UNIFYING VOTING THEORY RESULTS FROM NAKAMURA'S TO GREENBERG'S THEOREMS

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Abstract. Cycles, empty cores, intransitivities, and other complexities affect group decision and voting rules. Approaches to prevent these difficulties include the Nakamura number, Greenberg's theorem, and single peaked preferences. A simple, common explanation why these seemingly dissimilar conclusions hold, and new general results that significantly extend these assertions are developed.

1. Introduction

Political science and economic issues involving decision and voting methods that use paired comparisons can be plagued by cyclic outcomes, empty cores, and other difficulties. In response, a variety of seemingly dissimilar results (described below), such as the Nakamura number, Greenberg's theorem, and single peaked preferences (along with Sen's generalization), describe how to avoid cycles so that decisions can be made. But an understanding why these tools are successful tends to be obscured by technical proofs involving different mathematical techniques. Somewhat surprisingly, as shown here, all of these results have the same simple explanation. The main contributions develop new results (Thms. 9 and 10) to handle more general settings and unify the above approaches by including them as special cases.

As shown (Sect. 2), the culprits causing N-alternative paired comparison mysteries are profiles dominated by "ranking wheel configurations" ($\mathcal{RWC}_N$). By identifying a single source for difficulties, simpler arguments and extensions of major conclusions are obtained, new strengths and weaknesses are identified, and seemingly disparate results are related. The major assumption is that voters have complete, transitive preferences over the specified alternatives.

1.1. **General results.** A result capturing paired comparison difficulties is Arrow's Impossibility Theorem [1]. His theorem adopts the reductionist approach of dividing a complicated problem into parts, where answers for each part are combined to construct a general solution. Arrow does this by seeking the societal ranking for $N \geq 3$ alternatives in terms of each pair's societal rankings (Independence of Irrelevant Alternatives, IIA); these answers are to be assembled into a complete, transitive societal ranking. But as Arrow proves, no matter how cleverly each pair's societal ranking is determined, if the preferences of at least two agents must be used, settings exist where the paired comparison approach fails. A simple explanation of Arrow's result follows from the $\mathcal{RWC}_N$'s.

1.2. **Supermajority voting rules.** Supermajority voting rules require a winning proposition to receive at least a quota of q votes. An example from the US Senate is where, to avoid a filibuster, a vote must receive at least 60 of the 100 possible votes.

Definition 1. *A q-rule with n voters, denoted by (q, n), is where $q > \frac{n}{2}$ and a winning proposition must receive at least q of the n votes.*

While majority vote cycles require only three alternatives, exacting q-rules make it more difficult to create q-rule cycles. They can occur; Nakamura [6] determined the numbers of alternatives needed to construct, or avoid, these cycles.

Theorem 1. *(Namakura [6]) Nakamura's number for a (q, n) rule is $\nu(q, n) = \lceil \frac{n}{n-q} \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x. A q-rule cycle never occurs with $N < \nu(q, n)$ alternatives. With $N \geq \nu(q, n)$ alternatives, profiles exist with q-rule cycles.*

With the US Senate's q-rule, then, cycles can occur with $\nu(60, 100) = \lceil \frac{100}{100-60} \rceil = 3$ alternatives. A three-fourths rule requiring 75 votes is spared three-alternative cycles, but it admits $\nu(75, 100) = \lceil \frac{100}{25} \rceil = 4$ alternative cycles. In developing a simple explanation (by using $\mathcal{RWC}_N$'s) and a new interpretation for $\nu(q, n)$, it will follow that the likelihood of a $(75, 100)$ four-alternative cycle is so minuscule that it can be ignored, but $(60, 100)$ three-alternative, $(74, 100)$ four-alternative, or $(76, 100)$ five alternative cycles are more likely.

Cycles are also avoided by restricting profiles. A popular choice is Black's single peaked condition [2], where, for three alternatives, nobody has some alternative bottom ranked. Sen [10] extended Black's result to a best possible majority vote conclusion for any number of alternatives. But as general restrictions holding for all (q, n) rules had not been developed, a best possible result (Thm. 9) is derived here. This general assertion includes Nakamura's and Sen's results as special cases.

1.3. **Spatial voting.** Similar difficulties arise in spatial voting where each axis of an Euclidean space represents a different issue; e.g., a k-issue setting is modeled with points in $\mathbb{R}^k$. An agent's *ideal point* in $\mathbb{R}^k$ reflects the agent's views of the ideal combination of the level of each issue. Preferences often are modeled by the Euclidean distance of a proposal from an agent's ideal point where closer is better.

A *core point* is one that cannot be defeated with a specified voting rule. The *core* is the set of all core points; denote the (q, n) rule core by $\mathbb{C}(q, n)$. An example of a majority vote core is the well-known median voter theorem where, with a single issue and an odd number of voters, $\mathbb{C}(\lceil \frac{n}{2} \rceil, n)$ coincides with the median voter's ideal point. (Results describing when a q-rule cycle cannot occur, or $\mathbb{C}(q,n) \neq \emptyset$, are related; e.g., as a cycle allows each alternative to be beaten, $\mathbb{C}(q,n) = \emptyset$.) Greenberg determined when $\mathbb{C}(q,n) \neq \emptyset$ no matter where the voters' ideal points are located.

Theorem 2. *(Greenberg [4]) For k-issue spatial voting, let domain $\mathcal{D} \subset \mathbb{R}^k$ be a compact (i.e. closed and bounded), convex subset. A necessary and sufficient condition for $\mathbb{C}(q,n) \neq \emptyset$ for any positioning of the n ideal points in $\mathcal{D}$ is if $k \leq \nu(q,n) - 2$.*

With q-rules, then, the standard "single issue" assumption can be relaxed to allow up to $\nu(q,n) - 2$ issues. My extension of Greenberg's theorem (Thm. 10) includes Thms. 2 and 9 (hence Namamura's and Sen's conclusions) as special cases; it explains why the $\nu(q,n)$ term must be expected to appear in paired comparison results. The above results are extended, explained, and related with $\mathcal{RWC}_N$'s.

2. Ranking Wheel and a coordinate system

Profile space is divided into a subspace of *strongly transitive* "nicely behaving profiles" and an orthogonal subspace of "trouble creating profiles," characterized by $\mathcal{RWC}_N$'s. As there is nothing else, analyzing a paired comparison rule reduces to determining how the rule behaves on each component. This significant simplification makes it easier to understand conclusions and develop extensions. A useful rule of thumb is to expect that all negative conclusions reflect where $\mathcal{RWC}_N$ terms are unregulated, and all positive assertions restrict or modify certain $\mathcal{RWC}_N$'s. (Proofs are in the Appendix.)

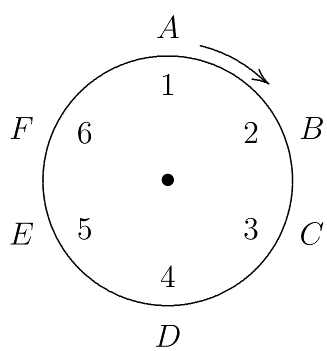


Figure 1. Creating a $\mathcal{RWC}_6$ profile.

2.1. Ranking wheel.

To define a N-alternative *"ranking wheel configuration"* ($\mathcal{RWC}_N$),[1] attach a freely rotating wheel to a surface. To borrow Saari's [8] terminology of a *ranking wheel*, place in an equally spaced manner the numbers 1, 2, …, N along the wheel's edge. As illustrated in Fig. 1, place the candidates' names on the surface

[1] In mathematical terms, this is the "Z_N-orbit" of the specified ranking.

in the order determined by a desired initial ranking; e.g., the j^{th} ranked candidate's name is positioned next to number j.

This initial position defines the first ranking; in Fig. 1 it is $A \succ B \succ C \succ D \succ E \succ F$. Rotate the wheel so that "1" is by the next candidate and read off the second ranking; in Fig. 1, moving "1" next to B defines the second ranking of $B \succ C \succ D \succ E \succ F \succ A$. Create rankings in this manner until each candidate is in first place precisely once; this defines N rankings. The $\mathcal{RWC}_6$ for Fig. 1 consists of the six rankings

$$\begin{array}{ll} A \succ B \succ C \succ D \succ E \succ F, & B \succ C \succ D \succ E \succ F \succ A, \\ C \succ D \succ E \succ F \succ A \succ B, & D \succ E \succ F \succ A \succ B \succ C, \\ E \succ F \succ A \succ B \succ C \succ D, & F \succ A \succ B \succ C \succ D \succ E \end{array} \tag{1}$$

As a $\mathcal{RWC}_N$ (e.g., Eq. 1) has each candidate in first, second, ..., last place precisely once, it is arguable that no candidate is favored; this completely tied outcome is satisfied by all positional rules. (Positional rules tally ballots by assigning specified points to candidates based on their ballot position; e.g., the plurality vote assigns a point only to a voter's first place candidate.) But a completely tied outcome *fails to hold* with majority votes over pairs!

To see this, start with $\{E, F\}$. The first five Eq. 1 rankings have $E \succ F$; only in the last ranking is $F \succ E$, so E beats F by a 5:1 vote. This same phenomenon happens with each adjacent pair of candidates, so Eq. 1 defines the cycle

$$F \succ A, \quad A \succ B, \quad B \succ C, \quad C \succ D, \quad D \succ E, \quad E \succ F$$

where each outcome has the decisive 5:1 tally. More generally, $\mathcal{RWC}_N$ defines a pairwise voting cycle over the N alternatives (given by adjacent entries in any ranking) where each has a (N-1):1 tally. Thus cyclic outcomes reflect both the cyclic construction of a $\mathcal{RWC}_N$ and the rules' myopic limitation of using only a small portion of available information from $\mathcal{RWC}_N$; e.g., such a rule cannot recognize whether global symmetries demonstrate that an outcome should be a tie.

Proposition 1. *With a $\mathcal{RWC}_N$ generated by $a_1 \succ a_2 \succ \cdots \succ a_N$, the tallies are*

$$N - s : s \text{ for } a_j \succ a_{j+s},\ 1 \leq s < \frac{N}{2} \text{ and } a_j \sim a_{j+s} \text{ if } s = \frac{N}{2}. \tag{2}$$

Here, a_{N+k} is identified with a_k. Notice how tallies tighten for $s \geq 2$ (Eq. 2); for this reason, comparisons of non-adjacently ranked alternatives in $\mathcal{RWC}_N$ play no role in what follows.

2.2. **Strongly transitive.** To define the "positive" class of profiles, let $\tau(X, Y)$ be the difference between the X and Y paired vote tallies. For example, the $\mathcal{RWC}_3$ defined by $A \succ B \succ C$ is the Condorcet triplet

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B, \tag{3}$$

with $\tau(A,B)=\tau(B,C)=\tau(C,A)=1$; the $\mathcal{RWC}_3$ for the reversed $C\succ B\succ A$ is

$$C\succ B\succ A,\quad B\succ A\succ C,\quad A\succ C\succ B; \tag{4}$$

with $\tau(A,B)=\tau(B,C)=\tau(C,A)=-1$. Similarly, with majority vote tallies of $A>B$ with 11:7, $B\succ C$ with 13:5, and $A\succ C$ with 15:3, the τ values are

$$\tau(A,B)=11-7=4,\ \tau(B,C)=13-5=8,\ \text{and}\ \tau(A,C)=15-3=12. \tag{5}$$

To characterize transitive outcomes by using τ, a triplet with $\tau(A,B)>0$ and $\tau(B,C)>0$ has a transitive ranking only if $\tau(A,C)>0$; e.g., Eq. 5 defines a transitive outcome; Eqs. 3 and 4 do not. The Eq. 5 example goes far beyond satisfying transitivity to satisfy the *equality* $\tau(A,B)+\tau(B,C)=\tau(A,C)$. Call such an outcome "strongly transitive."

Definition 2. *(Saari* [8, 9]*) A sequence of pairwise tallies of rankings over alternatives $\{X_j\}_{j=1}^N$ is "strongly transitivity" ($\mathcal{ST}_N$) if and only if all triplets satisfy the equality*

$$\tau(X_i,X_j)+\tau(X_j,X_k)=\tau(X_i,X_k). \tag{6}$$

The majority votes for the $\mathcal{ST}_4$ profile

(7)

No.	Ranking	No.	Ranking	No.	Ranking
3	$A\succ B\succ C\succ D$	2	$C\succ B\succ A\succ D$	1	$A\succ D\succ C\succ B$
4	$C\succ A\succ D\succ B$	4	$A\succ C\succ B\succ D$		

are $A\succ B$ by 12:2, $A\succ C$ by 8:6, $A\succ D$ by 14:0, $C\succ B$ by 11:3, $B\succ D$ by 9:5, and $C\succ D$ by 13:1, so

$$\tau(A,B)=10,\ \tau(A,C)=2,\ \tau(A,D)=14,\ \tau(B,C)=-8,\ \tau(B,D)=4,\ \tau(C,D)=12. \tag{8}$$

Using the Eq. 8 values with any triplet proves that Eq. 6 is satisfied.

The next theorem characterizes the $\mathcal{ST}_N$ profiles.

Theorem 3. *A N-candidate profile is $\mathcal{ST}_N$ if and only if for each $\mathcal{RWC}_N$, the number of voters with rankings from this $\mathcal{RWC}_N$ equals the number of voters with preference rankings from the $\mathcal{RWC}_N$ defined by reversing the original initial ranking.*

This means that a $\mathcal{ST}_N$ profile is "balanced" with respect to all $\mathcal{RWC}_N$'s (Thm. 3). To illustrate with Eq. 7, the first three rankings in the top line come from the $\mathcal{RWC}_4$ defined by $A\succ B\succ C\succ D$ and the remaining three come from the $\mathcal{RWC}_4$ defined by its reversal $D\succ C\succ B\succ A$. In the second line, the first four rankings are from $C\succ A\succ D\succ B$ and the last four are from the $\mathcal{RWC}_4$ defined by the reversed $B\succ D\succ A\succ C$.

Similarly, to create $\mathcal{ST}_N$ profiles, use a balanced number of voters with preferences from selective $\mathcal{RWC}_N$'s. As an $N=6$ example, let 5 voters prefer $A\succ B\succ C\succ D\succ$

$E \succ F$ and 3 prefer $C \succ D \succ E \succ F \succ A \succ B$ (from the $A \succ B \succ C \succ D \succ E \succ F$ $\mathcal{RWC}_6$) and balance this with 8 preferring $C \succ B \succ A \succ F \succ E \succ D$ (from the $\mathcal{RWC}_6$ defined by the reversed $F \succ E \succ D \succ C \succ B \succ A$). Adding another ranking, say 3 with $C \succ A \succ F \succ D \succ B \succ E$ requires a balance with 3 voters with rankings in the $\mathcal{RWC}_6$ defined by $E \succ B \succ D \succ F \succ A \succ C$, so let 2 have $D \succ F \succ A \succ C \succ E \succ B$ and 1 have $A \succ C \succ E \succ B \succ D \succ F$. By construction this 22 voter profile satisfies the balancing condition with respect to all $\mathcal{RWC}_6$'s, so it is $\mathcal{ST}_6$.

2.3. **Dividing the profile space.** With $N!$ ways to rank N alternatives, view profile space as a subset of $\mathbb{R}^{N!}$. By identifying each $\mathbb{R}^{N!}$ axis with a ranking, a profile can be represented as a $\mathbb{R}^{N!}$ vector where its components list the number of voters with each preference ranking. Illustrating with $N = 3$, by listing the six rankings in the

$$(A \succ B \succ C,\ A \succ C \succ B,\ C \succ A \succ B,\ C \succ B \succ A,\ B \succ C \succ A,\ B \succ A \succ C)$$

order, vector $(3, 0, 4, 0, 0, 5)$ is where 3 voters prefer $A \succ B \succ C$, 4 prefer $C \succ A \succ B$, and 5 prefer $B \succ A \succ C$, vector $(1, 0, 1, 0, 1, 0)$ represents Eq. 3, and $(0, 1, 0, 1, 0, 1)$ represents Eq. 4.

Profile space, $\mathbb{R}^{N!}$, is divided into orthogonal parts: The $\mathcal{ST}_N$ and $\mathcal{RWC}_N$ profiles. This means that *all profiles consist of $\mathcal{ST}_N$ and $\mathcal{RWC}_N$ components, but nothing else.* Because (as developed below) $\mathcal{ST}_N$ profiles never cause problems with standard rules, the $\mathcal{RWC}_N$ portions of profiles are completely responsible for the several centuries of mystery and debate about paired comparison outcomes.

Theorem 4. *For any $N \geq 3$, the span of ranking wheel configurations and strongly transitive configurations covers the full profile space. The space of $\mathcal{RWC}_N$ profiles has dimension $\frac{(N-1)!}{2}$, while that of $\mathcal{ST}_N$ profiles has dimension $(2N-1)\frac{(N-1)!}{2}$.*

Mathematical descriptions and added details of Thm. 4 are in the Appendix. For now, treat this decomposition as separating "well behaved" from "badly behaved" profile components. Assisting the use of Thm. 4 is that to have extreme cyclic tallies, *a profile must involve multiples of the same $\mathcal{RWC}_N$.* Otherwise some pair's tally is weakened by how it appears in a different $\mathcal{RWC}_N$ (Prop. 1) or in a $\mathcal{ST}_N$.

2.4. **A fundamental property.** A standard mystery is where a ranking changes by dropping an alternative. These difficulties are strictly caused by $\mathcal{RWC}_N$ terms.

Theorem 5. *Let $N \geq 3$. Dropping an alternative from a $\mathcal{ST}_N$ profile creates a $\mathcal{ST}_{N-1}$ profile; when X and Y are in both profiles, $\tau(X, Y)$ has the same value. But dropping an alternative from a $\mathcal{RWC}_N$ profile creates the sum of $\mathcal{RWC}_{N-1}$ and $\mathcal{ST}_{N-1}$ terms; it is a $\mathcal{RWC}_{N-1}$ with one ranking repeated.*

To illustrate, dropping alternative D from the $\mathcal{ST}_4$ profile in Eq. 7 results in

(9)

No.	Ranking	No.	Ranking	No.	Ranking
3	$A \succ B \succ C$	2	$C \succ B \succ A$	1	$A \succ C \succ B$
4	$C \succ A \succ B$	4	$A \succ C \succ B$		

To show that Eq. 9 is a $\mathcal{ST}_3$, the rankings from the Eq. 3 $\mathcal{RWC}_3$ are in the first column of Eq. 9; all other rankings come from the Eq. 4 $\mathcal{RWC}_3$. As there are seven rankings in each set, the Eq. 9 profile is $\mathcal{ST}_3$. In Eqs. 7 and 9, $\tau(A,B) = 10$, $\tau(A,C) = 2$, $\tau(B,C) = -8$.

However, dropping alternative D from the $\mathcal{RWC}_4$

$$A \succ B \succ C \succ D,\ B \succ C \succ D \succ A,\ C \succ D \succ A \succ B,\ D \succ A \succ B \succ C, \tag{10}$$

results in

$$A \succ B \succ C,\ B \succ C \succ A,\ C \succ A \succ B, \text{ and } A \succ B \succ C, \tag{11}$$

where the first three Eq. 11 terms define a $\mathcal{RWC}_3$, but the extra term does not. This last term is a combination of $\mathcal{RWC}_3$ and $\mathcal{ST}_3$ terms (Thm. 4), which is consistent with Thm. 5. (The $A \succ B \succ C$ decomposition is in the Appendix.) Illustrating typical problems with the plurality vote, the Eq. 10 outcome is a complete tie, but, because of the extra ranking, the Eq. 11 outcome is the different $A \succ B \sim C$.

With a standard paired comparison rule, say an agenda, its $\mathcal{ST}_N$ outcome remains unchanged after dropping an alternative or changing the order in which they are compared (Thm. 5). This need not be true if a profile has a $\mathcal{RWC}_N$ component. To illustrate, consider the five voter profile where four preferences are from the Eq. 10 $\mathcal{RWC}_4$ and the fifth is $A \succ B \succ C \succ D$. To see how the order of comparison matters, the $[C, B, A, D]$ agenda (the $\{B, C\}$ winner is advanced to be compared with A, that winner is compared with D for the final outcome) has D as the winner, while C wins with $[B, A, D, C]$. Dropping an alternative can also change the outcome; with $[D, A, C, B]$, A loses in the first vote and B is the winner, but dropping D makes A the winner with $[A, C, B]$.

So if a rule's ranking changes just by dropping an alternative, if cycles arise, or if the order of comparison matters, the problems are caused by $\mathcal{RWC}_N$ terms. As a corollary, to modify or eliminate these behaviors, a rule must be able to regulate and modify $\mathcal{RWC}_N$ components. But with the connected structure of $\mathcal{RWC}_N$ terms, this is impossible to achieve for a rule, such as an agenda, that stresses what happens with individual, separate pairs. This statement captures the essence of Arrow's Theorem, so it is discussed first.

3. Analyzing general results

A message from Thm. 4 is that, to analyze a paired comparison rule, first determine whether $\mathcal{ST}_N$ portions of a profile avoid difficulties. If so, then the $\mathcal{RWC}_N$ components cause all complexities suffered by the rule. By knowing which portion of a profile to emphasize, simple explanations may be found.

Arrow's Impossibility Theorem [1] is subject to Thm. 4 because it requires group decisions to be constructed in terms of paired comparisons.

(1) (Pareto) If everyone ranks a *pair* in the same manner, this common ranking is the pair's societal ranking.
(2) (IIA) The societal ranking of each *pair* depends only on how the voters rank this particular pair; all other information is irrelevant.

Thus an Arrovian rule adopts a reductionist philosophy of seeking a societal ranking by determining each pair's ranking. (Information about Arrow's theorem can be found in many references; e.g., [1].) Following the template of how to use Thm. 4, the next result shows that Arrow's conditions never cause problems with $\mathcal{ST}_N$ profiles.

Theorem 6. *For any $N \geq 3$ and $\mathcal{ST}_N$ profiles, a simple majority vote satisfies Arrow's* [1] *conditions for a social welfare function.*

The proof of Thm. 6 follows from Thm. 5, which ensures that paired outcomes for strongly transitive profiles agree with the outcomes after alternatives are dropped.

This result means that Arrow's negative assertion is caused by $\mathcal{RWC}_N$ portions of a profile (Thms. 4, 6). Knowing the source of problems makes conclusions easier to understand. For instance, detecting $\mathcal{RWC}_N$ components requires using more information from a profile than allowed by IIA; IIA restricts attention to single pairs. To illustrate the problem, consider voting rules that satisfy anonymity and neutrality (where, respectively, names of voters and alternatives do not matter). If such a rule satisfies IIA, then when finding the $\{A, B\}$ societal ranking, it is impossible for the rule to distinguish the Eq. 1 $\mathcal{RWC}_6$ from where five voters prefer $A \succ B \succ C \succ D \succ E \succ F$ and the last voter has the reversed ranking; IIA requires both profiles to have the same $\{A, B\}$ ranking. Whatever the non-tied $\{A, B\}$ societal ranking, neutrality and anonymity force the Eq. 1 outcome to be a cycle.

With general non-dictatorial rules, if the $\{A, B\}$ outcome for Eq. 1 is not $A \succ B$, then it becomes easy to design profiles with paradoxical outcomes. But to avoid a cycle, at some point the rule must accept the ranking of a particular single voter with contrary views; this makes the rule equivalent to being dictatorial. In other words, for a rule to recognize and react to a $\mathcal{RWC}_N$ portion of a profile, information from more than just individual pairs must be used: IIA prohibits this.

How to replace Arrow's negative assertion with positive conclusions now is obvious: Find ways to negate the impact of $\mathcal{RWC}_N$ portions of a profile. This, along with showing that Arrow's result does not mean as commonly accepted, is in [8, Chap. 2].

4. Supermajority votes: Nakamura's number

Nakamura sought the maximum number of alternatives, N, with which it is impossible to create a (q, n)-rule cycle. But by knowing that paired comparison cycles are caused by $\mathcal{RWC}_N$ profile components, his concern now is easy to answer; i.e., find the critical value of N so that appropriate multiples of a $\mathcal{RWC}_N$ never admit (q, n)-rule cycles.

To illustrate by finding this N value for the $(90, 100)$ rule, it follows from Thm. 4 that to create an 11-alternative cycle, use a $\mathcal{RWC}_{11}$. This choice yields a cycle with 10:1 tallies, so the $q = 90$ value requires using $\frac{90}{10} = 9$ copies of the $\mathcal{RWC}_{11}$. As a $\mathcal{RWC}_{11}$ has eleven voters, these copies define the preferences for 99 of the 100 voters; the last voter's preferences can be anything. Similarly with $N = 10$; as the $\mathcal{RWC}_{10}$ cycle has 9:1 tallies, $\frac{90}{9} = 10$ copies are needed. Each $\mathcal{RWC}_{10}$ has ten voters, so the preferences of all 100 voters are defined. With $N = 9$, the $\mathcal{RWC}_9$ cycle has 8:1 tallies, so $\lceil \frac{90}{8} \rceil = 12$ copies are needed. But twelve copies requires $12 \times 9 = 108$ of the 100 voters, so it is impossible to create a profile.

To summarize, a $\mathcal{RWC}_N$ profile defines a cycle assigning $N - 1$ points to each pair's winning alternative. A (q, n) rule cycle requires $\lceil \frac{q}{N-1} \rceil$ multiples of the N-voter $\mathcal{RWC}_N$. The full profile cannot have more than n-voters, so it must be that $\lceil \frac{q}{N-1} \rceil N \leq n$. By solving for N, it follows that *a (q, n) cycle can be created if and only if*

$$N \geq \lceil \frac{n}{n-q} \rceil. \tag{12}$$

These results explain and prove Thm. 1.

This computation provides a new interpretation of $\nu(q, n)$; it identifies which $\mathcal{RWC}_N$ (i.e., $N = \nu(q, n)$) and how many copies (i.e., $\lceil \frac{q}{\nu(q,n)-1} \rceil$) are needed to create worst case scenarios. For this reason, expect all (q, n) problems to be caused by $\mathcal{RWC}_{\nu(q,n)}$ terms.

Theorem 7. *For a (q, n) rule with $\nu(q, n)$ alternatives, a profile with $\lceil \frac{q}{\nu(q,n)-1} \rceil$ copies of a $\mathcal{RWC}_{\nu(q,n)}$ creates a q-rule cycle. A cycle cannot occur with $N < \nu(q, n)$ alternatives.*

4.1. **Likelihood.** While cycles can occur with $\nu(q, n)$ alternatives, are they robust or likely to arise? Answers, which depend on whether $\frac{n}{n-q}$ is an integer, suggest exercising caution when using Nakamura's number to anticipate voting difficulties.

Definition 3. *If $\frac{n}{n-q}$ is an integer for a (q,n) rule, call it a $\nu(q,n)$ bifurcation value.*

To motivate what follows, a $(150,200)$ rule has a $\nu(150,200)=4$ bifurcation value. To create a four-alternative cycle, use $\lceil\frac{q}{\nu(q.n)-1}\rceil = \frac{150}{3} = 50$ multiples of a $\mathcal{RWC}_4$ profile (Thm. 7). So let each of the $\mathcal{RWC}_4$ rankings

$$A\succ B\succ C\succ D,\ B\succ C\succ D\succ A,\ C\succ D\succ A\succ B,\ D\succ A\succ B\succ C$$

be supported by 50 voters. Here, A beats B, B beats C, C beats D, and D beats A where the winning candidate in each election receives *precisely* 150 of the 200 votes.

"Precisely" is emphasized because should even *one* of the two hundred voters change her preference ranking in any manner, the cycle disappears. If, for instance, a voter preferring $A\succ B\succ C\succ D$ just interchanges the ranking of her bottom two candidates to create the $A\succ B\succ D\succ C$ preferences, in the $\{C,D\}$ vote C receives 149 votes while D receives 51. By failing to reach the $q=150$ threshold, the societal ranking is $C\sim D$, leading to the $D\succ A, A\succ B, B\succ C$ and $C\sim D$ conclusion that (barely) avoids the cycle and identifies D as the sole candidate who cannot be beaten; i.e., $\mathbb{C}(300,400)=\{D\}$.

This example identifies a limitation in using $\nu(q,n)$ to indicate problems with cycles; $\nu(q,n)$ can represent highly delicate, unrealistic settings rather than a robust description of what to expect. In particular, with rare exceptions (only six profiles out of the more than a trillion trillions of possibilities), expect $\mathbb{C}(300,400)\neq\emptyset$ with four alternatives. Namely, *at a $\nu(q,n)$ bifurcation value, cycles with $\nu(q,n)$ alternatives are unlikely to arise.*

Quirks of this instability are captured by the related $(301,400)$ where a slightly larger q creates the larger $\nu(301,400)=5$ that avoids four-alternative cycles. It follows that worst case $(301,400)$ scenarios (Thm. 7) involve $\mathcal{RWC}_5$ profiles such as:

(13)

Number	Ranking	Number	Ranking
80	$A\succ B\succ C\succ D\succ E$	80	$B\succ C\succ D\succ E\succ A$
80	$C\succ D\succ E\succ A\succ B$	80	$D\succ E\succ A\succ B\succ C$
80	$E\succ A\succ B\succ C\succ D$		

Each pair's tally in the promised $A\succ B$, $B\succ C$, $C\succ D$, $D\succ E$, $E\succ A$ cycle is 320 to 80, which significantly exceeds $q=301$. So, instead of representing a highly delicate setting, the preference rankings of 19 of these voters can be changed in any manner, and the altered profile still supports the cycle.

This $\lceil\frac{q}{\nu(q,n)-1}\rceil - \frac{q}{\nu(q,n)-1} > 0$ gap admits still other supporting profiles. Instead of the Thm. 7 value of $\lceil\frac{q}{\nu(q,n)-1}\rceil$, using $\lceil\frac{q}{\nu(q,n)-1}\rceil - 8 = 68$ copies of a $\mathcal{RWC}_5$ provides $68\times 4 = 272$ votes for each pair's winner in a cycle. Although shy of the $q=301$ quota, only $68\times 5=340$ voters are used. From the 60 others, let 29 prefer $A\succ B\succ$

$C \succ D \succ E$, 29 have any ranking with $E \succ A$, and assign any ranking to the last two to create a supporting profile.

The larger the $\nu(q,n) - \frac{n}{n-q} > 0$ gap, the larger the set of supporting profiles. To illustrate with $\nu(300+x, 400) = 5$ for $1 \leq x \leq 20$, supporting examples involve $\mathcal{RWC}_5$ components. But as x approaches the next bifurcation value ($x = 20$ or $q = 320$), examples become more specialized, so cycles are more unlikely. Even stronger, at a bifurcation value, cycles are not a *generic property*, where "generic" means that the property is "expected;" e.g., the property holds almost everywhere and even after the preferences of any agent are changed.

Theorem 8. *At a $\nu(q,n)$ bifurcation value, $\frac{q}{\nu(q,n)-1} = n-q$, and the profile defining a q-rule cycle must consist of $n-q$ copies of a $\mathcal{RWC}_{\nu(q,n)}$. With any standard probability assumption over profiles, cycles are unlikely; the generic property is that $\mathbb{C}(q,n) \neq \emptyset$.*

5. Supermajority votes: Profile restrictions

Cycles can be avoided without restricting the number of alternatives by limiting which profiles are admissible. According to Thm. 4, this approach will succeed only by restricting profiles in a manner to prevent $\mathcal{RWC}_N$ terms from creating cycles.

As it turns out, profile restrictions and Nakamura avoid cycles in the same way: Both temper the strong $\mathcal{RWC}_N$ tallies that create cycles to a more benign $\mathcal{RWC}_{N-1}$ level. Namkamura does this by restricting the number of alternatives. As Prop. 2 shows, dropping s rankings from a $\mathcal{RWC}_N$ reduces some tallies to a $\mathcal{RWC}_{N-s}$ level.

Proposition 2. *Removing s rankings from a $\mathcal{RWC}_N$ defined by $a_1 \succ a_2 \succ \cdots \succ a_N$ causes s pairs of adjacently listed candidates to have an unanimous $a_j \succ a_{j+1}$ outcome. The common tally for all remaining $a_j \succ a_{j+1}$ outcomes is [N-(1+s)]:1.*

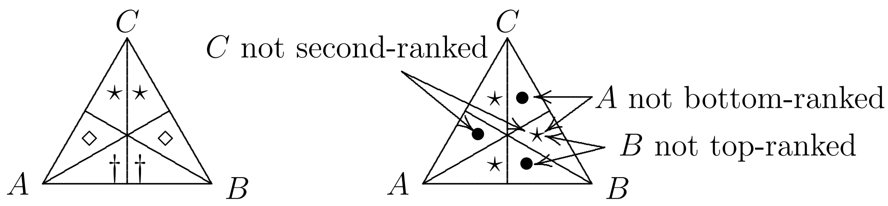
Proof: Let a_{j+1} be the top-ranked candidate in a removed ranking; only in this ranking is a_{j+1} is ranked above a_j in $\mathcal{RWC}_N$, so a_j beats a_{j+1} with an unanimous vote. There are precisely s such pairs. For all other pairs of adjacently ranked candidates, a_{j+1} beats a_j precisely once in the $N-s$ remaining rankings, so $a_j \succ a_{j+1}$ has a $(N-s)-1$:1 tally. □

5.1. **Ward's conditions.**

A well known profile restriction is Black's single-peaked condition [2], which, with three candidates, is where some candidate never is bottom ranked by even one voter. The "Condorcet-domain problem" (e.g., Fishburn [3], Monjardet [5]) extends Black's condition by seeking appropriate restrictions on N alternative rankings so that no matter how many voters are assigned to the admitted rankings, majority vote cycles never occur.

"Ward's conditions" (Ward [11]) solve the Condorcet-domain problem for three candidates. My description uses the (Saari [7]) geometric representation of profiles.

Here, a point's ranking in the Fig. 2a equilateral triangle is determined by its distance to each vertex; e.g., a point in the triangle with a dagger in the lower left corner is closest to A, next closest to B, and farthest from C, so it has the $A \succ B \succ C$ ranking.



a. Ward's conditions **b.** Explanation

Figure 2. Condorcet domains

Ward's constraints are:

(1) *No voter has a particular candidate top-ranked.* The Fig. 2a stars represent rankings where C is top-ranked. If no voter has either ranking, Ward's condition is satisfied. Rankings in the two regions sharing the vertex with a candidate's name have her top-ranked, so if no voter has rankings in either region adjacent to a particular vertex, this condition is satisfied.

(2) *No voter has a particular candidate middle-ranked.* The Fig. 2a diamonds are where C is middle-ranked. If no voter has either ranking, Ward's condition is satisfied. As "middle-ranked" involves diametrically opposite regions, if there are two ranking regions diametrically opposite each other (so the rankings reverse each other) without a single voter, this Ward condition is satisfied.

(3) Black's condition: *No voter has a particular candidate bottom-ranked.* The Fig. 2a daggers represent where C is bottom ranked. "Bottom-ranked" regions are the two regions farthest from the vertex assigned to the candidate, so if no voter has these rankings, Black's and Ward's conditions are satisfied.

Ward proved that if a profile satisfies one of these conditions, a majority vote cycle cannot occur. An explanation follows from Thm. 4 and Prop. 2. Namely, majority voting cycles require a profile to include a sufficiently large $\mathcal{RWC}_3$ multiple. The Eq. 3 $\mathcal{RWC}_3$ triplet in Fig. 2b is indicated by the bullets; the Eq. 4 $\mathcal{RWC}_3$ triplet has the stars. For a profile to contain a $\mathcal{RWC}_3$ triplet, it must have voters with rankings represented by *all three bullets*, or *all three stars.*

Ward's conditions succeed by *preventing a profile from including a $\mathcal{RWC}_3$ triplet.* To illustrate with Fig. 2b, if, for instance, no voter has B top-ranked, then the two ranking regions sharing the B-vertex are not assumed by any voter. One region has a bullet and the other a star, so this condition prohibits a profile from including either $\mathcal{RWC}_3$ triplet. Similarly, if C never is middle-ranked, then, as indicated in Fig. 2b, no voter has the ranking with the indicated bullet or the ranking represented by diametrically opposite region with a star, which disrupts the possibility of including a $\mathcal{RWC}_3$ triplet. The same analysis holds for the never bottom-ranked requirement,

Ward's conditions are the sharpest possible to ensure a Condorcet domain. To see why, if no voter's preference is in a starred region, say the one on the Fig. 2b right-hand side, then a profile cannot include the starred $\mathcal{RWC}_3$ triplet. To preclude the profile from including the other $\mathcal{RWC}_3$ triplet, exclude *any region* with a bullet. For each choice, the Fig. 2b arrows indicate which Ward condition is satisfied.

Sen [10] developed necessary and sufficient conditions for a Condorcet domain for $N \geq 4$ alternatives: When restricting a profile to each triplet, it must satisfy at least one of Ward's conditions. Sen's result is subsumed and extended by Thm. 9.

5.2. **General conditions.** As shown (Sect. 5.1), Ward's conditions ensure a core by prohibiting a profile from including a full $\mathcal{RWC}_3$ triplet. Theorem 9 extends these results to all (q, n) rules by suppressing (Prop. 2) the impact of $\mathcal{RWC}_{\nu(q,n)}$ terms.

Theorem 9. *For $r \geq 2$ alternatives, a necessary and sufficient condition for a set of ranking to avoid (q, n)-rule cycles and have $\mathbb{C}(q, n) \neq \emptyset$ independent of how voters are assigned to the rankings, is that, when restricted to each subset of $\nu(q, n)$ alternatives, at least one ranking is missing from each possible $\mathcal{RWC}_{\nu(q,n)}$.*

Nakamura's Theorem a special case because Thm. 9 trivially holds for $r < \nu(q, n)$; for $r \geq \nu(q, n)$, cycles can occur. Ward's result is a special case because, with $\nu(2, 3) = 3$, Thm. 9 requires dropping at least one ranking from each of the $\mathcal{RWC}_3$'s; this is Ward's requirement (Sect. 5.1). Sen's result is subsumed because for $n \geq 3$, the majority vote requires $\nu(\lceil \frac{n+1}{2} \rceil, n) = 3$; thus Thm. 9 requires applying Ward's condition to all triplets.

With $r > \nu(q, n)$ alternatives, one might expect to impose profile restrictions on $\mathcal{RWC}_r$ rather than the $\nu(q, n)$ parts. To explain with an example why the emphasis must be on $\mathcal{RWC}_{\nu(q,n)}$, consider $\nu(18, 24) = 4$ with the $r = 5$ alternatives A, B, C, D, E. An extremely severe profile restriction would allow at most one ranking to come from each $\mathcal{RWC}_5$. The following four rankings (each from a different $\mathcal{RWC}_5$), for instance, satisfy this tight constraint.

$$E \succ [A \succ B \succ C \succ D], \quad E \succ [B \succ C \succ D \succ A], E \succ [C \succ D \succ A \succ B],$$
$$E \succ [D \succ A \succ B \succ C].$$

But as these rankings embed the $\mathcal{RWC}_4$ generated by $[A \succ B \succ C \succ D]$, assigning $\frac{18}{3} = 6$ voters to each ranking creates a $(18, 24)$-rule cycle. With $r > \nu(q, n)$, then, restrictions on $\mathcal{RWC}_r$ structures are insufficient to prevent (q, n)-cycles; attention must be focussed on all subsets of $\nu(q, n)$ alternatives to prevent $\mathcal{RWC}_{\nu(q,n)}$ terms.

An extension of Ward's condition follows:

Corollary 1. *With $\nu(q,n)$ alternatives, a q-rule cycle never occurs if, for some integer $s \in \{1, 2, \ldots, \nu(q,n)\}$, there is an alternative that never is s^{th} ranked.*

In a $\mathcal{RWC}_N$, each alternative is ranked in each position once, so Cor. 1 ensures that a profile cannot contain a $\mathcal{RWC}_{\nu(q,n)}$. To see why this is not a necessary condition, with $\nu(q,n) = 4$ alternatives and for the six $\mathcal{RWC}_4$'s generated by $D \succ A \succ C \succ B, D \succ C \succ A \succ B, D \succ A \succ B \succ C, D \succ B \succ A \succ C, A \succ B \succ C \succ D$, and $A \succ C \succ B \succ D$, drop the generating ranking. This choice satisfies Thm. 9 but not Cor. 1.

6. When a core always exists

While Thm. 9 subsumes Nakamura's and Sen's results, the wording does not include Greenberg's result (Thm. 2). This is remedied by the following:

Theorem 10. *A necessary and sufficient condition for $\mathbb{C}(q,n) \neq \emptyset$ for any positioning of the n ideal points in a domain $\mathcal{D}$ is if it is impossible to construct a setting with $\beta = \nu(q,n)$ ideal points in $\mathcal{D}$ where $\mathbb{C}(\beta - 1, \beta) = \emptyset$.*

The close connection with Thm. 9, which excludes $\mathcal{RWC}_{\nu(q,n)}$ terms, is that Thm. 10 is equivalent to requiring that *a $\nu(q,n)$-person $\mathcal{RWC}_{\nu(q,n)}$ cannot be constructed in $\mathcal{D}$.* No conditions are imposed on the structure of $\mathcal{D}$, so nothing prevents $\mathcal{D}$ from consisting of preference rankings where a voter's ideal point is his preference ranking. By using different choices of $\mathcal{D}$, Thm. 10 subsumes Greenberg's theorem and Thm. 9 (which includes Nakamura's and Sen's results).

Theorem 10 simplifies the analysis by replacing (q,n) problems with $(\beta-1,\beta)$ settings. To illustrate by determining how many alternatives always allow $\mathbb{C}(200, 225) \neq \emptyset$, because $\nu(200, 225) = 5$, Thm. 10 reduces the analysis to finding the number of alternatives for which $\mathbb{C}(4, 5) \neq \emptyset$ always is true. A $\mathcal{RWC}_5$ requires five alternatives, so (Thm. 4, Props. 1 and 2) the answer is four; thus four or fewer alternatives ensures that $\mathbb{C}(200, 225) \neq \emptyset$. A related argument with any (q,n) shows how Nakamura's result is a special case of Thm. 10.

Similarly, to find profile restrictions so that $\mathbb{C}(200, 225) \neq \emptyset$ with ten alternatives, Thm. 10 converts the analysis to $(4, 5)$ rules. A five-voter $(4, 5)$ cycle is prevented only if the profile does not contain a $\mathcal{RWC}_5$ for any choice of five alternatives; this is the answer for both $(4, 5)$ and $(200, 225)$. In this manner, Thm. 9 is included in Thm. 10.

To include Greenberg's result (Thm. 2), the following proposition uses the fact that a Pareto point for coalition $\mathcal{C}$ is a point that, if altered in any manner, creates a poorer outcome for some member in $\mathcal{C}$. Denote the set of all Pareto point for a coalition by $\mathcal{P}(\mathcal{C})$. With (q,n) rules, $\mathcal{P}(\mathcal{C})$ is the convex hull defined by the ideal points of the agents in $\mathcal{C}$ and a minimal winning coalition $\mathcal{P}(\mathcal{C})$ contains q voters.

While the rankings in Prop. 3a represent Euclidean preferences (because they are often used), they can be generalized to continuous, convex utility functions with an ideal point as a bliss point.

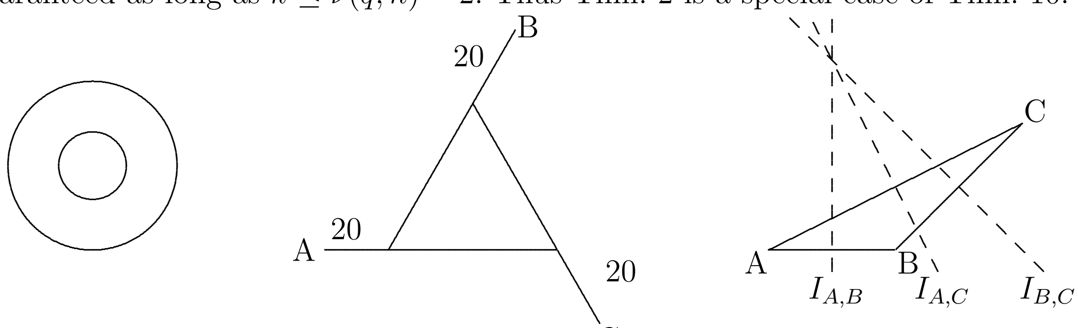
Proposition 3. *a. It is impossible to position N ideal points and N alternatives $\{a_j\}_{j=1}^N$ in $\mathbb{R}^{N-2}$ so that the associated rankings define a $\mathcal{RWC}_N$. It is possible to do so in $\mathbb{R}^{N-1}$.*

b. For (q,n), the core equals

$$\mathbb{C}(q,n) = \cap \mathcal{P}(\mathcal{C}) \tag{14}$$

where the intersection is over all minimal winning coalitions.

A way to avoid q-rule cycles in spatial voting is to restrict the number of issues, k, to prevent constructing a $\mathcal{RWC}_\beta$ from β ideal points (Thm. 10). Proposition 3a ensures this always can be done if $k \leq \beta - 2$. Therefore, a nonempty core is guaranteed as long as $k \leq \nu(q,n) - 2$. Thus Thm. 2 is a special case of Thm. 10.



a. $\mathbb{C}(22,30) \neq \emptyset$ **b.** $\mathbb{C}(40,60) = \emptyset$ **c.** $\mathbb{C}(60,100) \neq \emptyset$

Figure 3. Where $\mathbb{C}(q,n)$ is, and is not empty

Theorem 10 is more inclusive because it does not impose added restrictions on $\mathcal{D}$; e.g., $\mathcal{D}$ need not be a compact, convex set as required by Thm. 2. To illustrate, the Fig. 3a choice of $\mathcal{D}$ is an open annulus representing the selection of a point on an island with a lake in the middle. As this $\mathcal{D}$ is neither convex (the center is removed) nor compact (it is not closed), Thm. 2 does not apply. To prove that, say, $\mathbb{C}(22,30) \neq \emptyset$ no matter where the 30 ideal points are placed in the annulus, just use $\nu(22,30) = 4$ and show that the annulus' two dimensional nature prevents constructing a $\mathcal{RWC}_4$ (Thm. 10). This follows from Prop. 3a.

Perhaps only $\mathcal{D}$'s dimension is necessary. To prove this is false, consider a $(40,60)$ rule with the Fig. 3b one-dimensional triangle $\mathcal{D}$; e.g., a group wishes to select a sidewalk booth spot on a triangular block. As $\nu(40,60) = 3$, determine whether three points (Thm. 10) can be positioned in $\mathcal{D}$ so that $\mathbb{C}(2,3) = \emptyset$. To do so, place an ideal point at each vertex. A minimal winning coalition consists of two points, so each triangle edge is a Pareto set. As the intersection of these three legs is empty,

$\mathbb{C}(2,3) = \emptyset$ (Prop. 3b), which means that sixty ideal points can be positioned on $\mathcal{D}$ so that $\mathbb{C}(40,60) = \emptyset$.

Returning to the US Senate filibuster example where $\nu(60,100) = 3$, let preferences be given by the distance to each alternative and let $\mathcal{D}$ be the Fig. 4c triangle where the vertices are the three alternatives. As specified in Thm. 2, $\mathcal{D}$ is closed, convex, and its two dimensions exceed $\nu(60,100) - 2 = 1$. But as the $X \sim Y$ indifference lines, denoted by $I_{X,Y}$, show by allowing only four of the six rankings, it is impossible to position three ideal points in $\mathcal{D}$ to create a cycle. Thus (Thm. 10) it always is true that $\mathbb{C}(60,100) \neq \emptyset$. This setting differs from Thm. 2 by specifying the positions of the alternatives; this makes Thm. 10, rather than Thm. 2, the appropriate choice to determine whether a core must be nonempty.

7. Summary

Theorem 4 is the key that unlocks long standing mysteries about paired comparisons. This result separates profiles into components that never cause difficulties and $\mathcal{RWC}_N$ components that cause all of the troubles. By knowing what creates difficulties, this decomposition makes it easier to understand why problems occur, it unites seemingly dissimilar conclusions, and it permits several standard results to be extended. The emphasis of this paper is to unify and extend certain better known "worse case" conclusions. Elsewhere, the Thm. 4 structure is being used to extend other paired comparison assertions.

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8. Appendix

Proof of Prop. 1: To tally $\{a_j, a_{j+s}\}$ for the $\mathcal{RWC}_N$ generated by $a_1 \succ a_2 \succ \cdots \succ a_N$, as a_{j+s} is ranked above a_j in s of the $\mathcal{RWC}_N$ rankings, the outcome for $s < \frac{N}{2}$ is $a_j \succ a_{j+s}$ with tally $N - s{:}s$. If $s = \frac{N}{2}$, the outcome is a tie. $\square$

Proof of Thms. 3 and 4: While both theorems follow from statements in [9] (but here the more suggestive "strongly transitive" replaces the "Borda Profiles"), a shorter, simpler, more direct proof is provided. When expressing profiles in vector notion (Sect. 2.3), some components can be negative. For an interpretation, notice that both profiles $\mathbf{p}_1 = (4, 2, 1, 6, 5, 5)$ and $\mathbf{p}_2 = (10, 4, 0, 7, 2, 0)$ have 23 voters. Thus, vector $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1 = (6, 2, -1, 1, -3, -5)$ describes how preferences change moving from profile $\mathbf{p}_1$ to $\mathbf{p}_2$.

For $N = 3$, profiles that do not affect any positional or paired comparison outcomes are positive multiples of $\mathbf{E} = (1, 1, 1, 1, 1, 1)$, so $\mathbf{E}$ serves as an "origin". Because $\mathbf{E} - (1, 0, 1, 0, 1, 0)$ (from Eq. 3) equals $(0, 1, 0, 1, 0, 1)$, it follows from a vector analysis perspective that both vectors span the same profile component space. Thus, the direction of $\mathcal{RWC}_3$ terms in profile space is given by $\mathbf{c} = (1, -1, 1, -1, 1, -1)$; this difference between the Eq. 3 and 4 profiles represents moving preferences from one $\mathcal{RWC}_3$ to the other. The space of $\mathcal{RWC}_3$ profiles has dimension $\frac{(3-1)!}{2} = 1$, which agrees with Thm. 4. All vectors orthogonal to $\mathbf{c} = (1, -1, 1, -1, 1, -1)$ have dimension $3! - 1 = 5$, which also agrees with Thm. 4.

It remains to show that if vector $\mathbf{x} = (x_1, x_2, x_3, x_4, , x_5, x_6)$ is orthogonal to $\mathbf{c}$, it satisfies the balancing condition and is $\mathcal{ST}_3$. The orthogonality condition requires $x_1 + x_3 + x_5 = x_2 + x_4 + x_6$, which is the balancing condition. To show that the $\mathcal{ST}$ condition is satisfied, rewrite the orthogonality condition as

$$(x_1 - x_4) + (x_3 - x_6) = (x_2 - x_5). \tag{15}$$

Because $\tau(A, B) = (x_1 + x_2 + x_3) - (x_4 + x_5 + x_6)$ and $\tau(B, C) = (x_1 + x_5 + x_6) - (x_2 + x_3 + x_4)$, it follows that $\tau(A, B) + \tau(B, C) = 2(x_1 - x_4)$. Thus profile $\mathbf{x}$ is $\mathcal{ST}_3$ if and only if $2(x_1 - x_4) = \tau(A, C) = (x_1 + x_2 + x_6) - (x_3 + x_4 + x_5)$, which (by collecting terms) holds if and only if $(x_1 - x_4) + (x_3 - x_6) = (x_2 - x_5)$. As this is Eq. 15, the profile is $\mathcal{ST}_3$ if and only if it is orthogonal to $\mathbf{c}$ – the balancing condition. This completes the proofs for $N = 3$.

For $N \geq 3$, select a $\mathcal{RWC}_N$ defined by a complete transitive ranking $\mathbf{r}_N$, and compute the $\mathcal{RWC}_N$ defined by the reversal of $\mathbf{r}_N$. Let $\mathbf{E}_{\mathbf{r}_N}$ be the vector that has unity for any ranking that is in one of these two $\mathcal{RWC}_N$'s, and zero for all other components. Because every ranking in $\mathbf{E}_{\mathbf{r}_N}$ is accompanied by its reversal, for this profile $\tau(X,Y) = 0$ for all pairs. Because $\mathbf{E}_{\mathbf{r}_N}$ minus the vector representing one $\mathcal{RWC}_N$ is the vector representing the other one, the appropriate vector capturing this collection is the difference between these two vectors; represent it by $\mathbf{RWC}_N$. As there are $N!/N$ choices of $\mathcal{RWC}_N$, this combination defines $(N-1)!/2$ $\mathbf{RWC}_N$ directions for the cyclic terms. Thus, the vectors orthogonal to this space reside in a $N! - \frac{(N-1)!}{2} = \frac{(2N-1)(N-1)!}{2}$ dimensional space. This is the Thm. 4 assertion after it is shown that these vectors represent $\mathcal{ST}_N$ profiles.

The stronger, actual result asserts that various parts of such a profile are $\mathcal{ST}_N$, so the sum also is $\mathcal{ST}_N$.

Lemma 1. *If $\mathbf{x}$ is orthogonal to $\mathbf{RWC}_N$, define $\mathbf{x}_{RWC}$ to have the $\mathbf{x}$ component for each ranking represented in $\mathbf{RWC}_N$, and all other components are zero. Then $\mathbf{x}_{RWC}$ is $\mathcal{ST}_N$.*

So if a profile with n_1 voters preferring $A \succ B \succ C \succ D$, n_2 voters preferring $D \succ A \succ B \succ C$, n_3 voters preferring $A \succ C \succ D \succ B$ and n_4 voters preferring $D \succ C \succ A \succ B$ is orthogonal to the cyclic space, then the original profile, the profile consisting only of the first two types, and the profile consisting of only the last two types are all $\mathcal{ST}_4$.

To prove the lemma, suppose a $\mathbf{x}_{RWC}$ is given, where the goal is to prove for any X, Y, Z triplet that $\tau(X,Y) + \tau(Y,Z) = \tau(X,Z)$. Because the alternatives are not specified, we can assume that $X \succ Y \succ Z$ in the defining ranking $\mathbf{r}$ of the $\mathcal{RWC}_N$. Suppose that the coordinate system is such that there are x_j voters of the j^{th} ranking defined by the $\mathcal{RWC}_N$ and x_{j+N} of the reversal of this ranking. The orthogonality condition requires

$$\sum_{j=1}^{N}(x_j - x_{j+N}) = 0. \tag{16}$$

In computing the contributions of the j^{th} ranking to $\tau(X,Y), \tau(Y,Z), \tau(X,Z)$ values, if this ranking has the $X \succ Y \succ Z$ configuration, then all contributions are $x_j - x_{j+N}$. If the ranking has a $Y \succ Z \succ X$ configuration, the contributions are, respectively, $-(x_j - x_{j+N})$, $x_j - x_{j+N}, -(x_j - x_{j+N})$. The final possibility allowed by the ranking wheel construction is if the j^{th} ranking has a $Z \succ X \succ Y$ configuration; here the contributions are $x_j - x_{j+N}, -(x_j - x_{j+N}), -(x_j - x_{j+N})$. For the j^{th} ranking, the sum of the first two minus terms the third equals $x_j - x_{j+N}$. Summing over all j is $\sum_{j=1}^{N}(x_j - x_{j+N})$, which (Eq. 16) equals zero. But carrying out this summation by first determining the contribution to $\tau(X,Y)$, then $\tau(Y,Z)$, and then $-\tau(X,Z)$ terms, we have that $\tau(X,Y) + \tau(Y,Z) - \tau(X,Z) = 0$, which is what was needed to be proved.

Decomposing a profile: The decomposition of a profile into its $\mathcal{RWC}_N$ and $\mathcal{ST}_N$ components is based on the linear algebra, vector space representation. Illustrating with $N = 3$ and $(1, 0, 0, 0, 0, 0)$, which represents the unanimity $A \succ B \succ C$, the component in the $\mathbf{c}$ direction is its scalar product with the unit vector $\frac{\mathbf{c}}{\sqrt{6}}$, which is $6^{-1/2}$, so the component is $\frac{1}{6}\mathbf{c}$. Thus the strongly transitive part is $\frac{1}{6}(5, -1, -1, -1, -1, -1)$. (Fractional components have the same meaning as fractional appointments for faculty positions.)

Proof of Thm. 5: Consider the $\mathcal{RWC}_N$ generated by $a_1 \succ a_2 \succ \cdots \succ a_N$, and suppose a_k is dropped. The same $\mathcal{RWC}_N$ is generated by $a_k \succ a_{k+1} \succ \cdots \succ a_N \succ a_1 \succ \dots a_{k-1}$, so dropping a_k creates a $\mathcal{RWC}_{N-1}$ generated by $a_{k+1} \succ \cdots \succ a_N \succ a_1 \succ \dots a_{k-1}$ and another copy of this ranking. As illustrated with the above decomposition, the remaining ranking has $\mathcal{RWC}_N$ and $\mathcal{ST}_N$ components.

Now drop an alternative from a $\mathcal{ST}_N$ profile. Each $\tau(X, Y)$ value only depends on the relative $\{X, Y\}$ ordering in each ranking; it is not affected by any other alternatives, including the dropped one. Thus the $\tau(X, Y)$ value remains the same. This means that Eq. 6 is satisfied, so the new profile is $\mathcal{ST}_{N-1}$.

Proof of Thm. 8: If $\frac{n}{n-q} = \nu(q, n)$ is an integer, substituting it into $\frac{q}{\nu(q,n)-1}$ leads to $n - q$. As $\frac{q}{\nu(q,n)-1}$ specifies the number of copies of $\mathcal{RWC}_{\nu(q,n)}$, and as solving $\frac{n}{n-q} = \nu(q, n)$ for n leads to $\frac{q}{\nu(q,n)-1}\nu(q, n) = n$, it follows that in designing the profile, there are no extra voters. A simple computation shows that if this profile is altered in any way, the q-rule cycle disappears. The rest of the proof follows from the material prior to stating the theorem.

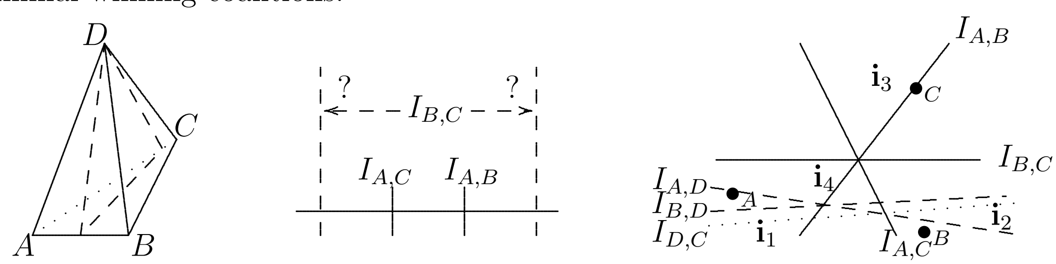
Proof of Thm. 9: Assume a set of rankings for $r \geq \nu(q, n)$ alternatives is given. If a set of $\nu(q, n)$ alternatives exists where a $\mathcal{RWC}_{\nu(q,n)}$ has all $\nu(q, n)$ rankings, then assign $\frac{q}{\nu(q,n)-1}$ voters to each ranking. Because of the structure, the appropriate number of times each candidate will defeat its adjacent partner in the $\mathcal{RWC}_{\nu(q,n)}$ (whether or not they are adjacent in the given ranking) forces the q-rule cycle.

In the opposite direction, assume when restricting a given set of rankings to any subset of $\nu(q, n)$ alternatives, at least one ranking from each $\mathcal{RWC}_{\nu(q,n)}$ is missing. If there are $r = \nu(q, n)$ alternatives, the conclusion follows from Prop. 2. With $r > \nu(q, n)$ alternatives, a cycle can occur with a $\mathcal{RWC}_r$. But because a $\mathcal{RWC}_{\nu(q,n)}$ can be constructed from the $\mathcal{RWC}_r$ by ignoring $r - \nu(q, n)$ variables (Thm. 5), it must be that rankings are removed from the $\mathcal{RWC}_r$, which can be assumed to be generated by $a_1 \succ a_2 \succ \cdots \succ a_r$. According to the proof of Prop. 2, if a_i is the bottom ranked alternative in a dropped ranking, then a_i is not in the cycle. If s rankings are removed, then these s alternatives (bottom ranked in each of he dropped rankings) are removed from a possible cycle. As the structure of the remaining alternatives creates a $\mathcal{RWC}_{r-s}$, the assumption requires $r - s < \nu(q, n)$, or $s = r - \nu(q, n) + \alpha$

where α is a positive integer. Thus the tallies for the alternatives creating a cycle (Prop. 2) are $r-(1-r-\nu(q,n)+\alpha):1$, or $(\nu(q,n)-1-\alpha):1$. This requires $\lceil\frac{q}{\nu(q,n)-1-\alpha}\rceil$ copies of each of of the $\nu(q,n)-\alpha$ rankings for a q-rule cycle. If a cycle could be created, $\lceil\frac{q}{\nu(q,n)-1-\alpha}\rceil(\nu(q,n)-\alpha)\leq n$. Solving for $\nu(q,n)$ leads to the contradiction $\nu(q,n)+\alpha\leq\nu(q,n)$. Hence a cycle cannot occur.

Proof of Thm. 10: If a $\mathcal{RWC}_{\nu(q,n)}$ can be constructed, then, just by assigning the required number of voters, $\lceil\frac{q}{\nu(q,n)-1}\rceil$, at each point, a (q,n)-cycle can be constructed so $\mathbb{C}(q,n)=\emptyset$. If a $\mathcal{RWC}_{\nu(q,n)}$ cannot be constructed, then, as computed in the description, $\mathbb{C}(q,n)\neq\emptyset$.

Proof of Prop. 3b: By definition of a Pareto point, if $\mathbf{p}\in\mathcal{P}(\mathcal{C})$ for a minimal winning coalition $\mathcal{C}$, then there is no alternative that $\mathcal{C}$ would select over $\mathbf{p}$. If $\mathbf{p}\in\cap\mathcal{P}(\mathcal{C})$ for all minimal winning coalitions, $\mathbf{p}$ cannot be beaten, so $\mathbf{p}\in\mathbb{C}(q,n)$. Conversely, if $\mathbf{p}\in\mathbb{C}(q,n)$, it cannot be beaten by any alternative. But if $\mathbf{p}\notin\mathcal{P}(\mathcal{C})$ for minimal winning coalition $\mathcal{C}$, then $\mathcal{C}$ can elect an alternative to $\mathbf{p}$. Thus, $\mathbf{p}\in\mathcal{P}(\mathcal{C})$ for all minimal winning coalitions.



a. The $I_{A,B}$ plane **b.** $N=3$ **c.** From $N=3$ to $N+1=4$

Figure 4. Proving Proposition 3

Proof of Prop. 3a: Position alternatives $\{a_j\}_{j=1}^N$ to define a $(N-1)$-dimensional simplex; each a_j defines a unique vertex. Each edge is defined by two vertices $\{a_i,a_j\}$; construct a plane orthogonal to this edge that passes through its midpoint. This plane, $I_{i,j}$, identifies all points where (with Euclidean preferences), $a_i\sim a_j$. (See Fig. 4a for the construction of the $A\sim B$, or $I_{A,B}$, plane.) These planes intersect to define $N!$ open regions; points in each region define a particular strict ranking of the N alternatives and all rankings are included. To create a $\mathcal{RWC}_N$, place the voter ideal points in the appropriate region. These ideal points also define a $(N-1)$-dimensional simplex where each face is determined by $(N-1)$ ideal points; this face is the Pareto set for this minimal winning coalition. As the faces do not have a common intersection, it follows (Prop. 3b) that the core is empty.

Induction argument: Different induction arguments can be used to complete the proof; all use the fact that a $\{X,Y\}$ pair changes ranking only by crossing $I_{X,Y}$. Start with $N=3$; suppose a $\mathcal{RWC}_3$ generated by $A\succ B\succ C$ is created where the alternatives and ideal points are on a line, where we can assume A is to the right of

C. When moving along the line among the ideal points $\mathbf{i}_{A\succ B\succ C}, \mathbf{i}_{B\succ C\succ A}$, and $\mathbf{i}_{C\succ A\succ B}$, to go from $\mathbf{i}_{A\succ B\succ C}$ to $\mathbf{i}_{B\succ C\succ A}$, $I_{B,C}$ cannot be crossed (if crossed, it would have to be re-crossed to preserve $B \succ C$), so the ideal points are separated by $I_{A,B}$ and $I_{A,C}$ in that order (Fig. 4b). Thus $I_{B,C}$ is either to the left of $I_{A,C}$ or to the right of $I_{A,B}$ (the two Fig. 4b dashed lines). But to go from $\mathbf{i}_{A\succ B\succ C}$ to $\mathbf{i}_{C\succ A\succ B}$, the path must cross $I_{B,C}$ and $I_{A,C}$ in that order, but never $I_{A,B}$. No matter which choice is selected for $I_{B,C}$, because $I_{A,B}$ must be crossed, the goal is geometrically impossible. With an added dimension (Fig. 2b), however, it is easy to construct such paths without crossing a forbidden $I_{X,Y}$.

With the induction hypothesis, assume the result holds for N alternatives; it must be shown it is impossible to construct a $\mathcal{RWC}_{N+1}$ defined by $a_1 \succ \cdots \succ a_N \succ a_{N+1}$ with $N+1$ alternatives and ideal points in $\mathbb{R}^{N-1}$. Label the rankings by $k = 1, \ldots, N+1$ with respective ideal points given by $\mathbf{i}_k$. (With $N = 3$ and Fig. 4c, 1 is the defining ranking $A \succ B \succ C \succ D$ and 4 is the last $D \succ A \succ B \succ C$.) By ignoring any alternative, the resulting rankings include a $\mathcal{RWC}_N$ (Thm. 5), so (induction hypothesis) a geometric representation of these N alternatives must define a $(N-1)$-dimensional simplex (for $N = 3$, a triangle) with the $(N+1)$ ideal points in appropriate sectors.

If a_k is dropped, the $\binom{N}{2}$ indifference planes meet in a common, the "not-a_k hub" denoted by $\tilde{a_k}$. (If they did not meet, they would define more than $N!$ open sectors where some represent non-transitive rankings. For $N = 3$, the three solid indifference lines in Fig. 4c meet at $\tilde{D}$ defining six sectors representing the six strict transitive rankings when D is not considered.) With $N+1$ alternatives, there are $N+1$ "$\tilde{a_k}$" hubs, which must be created (induction hypothesis), so a $a_j \sim a_k$ indifference plane, $I_{j,k}$, must meet $I_{j,x}$ and $I_{k,y}$ for $x, y \neq j, k$. (With $N = 3$, $I_{A,D}$ must meet $I_{A,B}$ and $I_{A,C}$ at, respectively, $\tilde{C}$ and $\tilde{B}$.)

With the structure for N alternatives established by ignoring a_{N+1}, determine where to position the $I_{N+1,k}$ planes as determined by changes in rankings. For each $I_{j,j+1}, j = 1. \ldots, N$, point $\mathbf{i}_{j+1}$ is on one side of this plane, and all other ideal points are on the other (Prop. 1). The following argument builds to a contradiction that both $\mathbf{i}_N$ and $\mathbf{i}_{N+1}$ are on one side of $I_{N,N+1}$, and the other $N-1$ points are on the other side.

Points $\mathbf{i}_1$ and $\mathbf{i}_{N+1}$ are "sector 1" – the sector with ranking 1 (Thm. 5) and defined by the $N-1$ boundaries $\{I_{j,j+1}\}_{j=1}^{N-1}$; the other ideal points are in sectors defined by $\mathcal{RWC}_N$. (For $N = 3$, $\mathbf{i}_1$ and $\mathbf{i}_4$ are in the $A \succ B \succ C$ sector 1, $\mathbf{i}_2$ is in $B \succ C \succ A$ and $\mathbf{i}_3$ is in $C \succ A \succ B$. Sector 1 is bounded by $I_{A,B}$ and $I_{B,C}$.) Only changes in the a_{N+1} ranking can occur in each of the $N!$ sectors, and the intersection of the $I_{N+1,k}$ planes with a sector can create at most $N+1$ regions. (For $N = 3$, the $I_{D,X}$ lines can create at most four regions in any sector; if they allowed five or more, extra regions would represent non-transitive rankings.) In sector 1, $\mathbf{i}_1$ and $\mathbf{i}_{N+1}$ have, respectively, a_{N+1} bottom and top ranked, so any path in this sector connecting $\mathbf{i}_1$ and $\mathbf{i}_{N+1}$ must cross all $\{I_{N+1,k}\}_{k=1}^{N}$ planes. (In Fig. 4c, to move from $D \succ A \succ B \succ C$ to $A \succ B \succ C \succ D$, D reverses rankings with each alternative, so each $I_{D,X}$ meets sector 1.) Except for $\mathbf{i}_1$, all ideal points have $a_{N+1} \succ a_1$, so points $\{\mathbf{i}_j\}_{j=2}^{N+1}$ are on the same $I_{N+1,1}$ side as $\tilde{a_{N+1}}$; thus $\mathbf{i}_{N+1}$ is on the hub side. In Fig. 4c, $I_{A,D}$ meets $I_{A,B}$ to identify $\tilde{C}$, and $I_{A,C}$ for $\tilde{B}$. All but $\mathbf{i}_1$ have $D \succ A$, so $\{\mathbf{i}_j\}_{j=2}^{4}$ are on the $\tilde{D}$ side of $I_{A,D}$.

Moving from $\mathbf{i}_{N+1}$ to $\mathbf{i}_1$, the first permissible change reverses $a_{N+1} \succ a_1$, so no other $I_{X,Y}$ can enter the sector 1 region defined by $I_{N+1,1}$ that contains $\mathbf{i}_{N+1}$. The ordering of how to reverse $a_{N+1} \succ a_k$ dictates how the $I_{N+1,k}$ planes intersect sector 1; moving from $\tilde{a_{N+1}}$ out, they follow the $k = 1, \ldots, N$ order. Point $\mathbf{i}_1$ is outside (i.e., away from $\tilde{a_{N+1}}$) of the last $I_{N+1,N}$ plane; i.e., $\mathbf{i}_1$ is on the side of $I_{N+1,N}$ away from the hub. While these $I_{N+1,k}$ planes cannot meet in the interior of the sector (or they would create more than N regions), some must intersect on the boundaries to create various hubs; e.g., $I_{N+1,2}$ meets $I_{N+1,1}$ on the $I_{1,2}$ boundary, and for $N > 3$, this line connects with at least two hubs.

Now consider the path from $\mathbf{i}_{N+1}$ (near $\tilde{a_{N+1}}$) to $\mathbf{i}_2$ in sector 2. This path crosses $I_{1,2}, I_{1,3}, \ldots, I_{1,N}$ in that order (to move a_1 to the bottom place). Similarly, at some position after crossing $I_{1,k}$, the path must cross $I_{N+1,k}$ (to move a_{N+1} to its final position) in the $I_{N+1,2}, I_{N+1,3}, \ldots, I_{N+1,N}$ order (to reverse rankings with the next adjacent alternative). Thus the $\{I_{N+1,k}\}_{k=2}^{N}$ planes meet sector 2, but they cannot meet each other in the interior of the sector. At the final stage, $\mathbf{i}_2$ is on the $I_{N+1,N}$ side away from $\tilde{a_{N+1}}$; i.e., $\mathbf{i}_1$ and $\mathbf{i}_2$ are on the $I_{N+1,N}$ side away from $\tilde{a_{N+1}}$.

Each $(N-1)$-dimensional plane $I_{N+1,k}$ is uniquely defined by $(N-1)$ of the $(N+1)$ possible hub points; only $\tilde{a_{N+1}}$ and $\tilde{a_k}$ are excluded. These hub points are determined by the intersection of certain edges of sectors k, $k = 1, \ldots, N-1$, with either $I_{N+1,1}$ or $I_{N+1,2}$; e.g., $\tilde{a_2}$ is the intersection of the sector 2 edge $a_3 \sim \cdots \sim a_N \sim a_1$ with $I_{N+1,1}$. In general, $\tilde{a_k}$ is the intersection of the k^{th}-sector's edge $a_{k+1} \sim a_{k+2} \sim \ldots a_N \sim a_1 \sim \cdots \sim a_{k-1}$ with $I_{N+1,1}$ if $k \neq 1$, and $I_{N+1,2}$ if $k = 1$. Thus, $I_{N+1,N}$ is uniquely defined by $\{\tilde{a_k}\}_{k=1}^{N-1}$. (In Fig. 4c, $\tilde{A}$ is far to the right where the dashed $I_{B,D}$ would meet the solid $I_{B,C}$ while $\tilde{B} = I_{A,D} \cap I_{A,C}$. As $\tilde{A}$, $\tilde{B}$ uniquely define $I_{D,C}$ (the dotted line), it follows that there must be two ideal points on each side, which is the desired contradiction. The rest of the proof leads to the same conclusion after handling the higher dimensional nature of $I_{N+1,N}$.)

Now consider the portion of the region $a_1 \succ a_2 \succ \cdots \succ a_{N-1}$ (with boundaries $I_{1,2}, I_{2,3}, \ldots, I_{N-2,N-1}$) on the $\tilde{a_{N+1}}$ side of $I_{N+1,1}$. This region, $\mathcal{R}$, contains the ideal points $\mathbf{i}_N, \mathbf{i}_{N+1}$. However, $I_{N+1,N}$ does not meet $\mathcal{R}$ because in sector 1, $I_{N+1,N}$ is separated from $\mathcal{R}$ by $I_{N+1,1}$, and none of its other defining hub points have this ranking. This means that $\mathbf{i}_N, \mathbf{i}_{N+1}$ are on one side of $I_{N+1,N}$ while at least $\mathbf{i}_1, \mathbf{i}_2$ are on the other. As this contradicts the tallies from Prop. 1, the proof is completed. (With an added dimension, the extra defining point for $I_{N+1,N}$ would separate $\mathbf{i}_N, \mathbf{i}_{N+1}$.)

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