A Note on Generalized Edges*

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Abstract

Although most relational data is represented via relatively simple dyadic (and occasionally hypergraphic) structures, a much wider range of relational formalisms is possible. These alternatives provide more natural representations for relationships with complex edge structure (e.g., those in which there are many distinct ways of participating in a given edge, in which edges require particular numbers of vertices to be realized, etc.). Despite their utility, no common framework for describing such "generalized graphs" currently exists. Here, I introduce one such formalism, which supports complex edges while taking conventional dyadic and hypergraphic edges as special cases. Examples of the use of this formalism to represent concepts from balance theory, interpersonal communication, and joint performance are provided.

Although the conventional definitions (and, to a lesser extent, notation) associated with simple graphs and digraphs are widely shared across fields, generalizations of the graphical concept are less standard. Here, I present a very general representation for relational systems, which I refer to as the *generalized graph*. The generalized graph includes standard dyadic graphs and hypergraphs as special cases, but also permits the description of more exotic relations (such as actor-behavior-object interactions, "channel-like" many-to-many communication streams, and multi-role collaborations). After introducing the formalism itself, I briefly describe some of its potential applications within the social sciences.

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1 Generalized Graphs

Definition 1. Given set of *vertices*, V, define an *endpoint set* to be tuple $\epsilon = \big(\epsilon_1, \ldots, \epsilon_s\big)$ such that $\epsilon_1, \ldots, \epsilon_s \subseteq V$. Let $|\cdot|$ denote cardinality. An endpoint set ϵ is then said to be *s-sided*, where $s = |\epsilon|$; further, $\epsilon_i \in \epsilon$ is said to be the *i*th *side* of ϵ, where $i \in 1, \ldots, s$.

Let $\cong$ be an equivalence relation on ϵ. The ordered pair $e = (\epsilon, \cong)$ is then said to form a *generalized edge* on V, with *orientation* $\cong$. Given a set of generalized edges, E, on V, we say that the ordered pair $G = (V, E)$ is the (generalized) *graph* formed by E on V. ■

The generalized graph includes a number of simpler notions as special cases. For instance:

- When $|\epsilon| = 2$, $|\epsilon_1| = |\epsilon_2| = 1 \;\forall\, e \in E$, then G is a conventional (dyadic) graph. Adding the condition that $\epsilon_1 \ncong \epsilon_2 \;\forall\; e \in E$ implies that G is a *digraph*; if, by contrast, $\epsilon_1 \cong \epsilon_2 \;\forall\; e \in E$, then we say that G is an *undirected graph*.

- A dyadic graph, G, is said to be *loopless* iff $\epsilon_1 \cap \epsilon_2 = \varnothing \;\forall\; e \in E$. More broadly, we say that a generalized edge is *loop-like* iff $\exists \epsilon_i, \epsilon_j \in \epsilon : \epsilon_i \cap \epsilon_j \neq \varnothing$, and a generalized graph is loopless when it contains no loop-like edges.

- G is said to be *multiplex* iff $\exists\; e_i, e_j \in E : i \neq j, e_i = e_j$. A graph which is multiplex is said to be a *multigraph*. An undirected graph which is loopless and which is not multiplex is said to be a *simple graph*.

- When $|\epsilon| = 1$, $|\epsilon_1| > 1 \;\forall\; e \in E$, then G is a (conventional) *hypergraph*. In general, we say that an edge, e, is *hypergraphic* (in the generalized sense) if $\exists \epsilon_i \in \epsilon : |\epsilon_i| > 1$. Such an edge is called a *hyperedge*. A graph which contains one or more hyperedges is also said to be hypergraphic, and is referred to as a generalized hypergraph.[1] (Clearly, the conventional hypergraph is a special case of the generalized hypergraph.)

2 Sample Applications

It should be emphasized that while the generalized graph is convenient in that it provides a simple way of expressing familiar concepts, the formalism

[1]In many cases, we may more loosely refer to such a graph as hypergraphic when it is *permitted* to contain hyperedges, even if all *realized* edges happen have sides of size 1. Thus, a digraph could be seen as a degenerate two-sided hypergraph.

also allows for the direct representation of relations which are difficult to describe dyadically. A few illustrative examples include the following:

- Various models within the social psychological literature (e.g., Affect Control Theory (Heise, 1987), balance theory (Heider, 1946)) are based on relations which are defined in non-dyadic terms. For instance, basic interactions within Affect Control Theory (ACT) are defined in terms of an "actor" who initiates a "behavior" towards an "object" (paralleling the usage of Newcomb (1953)).[2] Heiderian balance theory, similarly, is based on an interaction unit which simultaneously relates an observer (P), and alter (O), and an object (X). In both of these cases it is natural to think of interactions as generalized edges, with, $\forall e \in E$,

  1. $|\epsilon| = 3$
  2. $|\epsilon_i| = 1 \forall \epsilon_i \in \epsilon$
  3. $\epsilon_i \not\cong \epsilon_j, \epsilon_i \cap \epsilon_j = \varnothing \ \forall i, j : i \neq j$

  That is, the interactions can be directly represented as three-sided, non-hypergraphic, fully oriented, non-loop edges. The collection of all interactions within some specified scope (e.g., within a given conversation) then form a generalized graph. It should be emphasized that since a given entity can serve as observer/actor, alter, or object for different interactions, entities may not always participate in edges in the same way. The generalization of standard concepts such as degree to the multi-sided context is thus a potentially fruitful enterprise.

- While some communications systems are symmetric and point-to-point, many variations exist. Broadcast technologies, for instance, serve to allow one sender to – within a single transmission – direct a one-way communication to multiple receivers. With complex access controls, a particular channel may be constructed so as to be two-way for certain individuals and one-way for others (e.g., in the case of an announcement list to which only certain subscribers have posting privileges or a complaints file to which anyone can post, but which is readable by a limited number of persons). Direct representation of such complex communication channels is straightforward via generalized edges. For

[2]This "actor-behavior-object" (ABO) model has been subsequently extended by the inclusion of "settings" within which interactions take place (Smith-Lovin, 1987). Such extensions are easily represented by adding sides to the generalized edge structure.

instance, the announcement list described above might be constructed by the following edge constraints:

1. $|\epsilon| = 2$
2. $|\epsilon_i| \geq 1 \forall \epsilon_i \in \epsilon$
3. $\epsilon_1 \not\cong \epsilon_2$
4. $\epsilon_2 \subseteq \epsilon_1$

Where side 1 indicates the receiving end of the channel, and side 2 represents the transmission end. (Note that we have assumed that all transmitters are also receivers, and that every channel has both transmitters and receivers. These conditions could easily be relaxed.) A graph formed by such channels is thus a directed hypergraph, with loop-like generalized edges. Clearly, more complex channels (e.g., those incorporating a distinct "moderator" role) can be represented in this way as well.

- Certain types of production occur via a series of collaborative projects, in which multiple agents join to produce a particular good, and then disperse (possibly joining other agents to produce new goods). Examples of goods which are often produced in this manner include movies, structures (e.g., bridges, buildings), large research projects, and most complex engineered systems. Such "joint productions" generally involve some set of reasonably well-defined roles, each requiring the contribution of specialized labor; for instance, production of a modern motion picture involves the roles of "writer," "director," "actor," and "producer" (among others). While it is sometimes possible for one agent to fill multiple roles in a single production (e.g., to both write for and direct a single film), all roles must be filled for the production to occur. Given this, it is quite natural to represent such joint productions as multi-sided hypergraphs. For the sake of example, let us assume that movie production requires the four roles noted above, and that there can be only one director. Then we would require, for all $e \in E$,

  1. $|\epsilon| = 4$
  2. $|\epsilon_i| \geq 1 \forall \epsilon_i \in \epsilon$
  3. $|\epsilon_d| = 1$, where ϵ_d represents the edge side associated with direction
  4. $\epsilon_i \not\cong \epsilon_j \forall i, j : i \neq j, i, j \in 1, \ldots, 4$

Note that we implicitly allow the edges to be loop-like, reflecting the fact that agents may play multiple roles in the production process. We could, obviously, restrict this if necessary (e.g., disallowing loop-like edges altogether, limiting each agent to a maximum of n roles per edge, etc.). We could also relax the full orientation requirement, were there to be multiple roles which had to be filled, but which were exchangeable. For example, certain government procurement processes require that multiple firms be contracted to bid on a given project; each such firm could be seen as occupying one side of a joint production (the bidding process), although the sides are not fully oriented. Note that some orientation is still present, however, since the position of buyer (the State) is not exchangeable with that of bidder. Such constraints are easily represented via an appropriate choice of $\cong$.

3 Conclusion

It should go without saying (but rarely does) that choosing a representation for a phenomenon is a theoretical act. At best, employing a representation that does not adequately capture the important aspects of the system under study inhibits communication and makes theoretical progress difficult; at worst, it can lead one to erroneous conclusions (Butts, 2009). It is hoped that a more standard approach to generalized graphs will allow structural researchers to select representations that truly reflect their scientific understanding, rather than being limited to a small number of conventional options.

4 References

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