $\mathcal{RWC}_{N-s}$ level.

Proposition 2. *Removing s rankings from a $\mathcal{RWC}_N$ defined by $a_1 \succ a_2 \succ \cdots \succ a_N$ causes s pairs of adjacently listed candidates to have an unanimous $a_j \succ a_{j+1}$ outcome. The common tally for all remaining $a_j \succ a_{j+1}$ outcomes is [(N-s)-1]:1.*

To illustrate with $s = 3$ and the $\mathcal{RWC}_6$ in Eq. 1, if the three rankings on the right-hand side are dropped, the $A \succ B$, $C \succ D$, and $E \succ F$ outcomes now are unanimous while the $B \succ C$, $D \succ E$, and $F \succ A$ tallies become 2:1 ($\mathcal{RWC}_3$ tallies) rather than 5:1. So, a profile consisting of two copies of the remaining three Eq. 1 rankings cannot admit a (5, 6)-rule cycle because the $B \succ C$, $D \succ E$, and $F \succ A$ tallies are 4:2, which the rule treats as ties. For this profile $\mathbb{C}(5,6) = \{A, C, E\}$.

4.1. **Ward's conditions.** Black's single-peaked condition (Black, 1958) is a well known profile restriction where, with three candidates, some candidate never is bottom ranked by even one voter. The "Condorcet-domain problem" (e.g., Fishburn 1997, Monjardet, 2009) extends Black's condition by seeking appropriate restrictions on N-alternative rankings so that no matter how many voters are assigned to the admitted rankings, majority vote cycles cannot occur.

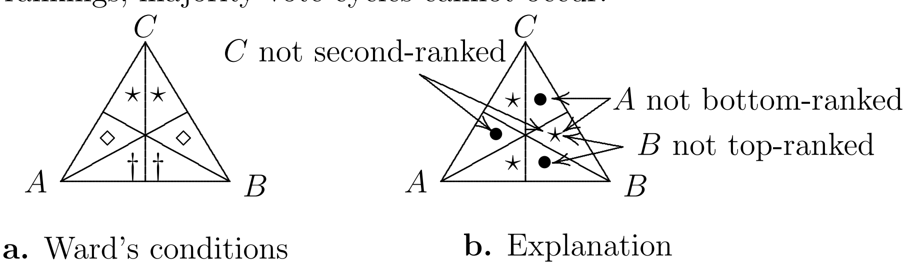


Figure 2. Condorcet domains

"Ward's conditions" (Ward, 1965) solve the Condorcet-domain problem for three candidates. The following description explaining why this is so uses the geometric representation of profiles introduced in (Saari, 1995). Here, a point's ranking in the Fig. 2a equilateral triangle is determined by its distance to each vertex; e.g., a point in the triangle with a dagger in the lower left corner is closest to A, next closest to B, and farthest from C, so it has the $A \succ B \succ C$ ranking.

Ward's constraints are:

(1) *No voter has a particular candidate top-ranked.* The Fig. 2a stars represent rankings where C is top-ranked. If no voter has either ranking, Ward's condition is satisfied. In general, rankings in the two regions sharing the vertex with a candidate's name have her top-ranked. So if no voter has rankings in either region adjacent to a particular vertex, this condition is satisfied.

(2) *No voter has a particular candidate middle-ranked.* The Fig. 2a diamonds are where C is middle-ranked. If no voter has either ranking, Ward's condition is satisfied. As "middle-ranked" involves diametrically opposite regions, if there are two ranking regions diametrically opposite each other (so the rankings reverse each other) without a single voter, this Ward condition is satisfied.

(3) Black's condition: *No voter has a particular candidate bottom-ranked.* The Fig. 2a daggers represent where C is bottom ranked. "Bottom-ranked" regions are the two regions farthest from the vertex assigned to the candidate, so if no voter has these rankings, Black's and Ward's conditions are satisfied.

Ward proved that if a profile satisfies any one of these conditions, a majority vote cycle cannot occur. A different explanation using Prop. 2 is that, for a profile to have a majority vote cycle, it must include an appropriate multiple of a complete $\mathcal{RWC}_3$. The Eq. 2 $\mathcal{RWC}_3$ triplet in Fig. 2b is indicated by stars; the Eq. 3 $\mathcal{RWC}_3$ triplet has bullets. For a profile to contain a full $\mathcal{RWC}_3$ triplet, it must have voters with rankings represented by *all three stars*, or *all three bullets*.

Ward's conditions succeed by *preventing a profile from including a $\mathcal{RWC}_3$ triplet.* To illustrate with Fig. 2b, if, for instance, no voter has B top-ranked, then the two ranking regions sharing the B-vertex are not assumed by any voter. One region has a bullet and the other a star, so this condition prohibits a profile from including either $\mathcal{RWC}_3$ triplet. Similarly, if C never is middle-ranked, then, as indicated in Fig. 2b, no voter has the ranking with the indicated bullet or the ranking represented by diametrically opposite region with a star, which makes it impossible to include a $\mathcal{RWC}_3$ triplet. The same analysis holds for the never bottom-ranked requirement,

To see why Ward's conditions are the sharpest possible to ensure a Condorcet domain, if no voter has his preference in a starred region, say the one on the Fig. 2b right-hand side, then a profile cannot include the starred $\mathcal{RWC}_3$ triplet. To preclude the profile from including the other $\mathcal{RWC}_3$ triplet, exclude *any region* with a bullet. For each choice, the Fig. 2b arrows indicate which Ward condition is satisfied.

Sen (1966) developed necessary and sufficient conditions for a Condorcet domain for $N \geq 4$ alternatives: When restricting a profile to each triplet, it must satisfy at least one of Ward's conditions. Sen's result is subsumed and extended by Thm. 5.

4.2. **General conditions.** Ward's conditions ensure a core by prohibiting a profile from including a full $\mathcal{RWC}_3$ triplet (Sect. 4.1). The elusive objective of finding appropriate profile restrictions for (q, n) rules finally is resolved in the same way.

Theorem 5. *For $r \geq 2$ alternatives, a necessary and sufficient condition for a set of ranking to avoid (q, n)-rule cycles and have $\mathbb{C}(q, n) \neq \emptyset$ independent of how voters are assigned to the rankings, is that, when restricted to each subset of $\nu(q, n)$ alternatives, at least one ranking is missing from each possible $\mathcal{RWC}_{\nu(q,n)}$.*

Because Thm. 5 includes all (q, n)-rules, it subsumes the classical results. Nakamura's Theorem, for instance, is a special case because Thm. 5 trivially holds for $r < \nu(q, n)$; for $r \geq \nu(q, n)$, cycles can occur. Ward's result is a special case because, with $\nu(2, 3) = 3$, Thm. 5 requires dropping at least one ranking from each $\mathcal{RWC}_3$; this is Ward's requirement (Sect. 4.1). Sen's result is subsumed because for $n \geq 3$, the majority vote requires $\nu(\lceil\frac{n+1}{2}\rceil, n) = 3$; thus Thm. 5 requires applying Ward's condition to all triplets. Implications for more general (q, n) rules follow immediately from the theorem.

With $r > \nu(q, n)$ alternatives, one might expect to impose profile restrictions on $\mathcal{RWC}_r$ terms rather than the $\nu(q, n)$ parts. To illustrate why the emphasis must be on $\mathcal{RWC}_{\nu(q,n)}$, consider $\nu(18, 24) = 4$ with the $r = 5$ alternatives A, B, C, D, E. An extremely severe profile restriction allows at most one ranking to come from each $\mathcal{RWC}_5$. The following four rankings (each from a different $\mathcal{RWC}_5$), for instance, satisfy this tight constraint.

$$E \succ [A \succ B \succ C \succ D], \quad E \succ [B \succ C \succ D \succ A], E \succ [C \succ D \succ A \succ B],$$
$$E \succ [D \succ A \succ B \succ C].$$

These rankings embed the $\mathcal{RWC}_4$ generated by $[A \succ B \succ C \succ D]$, so assigning $\frac{18}{3} = 6$ voters to each ranking creates a $(18, 24)$-rule cycle. With $r > \nu(q, n)$, then, restrictions on $\mathcal{RWC}_r$ structures are insufficient to prevent (q, n)-cycles; attention must be focussed on all subsets of $\nu(q, n)$ alternatives to prevent $\mathcal{RWC}_{\nu(q,n)}$ terms.

An extension of Ward's condition follows:

Corollary 1. *With $\nu(q, n)$ alternatives, a q-rule cycle never occurs if, for some integer $s \in \{1, 2, \dots, \nu(q, n)\}$, there is an alternative that never is s^{th} ranked.*

In a $\mathcal{RWC}_N$, each alternative is ranked in each position once, so Cor. 1 ensures that a profile cannot contain a $\mathcal{RWC}_{\nu(q,n)}$. But Cor. 1 is not a necessary condition for $N \geq 4$. To see why with $\nu(q, n) = 4$ alternatives and the six $\mathcal{RWC}_4$'s generated by $D \succ A \succ C \succ B, D \succ C \succ A \succ B, D \succ A \succ B \succ C, D \succ B \succ A \succ C, A \succ B \succ C \succ D$, and $A \succ C \succ B \succ D$, for each $\mathcal{RWC}_4$ drop the generating ranking. This choice satisfies Thm. 5, but it does not satisfy Cor. 1.

5. When a core always exists

While Thm. 5 subsumes Nakamura's and Sen's results, it fails to include Greenberg's result (Thm. 2). This is remedied by the following surprisingly general result.

Theorem 6. *A necessary and sufficient condition for $\mathbb{C}(q, n) \neq \emptyset$ for any positioning of the n ideal points in a domain $\mathcal{D}$ is if it is impossible to construct a setting with $\beta = \nu(q, n)$ ideal points in $\mathcal{D}$ where $\mathbb{C}(\beta - 1, \beta) = \emptyset$.*

What makes Thm. 6 unexpectedly general is that *no conditions are imposed on the structure of $\mathcal{D}$;* all structure is implicitly determined by whatever voting model is being considered. That is, all structures for $\mathcal{D}$ are determined by what it means for a model to position ideal points and make pairwise comparisons.

This freedom adds significantly to the applicability of the result. For instance, it allows $\mathcal{D}$ to consist of preference rankings where a voter's ideal point is his preference ranking. By using different choices of $\mathcal{D}$, Thm. 6 subsumes Greenberg's theorem and Thm. 5 (which includes Nakamura's and Sen's results). The close connection with Thm. 5, which excludes $\mathcal{RWC}_{\nu(q,n)}$ terms, is that Thm. 6 is equivalent to requiring that a $\nu(q,n)$*-person* $\mathcal{RWC}_{\nu(q,n)}$ *cannot be constructed in* $\mathcal{D}$.

A practical benefit of Thm. 6 is how it significantly simplifies the analysis by replacing (q,n) problems with simpler $(\beta-1,\beta)$ settings. For instance, to determine a limit on the number of alternatives that always allows $\mathbb{C}(200,225)\neq\emptyset$, because $\nu(200,225)=5$, Thm. 6 reduces the analysis to the much simplier problem of finding the number of alternatives for which it always is true that $\mathbb{C}(4,5)\neq\emptyset$. A $\mathcal{RWC}_5$ requires five alternatives, so (Props. 1 and 2) the answer is four; thus four or fewer alternatives ensure that $\mathbb{C}(200,225)\neq\emptyset$. A related argument with any (q,n) shows how Nakamura's result is a special case of Thm. 6.

Similarly, to find profile restrictions so that $\mathbb{C}(200,225)\neq\emptyset$ with ten alternatives, Thm. 6 reduces the analysis to examining the simpler $(4,5)$-rule. A $(4,5)$ cycle is prevented only if the profile does not contain a $\mathcal{RWC}_5$ when restricted to any choice of five of the ten alternatives; thus, this is the answer for both $(4,5)$ and $(200,225)$. In this manner, Thm. 5 is included in Thm. 6.

To include Greenberg's result (Thm. 2), we need Prop. 3, which uses the fact that a Pareto point for coalition $\mathcal{C}$ is a point that, if altered in any manner, creates a poorer outcome for some member in $\mathcal{C}$. Denote the set of all Pareto points for a coalition by $\mathcal{P}(\mathcal{C})$. With (q,n)-rules, $\mathcal{P}(\mathcal{C})$ is the convex hull defined by the ideal points of agents in $\mathcal{C}$. If the region does not admit straight connected lines, then the hyperplanes connecting the ideal points can be replaced with unique connecting surfaces to define the faces of $\mathcal{P}(\mathcal{C})$. A minimal winning coalition $\mathcal{P}(\mathcal{C})$ contains q voters. While the rankings in Prop. 3b represent Euclidean preferences (primarily because these preferences are so often used), they can be generalized to continuous, convex utility functions with an ideal point as a bliss point.

Proposition 3. *a. For* (q,n)*, the core equals*

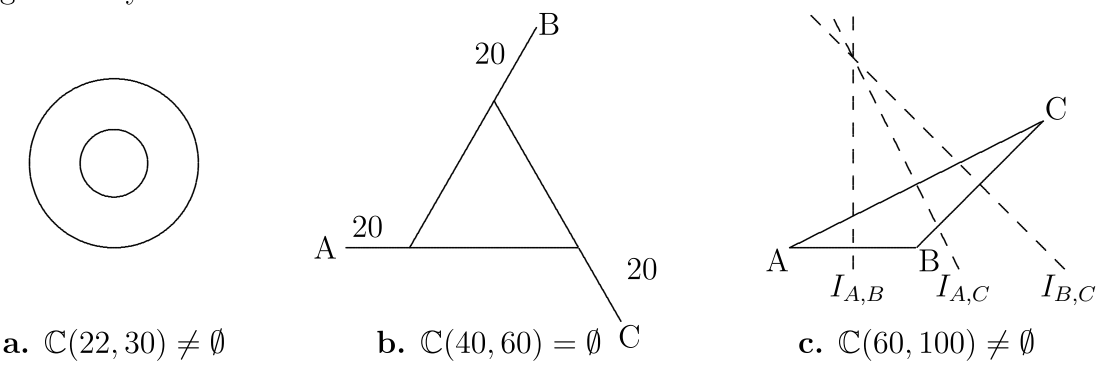
$$\mathbb{C}(q,n)=\cap\mathcal{P}(\mathcal{C}) \tag{7}$$

where the intersection is over all minimal winning coalitions.

b. It is impossible to position N *ideal points and* N *alternatives* $\{a_j\}_{j=1}^{N}$ *in* $\mathbb{R}^{N-2}$ *so that the associated rankings define a* $\mathcal{RWC}_N$*. This can be done in* $\mathbb{R}^{N-1}$*.*

A way to avoid q-rule cycles in spatial voting is to restrict the number of issues, k, to make it impossible to construct a $\mathcal{RWC}_\beta$ from β ideal points (Thm. 6). Proposition 3b ensures that this always can be done if $k\leq\beta-2$. Therefore, a nonempty core is guaranteed as long as $k\leq\nu(q,n)-2$. Thus Thm. 2 is a special case of Thm. 6.

By not imposing added restrictions on $\mathcal{D}$, Thm. 6 is more inclusive than Thm. 2; e.g., rather than being a compact, convex set as required by Thm. 2, $\mathcal{D}$ can be just about anything. In fact, Prop. 3a (used to find positive results) holds for any domain $\mathcal{D}$ that just includes the lines and hyperplanes connecting the ideal points and defining the Pareto sets. As such, convexity and compactness conditions can be significantly relaxed.



a. $\mathbb{C}(22, 30) \neq \emptyset$ **b.** $\mathbb{C}(40, 60) = \emptyset$ **c.** $\mathbb{C}(60, 100) \neq \emptyset$

Figure 3. Where $\mathbb{C}(q, n)$ is, and is not empty

To illustrate Prop. 3a, the Fig. 3a choice of $\mathcal{D}$ is an open annulus representing the selection of a point on an island with a lake in the middle. This $\mathcal{D}$ is neither convex (the center is removed) nor compact (it is not closed), so Thm. 2 does not apply. Now suppose no matter where the agents place their ideal points, their top preferences include the lake. To prove that, say, $\mathbb{C}(22, 30) \neq \emptyset$ no matter where the 30 ideal points are placed in the annulus, just use $\nu(22, 30) = 4$ and show that the annulus' two dimensions prevents constructing a $\mathcal{RWC}_4$ (Thm. 6). That this is true follows immediately from Prop. 3a because the intersection includes the lake.

This example and Thm. 2 suggest that, perhaps, only the dimension of $\mathcal{D}$ is needed. To prove this is false, consider a $(40, 60)$-rule with the Fig. 3b one-dimensional triangle $\mathcal{D}$; e.g., a group wishes to select a sidewalk-booth spot on a triangular block. As $\nu(40, 60) = 3$, the problem reduces to determining whether three points (Thm. 6) can be positioned in $\mathcal{D}$ so that $\mathbb{C}(2, 3) = \emptyset$. To do so, place an ideal point at each vertex. A minimal winning coalition consists of two points, so each triangle edge is a Pareto set. As the intersection of these three legs is empty, $\mathbb{C}(2, 3) = \emptyset$ (Prop. 3a), which means that sixty ideal points can be positioned on $\mathcal{D}$ so that $\mathbb{C}(40, 60) = \emptyset$. (In the same way, examples can be created where $\mathcal{D}$ is a collection of a finite number of points, so $\mathcal{D}$ is zero-dimensional.)

To indicate how Thm. 6 includes settings different from those addressed by Thm. 2, return to the US Senate filibuster example where $\nu(60, 100) = 3$. Let the voter preferences be given by the distance from a voter's ideal point to each alternative and let $\mathcal{D}$ be the Fig. 3c triangle where the three alternatives are located at the vertices. As specified in Thm. 2, $\mathcal{D}$ is closed, convex, and its dimension of two exceeds $\nu(60, 100) - 2 = 1$. Thus one might suspect from Thm. 2 that the 100 ideal points can be positioned so that $\mathbb{C}(60, 100) = \emptyset$, which would unleash the consequences of the McKelvey chaos theorem. But as the $X \sim Y$ indifference lines (the perpendicular bisector of the X-Y edge; denoted by $I_{X,Y}$) show by allowing only four of the six rankings, it is impossible to position three ideal points in $\mathcal{D}$ to create a cycle. Thus (Thm. 6) it always is true that $\mathbb{C}(60, 100) \neq \emptyset$.

To explain this example, the Fig. 3c setting differs from the Thm. 2 because it provides added information: The positions of the alternatives – for the Senate, the choice of the proposals – are specified. When using models with added information about the alternatives and/or voter preferences, Thm. 6, rather than Thm. 2, is the appropriate result to determine whether a core must be nonempty.

6. Summary

The fact that all paired comparison difficulties are caused *only by* $\mathcal{RWC}_N$ *terms* in a profile (Saari, 2000), is the key that unlocks several long standing mysteries. This result makes it easier to understand why problems occur, unite seemingly dissimilar conclusions, and extend several standard results. The main contribution made here is to unify and significantly extend certain well known conclusions into a single new result (Thm. 6) with a much wider range of applicability.

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7. Appendix

Proof of Prop. 1: To tally $\{a_j, a_{j+s}\}$ for the $\mathcal{RWC}_N$ generated by $a_1 \succ a_2 \succ \cdots \succ a_N$, as a_{j+s} is ranked above a_j in precisely s of the $\mathcal{RWC}_N$ rankings, the outcome for $s < \frac{N}{2}$ is $a_j \succ a_{j+s}$ with tally $N-s{:}s$. If $s=\frac{N}{2}$, the outcome is a tie.

Proof of Prop. 2: Let a_{j+1} be the top-ranked candidate in a removed ranking; only in this ranking is a_{j+1} is ranked above a_j in $\mathcal{RWC}_N$, so a_j beats a_{j+1} with an unanimous vote. There are precisely s such pairs. For all other pairs of adjacently ranked candidates, a_{j+1} beats a_j precisely once in the $N-s$ remaining rankings, so $a_j \succ a_{j+1}$ has a $(N-s)-1{:}1$ tally. These are the tallies expected from a $\mathcal{RWC}_{N-s}$

Proof of Thm. 3: The computations prior to stating Thm. 3 show that if a profile has $\lceil\frac{q}{\nu(q,n)-1}\rceil$ copies of a $\mathcal{RWC}_{\nu(q,n)}$, a cycle will arise where the alternatives satisfy the q threshold. Thus, the preferences for the remaining $n-\lceil\frac{q}{\nu(q,n)-1}\rceil\nu(q,n)$ voters can be selected in any desired manner. But if any ranking from the $\lceil\frac{q}{\nu(q,n)-1}\rceil$ copies of a $\mathcal{RWC}_{\nu(q,n)}$ is removed, to obtain the requisite tallies for each pair, at least two other rankings must be added with appropriate properties; this reduces by at least one the number of voters whose rankings can be selected in an arbitrary manner. For instance, with five copies of the Eq. 1 rankings used to create a six-alternative cycle where the winning alternative receives $q=25$ votes, if one copy of the $A \succ B \succ C \succ D \succ E \succ F$ ranking is removed, the winners of the $A \succ B, B \succ C, C \succ D, D \succ$

$E, E \succ F$ tallies have only 24 votes. To reach the $q = 25$ threshold, for each pair, another voter's ranking must be added with the pair's preference ranking. As it is easy to show, to add one more vote for each of these five rankings (and not using the removed ranking) requires adding at least two transitive rankings of special types. This reduces by at least one the number of voters who can be assigned arbitrary preference rankings.

If a cycle could be created with $N < \nu(q, n)$ alternatives, computations prove it would take $\lceil \frac{q}{N-1} \rceil$ copies of a $\mathcal{RWC}_N$, which involves $\lceil \frac{q}{N-1} \rceil N$ voters. But as $\frac{N}{N-1}$ is a decreasing function (i.e., the derivative of $\frac{x}{x-1}$ is negative), any $N < \nu(q, n)$ would require more than n voters to create the profile.

Proof of Thm. 4: If $\frac{n}{n-q} = \nu(q, n)$, substituting into $\frac{q}{\nu(q,n)-1}$ leads to $n-q$. As $\frac{q}{\nu(q,n)-1}$ specifies the number of copies of $\mathcal{RWC}_{\nu(q,n)}$, and as solving $\frac{n}{n-q} = \nu(q, n)$ for n leads to $\frac{q}{\nu(q,n)-1}\nu(q, n) = n$, it follows that there are no extra voters. A computation shows that if this profile is altered in any way, the q-rule cycle disappears. The rest of the proof follows from the material prior to the statement of Thm. 4.

The following lemma is needed for some of the remaining proofs.

Lemma 1. *Dropping an alternative from a $\mathcal{RWC}_N$ profile defines a $\mathcal{RWC}_{N-1}$ with one ranking repeated.*

To illustrate with Eq. 1, if F is dropped, what remains is the $\mathcal{RWC}_5$ generated by $A \succ B \succ C \succ D \succ E$ and an extra copy of this ranking.

Proof of Lemma 1: Consider the $\mathcal{RWC}_N$ generated by $a_1 \succ a_2 \succ \cdots \succ a_N$, and suppose a_k is dropped. The same $\mathcal{RWC}_N$ is generated by $a_k \succ a_{k+1} \succ \cdots \succ a_N \succ a_1 \succ \cdots \succ a_{k-1}$, so dropping a_k creates a $\mathcal{RWC}_{N-1}$ generated by $a_{k+1} \succ \cdots \succ a_N \succ a_1 \succ \cdots \succ a_{k-1}$ and another copy of this ranking.

Proof of Thm. 5: For a given set of rankings for $r \geq \nu(q, n)$ alternatives, if there is a set of $\nu(q, n)$ alternatives where a $\mathcal{RWC}_{\nu(q,n)}$ has all $\nu(q, n)$ rankings, then assign $\lceil \frac{q}{\nu(q,n)-1} \rceil$ voters to each ranking. Because of the structure, the appropriate number of times each candidate will defeat its adjacent partner in the $\mathcal{RWC}_{\nu(q,n)}$ (whether or not they are adjacent in the given ranking) forces the q-rule cycle.

In the opposite direction, assume when restricting a given set of rankings to any subset of $\nu(q, n)$ alternatives, at least one ranking from each $\mathcal{RWC}_{\nu(q,n)}$ is missing. With $r = \nu(q, n)$ alternatives, the conclusion follows from Prop. 2. With $r > \nu(q, n)$ alternatives, assume that a q-cycle can be created; if this is possible, it is because a cycle may occur with a $\mathcal{RWC}_r$. But this $\mathcal{RWC}_r$ cannot include all of its rankings; if it did, then because a $\mathcal{RWC}_{\nu(q,n)}$ can be constructed from the $\mathcal{RWC}_r$ by ignoring $r - \nu(q, n)$ variables (Lemma 1), we would have a contradiction to the hypothesis. This means that rankings are removed from the $\mathcal{RWC}_r$.

Assume this $\mathcal{RWC}_r$ is generated by $a_1 \succ a_2 \succ \cdots \succ a_r$. According to the proof of Prop. 2, if a_i is the bottom ranked alternative in a dropped ranking, then a_i is not in the cycle. If s rankings are removed, then these s alternatives (bottom ranked in each of the dropped rankings) are removed from a possible cycle. As the structure of the remaining alternatives creates a $\mathcal{RWC}_{r-s}$, the assumption that each $\mathcal{RWC}_{\nu(q,n)}$ is missing a ranking requires $r - s < \nu(q,n)$, or $s = r - \nu(q,n) + \alpha$ where α is a positive integer. Thus the tallies for the alternatives creating a cycle (Prop. 2) are $r - (1 - r - \nu(q,n) + \alpha) : 1$, or $(\nu(q,n) - 1 - \alpha) : 1$. This requires $\lceil \frac{q}{\nu(q,n)-1-\alpha} \rceil$ copies of each of of the $\nu(q,n) - \alpha$ rankings for a q-rule cycle. If a cycle could be created, $\lceil \frac{q}{\nu(q,n)-1-\alpha} \rceil (\nu(q,n) - \alpha) \leq n$. Solving for $\nu(q,n)$ leads to the contradiction $\nu(q,n) + \alpha \leq \nu(q,n)$. Hence a cycle cannot occur.

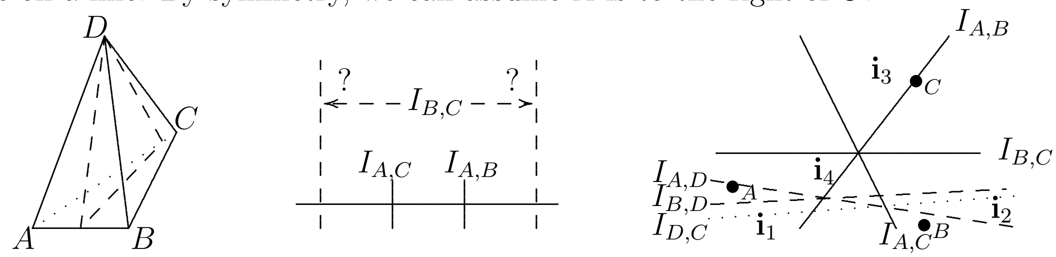
Proof of Thm. 6: If a $\mathcal{RWC}_{\nu(q,n)}$ can be constructed, then, by assigning $\lceil \frac{q}{\nu(q,n)-1} \rceil$ voters at each point, a (q,n)-cycle can be constructed so $\mathbb{C}(q,n) = \emptyset$. If a $\mathcal{RWC}_{\nu(q,n)}$ cannot be constructed, then, for each possible set of $\beta = \nu(q,n)$ agents, it is impossible to construct a $(\beta - 1, \beta)$-cycle. A direct computation of the kind used throughout this paper (with $\lceil \frac{q}{\beta} \rceil$ copies) shows that it is impossible to attain a q-rule cycle, so $\mathbb{C}(q,n) \neq \emptyset$.

Proof of Prop. 3a: By definition of a Pareto point, if $\mathbf{p} \in \mathcal{P}(\mathcal{C})$ for a minimal winning coalition $\mathcal{C}$, then there is no alternative that $\mathcal{C}$ would select over $\mathbf{p}$. If $\mathbf{p} \in \cap \mathcal{P}(\mathcal{C})$ for all minimal winning coalitions, $\mathbf{p}$ cannot be beaten, so $\mathbf{p} \in \mathbb{C}(q,n)$. Conversely, if $\mathbf{p} \in \mathbb{C}(q,n)$, it cannot be beaten by any alternative. But if $\mathbf{p} \notin \mathcal{P}(\mathcal{C})$ for minimal winning coalition $\mathcal{C}$, then $\mathcal{C}$ can elect an alternative to $\mathbf{p}$. Thus, $\mathbf{p} \in \mathcal{P}(\mathcal{C})$ for all minimal winning coalitions.

Proof of Prop. 3b: Position alternatives $\{a_j\}_{j=1}^N$ to define a $(N-1)$-dimensional simplex; each a_j defines a unique vertex. Each edge is defined by two vertices $\{a_i, a_j\}$; construct a plane orthogonal to this edge that passes through its midpoint. This plane, $I_{i,j}$, identifies all points where (with Euclidean preferences), $a_i \sim a_j$. (See Fig. 4a for the construction of the $A \sim B$, or $I_{A,B}$, plane.) These planes intersect to define $N!$ open regions; points in each region define a particular strict ranking of the N alternatives and each of the $N!$ rankings has its own region, To create a $\mathcal{RWC}_N$, place the voter ideal points in the appropriate region. Each set of $(N-1)$ ideal points defines a $(N-2)$-dimensional simplex (a face of the $(N-1)$-dimensional simplex. This face is the Pareto set for this minimal winning coalition. As the faces do not have a common intersection, it follows (Prop. 3a) that the core is empty.

Induction argument: Different induction arguments can complete the proof; all use the fact that a $\{X, Y\}$ pair changes ranking only by crossing $I_{X,Y}$. Start with $N = 3$ alternatives A, B, C; their positions define the $I_{X,Y}$ indifference lines. Suppose a

$\mathcal{RWC}_3$ generated by $A \succ B \succ C$ is created where the alternatives and ideal points are on a line. By symmetry, we can assume A is to the right of C.



a. The $I_{A,B}$ plane **b.** $N = 3$ **c.** From $N = 3$ to $N + 1 = 4$

Figure 4. Proving Proposition 3

When moving along the line among the ideal points $\mathbf{i}_{A\succ B\succ C}, \mathbf{i}_{B\succ C\succ A}$, and $\mathbf{i}_{C\succ A\succ B}$, to go from $\mathbf{i}_{A\succ B\succ C}$ to $\mathbf{i}_{B\succ C\succ A}$, notice that if a path can be found, a revision of this path does not cross the indifference line $I_{B,C}$. This is because if $I_{B,C}$ is crossed in one direction (to change the $B \succ C$ ranking to $C \succ B$), the path would have to recross $I_{B,C}$ to preserve $B \succ C$. This requires the $\mathbf{i}_{A\succ B\succ C}$ and $\mathbf{i}_{B\succ C\succ A}$ ideal points to be separated by $I_{A,B}$ and $I_{A,C}$ in that order (Fig. 4b). Thus $I_{B,C}$ is either to the left of $I_{A,C}$ or to the right of $I_{A,B}$ (the two Fig. 4b dashed lines). If it is to the right of $I_{A,B}$, then $\mathbf{i}_{A\succ B\succ C}$ has $I_{A,B}$ to the left of its position and $I_{B,C}$ to the right.

But to go from $\mathbf{i}_{A\succ B\succ C}$ to $\mathbf{i}_{C\succ A\succ B}$, the path must cross $I_{B,C}$ and $I_{A,C}$ in that order, but it can never cross $I_{A,B}$. No matter which choice is selected for $I_{B,C}$, it requires $I_{A,B}$ to be crossed with this second path; this contradiction means that the goal is geometrically impossible. With an added dimension (Fig. 2b), however, it is easy to construct such paths without crossing a forbidden $I_{X,Y}$.

With the induction hypothesis, assume the result holds for N alternatives; it must be shown that it is impossible to construct a $\mathcal{RWC}_{N+1}$ defined by $a_1 \succ \cdots \succ a_N \succ a_{N+1}$ with $N + 1$ alternatives and ideal points in $\mathbb{R}^{N-1}$.

Assume this is false; i.e., assume that the location of ideal points $\{a_j\}_{j=1}^{N=1}$ can be positioned in $\mathbb{R}^{N-1}$ so that ideal points $\{\mathbf{i}_j\}_{j=1}^{N+1}$ can be found to construct a $\mathcal{RWC}_{N+1}$. Label the rankings by $k = 1, \ldots, N+1$ with respective ideal points given by $\mathbf{i}_k$. (With $N = 3$ (and $N + 1 = 4$ alternatives) and Fig. 4c, the region with $\mathbf{i}_1$ has the ranking $A \succ B \succ C \succ D$ and the region with $\mathbf{i}_4$ represents the last $D \succ A \succ B \succ C$.) By ignoring any one of the $N + 1$ alternatives, the resulting rankings include a $\mathcal{RWC}_N$ (Lemma 1), so (induction hypothesis) a geometric representation of these N alternatives must define a $(N - 1)$-dimensional simplex (for $N = 3$, a triangle). This simplex is defined by N alternatives; all $(N + 1)$ ideal points are in appropriate sectors.

If a_k is dropped, the $\binom{N}{2}$ indifference planes meet in a common, the "not-a_k hub" denoted by $\tilde{a_k}$. (If they did not meet, they would define more than $N!$ open sectors where some represent non-transitive rankings. For $N = 3$ and dropping D, the three solid indifference lines in Fig. 4c meet at a point, call it $\tilde{D}$, defining six sectors representing the six strict transitive rankings when D is not considered.) With $N+1$ alternatives, there are $N+1$ "$\tilde{a_k}$" hubs, which must be created (induction hypothesis), so that the $a_j \sim a_k$ indifference plane, $I_{j,k}$, must meet $I_{j,x}$ and $I_{k,y}$ for each $x, y \neq j, k$. (With $N = 3$, $I_{A,D}$ must meet $I_{A,B}$ and $I_{A,C}$ at, respectively, $\tilde{C}$ and $\tilde{B}$.)

With the structure for N alternatives established by ignoring a_{N+1}, determine where to position the $I_{N+1,k}$ planes as determined by changes in rankings. For each $I_{j,j+1}, j = 1 \ldots, N$, point $\mathbf{i}_{j+1}$ is on one side of this plane, and all other ideal points are on the other (Prop. 1). What builds to a contradiction is that both $\mathbf{i}_N$ and $\mathbf{i}_{N+1}$ are on one side of $I_{N,N+1}$, and the other $N-1$ points are on the other side.

Points $\mathbf{i}_1$ and $\mathbf{i}_{N+1}$ are in "sector 1" – the sector with ranking 1 (Lemma 1) and defined by the $N-1$ boundaries $\{I_{j,j+1}\}_{j=1}^{N-1}$; the other ideal points are in sectors defined by $\mathcal{RWC}_N$. (For $N = 3$, $\mathbf{i}_1$ and $\mathbf{i}_4$ are in the $A \succ B \succ C$ sector 1, $\mathbf{i}_2$ is in $B \succ C \succ A$ and $\mathbf{i}_3$ is in $C \succ A \succ B$. Sector 1 is bounded by $I_{A,B}$ and $I_{B,C}$.) Only changes in the a_{N+1} ranking can occur in each of the $N!$ sectors, and the intersection of the $I_{N+1,k}$ planes with a sector can create at most $N+1$ regions. (For $N = 3$, the $I_{D,X}$ lines can create at most four regions in any sector; if they allowed five or more, extra regions would represent non-transitive rankings.) In sector 1, $\mathbf{i}_1$ and $\mathbf{i}_{N+1}$ have, respectively, a_{N+1} bottom and top ranked, so any path in this sector connecting $\mathbf{i}_1$ and $\mathbf{i}_{N+1}$ must cross all $\{I_{N+1,k}\}_{k=1}^{N}$ planes. (In Fig. 4c, to move from $D \succ A \succ B \succ C$ to $A \succ B \succ C \succ D$, D reverses rankings with each alternative, so each $I_{D,X}$ meets sector 1.) Except for $\mathbf{i}_1$, all ideal points have $a_{N+1} \succ a_1$, so points $\{\mathbf{i}_j\}_{j=2}^{N+1}$ are on the same $I_{N+1,1}$ side as $\tilde{a_{N+1}}$; thus $\mathbf{i}_{N+1}$ is on the hub side. In Fig. 4c, $I_{A,D}$ meets $I_{A,B}$ to identify $\tilde{C}$, and $I_{A,C}$ for $\tilde{B}$. All but $\mathbf{i}_1$ have $D \succ A$, so $\{\mathbf{i}_j\}_{j=2}^{4}$ are on the $\tilde{D}$ side of $I_{A,D}$.

Moving from $\mathbf{i}_{N+1}$ to $\mathbf{i}_1$, the first permissible change reverses $a_{N+1} \succ a_1$, so no other $I_{X,Y}$ can enter the sector 1 region defined by $I_{N+1,1}$ that contains $\mathbf{i}_{N+1}$. The ordering of how to reverse $a_{N+1} \succ a_k$ dictates how the $I_{N+1,k}$ planes intersect sector 1; moving from $\tilde{a}_{N+1}$ out, they follow the $k = 1, \ldots, N$ order. Point $\mathbf{i}_1$ is outside (i.e., away from $\tilde{a}_{N+1}$) of the last $I_{N+1,N}$ plane; i.e., $\mathbf{i}_1$ is on the side of $I_{N+1,N}$ away from the hub. While these $I_{N+1,k}$ planes cannot meet in the interior of the sector (or they would create more than N regions), some must intersect on the boundaries to create various hubs; e.g., $I_{N+1,2}$ meets $I_{N+1,1}$ on the $I_{1,2}$ boundary, and for $N > 3$, this line connects with at least two hubs.

Now consider the path from $\mathbf{i}_{N+1}$ (near $\tilde{a}_{N+1}$) to $\mathbf{i}_2$ in sector 2. This path crosses $I_{1,2}, I_{1,3}, \ldots, I_{1,N}$ in that order (to move a_1 to the bottom place). Similarly, at some position after crossing $I_{1,k}$, the path must cross $I_{N+1,k}$ (to move a_{N+1} to its final position) in the $I_{N+1,2}, I_{N+1,3}, \ldots, I_{N+1,N}$ order (to reverse rankings with the next adjacent alternative). Thus the $\{I_{N+1,k}\}_{k=2}^{N}$ planes meet sector 2, but they cannot meet each other in the interior of the sector. At the final stage, $\mathbf{i}_2$ is on the $I_{N+1,N}$ side away from $\tilde{a}_{N+1}$; i.e., $\mathbf{i}_1$ and $\mathbf{i}_2$ are on the $I_{N+1,N}$ side away from $\tilde{a}_{N+1}$.

Each $(N-1)$-dimensional plane $I_{N+1,k}$ is uniquely defined by $(N-1)$ of the $(N+1)$ possible hub points; only $\tilde{a}_{N+1}$ and $\tilde{a}_k$ are excluded. These hub points are determined by the intersection of certain edges of sectors k, $k = 1, \dots, N-1$, with either $I_{N+1,1}$ or $I_{N+1,2}$; e.g., $\tilde{a}_2$ is the intersection of the sector 2 edge $a_3 \sim \cdots \sim a_N \sim a_1$ with $I_{N+1,1}$. In general, $\tilde{a_k}$ is the intersection of the k^{th}-sector's edge $a_{k+1} \sim a_{k+2} \sim \dots a_N \sim a_1 \sim \cdots \sim a_{k-1}$ with $I_{N+1,1}$ if $k \neq 1$, and $I_{N+1,2}$ if $k = 1$. Thus, $I_{N+1,N}$ is uniquely defined by $\{\tilde{a_k}\}_{k=1}^{N-1}$. (In Fig. 4c, $\tilde{A}$ is far to the right where the dashed $I_{B,D}$ would meet the solid $I_{B,C}$ while $\tilde{B} = I_{A,D} \cap I_{A,C}$. As $\tilde{A}$, $\tilde{B}$ uniquely define $I_{D,C}$ (the dotted line), it follows that there must be two ideal points on each side, which is the desired contradiction. The rest of the proof leads to the same conclusion after handling the higher dimensional nature of $I_{N+1,N}$.)

Now consider the portion of the region $a_1 \succ a_2 \succ \cdots \succ a_{N-1}$ (with boundaries $I_{1,2}, I_{2,3}, \dots, I_{N-2,N-1}$) on the $\tilde{a}_{N+1}$ side of $I_{N+1,1}$. This region, $\mathcal{R}$, contains the ideal points $\mathbf{i}_N, \mathbf{i}_{N+1}$. However, $I_{N+1,N}$ does not meet $\mathcal{R}$ because in sector 1, $I_{N+1,N}$ is separated from $\mathcal{R}$ by $I_{N+1,1}$, and none of its other defining hub points have this ranking. This means that $\mathbf{i}_N, \mathbf{i}_{N+1}$ are on one side of $I_{N+1,N}$ while at least $\mathbf{i}_1, \mathbf{i}_2$ are on the other. As this contradicts the tallies from Prop. 1, the proof is completed. (With an added dimension, the extra defining point for $I_{N+1,N}$ would separate $\mathbf{i}_N, \mathbf{i}_{N+1}$.)

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