The Core with Positional Spatial Voting

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Abstract After introducing a way to analyze positional voting outcomes in a spatial voting context, 'core' type theorems are proved for these voting rules, and their runoffs, in the single issue setting.

While spatial voting has proved to be a valuable way to interject geometry and geometric intuition into the analysis of voting rules, the current methodology is essentially limited to pairwise comparisons — pairs of candidates, alternatives, or proposals. For example, McKelvey's (1979) seminal 'chaos' theorem for majority voting and Tataru's (1996, 1999) extension for q-rules are in terms of voting over pairs. My spatial voting result about the limits on the number of issues for which a core can persist after slightly perturbing the preferences (Saari, 1997, 2004) also is based on pairwise voting. A challenge, then, is to develop spatial voting approaches to handle the direct comparison of several alternatives.

Existing results in this direction include Schofield's (2008) insightful work on the positions adopted by political parties such as those from the US and Israel. But Schofield uses the spatial setting to capture how and why the policies of parties change, which differs from what I have in mind. An approach somewhat closer to my concern comes from Nurmi's recent research program to analyze paradoxes, including those by Simpson (1951), Anscombe (1976), and Ostrogorski (1970), in a spatial framework; in this manner the geometry helps to identify new connections and commonalities among these various puzzles. While this project is somewhat closer to my objective, it differs in that these mysteries emphasize the conflict between two groups of voters over a collection of single issues rather than a comparison among many alternatives. Anscombe paradox, for instance, is where a majority of voters can be on the losing side over a majority of issues. Thus the emphasis is on pairwise comparisons between the pitiful majority and the successful minority.

To the best of my knowledge, what is missing from this area is a spatial voting approach that can simultaneously compare several proposals. Topics that would benefit from such an approach include elections involving several candidates; here, rather than using pairwise comparisons, it is standard to use the plurality vote, or some other positional method. (A positional voting rule is where a ballot for n candidates is tallied by using specified weights $(w_1, w_2, \ldots, w_n = 0)$, $w_1 > 0, w_j \geq w_{j+1}$, where w_j points are assigned to the candidate ranked in the j^{th} *position*. So, $(1, 0, \ldots, 0)$ corresponds to the plurality vote, $(1, 1, \ldots, 1, 0)$ to what is called the 'antiplurality vote' (because this system is equivalent to voting *against* one candidate), and $(n-1, n-2, \ldots, 1, 0)$ to the Borda Count.)

With positional voting, all sorts of inconsistencies can occur; e.g., Nurmi (1999, 2002) is a master at creating clever examples involving not only these rules, but rules that are derived from them.[1] So, similar to Nurmi's program of using geometry to reexamine certain classical paradoxes involving pairwise comparisons, it would be valuable to compare positional voting rules in a spatial context. But to do so, an approach needs to be created to handle positional voting in spatial settings. This is an objective of this paper.

While methods to analyze spatial voting with positional rules do not appear to have been proposed before, the technique introduced here is very natural. Almost immediately this tactic leads to surprises; e.g., most positional rules turn out to be 'unstable' in the game theoretic sense of a 'core'. Along the way, I will indicate why a spatial analysis of positional methods is more complex than a study of pairwise comparisons.

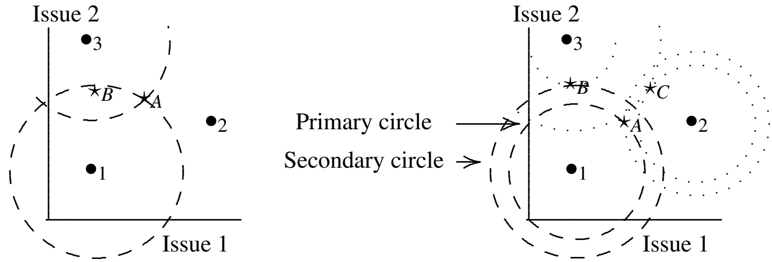
1. Positional Rules in a Spatial Context

Spatial voting models usually are based on Euclidean spaces and distances. So, with k issues, assign each to a coordinate direction in the Euclidean space $\mathbb{R}^k$; e.g., the two-dimensional space in Fig. 1a depicts a two issue setting. The coordinates of a point in $\mathbb{R}^k$ are intended to describe the level of support for each issue, so each voter's preferred position over the k issues can be represented by an 'ideal point'; these points are the bullets in Fig. 1. In Fig. 1a, for instance, the bullet for voter one shows that he has a moderate position on both issues; voter three has a similar opinion for issue 1, but takes a more extreme view for issue 2.

The position taken by a particular proposal (i.e., alternative, candidate, etc.) over these specified issues can also be represented by a point in $\mathbb{R}^k$.

[1] See Saari (2008) for an explanation of all three-candidate positional paradoxes.

Fig. 1 — Pairwise and positional voting



a. Pairwise comparisons *b.* Positional voting

The two proposals in Fig. 1a and the three in Fig. 1b are given by the stars.

The group outcome, of course, depends on what the voters want as determined by their preferences. These preferences typically are represented by the Euclidean distance between a proposal and a voter's ideal point where 'closer is better'. In Fig. 1a, for instance, the full circle is centered at voter one's ideal point and passes through proposal A. While this voter is indifferent between proposal A and anything that would be on the circle, he prefers any alternative, such as B, that is inside the circle. Thus this voter prefers $B \succ A$.

To use the Fig. 1a geometry to determine the outcome, notice that voter three also prefers $B \succ A$, while voter two clearly prefers $A \succ B$. According to the geometry, with a majority vote, the majority coalition of $\{1,3\}$ would elect B over A.

It is more difficult to see what outcome corresponds to the Fig. 1b preferences. But by carefully checking the dotted and dashed circles, it can be shown that these voter preferences define the majority vote cycle $C \succ A, A \succ B, B \succ A$ — an indecisive conclusion. But what would the Fig. 1b outcome be by using a positional rule?

To handle positional voting in a spatial framework, first normalize the weights to assign one point to a top-ranked candidate. So, to normalize the traditional $(w_1, w_2, 0)$ representation, divide all weights by w_1 to obtain

$$(\frac{w_1}{w_1}, \frac{w_2}{w_1}, 0) = (1, s, 0), \quad s \in [0,1]. \tag{1.1}$$

As illustrations, the plurality and antiplurality voting vectors of $(1,0,0)$ and $(1,1,0)$ already are in normalized form with the respective s values of $s = 0$ and $s = 1$. The Borda Count voting vector of $(2,1,0)$ has the normalized form of $(1, \frac{1}{2}, 0)$ with $s = \frac{1}{2}$, while $(3,2,0)$ has the normalized $(1, \frac{2}{3}, 0)$ form. Thus any three candidate positional method has an Eq. 1.1 form, and each $s \in [0,1]$ represents a class of positional rules.

To see how to determine the positional outcomes in a spatial voting context, notice the dashed circles in Fig. 1b. The smaller, or *primary circle*, which is centered about voter one's ideal point with radius determined by the closest point representing a candidate, indicates that voter one casts his first place vote for A. The interior of this circle identifies all positions this voter prefers to A; i.e., if there were such a point, it would receive his first place vote. The *secondary circle* is determined by the next closest position of a candidate; it identifies the voter's second place choice. So, in this figure, voter one's preferences are $A \succ B \succ C$, The dotted circles indicate that voter two's preference ranking is $B \succ C \succ A$ and voter three's is $C \succ A \succ B$.

The Fig. 1b geometry has each candidate on precisely one primary circle, so the plurality vote is a tie. But each candidate also is on precisely one secondary circle, so the antiplurality outcome also is a tie. Indeed, because each candidate's $(1, s, 0)$ tally equals

> [the number of primary circles] plus s times [the number of secondary circles] that contain the candidate's ideal point

this particular configuration defines a complete tie for all candidates independent of the choice of a positional method.

An advantage of using spatial voting is that the geometry can be used to indicate what a candidate can do to improve her chances. In the majority vote setting of Fig. 1a, for instance, A loses to B. But by slightly modifying her position directly toward voter one's ideal point so that the distance to the ideal point is reduced enough to cause the new dashed circle to exclude B's position, A would now beat B.

The same kind of geometric argument applies to positional voting. To see this with Fig. 1b, notice that A's position is very close to the voter three's secondary circle that passes through C. So, by slightly changing her position in a counterclockwise direction on voter one's primary circle, A can adopt a position closer to voter three's ideal point. In this way, her position, rather than C's, defines voter three's secondary circle. While this change does not affect the completely tied plurality outcome (each candidate remains on only one primary circle), it does change all other positional election outcomes. Namely, A now is on one primary circle and two secondaries, B is on one primary and one secondary, and C is only on one primary circle, so the $A : B : C$ positional tallies are $1 + 2s : 1 + s : 1$. Thus, with the exception of the plurality vote, all other positional rules have the $A \succ B \succ C$ election outcome. (The majority vote outcome over pairs remains a cycle.)

2. Median Voter and the Core

Paralleling results obtained with pairwise voting, there are many ways to use this geometry. As the above illustrations demonstrate, for instance, by casting voter preferences and candidate positions in a geometric setting, it may be possible for a losing candidate to determine how to modify her position to obtain a personally better outcome. But this ability to alter the outcome raises the issue as to whether there exists a proposal, or candidate's position, that cannot be beaten. This concept leads to the definition of a core. Analyzing the core for pairwise voting leads to the well known median voter theorem. But, when this core concept is applied to positional voting, surprises arise.

Definition 1 For a specified voting rule, a 'core point' is a point (e.g., candidate's position) that cannot be beaten by any other point (e.g., any other candidate's position). The core, denoted by $\mathscr{CORE}_{\text{voting rule}}$, is the collection of all of the voting rule's core points.

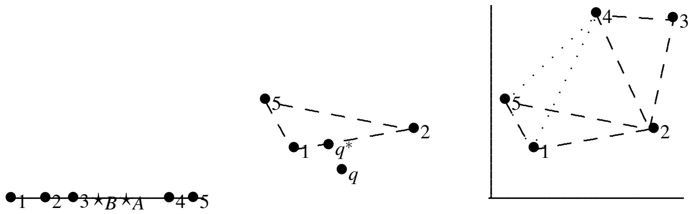
Had B's position in Fig. 1a been a core point, it would have been impossible for A to change her position to become the winner. Thus, core points constitute positions of stability for spatial voting. Before discussing the positional voting core, certain results about the core for majority voting over pairs will be reviewed. By doing so, some of the differences between how pairwise and positional voting rules are analyzed will become apparent; these differences cause the added complexity associated with positional rules.

With majority voting over pairs, a core point is a proposal that will never lose a majority vote to any other proposal. The first question to determine is whether core points exist and, if so, to characterize their properties.

To indicate what is involved, Fig. 2a depicts a five voter setting over a single issue. It is clear from the specified positions for candidates A and B that B beats A; this is because B receives the votes from the coalition $\{1,2,3\}$ while A receives votes only from $\{4,5\}$. But even though B's position is close to the average position of the voters, it is not a core point. The reason is that A could change her position to be between that of B and voter three's ideal point. In this manner, A would now command the votes from the winning coalition of $\{1,2,3\}$.

An extension of this argument shows that the core point in Fig. 2a is located precisely at voter three's ideal point — the median position. After all, if A claims this median voter position and B adopts any other spot, then A receives the votes of voter three and all voters whose ideal points are on the side opposite of that of B. A proof for this well-known result, known as the ‘median voter theorem’; is given below.

Fig. 2 — Majority voting core



a. Single issue *b.* Pareto points *c.* Pareto sets

The next question is to determine the core for the five voter setting in Fig. 2c. The approach to be used is motivated by the previous comments about which candidate commands the vote of which coalition. The following definitions are used to create some tools.

Definition 2 a. A Pareto point for a coalition of voters $\mathscr{C}$ is a point that, if changed in any manner, creates a comparatively poorer outcome for some member of the coalition. The Pareto set for the coalition, denoted by $\mathscr{P}(\mathscr{C})$, is the set of all of its Pareto points.

b. For a voting rule, a minimal winning coalition is one where, if all voters in the coalition vote in the same manner, it will win any election. But, if any member leaves the coalition, it no longer is a winning coalition.

To illustrate with Fig. 2c and majority voting, because there are five voters, any three candidate coalition is a minimal winning coalition. The Pareto set for one such coalition, $\mathscr{P}(\{1,2,5\})$, consists of all points in the triangle connecting the ideal points of these three voters. To see why this is so with the Fig. 2b diagram, select any point $\mathbf{q}$ that is not in the triangle. Next, find the point $\mathbf{q}^*$ on the triangle that is the closest to $\mathbf{q}$. It now follows from the Fig. 2b geometry that $\mathbf{q}^*$ is at least as close as $\mathbf{q}$ to each of the three ideal points and closer to at least one of them. (In Fig. 2b, $\mathbf{q}^*$ is closer than $\mathbf{q}$ to all of the ideal points.) Hence, $\mathbf{q}$ cannot be a Pareto point for this coalition. On the other hand, moving any point in the triangle increases the distance to at least one ideal point, so this becomes a poorer choice for that voter. With majority voting and Euclidean preferences, $\mathscr{P}(\mathscr{C})$ is the convex hull of the ideal points in $\mathscr{C}$.

This construction leads to the following result, which simplifies computing the core.

Theorem 1 Consider voting rules, such as the majority or plurality vote, where any coalition containing a minimal winning coalition is a winning coalition. Then,

$$\mathscr{CORE}_{rule} = \cap \mathscr{P}(\mathscr{C}) \tag{2.1}$$

where the intersection is over all minimal winning coalitions.

Proof The proof of this result, which probably is well known for the majority vote, is simple and immediate. If a point $\mathbf{q} \notin \cap \mathscr{P}(\mathscr{C})$, then there is at least one minimal winning coalition for which $\mathbf{q}$ is not in its Pareto set. By the definition of the Pareto set, there is a point $\mathbf{q}*$ that this coalition prefers over $\mathbf{q}$, and, because it is a winning coalition, these voters can enforce this $\mathbf{q}*$ outcome. Because $\mathbf{q}$ can be beaten, it is not a core point.

On the other hand, if $\mathbf{q}$ is in $\cap \mathscr{P}(\mathscr{C})$, it is in the Pareto set for all minimal winning coalitions. As such, any change in $\mathbf{q}$ will not be to the advantage of some voter in each coalition. As that voter will not support the change, there does not exist another point and a supporting winning coalition that will beat $\mathbf{q}$. Thus $\mathbf{q}$ is a core point. □

By applying Thm. 1 to Fig. 2c, it follows immediately that the core is empty. To see this, notice that $\mathscr{P}(\{1,2,5\}) \cap \mathscr{P}(\{2,3,4\}) = \{2\}$; i.e., the two triangles given by dashed lines intersect only in voter two's ideal point. So if the core exists, it is voter two's ideal point. On the other hand $\mathscr{P}(\{1,4,5\}) \cap \mathscr{P}(\{2,3,4\}) = \{4\}$ (the intersection of the dotted and the upper-right dashed triangles), so if the core exists, it must be voter four's ideal point. This contradiction immediately shows that the core is empty. This means that, whatever the proposal, there always exists another proposal that can beat it.

Before summarizing certain results, return to the single issue setting of Fig. 2a. With five voters, a minimal winning coalition has three voters. So, $\mathscr{P}(\{1,2,4\})$ is the interval connecting voter one's and four's ideal points while $\mathscr{P}(\{1,4,5\})$ is the full interval connecting voter one's and five's ideal points. Notice how, independent of whether voter three belongs to a coalition, the geometry forces his ideal point to belong to each winning coalition's Pareto set. Thus the core exists; it agrees with the median voter's ideal point.

More generally, consider a single issue with $n \geq 3$ voters. For the majority vote, a minimal winning coalition consists of $\frac{n}{2} + 1 = \frac{n+2}{2}$ voters if n is even and $\frac{n+1}{2}$ voters if n is odd. In either case, the restrictive geometry of a line (the single issue setting) requires the median of the voters' ideal points to be in each minimal winning coalition's Pareto set. After all, according to the definition of a median, if a coalition does not contain a voter whose ideal point is in the median, then the coalition must include voters with ideal points both to the right and to the left of the median voter. Consequently, any point in the median of the ideal points must be in the Pareto set for all winning coalitions. Thus, for a single issue, the majority vote core always exists; it consists of the region defined by the ideal points of the median voters. (As with the median in statistics, if n is odd, the core is a point and if n is even, the core, in general, is an interval.)

Extensions of these observations lead to delightful results. A major one, by McKelvey (1979), is that if the majority-vote core is empty, then it is possible to select any starting point $\mathbf{p}_s$ and concluding point $\mathbf{p}_c$ and be sure that there exists an agenda of winning propositions starting at $\mathbf{p}_s$ and ending at $\mathbf{p}_c$. Tataru (1996, 1999) extended this chaos result to q rules. Because $\mathbf{p}_c$ can be anything, it can be selected so that *all* voters prefer $\mathbf{p}_s$ to $\mathbf{p}_c$; namely, the final outcome could be something that all voters dislike.

These examples suggest that the existence of the core depends on the number of issues; i.e., the dimension of issue space. A general assertion for an odd number of voters is that the majority vote core usually is empty for two or more issues. By 'usually', I mean that if the core happens to exist, slightly changing the voter preferences will cause the core to disappear. For an even number of voters, a special kind of the majority vote core persists for two dimensions, but it usually is empty for more issues. The relationship connecting the number of issues with the persistence of a q-vote core is developed in Saari (1997, 2004).

3. The Core and Positional Rules

Results differ with positional rules. To indicate why this is so, suppose there are fifteen voters. With pairwise voting, any eight voters constitutes a minimal winning coalition. But with the plurality vote and three candidates, a winning coalition could have

> six voters if the other nine voters split as five and four between the remaining two candidates, or
>
> seven voters where each remaining candidate has the support of at least two voters, or
>
> eight voters where another candidate will obtain seven votes.

In other words, an important difference is that winning coalitions for pairs are determined by a single fixed number — a single parameter. But the winning coalitions associated with the plurality vote — or other rules used to simultaneously compare several alternatives — involve a more complicated computation that depends on the structure of other coalitions; i.e., on vectors of parameters.

What simplifies the analysis for majority voting over pairs is that if $\mathscr{C}$ is a winning coalition, then those voters who are not in $\mathscr{C}$ must be a losing coalition. This is not true for the plurality vote; e.g., with fifteen voters the coalition $\{1,2,3,4,5,6\}$ can be one winning coalition, and so can $\{10,11,12,13,14,15\}$. Because the Pareto sets for these two coalitions need not meet, we should not expect a core to exist.

These various possibilities are what makes it reasonable to expect negative core results for positional voting. The following theorem, along with its proof, captures this sense.

Theorem 2 Even for a single issue, with $n \geq 4$ voters and $p \geq 3$ proposals where $n > p$, the core for the plurality vote is empty.

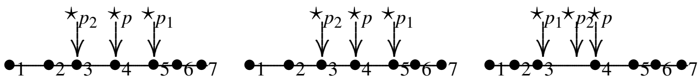
This result could be proved by showing that the intersection of the Pareto sets for the winning coalitions is empty. An instructive approach is to use the structures of winning coalitions (based on the remaining coalitions) to craft a constructive proof.

Proof The idea is captured by what happens with three proposals and $n > 3$ voters as represented in Fig. 3a. Select any point $\mathbf{p}$ on the single issue line. (In Fig. 3a, $\mathbf{p}$ is on voter four's ideal point.) To prove that $\mathbf{p}$ cannot be a core point, select the side of $\mathbf{p}$ with the largest number of ideal points (in Fig. 3a, both sides have three ideal points, so select the right side). If n is an even integer, the specified side has at least $\frac{n}{2} \geq 2$ such points; if n is an odd integer, there are at least $\frac{n-1}{2} \geq 2$ such points.

Select the position for one of the two other candidates, $\mathbf{p}_1$, to be on the first ideal point on the specified side of $\mathbf{p}$; clearly, $\mathbf{p}_1$ forms a plurality winning coalition $\mathscr{C}$. (In Fig. 3a, $\mathbf{p}_1$ is on voter five's ideal point, so $\mathscr{C} = \{5,6,7\}$.) Next, split the votes of the voters on the other side of $\mathbf{p}$. If no such voters exist, then place $\mathbf{p}_2$ between $\mathbf{p}$ and $\mathbf{p}_1$. If such voters do exist, place $\mathbf{p}_2$ on the ideal point of the first voter on the other side of $\mathbf{p}$. This construction limits $\mathbf{p}$ to be on at most one primary circle, but $\mathbf{p}_1$ is on several. Thus, $\mathbf{p}$ can be beaten. The proof for any number of candidates follows the same straddling construction. □

The fact the plurality core always is empty means it always is possible to find a new position that will beat any specified position. Thus we are left with the sense that the final outcome may reflect a game theoretic ma-

Fig. 3 — Positional voting coalitions



a. Plurality *b.* Centered *c.* Skewed

neuvering, or fatigue, rather than the voters' beliefs. This construction also makes it reasonable to suspect that the core is empty for all positional rules. This suspicion is close, but not quite correct; as asserted next, the positional voting core is empty with the possible exception of the Borda Count.

Theorem 3 For a single issue, three candidates, and any odd number of voters, the core for the $(1,s,0)$ positional rule is empty for $s \neq \frac{1}{2}$. For some arrangements of voter's ideal points the Borda Count ($s = \frac{1}{2}$) core exists, for other arrangements it does not.

Analyzing positional rules involves each candidate's disjoint coalitions defined by the primary and secondary circles; call them the *primary and secondary coalitions.* In the first, each voter has the same candidate top-ranked, in the second, each voter has the same candidate second ranked. (With more candidates, higher order coalitions are defined.) But what constitutes a winning coalition depends on the primary and secondary coalitions associated with the other candidates. Rather than using this structure, it suffices for the present purposes to use the insight gained from these coalitions to assemble a proof similar to that of Thm. 2

Proof Consider an odd number of voters given by $n > 3$, and select a point $\mathbf{p}$ to be a potential core point. While $\mathbf{p}$ can be any point, the argument will make it clear that, to have any chance to be a core point, $\mathbf{p}$ must be located at the median voter's ideal point.

There are two possible positioning for the other two candidates' points $\mathbf{p}_1$ and $\mathbf{p}_2$. The first is as represented in Fig. 3b where they straddle $\mathbf{p}$. Moreover, to siphon off first place votes, the worse case for $\mathbf{p}$ is to place $\mathbf{p}_1$ and $\mathbf{p}_2$ on the ideal points on each side of $\mathbf{p}$, or between $\mathbf{p}$ and these ideal points. By doing so, $\mathbf{p}$ belongs to at most one primary coalition, which is true only if $\mathbf{p}$ is on an ideal point. Assume that this is true. However, because $\mathbf{p}$ separates $\mathbf{p}_1$ and $\mathbf{p}_2$, her secondary coalition consists of all remaining voters. Thus the $(1,s,0)$ election tally for $\mathbf{p}$ is $1+(n-1)s$ points.

The primary coalition for each of the other two candidates consists of all voters on their respective sides of $\mathbf{p}$; let $\mathbf{p}_1$ be on the side with the largest number, N_1, of voters. Thus the primary coalition for $\mathbf{p}_2$ has $N_2 = n-1-N_1 \leq N_1$ voters. The only remaining second place vote that is not assigned to $\mathbf{p}$ is the voter who cast his first place vote for $\mathbf{p}$. To enhance $\mathbf{p}_1$'s tally, position $\mathbf{p}_1$ closer to this ideal point (which agrees with $\mathbf{p}$) than is $\mathbf{p}_2$. Thus, $\mathbf{p}_1$ receives $N_1 + s$ votes. The only way to reduce $\mathbf{p}_1$'s vote is for $\mathbf{p}$ to agree with the median voter's ideal point; here $N_1 = \frac{n-1}{2}$. Let this be the $\mathbf{p}$ location.

The number of votes each position receives is

$$\mathbf{p} \text{ receives } 1+(n-1)s, \quad \mathbf{p}_1 \text{ receives } \frac{n-1}{2}+s, \quad \text{and } \mathbf{p}_2 \text{ receives } \frac{n-1}{2}. \tag{3.1}$$

So, for $\mathbf{p}$ to be a core point, it must be that

$$1+(n-1)s \geq \frac{n-1}{2}+s, \text{ or } s \geq \frac{1}{2}\left[\frac{n-3}{n-2}\right] = \frac{1}{2}\left[1-\frac{1}{n-2}\right]. \tag{3.2}$$

For the Eq. 3.2 inequality to hold for all n, it must be that $s \geq \frac{1}{2}$. Namely, the $(1,s,0)$ core for $s < \frac{1}{2}$ is empty.

The remaining configuration has $\mathbf{p}_1$ and $\mathbf{p}_2$ on the same side of $\mathbf{p}$ where the goal is to position these points, as in Fig. 3c, to reduce the number of first and second place votes assigned to $\mathbf{p}$. To counter this effort, $\mathbf{p}$ must be at the median voter's ideal point.

With an odd number of voters, this positioning of $\mathbf{p}$ assigns it all of the first place votes on one side, or $\frac{n+1}{2}$ first place votes. To reduce the number of $\mathbf{p}$'s second place votes, place $\mathbf{p}_2$ between $\mathbf{p}_1$ and $\mathbf{p}$, i.e., $\mathbf{p}_1$ and $\mathbf{p}_2$ are on the same side of $\mathbf{p}$. Furthermore, initially place $\mathbf{p}_1$ on the first ideal point on the selected side of $\mathbf{p}$. (See Fig. 3c.) With this configuration, $\mathbf{p}_2$ picks up all second place votes. Thus, the number of votes for each $\mathbf{p}_j$ is

$$\mathbf{p} \text{ receives } \frac{n+1}{2}, \quad \mathbf{p}_1 \text{ receives } \frac{n-1}{2}, \quad \text{and } \mathbf{p}_2 \text{ receives } ns. \tag{3.3}$$

Because the $\mathbf{p}$ tally is larger than that of $\mathbf{p}_1$, only the $\mathbf{p}$ and $\mathbf{p}_2$ tallies need to be compared. So, for $\mathbf{p}$ to be a core point, it must be that

$$\frac{n+1}{2} \geq ns, \text{ or that } \frac{n+1}{2n} = \frac{1}{2}+\frac{1}{2n} \geq s. \tag{3.4}$$

For this expression to hold for all n, it follows that all $(1,s,0)$ rules with $s > \frac{1}{2}$ have an empty core.

It remains to analyze the $s = \frac{1}{2}$ setting to determine when a core does, or does not, exist. According to Eq. 3.4, when $s = \frac{1}{2}$, $\mathbf{p}_2$ is second ranked. So, to try to dethrone $\mathbf{p}$, select the $\mathbf{p}_1$ and $\mathbf{p}_2$ positions to increase $\mathbf{p}_2$'s tally; this requires converting some of $\mathbf{p}_2$'s second place tallies into first place tallies. To do so, move $\mathbf{p}_1$ farther from $\mathbf{p}$. In this manner, those ideal points that lie between $\mathbf{p}_2$ and the midpoint of $\mathbf{p}_1$ and $\mathbf{p}_2$ now are closer to $\mathbf{p}_2$. If there are k such points, then these k voters now cast their first place votes for $\mathbf{p}_2$ giving her a $k+(n-k)s$ tally. The key to the construction is to determine who gets these k voters' second place votes.

If each of these k voters who give first place votes to $\mathbf{p}_1$ also give their second place votes to $\mathbf{p}$, then, for $\mathbf{p}$ to be a core point, it must be (in comparing the votes for $\mathbf{p}$ and $\mathbf{p}_2$) that

$$\frac{n+1}{2}+ks \geq k+(n-k)s \text{ or } \frac{n-2k+1}{2} \geq (n-2k)s, \text{ or } \frac{1}{2}+\frac{1}{2(n-2k)} \geq s.$$

In other words, with these kinds of configurations of ideal points, the core for the Borda Count is the median voter's ideal point.

A configuration of ideal points that allow the Borda Count to have a core is where the ideal points tend to be clustered toward the center but spread out toward the extremes. Thus such a society has centralist views with a scattering of extremists. An example is where the distance difference between the j^{th} and $(j+1)^{th}$ ideal point on either side of the median point $\mathbf{p}$ is $2(j+1)$.

On the other hand, if the positioning of the ideal points is such that moving $\mathbf{p}_1$ trades first place for second place votes, then the Borda Count core will be empty. This is because the $\mathbf{p}$ and $\mathbf{p}_1$ tallies become $\frac{n+1}{2}$ to $k+(n-k)s$. So, for $\mathbf{p}$ to be the core,

$$\frac{n+1}{2} \geq k+(n-k)s \text{ or } \frac{1}{2}-\frac{k-1}{2(n-k)} \geq s. \tag{3.5}$$

Equation 3.5 quickly excludes the Borda Count from having a core.

For the Eq. 3.5 expression to hold, the ideal points must represent a highly divided political setting where the points are spread out from the center but accumulate at extreme positions on both sides. For instance, place the median voter's ideal point at the origin, the first ideal points on either side at ± 2, the second ideal points on either side at ± 3, and the remaining ideal points on either side bounded by ± 4. With this configuration and placing $\mathbf{p}_2$ at -3, as $\mathbf{p}_1$ moves, $\mathbf{p}$ does not pick up any first or second place votes, but $\mathbf{p}_1$ and $\mathbf{p}_2$ exchange them. Here, the core for any positional rule is empty. □

Extending these results to settings with an even number of voters is fairly immediate. Equation 3.3, for instance, is replaced with $\frac{n}{2} \geq ns$ with the same conclusion that $\frac{1}{2} \geq s$.

4. Runoff Elections

The last topic considered here involves 'positional-runoff elections'. This is where the three alternatives are ranked with a specified $(1,s,0)$ positional method. Then, based on some criterion, the two top-ranked candidates are advanced to a majority vote runoff. The question explored here is whether results change with different criteria.

For lack of a better name, let me call the first considered criterion 'at least average'. This is where any candidate receiving less than the average on the positional vote is dropped; the remaining candidates are advanced to the runoff. The result (Thm. 4) is that only the Borda runoff has a nonempty core.

It is reasonable to search for conditions that allow other positional runoffs to have a core. For instance, what about the usual 'majority vote' criterion where the two top-ranked candidates are advanced only if someone does not obtain a majority vote. Again, the Borda runoff has a core with this condition, but it does not expand the list of successful positional runoffs. A reason is that with larger s values, the tally does not provide a sufficient distinction between a voter's first and second place choices. This lack of distinction is precisely what can be used to disqualify any point as a core point.

A way to expand the list of positional runoffs is to require a runoff to occur no matter what is the tally on the first stage. Namely, *always* advance the two top ranked candidates to a runoff. With this condition, (Thm. 5) the positional runoff core exists for $\frac{1}{2} \leq s \leq 1$.

Theorem 4 For a single issue and any odd number of voters, with the 'at least average' rule, only the Borda runoff rule has a core. The core agrees with the position of the median voter's ideal point.

Proof What needs to be proved is that for any configuration of voter preferences along the line (representing the single issue), if **p** located at the median voter's ideal point, then with the Borda Count, **p** is advanced to the runoff. Here, standard results from pairwise comparisons ensure the **p** cannot be beaten in a majority vote. It also must be shown that for positional rules that differ from the Borda Count, settings can be created where **p** does not receive an average vote.

The first step is to show for $s < \frac{1}{2}$ that any selected point **p** can be bottom-ranked with a careful positioning of the other two points. (If a candidate is bottom-ranked, she cannot have at least the average tally.) The earlier arguments show that, to have any chance, **p** must be located at the median voter's ideal point.

According to Eq. 3.1, with the Fig. 3b positioning where $\mathbf{p}$ is at the median voter's ideal point, $\mathbf{p}$ is at least second ranked if and only if

$$1+(n-1)s \geq \frac{n-1}{2} \quad \text{or if} \quad s \geq \frac{n-3}{2(n-1)} = \frac{1}{2} - \frac{1}{n-1}.$$

In other words, this inequality is satisfied for any choice of n only if $s \geq \frac{1}{2}$. In turn, this means that the core is empty for any positional runoff rule where $s < \frac{1}{2}$. In particular, beyond the 'at least average' approach, this comment includes the standard plurality-runoff used in many elections including the French presidential elections.

Moving $\mathbf{p}_1$ and $\mathbf{p}_2$ away from $\mathbf{p}$ on their respective sides only enhances $\mathbf{p}$'s tally (by converting second place to first place votes), so nothing in this configuration affects $\mathbf{p}$'s status for advancing to the runoff. Thus, it remains to consider the Fig. 3c settings where $\mathbf{p}_1$ and $\mathbf{p}_2$ are on the same side of $\mathbf{p}$ in an attempt to hurt $\mathbf{p}$ by increasing the tallies for the other candidates. However, even if this process prevents $\mathbf{p}$ from gaining any added votes, $\mathbf{p}$ has at least $\frac{n+1}{2}$ points. As the sum of points assigned to all three candidates is $n(1+s)$, the average vote share is $\frac{n(1+s)}{3}$. So, for $\mathbf{p}$ to receive at least the average tally, then $\frac{n+1}{2} \geq \frac{n}{3} + \frac{ns}{3}$, or $\frac{n}{6} + \frac{1}{2} \geq s\frac{n}{3}$. In other words, only for those values of $s \geq \frac{1}{2}$ that satisfy

$$\frac{1}{2} + \frac{3}{2n} \geq s \tag{4.1}$$

will $\mathbf{p}$ receive at least an average tally for the positional outcome. With any number of voters, the only such rule is the Borda runoff. Thus, only the Borda runoff has a non-empty core with the 'at least average criterion'. This is true for any configuration of the voters' ideal points on the line. □

For intuition about Thm. 4, notice that if $s = 1$, then $\mathbf{p}_2$'s tally of ns is precisely half of all of the votes cast. Because the $\mathbf{p}$ and $\mathbf{p}_1$ tallies are, initially, close to each other, it follows that there are settings where neither can manage to reach the one-third threshold. Clearly, the same holds for values of s near unity. By using the described strategy to increase $\mathbf{p}_2$'s vote with the positioning of $\mathbf{p}_1$, the $\mathbf{p}_2$ tally exceeds half of the total vote.

The argument also shows that, for larger values of s, the majority criterion for a runoff need not have a core. The answer, however, changes by using the condition that the two top-ranked candidates *must* be advanced to a runoff.

Theorem 5 For a single issue and any odd number of voters, if the two top-ranked candidates are advanced to a runoff, then the positional runoffs for all $s \geq \frac{1}{2}$ rules have the median voter's ideal point as the core.

Proof According to the proof of Thm. 4, we only need to check that if $s \geq \frac{1}{2}$, then $\mathbf{p}$ is ranked in the top two. It follows from Eq. 3.3 that with an odd number of voters, the tally for $\mathbf{p}$ exceeds that of $\mathbf{p}_1$ with the initial positioning. Moving the $\mathbf{p}_1, \mathbf{p}_2$ points is an attempt to increase $\mathbf{p}_2$'s tally, but this is at the expense of $\mathbf{p}_1$'s tally (as some of her first place votes are transferred to $\mathbf{p}_2$). Therefore, $\mathbf{p}$ always will have at least a second place standing. (But, according to Eq. 4.1, $\mathbf{p}$'s tally may be below the 'average'.) As the median voter cannot be beaten in a majority vote runoff, it follows that for any $s \geq \frac{1}{2}$, the core for the positional runoff exists and it is the ideal point of the median voter. □

5. Concluding Thoughts, Other Issues

According to these results and with the exception of the Borda Count, whatever positional rule is being used and whatever position a candidate assumes, it is possible to 'game' the setting so that this specified position will be beaten. In other words, these positional rules do not provide a desired sense of stability for election outcomes.

Of course, it is reasonable to wonder whether it matters if the core is empty. While I have yet to prove any results, based on experimenting with some examples involving two issues, I fully expect that a 'chaos' theorem of the kind McKelvey developed for pairwise voting can be found. As true with McKelvey's result, such a conclusion would imply the danger that the final 'winner' of a series of positional elections (e.g., after amendments, etc.) could fail to reflect the interests of *any* of the voters.

My mention of the McKelvey result underscores the fact that the theorems developed here constitute only a start in the exploration of positional voting in a spatial context. After all, it is reasonable to investigate whether different pairwise voting results have parallel statements for positional methods. As examples, what happens with more issues? Instead of defining the core in terms of a specified number of candidates, how about allowing new candidates to join in. Namely, fix the locations of the original candidates; the only new variable is the position of a new candidate. Which positional methods would allow a core in the sense that the original winner remains the winner?

Beyond examining possible parallels with pairwise voting, there are all sorts of issues with positional methods and the associated rules. For instance, while a geometric approach for positional voting was introduced in this article, it was not used to examine the kinds of concerns that Nurmi (1999, 2002) raises. Ah, the delight of this area: it is so easy to find so many new issues that are of pragmatic, practical importance.

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