The Dynamics of Costly Signaling

Elliott Wagner

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Abstract

Costly signaling and screening are two mechanisms through which the honesty of signals can be secured in equilibrium. This paper explores the dynamics of one such signaling game: Spence's model of education. It is found that separating equilibria are unlikely to emerge under either the replicator or best response dynamics, but that partially communicative mixed equilibria are quite important dynamically. After relating these results to traditional refinements, it is suggested that these mixtures may play significant, and underappreciated, roles in the explanation of the emergence and stability of information transfer.

1 Introduction

How can the honesty of communication between two agents be ensured when their interests do not coincide? This is one way of framing the question Spence (1973) posed in his seminal paper on about job market signaling. His famous answer was that a particular structure of costly signaling can guarantee honesty in equilibrium. However, a complication typical of signaling models in this tradition is that they often have an infinity of equilibrium outcomes. Such an abundance of equilibria makes equilibrium selection a daunting task. Economists seeking to address this issue have often posited equilibrium refinements with the aim of identifying only a small number believable equilibrium outcomes.[1]

This paper takes a different approach to equilibrium selection in Spence's original model of education. Instead of applying an equilibrium refinement, Spence's game will be embedded into two common game dynamics: the replicator and best response dynamics. These dynamics arise from very different modeling assumptions. The replicator dynamic is a paradigm example of an unsophisticated process of imitation, whereas the best response dynamic is an archetype of myopic rational behavior. Nonetheless, both have been proposed as models of learning in games (Schlag, 1998; Gilboa and Matsui, 1991). In order to study equilibrium selection and maintenance, the dynamic stability of the equilibria in Spence's model and the sizes of the basin of attraction of each attractor will be investigated under both these dynamics. Section 2 reviews Spence's game and section 3 carries out this study into the dynamics. It is found that mixed equilibria, which are largely ignored in the signaling literature, play a very important role in the emergence and stability of information transfer. Since both dynamics are most easily interpreted as models of large populations, these mixtures are naturally interpreted as polymorphic market states. Additionally, it is found that when two separating equilibria are present in the model, only the separating equilibrium that yields senders the highest payoff – called the Riley equilibrium by Nöldeke and Samuelson (1997) – is stable under the best response dynamic. It also has the largest basin of attraction, at least out of other separating possibilities, under the replicator dynamic.

These results are discussed and connected to the literature in section 5. In brief, it is shown that for a wide class of parameter settings, there exists a dynamically stable mixed equilibrium. This equilibrium is equivalent to the mixed sequential equilibrium of the Spence variant from Cho and Kreps (1987) and the two period cycle found by Nöldeke and Samuelson (1997). This mixture attracts a large portion of initial conditions under both dynamics studied. It is suggested that this class of mixed equilibria may play significant, and under-

[1] Nöldeke and Samuelson (1997) is the exception. They studied an explicit dynamic process tailored for Spence's model in which the receivers revise their beliefs upon observing the quality of hired workers and the signals these workers sent. Their findings are surveyed in section 2 and connected to the results of this study in section 5

appreciated, roles in the explanation of information markets and costly signaling in general.

2 Job market signaling

The standard presentation of Spence's (1973) model of education takes the form of a signaling game with two players: a worker (the sender) and an employer (the receiver); see, e.g., Fudenberg and Tirole (1991), Gibbons (1992), or Osborne and Rubinstein (1994) for textbook accounts. The worker knows her ability level θ, but her prospective employer does not. After observing θ, the worker chooses some level of education $e \in \mathbb{R}^+$ to purchase. The worker incurs a cost $c(\theta, e)$ obtaining this education. It is assumed that the value of the worker to the employer is θ, and that, after observing the worker's message (choice of e), the employer pays the worker a wage w that is equal to the employer's expectation of θ. To model this assumption the employer is frequently presumed to seek to minimize the quadratic difference between the wage and θ, hence the payoff to the employer is given as $-(w-\theta)^2$. The payoff to the employee is $w - c(\theta, e)$.

To simply analysis, it is typically assumed that employees are one of two types. That is, $\theta \in \{\theta_L, \theta_H\}$. Denote the probability of a worker being of these types p_L and p_H (with $p_H = 1 - p_L$). And finally, following Spence's original article, let $c(\theta, e) = \frac{e}{\theta}$, $\theta_L = 1$, and $\theta_H = 2$, so that $c(\theta_L, e) = e$ and $c(\theta_H, e) = \frac{e}{2}$. These assumptions are not necessary for the equilibrium analysis of Spence's model, but they make the dynamical analysis of the next section tractable. So, for ease of exposition they're made now.

It is well known that this game has an infinite number of perfect Bayesian equilibria. Coupled with the appropriate beliefs, they come in three flavors: pooling, separating, and hybrid. In a pooling equilibrium both types of worker send the same message e^* and the employer offers the wage $w^* = p_L + 2p_H = 1 + p_H$ upon the receipt of this message. In this sort of equilibrium, no information transfer takes place. In a separating equilibrium, type θ_H workers send message $e^* \in [1, 2]$ and type θ_L workers send $e_L = 0$. The employer then offers the wage 2 upon receipt of message e^* and 1 upon receipt of message e_L. Education level functions as a perfect indicator of worker productivity in a separating equilibrium. The insight being, of course, that education need not improve worker productivity in order to garner the more educated higher wages in equilibrium.

Hybrid equilibria are mixtures in which one type of worker chooses one level of education with certainty and the other type randomizes between pooling and separating. In these mixed equilibria, education level carries some information, but information transfer is imperfect. Hybrid equilibria are often ignored in discussions of Spence's signaling game,[2] but the dynamic analysis presented in the rest of this paper suggests that they may play an important role in the informational structure of markets.

So there are three types of equilibria and a continuum of each type. Many refinements have been proposed to limit this variety to just the reasonable equilibria. Indeed, the signaling and screening refinement literature is so immense that it is impossible to summarize even a small fraction of it here. I will, however, mention two influential refinements. Riley (1979) showed (in a more abstract model) that there is a unique reactive equilibrium that Pareto dominates the family of consistent price functions, the implicit suggestion being that if a market finds its way to a separating equilibrium, that equilibrium will be the one that provides the high quality senders the greatest payoff. In terms of Spence's model as described above, this is the separating equilibrium with $e^* = 1$. Nöldeke and Samuelson (1997) call this the Riley equilibrium. In a different approach to this issue, Cho and Kreps (1987) suggested refining out-of-equilibrium beliefs. Their Intuitive Criterion, as applied to Spence's model, has a great deal of bite. It eliminates all equilibrium outcomes with the exception of the Riley equilibrium.

Nöldeke and Samuelson (1997) proposed an alternative approach to equilibrium selection in Spence's model. Finding inspiration in Spence's remark that certain equilibrium outcomes

[2] They are not mentioned in either Fudenberg and Tirole (1991) or Osborne and Rubinstein (1994), and are not discussed in Spence (1973). Similarly, in his expansive survey of screening and signaling research, Riley (2001) does not cover these mixtures. Gibbons (1992) does provide a discussion.

would not survive a dynamic process of belief and strategy revision coupled with arbitrarily small perturbations, they investigated such a dynamic model explicitly. They found that the dynamic process leads to one of three types of recurrent set. One such set contains the separating Riley equilibria. Another contains pooling equilibria. The third contains a two period cycle. The cycles consists of type θ_H senders always sending the message e^* and type θ_L senders switching between sending e^* and e_L on each iteration of the dynamic. It is obvious that such a cycle corresponds to the hybrid equilibria described above, and Nöldeke and Samuelson show that this cycle exists if an only if there is a mixed sequential equilibrium of the Cho and Kreps (1987) varient of Spence's model.

3 Pruning Spence's game

The previous section reviewed how it is that message cost structure can allow honest signaling in equilibrium. However, important dynamic questions are left unanswered. For example, how likely is it that a system of senders and receivers ends up at a separating equilibrium instead of at pooling or at a mixture? Also, is it always the case that Spence's model converges to one of the Nash equilibria? Nöldeke and Samuelson's discrete-time revision protocol did not always converge, but what about more run-of-the-mill dynamical systems? Such dynamics are known to exhibit complex behavior in some circumstances. For example, neither the replicator dynamic nor the best response dynamic is guaranteed to converge to the mixed Nash equilibrium of rock-paper-scissors. Instead, play may spiral outward toward the boundary of phase space under the replicator dynamic or end up oscillating endlessly in a stable limit cycle under the best response dynamic (Gaunersdorfer and Hofbauer, 1995). And some simple models of price adjustment admit literally any dynamic behavior including limit cycles and chaos (Saari, 1991). In any case, results of this sort demonstrate the importance of understanding out-of-equilibrium play and how it can lead – or not – to equilibrium.

However, in order to study the dynamics here in detail, it is necessary to prune the

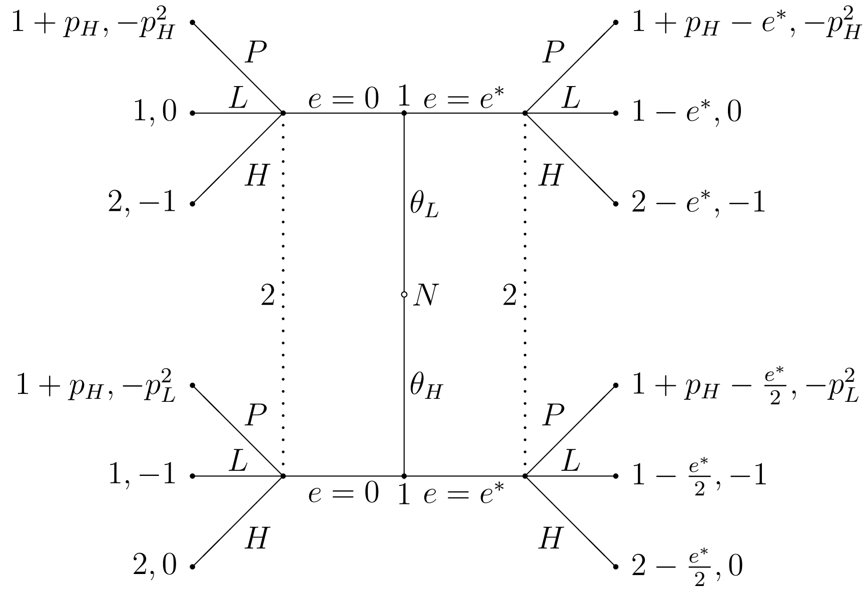


Figure 1: The extensive form representation of Spence's game in which workers choose from only two levels of education – 0 and e^* – and employers either act as though senders are pooling (P) or act as though the message indicates low quality (L) or act as though the message indicates high quality (H).

strategy space of Spence's game because there is not a thoroughly developed theory of adaptive dynamics for games, and in particular Bayesian games, with infinite strategy spaces. If the important dynamical question is understanding how a system evolves to either pooling, separating, partial communication, or non-convergence, then an obvious way to shrink the strategy space is to consider workers who have a choice to either always send message $e_L = 0$, separate honestly (i.e. type θ_L workers send e_L and type θ_H workers send e^*), or always send message e^*. Call these strategies *Low*, *Sep*, and *High* respectively. Likewise, we can limit the employer's strategies to acting as through the messages are meaningless (i.e. offering the pooling wage regardless of message received) and acting as though the messages correctly identify sender types (i.e., offer 1 is e_L is received and offer 2 if e^* is received). Call the former strategy *Pool* and the latter *Sep*. This pruned extensive form game is shown in Figure 1.

Both the replicator dynamic and the best response dynamic are infinite population models, and payoffs to strategy types are given by the type's expected payoff when matched with a random member of the population. Therefore, we can now focus analysis on the 3×2

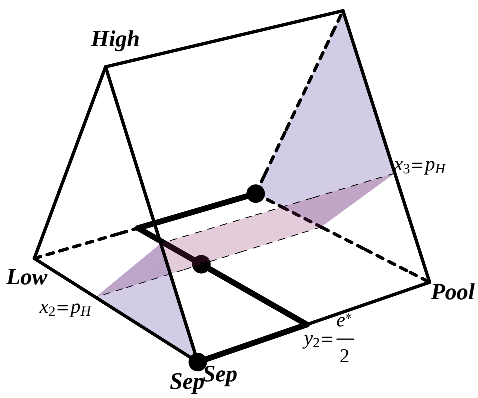
	Pool	*Sep*
Low	$1 + p_H, -p_L p_H$	$1, -p_H$
Sep	$1 + p_H - \frac{p_H e^*}{2}, -p_L p_H$	$1 + p_H - \frac{p_H e^*}{2}, 0$
High	$1 + p_H - p_L e^* - \frac{p_H e^*}{2}, -p_L p_H$	$2 - p_L e^* - \frac{p_H e^*}{2}, -p_L$

Table 1: The pruned Spence signaling game.

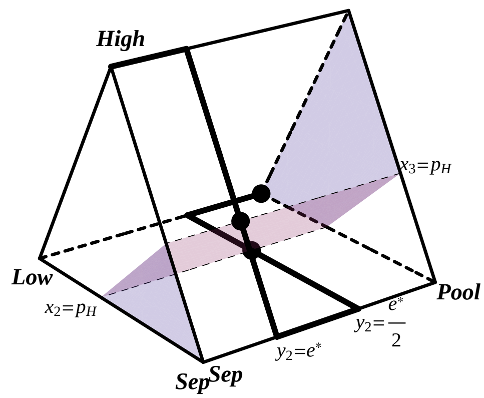
normal game shown in Table 1 in which the payoffs are the expectations of payoffs from the extensive form game. Notice that if the receiver plays *Pool*, the sender's unique best response is to play *Low*. Likewise, the receiver's best response to *Low* is to play *Pool*. Thus, the profile (*Low*, *Pool*) is a strict Nash equilibrium. It corresponds to a pooling equilibrium in Spence's original game; workers don't purchase education and employers don't listen to signals.

Similarly, the receiver's unique best response to *High* is to play *Pool*. However, the receiver's unique best response to *Sep* is to play *Sep*. All of these best response relationships are independent of e^*. To determine the equilibrium structure of this game, It only remains to determine sender's best response to the receiver's playing the pure strategy *Sep*. *Sep* will be the sender's unique best response just in case $6 - e^* > 4$ and $6 - e^* > 8 - e^*$. These conditions are satisfied if and only if $1 < e^* < 2$. Accordingly, when $1 < e^* < 2$, the profile (*Sep*, *Sep*) corresponds to a separating equilibrium in the full game.

On the other hand, if $0 < e^* < 1$ then this separating profile is not an equilibrium. However, an important mixed equilibrium exists for these values of e^*. The profile in which the sender randomizes between *Sep* and *High* with probabilities p_L and p_H, and the receiver randomizes between *Pool* and *Sep* with probabilities $1 - e^*$ and e^* respectively is a Nash equilibrium when $0 < e^* < 1$. It corresponds to a hybrid equilibrium in the original game in which high productivity workers send message e^* with certainty and low productivity workers randomize between separating from and pooling with the high type. The mixed strategy space and best response correspondences for both cases are drawn in Figure 2.



(a)



(b)

Figure 2: Best response correspondences for the pruned Spence signaling game with (a) $1 < e^* < 2$ and (b) $0 < e^* < 1$. The sender's best reply is shown by the thick line. The receiver's best reply is shown by the translucent surface. x_2 signifies the probability that the sender plays *Sep*, x_3 the probability that the sender plays *High*, and y_2 the probability that the receiver plays *Sep*. Nash equilibria are highlighted by black dots.

These correspondences are crucial for analyzing the best response dynamic below.

Although this 3×2 game has pruned out a continuum of sending strategies and a continuum of receiving strategies, it still captures the spirit of Spence's model. Pooling is an equilibrium regardless of the value of e^*. And separating can be an equilibrium for e^* set sufficiently high. Thus, just like in Spence's model, a costly education can signal high quality and secure high wages even though education itself may not increase productivity. Now that we have a two player normal form game that retains some of the structure of Spence's original model, it is possible to proceed in analyzing the dynamics of job market signaling.

4 Dynamics

The two adaptive dynamics applied here are the two population replicator and best response dynamics. The first population is the population of workers. They choose from three pure strategies. Denote the proportion that chooses each strategy *Low*, *Sep*, and *High* as x_1, x_2, and x_3. The second population consists of employers. Let y_1 and y_2 be the proportions of the population that play *Pool* and *Sep*. Because $x_1 + x_2 + x_3 = 1$ and $y_1 + y_2 = 1$, the dynamics for this system lives in the five dimensional space $\Delta^3 \times \Delta^2$ where Δ^n is the $n-1$ dimensional simplex $\{(p_1, \ldots, p_n) \,|\, p_i \geq 0, \sum p_i = 1\}$. Coordinates in phase space will be written (x_2, x_3, y_2).[3]

The replicator dynamic for the pruned game is given by the three differential equations

$$\begin{aligned} \dot{x}_2 &= x_2 \left[(Ay)_2 - x \cdot Ay \right] \\ \dot{x}_3 &= x_3 \left[(Ay)_3 - x \cdot Ay \right] \\ \dot{y}_2 &= y_2 \left[(Bx)_2 - y \cdot Bx \right] \end{aligned} \tag{RE}$$

where A is the sender's 3×2 payoff matrix and B is the receiver's 2×3 payoff matrix. Although this dynamic was originally formulated by Taylor and Jonker (1978) to model natural selection in an asexually reproducing population, it also provides a model of cultural learning in economic situations. In this context, the equations give the fluctuations in strategy distributions as agents imitate successful members of their population. In other words, these equations describe large populations of employers and workers in which individual agents, when called on to revise their strategy choice, choose to imitate a more prosperous player.[4]

An advantage of focusing on the replicator dynamic is that stability analysis of its rest points can provide information about the stability of these points under all two population uniformly monotone selection dynamics. This family of game dynamics is characterized by a positive linear correlation between relative growth rates and payoff differences. Since a rest point is asymptotically stable for the replicator dynamic if and only if it is asymptotically stable under every two population uniformly monotone selection dynamic (Cressman, 2003, Theorem 3.5.3), studying this process provides an understanding of the behavior of this larger class of selection dynamics.

[3]It is convenient here to work directly with x_2, x_3, and y_2 instead of x_1 or y_1.

[4]See Weibull (1997) for a survey of imitative dynamics and their relationship with the replicator dynamic.

The best response dynamic for the pruned game is written as

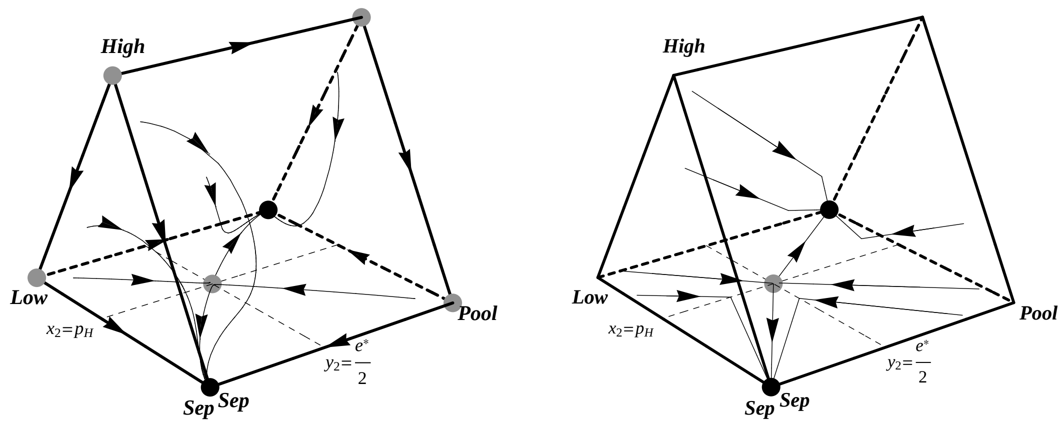
$$\begin{aligned} \dot{x}_2 &= BR(y) - x_2 \\ \dot{x}_3 &= BR(y) - x_3 \\ \dot{y}_2 &= BR(x) - y_2 \end{aligned} \tag{BR}$$

where $BR(y) = \{\hat{x} \in \Delta^3 \,|\, \hat{x}\cdot Ay \geq x\cdot Ay \text{ for all } x \in \Delta^3\}$ and $BR(x)$ is defined similarly.[5] The usual interpretation of this dynamic is that a small fraction of each large population revises their strategy at each time interval. Upon revision, they choose a best reply to the current state. A complication in analyzing this system is that it is not in general differentiable at rest points (Nash equilibria) because it is at these points where BR abruptly changes and is often many-valued. However, an analytic virtue of this dynamic is that piecewise linear solutions can be constructed from any initial condition. This is due to the fact that, at every state, the best response dynamic moves the population in a straight line toward the current best reply profile; see, e.g. Hofbauer and Sigmund (1998) or Cressman (2003). This fact will be used in the constructions below.

4.1 When separating is an equilibrium

When $1 < e^* < 2$ the dynamics of the pruned game shown in Table 1 are straightforward. There are two strict Nash equilibria: $(Low, Pool)$ and (Sep, Sep). Therefore, the corresponding states $(0, 0, 0)$ and $(1, 0, 0)$ are asymptotically stable under both dynamics. Furthermore, the pure sending strategy $High$ is strictly dominated by the pure strategy Sep. Pure strategies that are strictly dominated by other pure strategies are driven to extinction by both dynamics. Thus every initial condition in the interior of phase space is brought to the $x_3 = 0$ boundary face. And, because there are two sinks on this face, index theory asserts that there must be a saddle between them. This saddle is the mixed Nash equilibria at

[5] Since BR is a set-valued function, the best response dynamic is not technically a dynamical system. Instead it is a differential inclusion.



(a) (b)

Figure 3: Phase portraits showing the dynamics of the pruned Spence signaling game with $1 < e^* < 2$ for (a) the replicator dynamic and (b) the best response dynamic. Black and grey dots indicate stable and unstable rest points respectively.

location $(p_H, 0, \frac{e^*}{2})$. It is easily confirmed unstable through linearization (for the replicator dynamic) or inspection of the best response correspondences (for the best response dynamic). Phase portraits for both systems are illustrated in Figure 3. Additionally, since asymptotically stable rest points in the replicator dynamic are asymptotically stable for every two population uniformly monotone selection dynamic, we can immediately conclude that the pooling and separating equilibria are asymptotically stable under all such dynamics.

There is nothing too surprising about these results. Depending on the initial conditions of the system, the dynamics carry it to either the pooling or the separating equilibrium. The only perhaps unexpected features of these systems are the potentially very small basins of attraction for the separating equilibria. For the replicator dynamic it is necessary to use numerical integration to estimate the proportion of phase space that converges to each of the attractors. But, for at least some values of p_H that make the geometry relatively simple, the size of the basin of attraction for separating under the best response dynamic can be found analytically. The basin of attraction for separating is the portion of phase space contained

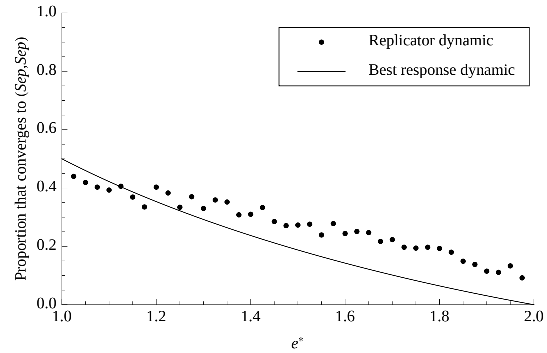


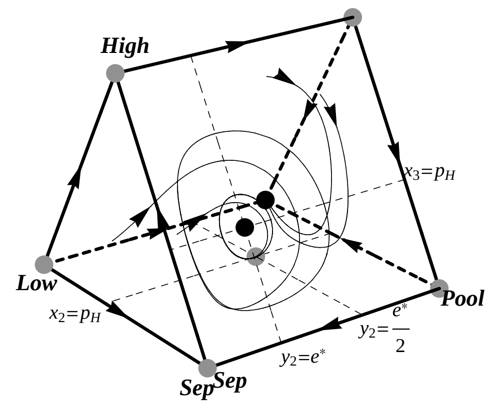
Figure 4: The proportion of phase space that converges to separating under both dynamics with $p_H = .5$. Each data point for the replicator dynamic is the average of $1,000$ randomly chosen initial conditions. The volume of space that leads to separating under the best response dynamic when $p_H = .5$ is $\frac{(2-e^*)(p_H-1)(p_H e^{*2}-e^*(1+2p_H)-3p_H)}{3e^*}$.

within the two two-dimensional separatrixes that lead directly to the unstable rest point. A chart of the proportion of phase space that leads to separating under both dynamics is shown in Figure 4.

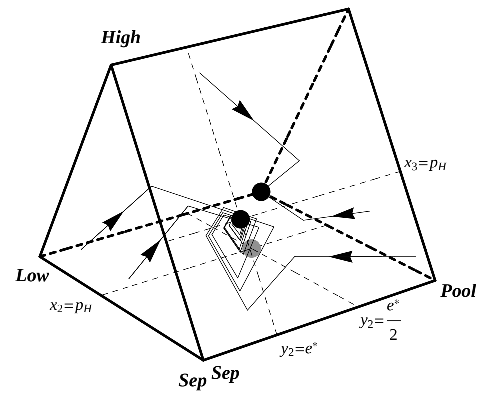
4.2 When separating is not an equilibrium

When $0 < e^* < 1$ the dynamics become more complex. $(Low, Pool)$ remains a strict Nash equilibrium and hence asymptotically stable. There are also two other rest points, each mixtures. One, at coordinates $(p_H, 0, \frac{e^*}{2})$ is unstable. The other rest point lies at $H = (p_L, p_H, e^*)$. This rest point corresponds to a hybrid equilibrium in which the high quality workers always send e^* and the low quality workers flip a biased coin to determine whether to send e^* or 0. Unfortunately, establishing the stability of this state is nontrivial. Linear stability analysis is not feasible for the best response dynamic because it is not differentiable here. And it is inconclusive for the replicator dynamic because the the rest point is not hyperbolic. However, the point H is indeed stable under the replicator dynamic.

Theorem 1. *The hybrid equilibrium* $H = (1 - p_H, p_H, e^*)$ *is neutrally stable under the replicator dynamic when* $0 < e^* < 1$.



(a)



(b)

Figure 5: Phase portraits showing the dynamics of the pruned Spence signaling game with $0 < e^* < 1$ for (a) the replicator dynamic and (b) the best response dynamic. Black and grey dots indicate stable and unstable rest points respectively.

Hirsch et al. (2004) call rest points like H "spiral centers." Initial conditions near H quickly spiral toward the $x_1 = 0$ boundary face. Then, once on this face, they cycle endlessly in closed periodic orbits centered on H. Thus, the hybrid equilibrium H is neutrally stable. Figure 5 shows a phase portrait for this system. Unfortunately, since H is not a hyperbolic rest point, nothing can be concluded about the stability of H under all uniformly monotone selection dynamics. A perturbation to the dynamic will change the system's qualitative behavior, sending orbits, for instance, either spiraling into or away from H.

The stability of H under the best response dynamic is not as delicate as under the replicator, however it is also not straightforward to demonstrate. But, as might be expected by analogy to matching pennies, H is indeed asymptotically stable under the best response dynamic.

Theorem 2. *The hybrid equilibrium $H = (1 - p_H, p_H, e^*)$ is asymptotically stable under the best response dynamic.*

The system's phase portraits are shown in Figure 5. Since H is not an attractor under the replicator dynamic, we cannot ask how much of phase space is attracted to H. But,

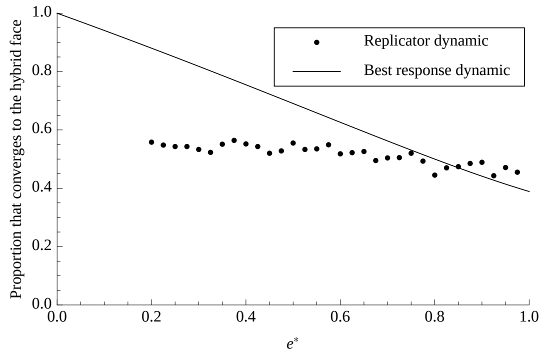


Figure 6: The proportion of phase space that converges to the $x_1 = 0$ boundary face under both dynamics with $p_H = .5$. Each data point for the replicator dynamic is the average of 1,000 randomly chosen initial conditions. The volume of space that leads to H under the best response dynamic when $\frac{48-64e^*+23e^{*2}}{48-36e^*+6e^{*2}}$.

since the linearization of H does have one negative eigenvalue, it is possible to investigate how much of phase space is attracted onto the $x_1 = 0$ boundary face. And, once again, it is possible to solve for the exact proportion of phase space that is attracted to H under the best response dynamic (at lease for some values of p_H for which the geometry is not too complex). Figure 6 shows the sizes of these basins of attraction for H under both dynamics. Notice that, for all values of e^*, a greater fraction of phase space ends in either at H or in oscillations centered on H than ended at the separating equilibrium above.

4.3 Separating vs. hybrid equilibria

So, hybrid equilibria are stable under both dynamics and have a large influence on the emergence of costly signaling. But so far this equilibrium type has only been pitted against pooling. Do hybrid equilibria outperform separating? To approach this question it is possible to study a slightly larger strategic form game in which pooling, separating, and hybrid equilibria all coexist in phase space. This enlarged game is shown in Table 2. In the interaction captured by this expanded game, each worker sends one of three messages: 0, e_1^*, or e_2^* with $e_1^* < e_2^*$. The natural sending strategies include separating with high quality workers sending e_1^* ($Sep_{e_1^*}$) and separating with those workers sending e_2^* ($Sep_{e_2^*}$). The

	$Pool$	$Sep_{e_1^*}$	$Sep_{e_2^*}$
Low	$\frac{3}{2}, -\frac{1}{4}$	$1, -\frac{1}{2}$	$1, -\frac{1}{2}$
$Sep_{e_1^*}$	$\frac{3}{2} - \frac{e_1^*}{4}, -\frac{1}{4}$	$\frac{3}{2} - \frac{e_1^*}{4}, 0$	$1 - \frac{e_1^*}{4}, -\frac{1}{2}$
$High_{e_1^*}$	$\frac{3}{2} - \frac{3e_1^*}{4}, -\frac{1}{4}$	$2 - \frac{3e_1^*}{4}, -\frac{1}{2}$	$1 - \frac{3e_1^*}{4}, -\frac{1}{2}$
$Sep_{e_2^*}$	$\frac{3}{2} - \frac{e_2^*}{4}, -\frac{1}{4}$	$\frac{3}{2} - \frac{e_2^*}{4}, -\frac{1}{2}$	$\frac{3}{2} - \frac{e_2^*}{4}, 0$

Table 2: The expanded pruned Spence signaling game with $p_H = \frac{1}{2}$.

corresponding receiver strategies are then interpreting all messages less than e_1^* as originating from a low quality worker ($Sep_{e_1^*}$) and interpreting all messages less than e_2^* as indicating low productivity ($Sep_{e_2^*}$).

Both the replicator and best response dynamics for this game live in the five dimensional space $\Delta^4 \times \Delta^3$. For $0 < e_1^* < 1 < e_2^* < 2$, there are three stable rest points corresponding to the types of equilibria. The pooling profile $(Low, Pool)$ and the separating profile $(Sep_{e_2^*}, Sep_{e_2^*})$ are strict Nash equilibria and hence asymptotically stable states. There is also a hybrid equilibrium in which the sender randomizes between $Sep_{e_1^*}$ and $High_{e_1^*}$ and the receiver randomizes between Low and $Sep_{e_1^*}$. This equilibrium is neutrally stable under the replicator dynamic and asymptotically stable under the best response dynamic. In fact, the replicator dynamics from sections 4.1 and 4.2 above are recaptured on boundary faces of the replicator dynamic for the expanded game in Table 2.

It is convenient to use the logit dynamic to estimate the proportion of the space attracted to each rest point under the best response dynamic. For small values of η, this dynamic, given here by

$$\dot{x}_i = \frac{\exp\left(\frac{(Ay)_i}{\eta}\right)}{\sum_{j=1}^{3} \exp\left(\frac{(Ay)_j}{\eta}\right)}, \quad \dot{y}_i = \frac{\exp\left(\frac{(Bx)_i}{\eta}\right)}{\sum_{j=1}^{2} \exp\left(\frac{(Bx)_j}{\eta}\right)}$$

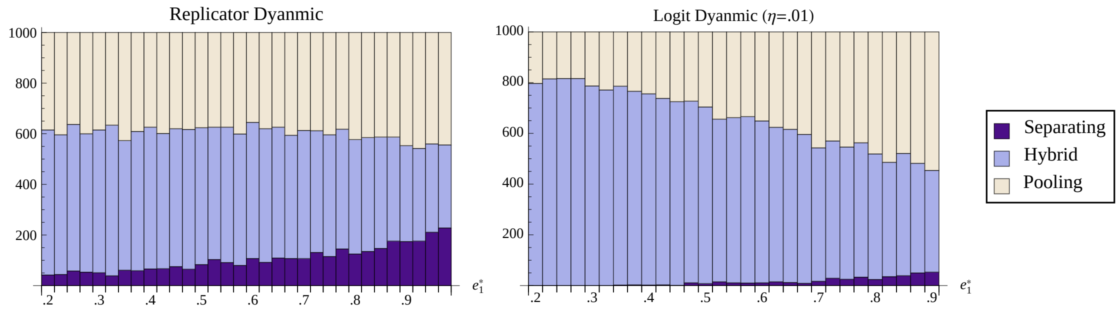


Figure 7: The number of randomly chosen initial conditions that converged to pooling, separating, and the hybrid face under both the replicator dynamic and the logic dynamic with $\eta = .01$. e_2^* is held fixed at 1.025 while e_1^* is varied.

approximates the best response dynamic (Fudenberg and Levine, 1998). Figure 7 shows the number of randomly chosen initial conditions that converged to the pooling, separating, and hybrid equilibria. Notice that for all values of e_1^*, the hybrid face attracts a larger portion of phase space than the separating equilibrium. Indeed, even under the best of conditions, perfect communication seems a relatively unlikely outcome of the dynamic process. Most initial states lead to partial information transfer or to no communication at all.

4.4 Riley equilibria

It is also possible to use the expanded game from Table 2 to investigate which of two separating equilibria the dynamics select. When $1 < e_1^* < e_2^* < 2$, the sending strategy $High_{e_1^*}$ can be excluded to ease analysis since it is strictly dominated by $Sep_{e_1^*}$. This leaves three sending strategies – Low, $Sep_{e_1^*}$, and $Sep_{e_2^*}$ – with two strict Nash equilibria – $(Low, Pool)$ and $(Sep_{e_1^*}, Sep_{e_1^*})$. These states are asymptotically stable under both the replicator and best response dynamics (and hence also under every two population uniformly monotone selection dynamic). There is also a Nash equilibrium component in which the sender plays $Sep_{e_2^*}$ and the receiver plays $Sep_{e_2^*}$ with probability $q > \frac{1}{2}(e_2^* - e_1^*)$ and $Sep_{e_1^*}$ with probability $1 - q$.

Although this Nash component constitutes a set of connected rest points under the best response dynamic, it is unstable. To see why, Figure 8 shows a phase portrait for a reduced

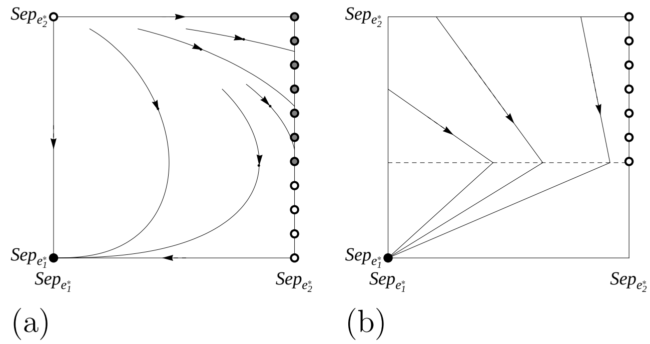


Figure 8: Phase portraits for the game in Table 2 limited to the sending and receiving strategies $Sep_{e_1^*}$ and $Sep_{e_2^*}$ for (a) the replicator dynamic and (b) the best response dynamic. The horizontal direction shows the sending population and the vertical direction shows the receiving. Black, grey, and white dots indicate asymptotically stable, neutrally stable, and unstable rest points respectively.

system without the strategies *Low* and *Pool*. Because the strategy profile $(Sep_{e_2^*}, Sep_{e_2^*})$ is unstable, the best response dynamic provides a rather strong device for equilibrium selection. If the system evolves to a separating equilibria, that separating equilibria will be the one in which senders send the cheapest possible signal. In other words, the best response dynamic here predicts that separating equilibria will be what Nöldeke and Samuelson (1997) call a Riley equilibrium. Namely, the separating equilibrium that guarantees the high quality senders the highest payoff out of all possible separating equilibria.

This Nash component is also part of a line of rest points connecting the receiver strategies $Sep_{e_1^*}$ and $Sep_{e_2^*}$ under the replicator dynamic. Here, unlike the flow for the best response dynamic, every point on the interior of this component is neutrally stable. Figure 8 provides some visual intuition, and Figure 9 shows the fraction of randomly chosen initial conditions that ended at each attracting set for the replicator dynamic. Notice that, although the stable line can have a rather large basin of attraction, it is still true that the separating state with the cheaper signal attracts a larger proportion of the space. So although the replicator dynamic does not provide as strong of prediction as the best response dynamics, the results suggest that the Riley equilibrium is perhaps the most plausible separating outcome.

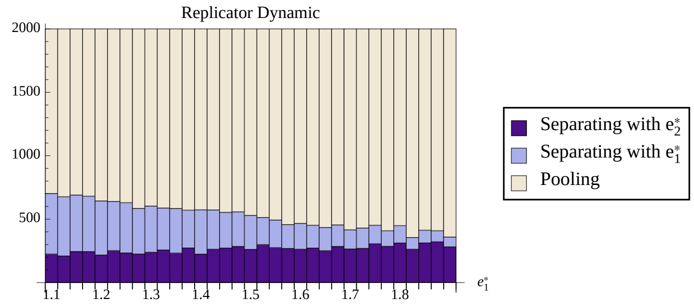


Figure 9: The number of randomly chosen initial conditions that converged to each stable set under the replicator dynamic. For all trials, $e_2^* = 1.9$ and $p_H = .5$.

5 Discussion

The previous section showed that the ordinary predictions of equilibrium refinement theory are not validated by dynamic analysis in two respects. First, contrary to influential refinements such as the Intuitive Criterion, pooling is a likely result of both adaptive processes investigated above. As much as optimists may hope to rule out such uninspired states in which no information is conveyed, these non-communicative states may be the norm rather than the exception. Second, mixtures may be more likely than previously thought. Refinements often exclude hybrid equilibria from being considered rational solutions to Spence's game. However, the results above demonstrate that mixtures are likely outcomes of dynamic processes. Although the hybrid is not asymptotically stable under the replicator dynamic, its boundary face can attract a large proportion of initial conditions. And even more strikingly, under the best response dynamic, the hybrid is asymptotically stable and, for small values of e^*, attracts almost all of phase space.

On the other hand, some of the predictions of more traditional game-theoretic analysis are recaptured in the dynamic approach. For instance, these results provide further evidence for the claim that the Riley equilibrium is the most plausible separating equilibrium. Additionally, the stable hybrid seen above mirrors some similar equilibria in other screening and signaling models. For instance, the hybrid equilibrium is stable under the same conditions ($0 < e^* < 1$) in which there is a mixed sequential equilibrium of Cho and Kreps's (1987)

Spence variant in which two employers hold a Bertrand-style auction to set the wage. And the conditions for the stable hybrid are the same as those for which the Rothschild and Stiglitz (1976) one-shot screening game has no Nash equilibrium (see Riley, 2001). And these conditions are the same as those that generate the stable two-period cycle under the Spencian dynamic from Nöldeke and Samuelson (1997). Indeed, this cycle is the direct counterpart to the stable hybrid and the oscillations around it described above. The variety of models in which this mixture appears, and its stability under three very different adaptive processes – the replicator, best response, and Spencian dynamics – suggests that it may play a very important role in both out-of-equilibrium and in-equilibrium signaling behavior.

It is worth noting that the message e^* is not equally informative at each possible stable hybrid. In the three dimensional systems from sections 4.1 and 4.2, the stable hybrid is located at (p_L, p_H, e^*). Perhaps an unintuitive consequence of this point's location is that the frequency of deception is not influenced by the signaling level e^*. That is, the likelihood that a costly signal originated from a low quality worker does not depend on the cost of the signal. A consequence of this observation is that firms (at least in this model) are unable to deter dishonest signaling by increasing the level of education required at the hybrid profile. A similar oddity is that the attention the firms pay to the signal e^* does not depend on the signal's reliability. Instead it depends solely on the signal's strength.

As for applicability of these results to actual social interactions, it is not implausible that hybrid equilibria play a more important role than they are credited for. The study of costly signaling goes back at least to Veblen (1899), who brought attention to the American leisure class's predilection to flaunt wealth through ornate silver utensils, flamboyant homes, and other methods of conspicuous consumption. Veblen's famous thesis held that these members of the upper crust were investing in costly signals to demonstrate their prestige. But status signals are not always honest. The system Veblen described, no less than the system of consumerism and status signaling today, was not one at a separating equilibrium. It seems that a better description is that it was out of equilibrium (perhaps in oscillations centered on a mixture) or in equilibrium, but a partially reliable hybrid rather than one of perfect communication. Similarly, it is unrealistic to maintain that education as a job market signal is perfectly communicative; it is not always the case that only the most productive individuals invest in education. More realistic is the out-of-equilibrium cycling or hybrid picture. The dynamic analysis above shows how it is possible for such real-life market states to be reached. If the goal of costly signaling research is to explain how information is transfered through competitive markets, then it is reasonable to suspect that it is the hybrid equilibrium type that likely does the explanatory work. Refinements have been too quick to exclude such mixtures.

The relevance of this moral may extend beyond economics. Spence's model of job market signaling bears a remarkable resemblance to the structure of costly signaling models from biology. Zahavi (1975) proposed that extravagant characteristics, such as the peacock's tail, evolved because they honestly signal quality to prospective mates. According to this theory of sexual signaling, the peacock can be thought of as investing – at a potentially high cost – in the production of a gaudy tail in order to win access to peahens. Thus, the peacock's tail plays the role of education in Spence's model. Grafen (1990) spelled out Zahavi's proposal with a mathematical framework that is structurally similar to Spence's game. However, Zahavi's costly signaling hypothesis has recently been questioned. Arguments leveled against it include the charge that signaling equilibria leave all participants worse off (Bergstrom and Lachmann, 1997) and the observation that signal cost is not necessary in equilibrium if there are costs imposed on out of equilibrium play (Lachmann et al., 2001).

In one respect, the above analysis suggests another criticism of the costly signaling hypothesis. Perhaps it is the case that, from a dynamic point of view, costly signaling is just a very unlikely outcome of the evolutionary process. Of course, this research does not immediately transfer to biological models, but studying the dynamics of such systems would an interesting next step. However, in another respect the preceding analysis might vindicate one aspect of the costly signaling hypothesis. Game theoretic modeling in biology is centered around static analysis and the evolutionary stable strategy (ESS) refinement in particular (Maynard Smith and Price, 1973). The ESS concept is unambiguous in single population models, but in multi-population models – like those that naturally arise in studies of signaling – there is some debate about how to proceed. A common position maintains that the correct interpretation of a multi-population ESS is simply a strict Nash equilibrium (Weibull, 1997). Hybrid equilibria, however, are mixtures and thus not strict Nash. Consequently, biologists have not payed much attention to polymorphic outcomes of multi-population models. But, due to the deep similarities between Spence and Grafen's games, it is likely that Grafen's may admit mixed equilibria and that, dynamically, these mixtures may be crucial for understanding out-of-equilibrium behavior and perhaps even likely outcomes of the evolutionary process. Before biologists discount the costly signaling hypothesis too much, perhaps it would be wise to investigate such possible mixtures in which messages can be low cost and information transmission is partial.

The conclusion that partial communication may be the most likely outcome of a social interaction has some precedent in the economic literature. Crawford and Sobel (1982) showed that, in the context of a cheap talk game, signals in equilibrium are more informative the more closely the sender and receiver's preferences are aligned. In other words, in cheap talk games in which interests diverge, we should not expect perfect communication, but instead only partially informative messages. Of course, in Spence's model signals are costly, and it is the cost structure of the signals the makes perfect communication an equilibrium whereas it is not in Crawford and Sobel's cheap talk game (unless the agents' preferences are perfectly aligned). However, the above results demonstrate that hybrids, not separating equilibria, are the most likely communicative strategies to arise dynamically. Just like in Crawford and Sobel's model we see messages that do not fully reveal the state of the world. But in Spence's game the signal noise derives from a different mechanism. The noise in Crawford and Sobel's model stems from the senders deterministically attaching the same message to several states. On the other hand, the noise in Spence's model stems from randomization. At the hybrid H,

e_L means "low," but e^* may mean either "low" or "high." The point being that in competitive social interactions in which parties have the opportunity to communicate private information (markets, status signaling, sexual signaling, etc.) we should not necessarily expect perfect information transfer (even if such communication may be an equilibrium outcome). Instead we should expect perhaps two dimensions of strategic noise: partial pooling like seen in Crawford and Sobel's game and randomization like seen in the mixtures of Spence's model.

6 Conclusion

I began by asking how the honesty of communication can be guaranteed when agents' interests diverge. This paper has sought to address this question by investigating how honest communication can emerge through a dynamic process. It turns out that perfectly honest communication is not too likely an outcome, even if it is a possible equilibrium result. If communication does happen to emerge from an out-of-equilibrium market, the model predicts that such communication will likely be partial. Of course, these results are not definitive; there is a great deal of work left to do.

From a purely technical standpoint, the model is unsatisfactory because it immediately limits the sender to choosing a strategy from a finite set of education levels. The way to address this issue would be to study costly signaling with a dynamic over infinite strategy spaces such as a modification of the one developed by Oechssler and Riedel (2001). This would, however, be a nontrivial application of such dynamics because the underlying game being played is Bayesian. A second purely formal extension would be to perturb the replicator dynamic slightly, thus destroying the closed periodic orbits from section 4.2 and breaking the lines of rest points seen in section 4.4. Doing so would perhaps yield stronger equilibrium predictions from the replicator dynamic.

From a slightly more applied point of view, it would be fascinating to apply this dynamic framework to the Cho and Kreps (1987) game in which employers hold an auction to set the market wage. Here I followed Spence (1973) in assuming that the firm sets the wage equal to the expected value. This assumption dramatically lowers the dimensionality of phase space, but leaves the question as to how competitive wages become established unanswered. Directly studying the dynamics of Cho and Kreps's model would address this issue. Another possible line of research would involve increasing the number of worker types. Once again, following Spence in assuming that there are only two types of worker lowers the dimensionality of the system. However, since increasing the number of types poses problems for the evolution of communication even when interests are perfectly aligned (Huttegger et al., 2009), it is likely that such an alteration to the model above will pose additional hurdles to the emergence of separating strategies. Such an extension would likely make the predictive differences between refinement theory and the dynamic approach even more salient.

Lastly, since this research suggests the importance of mixed equilibria – or equivalently, polymorphic market outcomes – one might look for these states in empirical studies of screening and signaling. Despite the large number of attempts to test the theory (see Riley, 2001, for a survey), to the best of my knowledge, no studies have incorporated polymorphic market states into their research. Including such partially communicative screening or signaling equilibria could make for richer empirical investigations.

A Proofs

Proof of Theorem 1. At H, the Jacobian matrix of the replicator dynamic for the pruned Spence signaling game reduces to

$$J = \begin{pmatrix} \frac{1}{2}(p_H^2 e^* - p_H e^*) & \frac{1}{2}(p_H^2 e^* - p_H e^*) & -p_H^3 + 2p_H^2 - p_H \\ -\frac{1}{2}e^* p_H^2 & -\frac{1}{2}e^* p_H^2 & p_H^3 - 2p_H^2 + p_H \\ e^* p_H - e^{*2} p_H & -e^* + e^{*2} + 2e^* p_H - 2e^{*2} p_H & 0 \end{pmatrix}$$

The characteristic equation $\det(J - \lambda I) = 0$ has solutions $\lambda_1 = -\frac{1}{2} p_H e^*$ and

$$\lambda_{2,3} = \pm\sqrt{-p_H e^* \left(e^* - 1\right) \left(p_H - 1\right)^3}$$

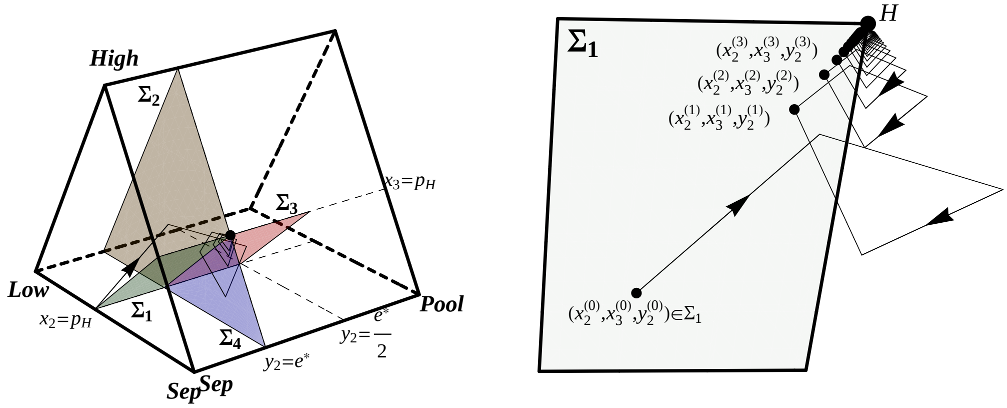
These solutions are the eigenvalues of J. By hypothesis, $0 < p_H, e^* < 1$. Therefore $(e^* - 1) < 0$ and $(p_H - 1) < 0$, which makes $(e^* - 1)(p_H - 1)^3 > 0$. Consequently, λ_1 is real and negative, and both λ_2 and λ_3 are purely imaginary.

From the Center Manifold Theorem (Carr, 1981; Guckenheimer and Holmes, 1983) it follows that there exists a stable manifold tangent to λ_1's eigenspace and a center manifold tangent to the eigenspace corresponding to λ_2 and λ_3. According to the Center Manifold Theorem, since J has no eigenvalues with positive real part, the stability of rest point H depends solely on the dynamics upon the center manifold.

The center eigenspace here is simply the $x_1 = 0$ boundary face of phase space (indeed, the center manifold is identical to this boundary face). From inspecting the pruned game, it is easy to see that this face is characterized by a best response cycle. Indeed, the dynamics on this face are the dynamics of matching pennies. Under the two population replicator dynamic, it is well known that matching pennies yields closed periodic orbits centered on the mixed Nash equilibrium (Weibull, 1997; Hofbauer and Sigmund, 1998; Cressman, 2003). Thus, this is also the behavior of the system on the center manifold. Since the mixed Nash is neutrally stable in matching pennies, the hybrid equilibrium is neutrally stable in the pruned Spence signaling game. □

Proof of Theorem 2. This result will be shown by the construction of a first return map from Σ_1 to Σ_1 such that iteration of the map leads to H. Let

$$\Sigma_1 = \left\{(x_2, x_3, y_2) \in \mathbb{R}^3 \,|\, x_2 = p_H + \alpha x_3, 0 < x_3 < p_H, e^* < y_2 < 1\right\}$$



(a) (b)

Figure 10: The surfaces $\Sigma_1, \Sigma_2, \Sigma_3$, and Σ_4 are illustrated in (a) along with a solution orbit starting from Σ_1. The iteration of the first return map and convergence to H is demonstrated in (b).

if $p_H > \frac{e-1}{e-2}$ and

$$\Sigma_1 = \left\{ (x_2, x_3, y_2) \in \mathbb{R}^3 \,|\, x_2 = p_H + \alpha x_3, 0 < x_3 < p_H, e^* < y_2 < \frac{e(p_H - 1)}{2p_H - 1} \right\}$$

otherwise. This adjustment when $p_H < \frac{e-1}{e-2}$ is necessary to guarantee that a point on Σ_1 returns to Σ_1 instead of converging to the pooling equilibria. Likewise let

$$\Sigma_2 = \left\{ (x_2, x_3, y_2) \in \mathbb{R}^3 \,|\, 0 < x_2 < 1 - p_H, \max\left[0, \frac{x_2 - p_H}{\alpha}\right] < x_3 < 1 - x_2, y_2 = e^* \right\}$$
$$\Sigma_3 = \left\{ (x_2, x_3, y_2) \in \mathbb{R}^3 \,|\, x_2 = p_H + \alpha x_3, 0 < x_3 < p_H, \frac{e^*}{2} < y_2 < e^* \right\}$$
$$\Sigma_4 = \left\{ (x_2, x_3, y_2) \in \mathbb{R}^3 \,|\, p_H < x_2 < 1, 0 < x_3 < \min\left[1 - x_2, \frac{x_2 - p_H}{\alpha}\right], y_2 = e^* \right\}$$

with $\alpha = \frac{1-2p_H}{p_H}$. These surfaces (shown in Figure 10) are the locations at which the solution trajectories to the best response dynamic abruptly change direction.

Due to the piecewise linear nature of the solution orbits, every orbit leaving each of these surfaces travels in a straight line toward the current best response profile. For instance, a solution from an initial condition on Σ_1 will travel in a line toward $(0, 1, 0)$ until it intersects

Σ_2 at which point the solution changes direction to point to $(1,0,0)$ until intersecting Σ_3, and so on. Working all this out, it is easy to see that an initial condition on Σ_1 at coordinates (x_2, x_3, y_2) will intersect Σ_2 at coordinates (x_2', x_3', e^*) given by the linear equation

$$(x_2', x_3', e^*)^T = (-p_H - \alpha x_3, 1 - x_3, -y_2)^T \left(\frac{y_2 - e^*}{y_2}\right) + (p_H + \alpha x_3, x_3, y_2)^T$$

Then the solution will be carried to Σ_3 with an intersection at

$$(x_2'', x_3'', y_2'')^T = (1 - x_2', -x_3', -e^*)^T \left(\frac{p_H - x_2' + \alpha x_3'}{1 - x_2' + \alpha x_3'}\right) + (x_2', x_3', e^*)^T$$

Next, the solution aims to Σ_4 with an intersection at

$$(x_2''', x_3''', e^*)^T = (1 - x_2'', -x_3'', 1 - y_2'')^T \left(\frac{e^* - y_2''}{1 - y_2''}\right) + (x_2'', x_3'', y_2'')^T$$

Finally, the solution returns to Σ_1 at the location given by

$$(x_2'''', x_3'''', y_2'''')^T = (-x_2''', 1 - x_3''', 1 - e^*)^T \left(\frac{p_H - x_2''' + \alpha x_3'''}{\alpha(x_3''' - 1) - x_2'''}\right) + (x_2''', x_3''', e^*)^T$$

Putting the four linear components together, the first return of a state on Σ_1 with coordinates $x_3^{(n)}$ and $y_2^{(n)}$ will be at the location given by the two dimensional map

$$\begin{aligned} x_3^{(n+1)} &= \frac{p_H(e^* - e^* x_3^{(n)} + e^{*2} x_3^{(n)} - y_2^{(n)})}{e^* - e^* p_H + e^{*2} p_H - y_2^{(n)}} \\ y_2^{(n+1)} &= \frac{e^*(e^* + e^* p_H y_2^{(n)} - p_H y_2^{(n)} - y_2^{(n)})}{e^* - e^* p_H + e^{*2} p_H - y_2^{(n)}} \end{aligned}$$

As this fractional linear recurrence is iterated, we see that the coordinates of the n^{th} return to Σ_1 are

$$x_3^{(n)} = \frac{np_H(e^* - y_2^{(0)}) - p_H e^* x_3^{(0)} + e^{*2} x_3^{(0)}}{n(e^* - y_2^{(0)}) - e^* p_H + e^{*2} p_H}$$
$$y_2^{(n)} = \frac{ne^*(e^* - y_2^{(0)}) - e^* p_H y_2^{(0)} + e^{*2} p_H y_2^{(0)}}{n(e^* - y_2^{(0)}) - e^* p_H + e^{*2} p_H y_2^{(0)}}$$

Obviously, $\lim_{n\to\infty} x_3^{(n)} = p_H$ and $\lim_{n\to\infty} y_2^{(n)} = e^*$, so any initial condition on Σ_1 will be taken to H under the best response dynamic. Through a simple geometric argument one can show that the dynamic will take a small perturbation off H to one of these Σ_i and then to Σ_1. Therefore, the point H is asymptotically stable. □

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