Deriving scientific or geometric laws from thought experiments, via meaningfulness, with an application to the Pythagorean Theorem

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January 23, 2012

Abstract

Typically, the mathematical expression of a scientific law does not depend of the units of the variables. We formalize this type of invariance by a general property called meaningfulness. In this context, abstract constraints formalizing thought experiments may be sufficient for the derivation of a law. We give an illustration in geometry by deriving the Pythagorean Theorem (up to the exponent).

The standard mathematical expressions of scientific laws typically do not depend upon the units of measurement. The most important rationale for this convention is that measurement units do not appear in nature[1]. Thus, any mathematical model or law whose form would be fundamentally altered by a change of units would be a poor representation of the empirical world. As far as I know, however, there is no agreed upon formalization of this type of invariance of the form scientific laws, even though there has been some proposals (see Falmagne and Narens, 1983; Narens, 2002; Falmagne, 2004).

Expanding on the just cited papers, I propose here a general condition of 'meaningfulness' constraining *a priori* the form of any function describing a scientific or geometric law expressed in terms of ratio scales variables such as mass, length, or time[2].

The interest of such a meaningfulness condition from a philosophy of science viewpoint is that, in its context, abstract constraints on the function, formalizing 'thought experiments', may yield the exact form of a law, possibly up to some real valued parameters. We will illustrate this point here with an example in geometry.

A case of abstract constraint on a real, positive valued function G of two real positive variables, is the condition formalized by the equation

$$G(G(y,r),t) = G(G(y,t),r). \tag{1}$$

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[1]The only exception is the counting measure, as in the case of the Avogadro number.

[2]The results can be extended to other cases, in particular interval scales.

where G is strictly monotonic and continuous in both real variables. An interpretation of $G(y,r)$ in Equation (1) is that the second variable r in modifies the state of the first variable y, creating an effect evaluated by $G(y,r)$ in the same measurement variable as y. The left hand side of (1) represents a one-step iteration of this phenomenon, in that $G(y,r)$ is then modified by t, resulting in the effect $G(G(y,r),t)$. Equation (1), which is referred to as the 'permutability' condition (in Aczél, 1966), formalizes the concept that the order of these two modifiers r and t is irrelevant.

Under fairly general conditions making empirical sense, the permutability condition (1) implies the existence of a representation

$$G(y,r) = f^{-1}(f(y) + g(r)), \tag{2}$$

where f and g are real valued, strictly monotonic continuous functions. We prove this fact here, slightly generalizing results of Hosszú (1962c,b,a) (cf. also Aczél, 1966). It is clear that the representation (2) implies (1): we have

$$\begin{aligned}
G(G(y,r),t) &= f^{-1}(f^{-1}(f(G(y,r)) + g(t))) && \text{(by (2))}\\
&= f^{-1}(f^{-1}(f(f^{-1}(f(y) + g(r))) + g(t))) && \text{(by (2) again)}\\
&= f^{-1}(f(y) + g(r) + g(t)) && \text{(simplifying)}\\
&= f^{-1}(f(y) + g(t) + g(r)) && \text{(by commutativity)}\\
&= G(G(y,t),r) && \text{(by symmetry)}\,.
\end{aligned}$$

We will also use a more general condition, called 'quasi permutability', which is defined by the equation

$$M(G(y,r),t) = M(G(y,t),r) \tag{3}$$

and lead to the representation

$$M(y,r) = m((f(y) + g(r))\,. \tag{4}$$

The combined consequences of permutability or quasi permutability and meaningfulness are powerful ones. For instance, if we suppose that the function G is symmetric, that is,

$$G(y,r) = G(r,y), \tag{5}$$

a fact that can typically also be deduced from *a priori* considerations, then, under sensible continuity and solvability conditions, G has necessarily the form

$$G(y,x) = \left(y^{\theta} + x^{\theta}\right)^{\frac{1}{\theta}}. \tag{6}$$

Note that this form generalizes that of the Pythagorean Theorem (which obtains when $\theta = 2$.) This result is established by Theorem 15. Many of the mathematical tools used in our arguments are borrowed from functional equations (as in Aczél, 1966, 1987).

We begin by stating some fundamental definitions. We then described a few examples of laws, taken from physics and geometry, in which the permutability condition naturally applies. The following section is devoted to some preparatory lemmas or instrumental results. We recall there some basic functional equation facts. The last three section contain the main results of the paper.

Basic Concepts and Examples

1 Definition. We write $\mathbb{R}_+$ and $\mathbb{R}_{++}$ for the nonnegative and the positive reals. Let J, J', and H be real nonnegative intervals of positive length. A *(numerical) code* is a function $M : J \times J' \xrightarrow{\text{onto}} H$ which is strictly increasing in the first variable, strictly monotonic in the second one, and continuous in both. A code M is *solvable* if it satisfies the following two conditions.

[S1] If $M(x,t) < p \in H$, there exists $w \in J$ such that $M(w,t) = p$.

[S2] The function M is *1-point right solvable*, that is, there exists a point $x_0 \in J$ such that for every $p \in H$, there is $v \in J'$ satisfying $M(x_0, v) = p$. In such a case, we may say that M is *x_0-solvable*.

By the strict monotonicity of M, the points w and v of [S1] and [S2] are unique.

Two functions $M : J \times J' \to H$ and $G : J \times J' \to H'$ are *comonotonic* if

$$M(x,s) \leq M(y,t) \iff G(x,s) \leq G(y,t), \qquad (x,y \in J; s,t \in J'). \tag{7}$$

In such a case, the equation

$$F(M(x,s)) = G(x,s) \qquad (x \in J; s \in J') \tag{8}$$

defines a strictly increasing continuous function $F : H \xrightarrow{\text{onto}} H'$. We may say then that G is *F-comonotonic* with M.

We turn to the key condition of this paper.

2 Definition. A function $M : J \times J' \longrightarrow H$ is *quasi permutable* if there exists a function $G : J \times J' \to J$ co-monotonic with M such that

$$M(G(x,s),t) = M(G(x,t),s) \qquad (x,y \in J; s,t \in J'). \tag{9}$$

We say in such a case that M is *permutable with respect to G*, or *G-permutable* for short. When M is permutable with respect to itself, we simply say that M is *permutable*, a terminology consistent with Aczél (1966, Chapter 6, p. 270).

We mention the straightforward consequence:

3 Lemma. *A function $M : J \times J' \to H$ is G-permutable only if G is permutable.*

PROOF. Indeed, suppose that G is F-comonotonic with M. For any $x \in J$ and $s,t \in J'$, we get $G(G(x,s),t) = F(M(G(x,s),t)) = F(M(G(x,t),s)) = G(G(x,t),s)$. □

Many scientific laws embody permutable or quasi permutable numerical codes. We give five quite different examples below.

4 Examples. (a) THE LORENTZ-FITZGERALD CONTRACTION. This term denotes a phenomenon in special relativity, according to which the apparent length of a rod measured by an observer moving at the speed v with respect to that rod is a decreasing function of v, vanishing as v approaches the speed of light. This function is specified by the formula

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}, \tag{10}$$

in which $c > 0$ denotes the speed of light, ℓ is the actual length of the rod (for an observer at rest with respect to the rod), and $L : \mathbb{R}_+ \times [0, c[\xrightarrow{\text{onto}} \mathbb{R}_+$ is the length of the rod measured by the moving observer.

The function L is a permutable code. Indeed, L satisfies the strict monotonicity and continuity requirements, and we have

$$L(L(p, v), w) = p\left(1 - \left(\frac{v}{c}\right)^2\right)^{-\frac{1}{2}}\left(1 - \left(\frac{w}{c}\right)^2\right)^{-\frac{1}{2}} = L(L(p, w), v). \tag{11}$$

It is also clear that L satisfies the solvability Condition [S1]; but, as Condition [S2] does not hold, this code is not solvable. However, for any $\ell_0 > 0$, the restriction L^{ℓ_0} of L to $[0, \ell_0] \times [0, c[$ is an ℓ_0-solvable permutable code satisfying [S1], as is easily checked. Ultimately, whatever result is obtained for any L^{ℓ_0} can be extended to $L = \lim_{\ell_0 \to \infty} L^{\ell_0}$. A similar observation holds for the next example and for Example (d).

(b) BEER'S LAW. This law applies in a class of empirical situations where an incident radiation traverses some absorbing medium, so that only a fraction of the radiation goes through. In our notation, the expression of the law is

$$I(x, y) = x\, e^{-\frac{y}{c}}, \qquad (x, y \in \mathbb{R}_+,\ c \in \mathbb{R}_{++} \text{ constant}) \tag{12}$$

in which x denotes the intensity of the incident light, y is the concentration of the absorbing medium, c is a reference level, and $I(x, y)$ is the intensity of the transmitted radiation. The form of this law is similar to that of the Lorentz-FitzGerald Contraction and the same arguments apply. Thus, the function $I : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$ is also a permutable code satisfying [S1], and for any $x_0 > 0$, its restriction to $[0, x_0]$ is a solvable permutable code.

(c) THE MONOMIAL LAWS. Consider the code $M : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{\text{onto}} \mathbb{R}_+$ with exponent parameters $\xi > 0$, $\theta > 0$ and $\nu > 0$ defined by the equation

$$M(x, s) = \xi x^\theta s^\nu. \tag{13}$$

The code M is permutable with respect to the code

$$G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ : (x, s) \mapsto \frac{x s^{\nu/\theta}}{\xi}.$$

Indeed, M is co-monotonic with G and

$$M(G(x, s), t) = \xi G(x, s)^\theta t^\nu = \xi\left(\frac{x s^{\nu/\theta}}{\xi}\right)^\theta t^\nu = \xi^{1-\theta} x^\theta s^\nu t^\nu = M(G(x, t), s).$$

It is clear that both M and G are solvable.

(d) Equation (13) describes not only many physical laws, but also some fundamental formulas of geometry, such as the volume $C(\ell, r)$ of a cylinder of radius r and height ℓ, for example. In this case, we have

$$C(\ell, r) = \ell\pi r^2, \tag{14}$$

which is permutable. We have

$$C(C(\ell, r), v) = C(\ell\pi r^2, v) = \ell\pi r^2 \pi v^2 = C(C(\ell, v), r).$$

Thus, the constant ξ of (13) is equal to π. We give another geometric example below.

(e) THE PYTHAGOREAN THEOREM. The function

$$P(x, y) = \sqrt{x^2 + y^2} \qquad (x, y \in \mathbb{R}_{++}), \tag{15}$$

representing the length of the hypothenuse of a right triangle in terms of the lengths of its sides, is a permutable code. We have indeed

$$P(P(x, y), z) = \sqrt{P(x, y)^2 + z^2} = \sqrt{x^2 + y^2 + z^2} = P(P(x, z), y).$$

The other conditions are clearly satisfied, and so is Condition [S1]. Condition [S2] would be achieved by taking an appropriate restriction of the function P as in the case of Examples 4(a) and (b). Notice that the code P is a symmetric function: we have $P(x, y) = P(y, x)$ for all $x, y \in \mathbb{R}_{++}$. Equation (15) specializes the result of Theorem 15.

We go back to the Pythagorean Theorem later on in this paper (see Subsection 17), and show directly, by an elementary geometrical argument, that the length $P(x, y)$ of the hypothenuse of a right triangle[3] with leg lengths x and y, is permutable and also satisfies a special kind of quasi permutability. Using our Theorem 15, this implies that the equation

$$P(x, y) = \left(x^\theta + y^\theta\right)^{\frac{1}{\theta}} \qquad (x, y \in \mathbb{R}_{++})$$

must hold for some $\theta \in \mathbb{R}_{++}$. We prove thus there a generalization of the Pythagorean Theorem.

Preparatory Results

We mentioned that several of our results were obtained by functional equation arguments. For completeness, we recall in this section, without proofs, a few elementary results instrumental to our purpose. The classic treatise on the subject is Aczél (1966, see p. 141). For a short, more recent account, see Aczél (1987).

[3]Thus, this P is not defined by (15).

5 Theorem. *Suppose that h, k and m are three real valued continuous functions on $\mathbb{R}$ satisfying the equation*

$$h(x+y) = k(x) + m(y) \qquad (x, y \in \mathbb{R}). \tag{16}$$

Then there exists three constant ξ, θ_1 and θ_2 such that, for all $x \in \mathbb{R}$,

$$h(x) = \xi x + \theta_1 + \theta_2 \tag{17}$$
$$k(x) = \xi x + \theta_1 \tag{18}$$
$$m(x) = \xi x + \theta_2 \,. \tag{19}$$

The same result holds, with $\xi > 0$, if "continuous" is replaced by "increasing" in the statement of the theorem. We refer to the triple of equations (17)-(18)-(19) as the *solution* of the *Pexider Equation* (16). There are three other, similar Pexider equations.

With $h > 0$, these are[4]:

Equation	Domain	Solution	
$h(x+y) = k(x)m(y)$	$x, y \in \mathbb{R}$	$\begin{cases} h(x) = \theta_1\theta_2 e^{\xi x} \\ k(x) = \theta_1 e^{\xi x} \\ m(x) = \theta_2 e^{\xi x}, \end{cases}$	(20)
$h(xy) = k(x) + m(y)$	$x, y \in \mathbb{R}_{++}$	$\begin{cases} h(x) = \xi \ln x + \theta_1 + \theta_2 \\ k(x) = \xi \ln x + \theta_1 \\ m(x) = \xi \ln x + \theta_2, \end{cases}$	(21)
$h(xy) = k(x)m(y)$	$x, y \in \mathbb{R}_{++}$	$\begin{cases} h(x) = \xi_1\xi_1 x^\theta \\ k(x) = \xi_1 x^\theta \\ m(x) = \xi_2 x^\theta \,. \end{cases}$	(22)

When $k = m = h$ in (16) (the three functions are identical), one obtains the *Cauchy equation*, with solution $h(x) = \xi x$. The three other Pexider equations generate three corresponding variants of the Cauchy equation, with straightforward solutions.

6 Remark. The solutions given for the Pexider equations (16), (20), (21) and (22) are also valid when the domain of the equation is an open connected subset of $\mathbb{R}^2$ rather than $\mathbb{R}^2$ itself. Indeed, Aczél (1987, see also Aczél, 2005, Chudziak and Tabor, 2008, and Radó and Baker, 1987) has shown that, in such cases, this equation can be extended to the real plane. A similar (but not identical) remark applies to the Cauchy equation and its variants. In this case, however, some technical issues have to be taken care of: the domain should not be "too small"[5]. In all the cases considered in this paper, the domain of the functional equation under consideration satisfies the required constraints. In our proofs, we omit the detailed verification. When using such results in the sequel, we simply refer to the Pexider equation or to the Cauchy equation. The particular variant under consideration will be clear from the context.

[4]The assumption that $h > 0$ is not necessary but simplifies the exposition.

[5]Condition (iii) in Lemma 8 suggests the relevant constraint for this case. This condition ensures that the domain of the operation $\bullet$ is not "too small."

The main step in our developments is based on the following construction.

7 Definition. Suppose that $G : J \times J' \to J$ is a code that is x_0-solvable in the sense of Condition [S2]. Define the operation $\bullet$ on J by the equivalence

$$x \bullet y = G(x, v) \quad \iff \quad G(x_0, v) = y \qquad (x, y \in J; v \in J'). \tag{23}$$

We show in this section that a solvable code G is permutable if and only if it has an additive representation

$$G(y, v) = f^{-1}(f(y) + g(v)) \qquad (x, y \in J; v \in J') \tag{24}$$

where $f : J \to \mathbb{R}_+$ and $g : J' \to \mathbb{R}_+$ are continuous functions with f strictly increasing and g strictly monotonic.

The basic tool lies in the following lemma (for a proof, see Falmagne, 1975).

8 Lemma. *Let J be a real non degenerate interval. With $R \subseteq J \times J$, let $\bullet : R \to J$ be a non necessarily closed operation on J. We write xRy to mean that $x \bullet y$ is defined. Suppose that the triple $(J, \bullet, \leq)$, where $\leq$ is the inequality of the reals, satisfies the following five independent conditions:*

(i) *yRx if xRy, and when yRx, then $y \bullet x = x \bullet y$;*

(ii) *whenever yRx, wRz, wRy', $z'Rx$, yRy' and $z'Rz$, then*

$$(y \bullet x = w \bullet z) \text{ and } (w \bullet y' = z' \bullet x) \quad \text{imply} \quad y \bullet y' = z' \bullet z;$$

(iii) *there exists $x \in J$ such that xRx and $x \bullet xRx$;*

(iv) *if $y \bullet x < z$, then $y \bullet w = z$ for some w in J;*

(v) *for every x, y and z in J, with $x < y$, the set $N(x, z; y) = \{n \in \mathbb{N}^+ \mid x_y^n \leq z\}$ is finite, where the sequence (x_y^n) is defined recursively as follows:*

  (a) *$x_y^1 = x$;*

  (b) *if x_y^{n-1} is defined and x' exists such that $y \bullet x_y^{n-1} = x \bullet x'$ then $x_y^n = x'$.*

Then, there exists a strictly increasing function $f : J \to J$ such that

$$f(x \bullet y) = f(y) + f(y).$$

(For a proof, see Falmagne, 1975).

9 Lemma. *Let $G : J \times J' \to J$ be a solvable, permutable code. Then, the triple $(J, \bullet, \leq)$, with the operation $\bullet$ defined by (23), satisfies Conditions (i)-(v) of Lemma 8. Moreover, the operation $\bullet$ is associative, strictly increasing and continuous in both variables.*

PROOF. Take any $x, y \in J$ with

$$G(x_0, r) = x \tag{25}$$

and

$$G(x_0, v) = y. \tag{26}$$

(i) By (23), (25), (26) and the permutability of G, we get successively,

$$y \bullet x = G(y, r) = G(G(x_0, v), r) = G(G(x_0, r), v) = G(x, v) = x \bullet y.$$

(ii) Suppose that

$$(y \bullet x = w \bullet z) \text{ and } (w \bullet y' = z' \bullet x). \tag{27}$$

With (25), (26) and

$$G(x_0, s) = z,\ G(x_0, t) = w,\ G(x_0, v') = y',\ G(x_0, s') = z', \tag{28}$$

we get from (27)

$$G(y, r) = G(w, s) \tag{29}$$
$$G(w, v') = G(z', r)\,. \tag{30}$$

Equation (29) gives

$$G(G(y, r), v') = G(G(w, s), v'),$$

which yields successively

$$\begin{aligned} G(G(y, v'), r) &= G(G(w, v'), s) && \text{(by permutability)} \\ &= G(G(z', r), s) && \text{(by (30))} \\ &= G(G(z', s), r) && \text{(by permutability)}, \end{aligned}$$

so

$$G(G(y, v'), r) = G(G(z', s), r).$$

By the strict monotonicity of G in the first variable, we obtain $G(y, v') = G(z', s)$ and thus $y \bullet y' = z' \bullet z$.

(iii) By the solvability condition [S2], there exists $x \in J$ such that, with $G(x_0, r) = x$, we have both

$$x \bullet x = G(x, r) \in J \quad \text{and} \quad (x \bullet x) \bullet x = G(G(x, r), r) \in J.$$

(iv) If $x \bullet y < z$, then $y \bullet x = G(y, r) < z \in J$ by commutativity, (25), and the definition of $\bullet$. Applying [S1], we get $G(w, r) = z$ for some $w \in J$. Using again (25), we obtain $x \bullet w = z$.

(v) We first show that the sequence (x_y^n) defined by (a) and (b) is strictly increasing. We proceed by induction. Since $x < y$ by definition, we get from (25) and (26)

$$x = G(x_0, r) < G(x_0, v) = y,$$

with the function G strictly monotonic in its second variable. In the sequel, we suppose that G is strictly decreasing in its second variable; so,

$$v < r\,. \tag{31}$$

The proof is similar in the other case. The following equalities hold by the definitions of x_y^1, x_y^2 and commutativity:

$$y \bullet x_y^1 = y \bullet x = G(y, r) = x \bullet y = x \bullet x_y^2 = x_y^2 \bullet x = G(x_y^2, r).$$

From $G(y,r) = G(x_y^2, r)$, we get $x_y^2 = y$ and $x_y^1 < x_y^2$. Assuming that $x_y^{n-1} < x_y^n$, we get $y \bullet x_y^n = x \bullet x_y^{n+1}$ by the definition of the term x_y^{n+1} in Condition (v) (b) of Lemma 8, and by commutativity

$$x_y^n \bullet y = G(x_y^n, v) = x_y^{n+1} \bullet x = G(x_y^{n+1}, r),$$

yielding $G(x_y^n, v) = G(x_y^{n+1}, r)$. Since $v < r$ and G is decreasing in its second variable

$$G(x_y^{n+1}, v) > G(x_y^{n+1}, r) = G(x_y^n, v),$$

and so

$$x_y^n < x_y^{n+1}$$

because G is strictly increasing in its first variable. By induction, the sequence (x_y^n) is strictly increasing.

Suppose that the set $N(x, z; y)$ of Condition (v) is not finite. Thus, the point z is an upper bound of the sequence (x_y^n). Because this sequence is increasing and bounded above, it necessarily converges. Without loss of generality, we can assume that we have in fact $\lim_{n\to\infty} x_y^n = z$. Since

$$y \bullet x_y^{n-1} = x \bullet x_y^n < x \bullet z$$

for all $n \in \mathbb{N}$, the solvability Condition (iv) implies that there is some $z' \in J$ such that $y \bullet z' = x \bullet z$, with necessarily $z' < z$. There must be some $m \in \mathbb{N}$ such that $z' < x_y^m < z$. We obtain thus

$$x \bullet z = y \bullet z' < y \bullet x_y^m = x \bullet x_y^{m+1}$$

and so $z < x_y^{m+1}$, in contradiction with $\lim_{n\to\infty} x_y^n = z$, with (x_y^n) an increasing sequence. We conclude that the set $N(x, z; y)$ must be finite for all x, y and z in J, with $x < y$. We conclude that the Conditions (i)-(v) of Lemma 8 are satisfied.

To prove that $\bullet$ is associative, we take any x, y and z in J. Using again $G(x_0, r) = x$, $G(x_0, v) = y$ and $G(x_0, s) = z$, we have

$$\begin{aligned}
x \bullet (y \bullet z) &= G(y \bullet z, r) && \text{(since } G(x_0, r) = x) \\
&= G(G(y, s), r) && \text{(since } G(x_0, s) = z) \\
&= G(G(y, r), s) && \text{(by permutability)} \\
&= G(x \bullet y, s) && \text{(since } G(x_0, r) = x) \\
&= z \bullet (x \bullet y) && \text{(since } G(x_0, s) = z) \\
&= (x \bullet y) \bullet z && \text{(by commutativity).}
\end{aligned}$$

Finally, since for all $x, y \in J$, we have

$$x \bullet y = G(y,r) = y \bullet x = G(x,v),$$

it is clear that the operation $\bullet$ is continuous and strictly increasing in both variables.

$\square$

We come to the key result of this section, which generalizes those of Hosszú (1962c,b,a) (cf. also Aczél, 1966).

10 Theorem. (i) *A solvable code $M : J \times J' \to H$ is quasi permutable if and only if there exists three continuous functions $m : \{f(y) + g(r) \,|\, x \in J, r \in J'\} \to H$, $f : J \to \mathbb{R}$, and $g : J' \to \mathbb{R}$, with m and f strictly increasing and g strictly monotonic, such that*

$$M(y,r) = m(f(y) + g(r)). \tag{32}$$

(ii) *A solvable code $G : J \times J' \to J$ is a permutable code if and only if, with f and g as above, we have*

$$G(y,r) = f^{-1}(f(y) + g(r)). \tag{33}$$

(iii) *If a solvable code $G : J \times J \to J$ is a symmetric function—that is, $G(x,y) = G(y,x)$ for all $x, y \in J$— then G is permutable if and only if there exists a strictly increasing and continuous function $f : J \to J$ satisfying*

$$G(x,y) = f^{-1}(f(x) + f(y)). \tag{34}$$

(iv) *If the code G in (33) is differentiable in both variables, with non vanishing derivatives, then the functions f and g are differentiable. This differentiability result also applies to the code G and the function f in (34).*

Our argument for establishing (i) and (ii) is essentially the same as that in Aczél (1966, p. 271-273) but, because our solvability conditions [S1]-[S2] are weaker, relies on Lemma 8 rather than on the representation in the reals of an ordered Archimedean group[6].

PROOF. (i)-(ii) Suppose that the code M of the theorem is permutable with respect to a F-comonotonic code G. By Lemma 3, the code G is permutable. Defining the operation $\bullet : J \times J' \to J$ by

$$y \bullet x = G(y,r) \quad \Longleftrightarrow \quad G(x_0, r) = x, \tag{35}$$

it follows from Lemma 9 that the triple $(J, \bullet, \leq)$ satisfies Conditions (i)-(v) of Lemma 8, with the operation $\bullet$ associative and continuously increasing in both variable. Accordingly, there exists a continuous, strictly increasing function $f : J \to J$ such that

$$f(y \bullet x) = f(y) + f(x). \tag{36}$$

[6] Cf. Hölder (1901).

Defining the strictly monotonic function $\psi : J' \to J$ by

$$\psi(s) = G(x_0, s),$$

we get from (35) and (36),

$$f(y \bullet x) = f(G(y,r)) = f(y \bullet G(x_0, r)) = f(y) + f(\psi(r)),$$

and thus

$$G(y,r) = f^{-1}(f(y) + f(\psi(r))),$$

or with with $g = f \circ \psi$,

$$G(y,r) = f^{-1}(f(y) + g(r)). \tag{37}$$

(Notice that $f(y) + g(r) \in J$.) Because G is F-comonotonic with M, and F maps H onto J, we obtain

$$M(y,r) = F^{-1}(G(y,r)) = (F^{-1} \circ f^{-1})(f(y) + g(r)),$$

or, with $m = F^{-1} \circ f^{-1}$,

$$M(y,r) = m(f(y) + g(r)) \qquad (y \in J; r \in J'; f(y) + g(r) \in J). \tag{38}$$

It is clear that the functions f and g in (37) and the functions m, f and g in (38) are continuous, with the required monotonicity properties. This proves the necessity part of (i). The sufficiency is straightforward.

(ii) This was established in passing: cf. Eq. (37).

(iii) From (ii), we get by the symmetry of G

$$G(x,y) = f^{-1}(f(x) + g(y)) = G(y,x) = f^{-1}(f(y) + g(x))$$

yielding

$$f(x) - g(x) = f(y) - g(y) = K$$

for some constant K and all $x, y \in J$. We have thus $g(x) = f(x) - K$ for all $x \in J$. Since $g^{-1}(t) = f^{-1}(t + K)$, we obtain

$$g^{-1}(g(x) + g(y)) = f^{-1}(g(x) + g(y) + K) = f^{-1}(f(x) + g(y)) = G(x,y).$$

Defining $h = g$, we obtain (34).

(iv) If the code G in (33) is differentiable with non vanishing derivatives, then, for every $r \in J$, the inverse G_r^{-1} of G in the first variable is differentiable. From (33), we get $f(x) = G_r^{-1}(x) + g(r)$ with $G_r^{-1}(x) = y$. So, f is differentiable, and since, from (33) again,

$$g(r) = f(G(y,r)) - f(y)$$

with f differentiable and G differentiable in the second variable, g is also differentiable. The differentiability of f in (34) is immediate. □

We mention in passing a simple uniqueness result concerning our basic representation equation (33).

11 Lemma. *Suppose that the representation $G(y, r) = f^{-1}(f(y) + g(r))$ of Theorem 10(ii) holds for some code G, with f and g satisfying the stated continuity and monotonicity conditions. Then we also have $G(y, r) = (f^*)^{-1}(f^*(y) + g^*(r))$ for some continuous functions f^* and g^*, respectively co-monotonic with f and g, if and only if $f^* = \xi f + \theta$ and $g^* = \xi g$, for some constants $\xi > 0$ and θ.*

PROOF. (Necessity.) Suppose that

$$(f^*)^{-1}(f^*(y) + g^*(r)) = f^{-1}(f(y) + g(r)).$$

Then, with $z = f(y)$ and $s = g(r)$ and applying f^* on both sides, we get

$$(f^* \circ f^{-1})(z) + (g^* \circ g^{-1})(s) = (f^* \circ f^{-1})(z + s), \tag{39}$$

a Pexider equation. It is clear that $(f^* \circ f^{-1})$ and $(g^* \circ g^{-1})$ are strictly increasing and continuous and that (39) is defined on an open connected subset of $\mathbb{R}^2_+$. By Theorem 5 with $h = (f^* \circ f^{-1})$ and $k = m = (g^* \circ g^{-1})$, we get $(f^* \circ f^{-1})(z) = \xi z + \theta$ and $(g^* \circ g^{-1})(s) = \xi s$, $\xi > 0$, and so $f^*(y) = \xi f(y) + \theta$ and $g^*(r) = \xi g(r)$.

(Sufficiency.) If $f^* = \xi f + \theta$ and $g^* = \xi g$, with $\xi > 0$, then

$$\begin{aligned}(f^*)^{-1}(f^*(y) + g^*(r)) &= f^{-1}\left(\frac{f^*(y) + g^*(r) - \theta}{\xi}\right)\\ &= f^{-1}\left(\frac{\xi f(y) + \theta + \xi g(r) - \theta}{\xi}\right)\\ &= f^{-1}(f(y) + g(r)).\end{aligned}$$

□

Meaningful Collection of Codes

Our main goal in this paper is to axiomatize a particular type of invariance for scientific or geometric laws. The consequence of this axiomatization should be that the form of an expression representing a scientific law should not be altered by changing the units of the variables. The next definition, which was already used by Falmagne (2004) (see also Falmagne and Narens, 1983) applies to codes regarded as functions of two variables. The extension to the general case of n ratio scales variables is straightforward.

We illustrate the definition by our Example 4(a) involving the Lorentz-FitzGerald Contraction, which we expressed by the equation

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}. \tag{40}$$

The trouble with this notation is its ambiguity: the units of ℓ, which denotes the length of the rod, and of v, for the speed of the observer, are not specified. Writing $L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to to 70 meters and 3 kilometers per second, respectively. Such a parenthetical reference is standard in a scientific context, but is not instrumental for our purpose, which is to express, formally, an invariance with respect to any change in the units[7].

To rectify the ambiguity, we propose to interpret $L(\ell, v)$ as a shorthand notation for $L_{1,1}(\ell, v)$, in which ℓ and L on the one hand, and v on the other hand, are measured in terms of two particular initial or 'anchor' units fixed arbitrarily. Such units could be m (meter) and km/sec, if one wishes. Describing the phenomenon in terms of other units amounts to multiply ℓ and v in any pair (ℓ, v) by some positive constants α and β, respectively. At the same time, L also gets to be multiplied by α, and the speed of ligh c by β. Doing so defines a new function $L_{\alpha,\beta}$, which is different from $L = L_{1,1}$ if either $\alpha \neq 1$ or $\beta \neq 1$ (or both), but carries the same information from an empirical standpoint. For example, if our new units are km and m/sec, then the two expressions

$$L_{10^{-3},10^{3}}(.007, 3000) \quad \text{and} \quad L(70,3) = L_{1,1}(70,3),$$

while numerically not equal, should describe the same empirical situation. The appropriate definition of $L_{\alpha,\beta}$ is clear: we should write

$$L_{\alpha,\beta}(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{\beta c}\right)^2}. \tag{41}$$

The connection between L and $L_{\alpha,\beta}$ is thus

$$\begin{aligned}
\frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) &= \left(\frac{1}{\alpha}\right)\alpha\ell\sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} \\
&= \ell\sqrt{1 - \left(\frac{v}{c}\right)^2} \\
&= L(\ell, v).
\end{aligned}$$

This immediately implies, for any α, β, ν and μ in $\mathbb{R}_{++}$,

$$\frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu}L_{\nu,\mu}(\nu\ell, \mu v), \tag{42}$$

which is the invariance equation we were looking for, in this case, and which is generalized as Equation (45) in the next definition.

12 Definition. Let $[a, a'[$, $[b, b'[$ and $]d, d'[$ be three real non negative intervals and let

$$\mathcal{M} = \{M_{\alpha,\beta,\gamma} \mid \alpha, \beta, \gamma \in \mathbb{R}_{++}\} \tag{43}$$

be a 3-parameter collection of functions

$$M_{\alpha,\beta,\gamma} : [\alpha a, \alpha a'[\times [\beta b, \beta b'[\xrightarrow{\text{onto}} [\gamma d, \gamma d'[, \tag{44}$$

[7]A relevant point is made by Suppes (2002, see "Why the Fundamental Equations of Physical Theories Are not Invariant", p. 120).

strictly monotonic and continuous in both variables. We say that the collection $\mathcal{M}$ is *meaningful* if for all choices of parameters $\alpha, \beta, \gamma, \mu, \nu$ and η in $\mathbb{R}_{++}$, we have

$$\frac{1}{\gamma} M_{\alpha,\beta,\gamma}(\alpha x, \beta r) = \frac{1}{\eta} M_{\mu,\nu,\eta}(\mu x, \nu r), \qquad (x \in [a, a'[\,;\ r \in [b, b'[\,). \tag{45}$$

The parameters α, β and γ in a function $M_{\alpha,\beta,\gamma}$ in $\mathcal{M}$ stand for the units of the three ratio scales used for the two input variables and the output variable. Thus, (45) links two functions in $\mathcal{M}$ computed with possibly different units. The family $\mathcal{M}$ is called a *self-transforming collection* or *ST-collection* if $[a, a'[= [d, d'[$ and $\alpha = \gamma$ for any function $M_{\alpha,\beta,\gamma}$ in $\mathcal{M}$. (Thus, the first input variable and the output variable of a function in $\mathcal{M}$ are measured in the same measurement scale and with the same unit.) We use then the abbreviation $M_{\alpha,\beta} = M_{\alpha,\beta,\alpha}$. It is clear that the meaningfulness condition formalized by (45) also applies to self-transforming collections. Note that the intervals $[a, a'[$, $[b, b'[$ and $]d, d'[$ may be unbounded above: we may have, for example, $a' = \infty$, in which case $\alpha a'$ in (44) must be taken to mean ∞. The invariance property formalized by (45) is called the *meaningfulness* of $\mathcal{M}$. By convention, we write $M = M_{1,1,1}$ (or $M = M_{1,1}$ in the case of self-transforming collection). When the functions in $\mathcal{M}$ are codes in the sense of Definition 1, then M may be referred to as the *initial code*.

Note that (45) implies

$$\frac{1}{\gamma} M_{\alpha,\beta,\gamma}(\alpha x, \beta r) = M(x, r), \qquad (x \in [a, a'[\,;\ r \in [b, b'[\,). \tag{46}$$

13 Remark. The collection of Example 4(d) containing all the codes

$$C_{\alpha,\alpha,\alpha^3} : (\alpha\ell, \alpha r) \mapsto \alpha\ell\pi(\alpha r)^2$$

computing the volume of a cylinder is not a self-transforming collection: the unit of the output variable differs from that of the first variable. The meaningfulness condition (45) applies in this case. We have, for any α and β in $\mathbb{R}_{++}$,

$$\frac{1}{\alpha^3} C_{\alpha,\alpha,\alpha^3}(\alpha\ell, \alpha r) = \left(\frac{1}{\alpha^3}\right)(\alpha\ell)\pi(\alpha r)^2 = \left(\frac{1}{\beta^3}\right)(\beta\ell)\pi(\beta r)^2 = \frac{1}{\beta^3} C_{\beta,\beta,\beta^3}(\beta\ell, \beta r).$$

The argument used in our discussion of Example 4(d) shows that all such codes are permutable. Thus all the codes in a collection $\mathcal{M}$ may be permutable without $\mathcal{M}$ being a ST-collection.

The meaningfulness condition just introduced is a powerful one. In particular, it enables some properties of one of the functions in $\mathcal{M}$ to extend to all the others in that collection. The next lemma illustrates this point.

14 Lemma. *If one of the functions in a meaningful collection $\mathcal{M}$ is a code in the sense of Definition 1, then all the functions in $\mathcal{M}$ are codes. If some code $M_{\alpha,\beta,\gamma}$ in a meaningful collection of codes $\mathcal{M}$ satisfies any of the following five properties:*

(i) *$M_{\alpha,\beta,\gamma}$ is solvable;*

(ii) $M_{\alpha,\beta,\gamma}$ *is differentiable in both variables;*

(iii) $M_{\alpha,\beta,\gamma}$ *is quasi permutable;*

(iv) $M_{\alpha,\beta,\gamma}$ *is a symmetric function, with* $\alpha = \beta$*;*

(v) $\mathcal{M}$ *is a self-transforming collection and* $M_{\alpha,\beta} = M_{\alpha,\beta,\beta}$ *is permutable;*

then all the codes in $\mathcal{M}$ *satisfy the same property. Moreover, if* $M_{\alpha,\beta} = M$*, so that* $M(x,r) = f^{-1}(f(x) + g(r))$ *by Theorem 10(ii), then for any* μ, η *we have*

$$M_{\mu,\eta}(x,r) = \eta f^{-1}\left(f\left(\frac{x}{\mu}\right) + g\left(\frac{r}{\eta}\right)\right). \tag{47}$$

PROOF. Without loss of generality, we suppose that $\alpha = \beta = \gamma = 1$, and we write $M = M_{1,1,1}$. Because the family $\mathcal{M}$ is meaningful, we have, for all positive real numbers μ, ν and η,

$$M_{\mu,\nu,\eta}(x,r) = \eta M\left(\frac{x}{\mu}, \frac{u}{\nu}\right) \qquad (x \in [\alpha a, \alpha a'[; r \in [\beta b, \beta b'[). \tag{48}$$

Suppose that M is a code. By definition, M strictly increasing and continuous in both variables and so is $M_{\mu,\nu,\eta}$ by (48); thus, $M_{\mu,\nu,\eta}$ is a code.

(i) Suppose that the code M is solvable. If $M_{\mu,\nu,\eta}(x,r) < p$, for some code $M_{\mu,\nu,\eta}$ in $\mathcal{M}$, then $M(\frac{x}{\mu}, \frac{u}{\nu}) < \frac{p}{\eta}$ follows from (48). As the code M satisfies [S1], there must be some $w \in [b, b'[$ such that $M(\frac{x}{\mu}, w) = \frac{p}{\eta}$. Defining $t = \nu w$, we get

$$M_{\mu,\nu,\eta}(x,t) = \eta M\left(\frac{x}{\mu}, \frac{t}{\nu}\right) = p.$$

Thus, the code $M_{\mu,\nu,\eta}$ also satisfies [S1]. Since M satisfies [S2], there exists some $x_0 \in [a, a'[$ such that M is x_0-solvable. Define $y_0 = \mu x_0 \in [\mu a, \mu a'[$ and take any q in the range of the function $M_{\mu,\nu,\eta}$. This implies that $\frac{q}{\eta}$ is in the range of M, and by [S2] applied to M, there is some w such that $M(x_0, w) = \frac{q}{\eta}$ or, equivalently with $v = \beta w$,

$$q = \eta M\left(\frac{y_0}{\mu}, \frac{v}{\nu}\right) = M_{\mu,\nu,\eta}(y_0, v),$$

by the meaningfulness of the family $\mathcal{M}$. Thus, $M_{\mu,\nu,\eta}$ is y_0-solvable.

(ii) The differentiability of $M_{\mu,\nu,\eta}$ results from that of M via (48).

(iii) Suppose now that M is quasi permutable. (We do not assume here that $\mathcal{M}$ is a transformation family.) Thus, there exists a code $G : [a, a'[\times [b, b'[\to [a, a'[$ comonotonic with M such that

$$M(G(x,s),t) = M(G(x,t),s) \qquad (x, y \in [a, a'[; s, t \in [b, b'[). \tag{49}$$

For any pair of parameters (μ, ν), define the function $G_{\mu,\nu} : [a, a'[\times [b, b'[\to [a, a'[$ by the equation

$$G_{\mu,\nu}(x, r) = \mu G\left(\frac{x}{\mu}, \frac{u}{\nu}\right). \tag{50}$$

Thus, $G_{\mu,\nu}$ is comonotonic with $M_{\mu,\nu,\eta}$ and we have successively

$$\begin{aligned} M_{\mu,\nu,\eta}(G_{\mu,\nu}(x,r), v) &= \eta M\left(\frac{1}{\mu}G_{\mu,\nu}(x,r), \frac{v}{\nu}\right) && \text{(by C-meaningfulness)} \\ &= \eta M\left(G\left(\frac{x}{\mu}, \frac{u}{\nu}\right), \frac{v}{\nu}\right) && \text{(by the definition of } G_{\mu,\nu}) \\ &= \eta M\left(G\left(\frac{x}{\mu}, \frac{v}{\nu}\right), \frac{y}{\nu}\right) && \text{(by the permutability of } G) \\ &= M_{\mu,\nu,\eta}(G_{\mu,\nu}(x,v), r) && \text{(by symmetry).} \end{aligned}$$

Consequently, any code $M_{\mu,\nu,\eta}$ is $G_{\mu,\nu}$-permutable.

(iv) This results immediately from the definition of the meaningfulness of a collection.

(v) Suppose that $\mathcal{M}$ is a transformation family and that M is permutable. We have thus, for any $M_{\mu,\nu} = M_{\mu,\nu,\eta}$,

$$\begin{aligned} \frac{1}{\mu}M_{\mu,\nu}\left(M_{\mu,\nu}(x,r), v\right) &= M\left(\frac{1}{\mu}M_{\mu,\nu}(x,r), \frac{v}{\nu}\right) && \text{(by the meaningfulness of } \mathcal{M}) \\ &= M\left(M\left(\frac{x}{\mu}, \frac{r}{\nu}\right), \frac{v}{\nu}\right) && \text{(by the meaningfulness of } \mathcal{M}) \\ &= M\left(M\left(\frac{x}{\mu}, \frac{v}{\nu}\right), \frac{r}{\nu}\right) && \text{(by the permutability of } M) \\ &= \frac{1}{\mu}M_{\mu,\nu}\left(M_{\mu,\nu}(x,v), u\right) && \text{(by symmetry).} \end{aligned}$$

We have thus $M_{\mu,\nu}\left(M_{\mu,\nu}(x,r), v\right) = M_{\mu,\nu}\left(M_{\mu,\nu}(x,v), r\right)$ and so $M_{\mu,\nu}$ is permutable.

Equation (47) results from Equation (33) of Theorem 10(ii) and Equation (48).

This completes the proof of the theorem. $\square$

Main Result

This result concern meaningful self-transforming collections of solvable, quasi permutable and symmetric codes. We recall that a code M is G-permutable only if G is permutable (Lemma 3). Also, by Theorem 10(iii), a solvable, permutable and symmetric code $G : J \times J \to J$ has a representation

$$G(y, r) = h^{-1}(h(y) + h(r)), \tag{51}$$

with h strictly increasing and continuous.

15 Theorem. *Suppose that* $\mathcal{G} = \{G_\nu\}$ *is a meaningful self-transforming collection of codes, with* $G_\nu : \mathbb{R}_{++} \times \mathbb{R}_{++} \overset{onto}{\longrightarrow} \mathbb{R}_{++}$. *Moreover, suppose that one of these codes is solvable, permutable with respect to the initial code* G, *symmetric, strictly increasing in both variables, and differentiable with continuous non vanishing derivatives. Then we necessarily have for some* $\theta \in \mathbb{R}_{++}$:

$$G(y,x) = \left(y^\theta + x^\theta\right)^{\frac{1}{\theta}} \qquad (y, x \in \mathbb{R}_{++}) \tag{52}$$

where G *is the initial code.*

Equation (52) is of course a generalization of the defining equation of the Pythagorean Theorem, which applies when $\theta = 2$. The assumption of differentiability makes for a short proof. In all likelihood, it can be dispensed with.

PROOF. By Lemma 14, all the codes in $\mathcal{G}$ are solvable, permutable, and symmetric. Thus, Statement (iii) of the representation Theorem 10 holds for the code G. By Theorem 10(iii), we have

$$G(y,x) = f^{-1}(f(y) + f(x)). \tag{53}$$

We get successively

$$\begin{aligned}
G_\nu(G(\nu y, \nu x), \nu z) &= G_\nu(f^{-1}(f(\nu y) + f(\nu x)), \nu z) && \text{(by Theorem 10(iii))}\\
&= \nu G\left(\frac{1}{\nu} f^{-1}(f(\nu y) + f(\nu x)), z\right) && \text{(by meaningfulness)}\\
&= \nu f^{-1}\left(f\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu x)\right)\right) + f(z)\right) && \text{(by Theorem 10(iii))}\\
&= \nu f^{-1}\left(f\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu z)\right)\right) + f(x)\right) && \text{(by quasi permutability).}
\end{aligned}$$

Equating the last two r.h.s's above and simplifying (applying $\frac{1}{\nu} f$ on both sides) gives

$$f\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu x)\right)\right) + f(z) = f\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu z)\right)\right) + f(x). \tag{54}$$

Differentiating (54) with respect to x and z gives the two equations:

$$f'\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu x)\right)\right) \frac{1}{\nu} \left(f^{-1}\right)'\left(f(\nu y) + f(\nu x)\right) f'(\nu x)\nu = f'(x) \tag{55}$$

$$f'(z) = f'\left(\frac{1}{\nu} f^{-1}\left(f(\nu y) + f(\nu x)\right)\right) \frac{1}{\nu} \left(f^{-1}\right)'\left(f(\nu y) + f(\nu x)\right) f'(\nu z)\nu. \tag{56}$$

Switching the two sides in (56), dividing (55) by (56), and simplifying gives

$$\frac{f'(\nu x)}{f'(\nu z)} = \frac{f'(x)}{f'(z)},$$

that is, with $z = 1$ and $h(\nu) = \frac{f'(\nu)}{f'(1)}$,

$$f'(\nu x) = f'(x)h(\nu),$$

a Pexider equation with solution $f'(x) = \kappa x^{\zeta}$ and $h(\nu) = \nu^{\zeta}$. We obtain thus with $\xi = \frac{\kappa}{\theta}$ and $\theta = \zeta - 1$, $f(x) = \xi x^{\theta}$, and thus

$$G(y, x) = f^{-1}(f(y) + f(x)) = \left(\frac{\xi y^{\theta} + \xi x^{\theta}}{\xi}\right)^{\frac{1}{\theta}} = \left(y^{\theta} + x^{\theta}\right)^{\frac{1}{\theta}}.$$

□

Application to the Pythagorean Theorem

We use Theorem 15 to obtained a generalized form of the Pythagorean Theorem (we do not specify the exponent). We suppose that the length $P(x, y)$ of the hypotenuse of a right triangle with leg lengths $x > x_0$ and $y > x_0$ (for some $x_0 > 0$) is a symmetric solvable code[8]; thus $P : [x_0, \infty[\times\mathbb{R}_+ \to [x_0, \infty[$. We take the function P to be the initial code of a family of codes $\{P_\alpha\}$. We establish the permutability and the quasi permutability of the code P with respect to P_α, for any $\alpha > 0$, by an elementary geometric argument.

16 The Permutability of P. A right triangle $\triangle ABC$ with leg lengths x and y and hypothenuse of length $P(x, y)$ is represented in Figure 1A. Thus $AB = x$, $BC = y$ and $P(x, y) = AC$. Another right triangle $\triangle ACD$ is defined by the segment $\overline{CD}$ of length z, which is perpendicular to the plane of $\triangle ABC$. The length of the hypothenuse $\overline{AD}$ of $\triangle ACD$ is thus $P(P(x, y), z) = AD$. Still another right triangle $\triangle EAB$ is defined by the perpendicular $\overline{AE}$ to the plane of $\triangle ABC$. We choose E such that $AE = z = CD$; we have thus $EB = P(x, z)$. Since $\overline{AE}$ is perpendicular to the plane of $\triangle ABC$ and $\triangle ABC$ is a right triangle, $\overline{EB}$ is perpendicular to $\overline{BC}$. The lines $\overline{BC}$ and $\overline{BE}$ are perpendicular. (Indeed, the perpendicular L at the point B to the plane of triangle $\triangle ABC$ is coplanar with $\overline{AE}$. So, as $\overline{BC}$ is perpendicular to both $\overline{AE}$ and L, it must be perpendicular to to the plane of $\triangle EAB$, and so it must be perpendicular to $\overline{EB}$.) Accordingly, $EC = P(P(x, z), y)$ is the length of the hypothenuse of the right triangle $\triangle EBC$. It is clear that, by construction, the four points A, C, D and E are coplanar. They define a rectangle whose diagonals $\overline{AD}$ and $\overline{EC}$ must be equal. So, we must have $P(P(x, y), z) = P(P(x, z), y)$, establishing the permutability of the code P.

17 The Quasi Permutability of P. For any positive real number α, the triangle $\triangle A'B'C'$ pictured in Figure 1B, with $C' = c$, A collinear with $A'C'$, B collinear with $B'C'$, and $A'B' = \frac{x}{\alpha}$, $B'C' = \frac{y}{\alpha}$ and $A'C' = \frac{P(x,y)}{\alpha}$, is similar to the triangle $\triangle ABC$ also represented in Figure 1B. So, we have

$$P\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) = \frac{P(x, y)}{\alpha}. \tag{57}$$

[8]Cf. our discussion of Condition [S2] in the context of Example 4(e).

The function P is the initial code of the meaningful family of codes $\{P_\alpha\}$. For the code P_α in that family, we get

$$\begin{aligned}
P_\alpha\left(P(x,y),z\right) &= \alpha P\left(\frac{P(x,y)}{\alpha},\frac{z}{\alpha}\right) && \text{(by meaningfulness)}\\
&= \alpha P\left(P\left(\frac{x}{\alpha},\frac{y}{\alpha}\right),\frac{z}{\alpha}\right) && \text{(by Equation (57))}\\
&= \alpha P\left(P\left(\frac{x}{\alpha},\frac{z}{\alpha}\right),\frac{y}{\alpha}\right) && \text{(by the permutability of } P\text{)}\\
&= \alpha P\left(\frac{P(x,z)}{\alpha},\frac{y}{\alpha}\right) && \text{(by Equation (57))}\\
&= P_\alpha\left(P(x,z),y\right) && \text{(by meaningfulness).}
\end{aligned}$$

We conclude that any code $P_a a$ in the family $\{P_\alpha\}$ is quasi permutable with respect to the initial code P.

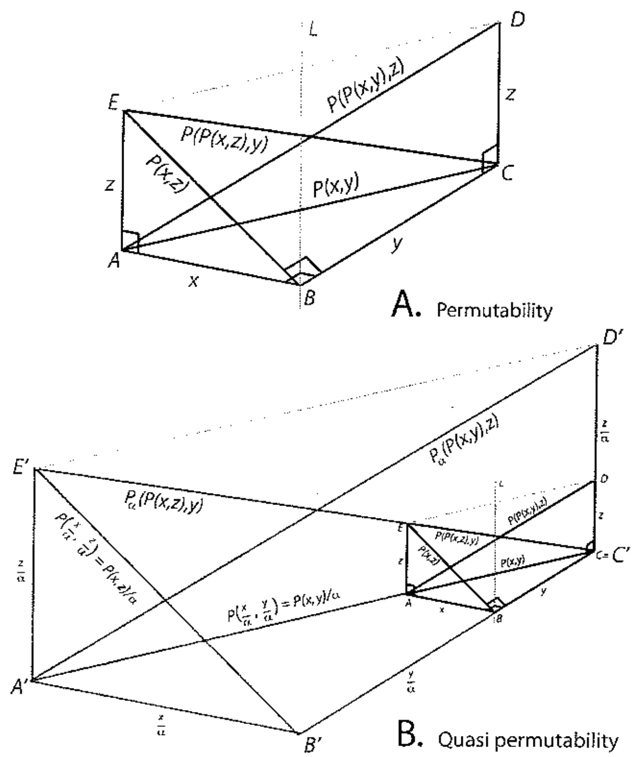

Figure 1: A. The upper graph illustrates the permutability condition formalized by the equation $P(P(x,y),z) = P(P(x,z),y)$.
B. The lower graph shows that the quasi permutability condition formalized by the equation $P_\alpha(P(x,y),z) = P_\alpha(P(x,z),y)$ only involves a rescaling of all the variables pictured in Figure 1A, resulting in a similar figure, with the rectangle $A'B'C'D'$ similar to the rectangle $ABCD$. The measures of the two diagonals of the rectangle $A'B'C'D'$ are $P_\alpha(P(x,y),z)$ and $P_\alpha(P(x,z),y)$.

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