Connections and Implications of the Ostrogorski Paradox for Spatial Voting Models

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Abstract

The basic theme of this paper is to connect spatial voting concepts to the troubling Ostrogorski and Anscombe paradoxes (where a majority of the voters can be on the losing side over a majority of the issues). By doing so, answers to longstanding concerns are obtained; e.g., Kelly conjectured that an Ostrogorski paradox ensures that no candidate can be a Condorcet winner; we prove this is true for any number of issues and establish a connection with McKelvey's spatial "chaos theorem." Other results include explaining why these paradoxes occur (i.e., "issue-by-issue" voting strips any intended connection a voter might have among the issues), showing that if an Ostrogorski paradox occurs, then all possible spatial voting representations have an empty core, firmly establishing the long-suspected connection between the Ostrogorski paradox and the lack of a Condorcet winner in paired comparison voting, provide new supermajority conclusions for the Ostrogorski paradox, and introduce a new class of these paradoxes in terms of when a party can offer a stance on an issue.

1 Introduction

Spatial models occupy an important position in modern social choice theory. From the early applications to party competition and electoral equilibrium they have spread to the study of inter-institutional power in the European Union (EU) and cabinet coalitions in multiparty systems (Downs 1957; Napel and Widgrén 2004; Napel and Widgrén 2006; Laver and Shepsle 1996). They have applications in expert systems by advising voters

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how to make choices in elections. The models use assumptions that allow modelers to assign voters or decision makers to points in a policy space, which often is a multi-dimensional Euclidean space. Similarly, the decision alternatives, or candidates, are typically positioned in the policy space as points or probability distributions over points. Work on spatial models has produced a variety of results ranging from the existence of stable outcomes (equilibria) of various various kinds (McKelvey and Schofield 1987; Saari 1997, Saari and Asay 2009) to power distributions among voters (Steunenberg et al. 1999) and suggestions for institutional design (Shepsle and Weingast 1987).

We offer a new connection by identifying spatial models with well-known puzzles such as the Ostrogorski paradox. In this manner, new themes are introduced while several mysteries and conjectures are answered. We also question the above standard condition whereby a voter prefers the candidate (or policy position) whose stance is closest to his. As we show, plausible variations in this basic assumption have implications concerning the prospects for reasonably stable outcomes in social choice, which may be better than suggested by the above mentioned results. After describing a variety of paradoxical settings, we explain why they all occur.

The particular class of aggregation paradoxes emphasized here bears the name of Ostrogorski, a Russian diplomat and political theorist whose magnum opus (Ostrogorski 1902) appeared in the opening years of the 20th century (see also Rae and Daudt (1976) and Bezembinder and Van Acker (1985)). This paradox is introduced and analyzed in the next section, and its variant—the exam paradox (Nermuth 1992)—is discussed in the section that follows it. We then answer some conjectures, connect these issues with the core and "chaos theorem" of spatial voting, and consider the implications that these paradoxes have for spatial modeling of individual preferences. To indicate the generality of certain answers, Sect. 9 considers another aggregation paradox, viz. Simpson's paradox (Simpson 1951).

Aggregation paradoxes have been the focus of scholarly attention for some time. As an illustration, the notion behind "Simpson's Paradox" was recognized by Cohen and Nagel (1934, 449) nearly two decades before Simpson's important article. In 1940's and 1950's important contributions were made by Kenneth O. May (1946; 1947). Of relevance to this current paper is the pioneering work (May, 1954) demonstrating how "cyclic preferences" make sense should an individual's preferences be determined by multiple criteria of performance. One of our points is related to May's: As we show, the basic tenet underlying many spatial models, which is that voter preferences have a particular spatial representation, is far from innocuous. Situations exist where one might expect rational individuals to choose between two alternatives the one that is further away from the individuals' optimum point.

2 Ostrogorski's paradox

Consider an election involving 5 voters, 2 parties, and 3 issues where each voter views the issues to be of equal importance and no other considerations influence the voters' opinions about the parties. Consider two ways of determining the dominant party: (1)

issue	*issue 1*	*issue 2*	*issue 3*	*the voter votes for*
voter A	X	X	Y	X
voter B	X	Y	X	X
voter C	Y	X	X	X
voter D	Y	Y	Y	Y
voter E	Y	Y	Y	Y
winner	Y	Y	Y	?

Table 1: Ostrogorski's paradox

Each voter prefers the party that is closer to his/her (hereinafter his) opinion on more issues; (2) For each issue, the winner is the party that receives more votes than its competitor; the dominant party is the one winning on more issues than the other. In a nutshell, Ostrogorski's paradox occurs when the outcome differs in these two cases.

The Table 1 distribution of opinions about parties X and Y provides a strong version of the paradox because not only do the results differ under procedures (1) and (2), but the winner under (2) is unanimous. Replacing any one Y with an X creates a weaker version of the paradox where "just" a majority differs under (1) and (2).

One might worry whether the qualitative nature of Table 1 is misleading; perhaps "closeness" of voters to parties should be captured in a more precise manner. To address this concern, represent each issue with a coordinate direction in a 3-dimensional Euclidean space $\mathbb{R}^3$ where $X = (4, 4, 4)$ and $Y = (3, 3, 3)$. Represent each voter's stance for the three issues with an ideal point in $\mathbb{R}^3$; let A's views be at $(4, 5, 3.2)$, B at $(5, 3.2, 4)$, C at $(3.2, 4, 5)$, D at $(2, 2, 2)$ and E at $(1, 1, 1)$. By comparing these component values with those of X and Y, the Table 1 conclusion of a majority support for Y for each issue follows. In contrast, each of A, B, and C is closer to X than to Y; the common distance from A, B, or C to X given by the sum of distances of each issue (called the l_1 distance) is $0.8 + 0 + 1 = 1.8$, which is smaller than the common l_1 distance of any of these three points to Y which is $0.2 + 1 + 2 = 3.2$. The conclusion holds with other distances; e.g., the common Euclidean distance to X is $\sqrt{(0.8)^2 + 0^2 + 1^2} = \sqrt{1.64}$ while that to Y is the larger $\sqrt{0.2^2 + 1^2 + 2^2} = \sqrt{5.04}$.

To introduce our concern about the appropriate way to represent a voter's preferences, in Table 1 replace "voter" with "criterion" and imagine an individual trying to determine whether to support candidate X or Y. The criteria may be relevant educational background, political experience, negotiation skills, political connections, etc. The issues might be education, economy, and foreign policy. Each row and column table entry indicates our voter's preference of a candidate for that criterion and issue. While there are at least two natural ways for our voter to make his choice, if all issues and criteria are deemed of equal importance, our voter's decision is ambiguous: emphasizing criteria (row-column aggregation) with the majority principle suggests supporting X; stressing issues (column-row aggregation) yields Y.

Geometrically, our voter's views can be represented with the above values for the

issue	*issue 1*	*issue 2*	*issue 3*
voter A	X	X	Y
voter B	Y	Y	Y
voter C	Y	X	X
voter D	X	Y	X
voter E	X	Y	X
winner	X	Y	X

Table 2: Anscombe's paradox

locations of the criteria over issues; the l_1 distance of his opinions from Y is $3(3.2)+3+6 = 18.6$ while that from X is $3(1.8) + 6 + 9 = 20.4$. Using the Euclidean metric, the distance from Y is $\sqrt{3(5.04) + 3 + 12} = \sqrt{30.04}$ while the distance from X is the larger $\sqrt{3(1.64) + 12 + 27} = \sqrt{43.92}$. With either metric, then, this individual's views are closer to that of candidate Y than of candidate X. While the standard spatial voting assumption has this individual selecting the closest candidate Y, is this the correct choice? In reality, it can not be inferred in a pairwise comparison between X and Y whether our voter would always accept Y. In fact, by resorting to the reasonable principle of basing his choice on the criterion-wise performance of candidates, he will vote for X because X outperforms Y on three criteria, while Y beats X on only two. This assertion corresponds to common usage where a voter justifies his support for a candidate in terms of her education, experience, skills, and so forth.

Because Y is closer to the individual's beliefs on each dimension, these problems cannot be resolved by assigning salience weights to issue dimensions. Strategic considerations—which may underly occasional votes against preferences—do not enter into the calculus dictating the choice of X rather than Y because there are only two alternatives and the other voters' ideal points are not known.

3 The Anscombe and exam paradoxes

Closely related to Ostrogorski's paradox is one described by Anscombe (1976). In a nutshell, it says that a majority of voters could be in a minority (i.e., on the losing side) on a majority of issues involving dichotomous choices. In Table 2, which illustrates this paradox, voters A, B and C are on the losing side on a majority of issues: A on issues 2 and 3, B on issues 1 and 3, and C on issues 1 and 2.

Table 2 is *not* an Ostrogorski's paradox, so these paradoxes are not equivalent. (Ostrogorski is a specialized Anscombe paradox.) Using this table to model our voter who is trying to determine which candidate to support, Anscombe's paradox captures the amusing but not uncommon situation where a voter's candidate of choice, as based on criteria, could disappoint the voter with her stance over issues.

But while an Anscombe paradox need not be an Ostrogorski paradox, it can always be converted into one with appropriately selected parties U and V (Nurmi and Meskanen

criteria	issue 1	issue 2	issue 3	issue 4	average	score
criterion 1	1	1	2	2	1.5	1
criterion 2	1	1	2	2	1.5	1
criterion 3	1	1	2	2	1.5	1
criterion 4	2	2	3	3	2.5	2
criterion 5	2	2	3	3	2.5	2

Table 3: X's distances from the voter's ideal point

2000).

Theorem 1 *For any Anscombe paradox based on specified positions of parties X and Y, there exist parties U and V (created by adopting various positions from parties X and Y) where the Anscombe paradox becomes an Ostrogorski paradox.*

Proof: For an Anscombe paradox, let $\mathcal{M}$ be the set of voters in the majority where each is on the losing side for a majority of issues, and let $\mathcal{L}$ be the set of these issues. For each issue not in $\mathcal{L}$, let the stance of party U and V be, respectively, that of the winning and losing side. For each issue in $\mathcal{L}$, let the stance of the party U and V be, respectively, the stance of the losing and winning side.

By construction, each member of $\mathcal{M}$ agrees with party U over a majority of issues, so $\mathcal{M}$ constitutes the majority party. Also by construction, each member of $\mathcal{M}$ is on the losing side of a majority of issues. Thus an Ostrogorski paradox is created. If the original Anscombe paradox is a strong one (i.e., a majority of voters are on the losing side of all issues), then so is the corresponding Ostrogorski paradox. □

To illustrate this Anscombe and Ostrogorski connection, Table 2 is not an Ostrogorski paradox because voters A and C support party X while voter B supports party Y. But with the U and V parties, these voters' preferences over issues become, respectively, (V, U, U), (U, V, U), (U, U, V), so all three support party U. (The other two voters unanimously support party V.) This change from X and Y to U and V is not dissimilar to how parties are formed and/or respond to changing circumstances.

The exam paradox introduced and analyzed by Nermuth (1992) generalizes Ostrogorski's paradox by positioning it in a domain where the proximity of alternatives to ideal points assumes degrees rather than dichotomous values. (This approach is similar to our example using points in $\mathbb{R}^3$ but where the l_1 and Euclidean distances are replaced with a related mathematical representation.) As an illustration of Nermuth's example, consider four issues and five criteria. One of two competitors, X, is located at the distances from the voter's ideal point in a multi-dimensional space given in Table 3. The score of X on each criterion is simply the arithmetic mean of its distances rounded to the nearest integer; with a tie, round down to the nearest integer. X's competitor Y, in turn, is located in the space as indicated in Table 4.

criteria	issue 1	issue 2	issue 3	issue 4	average	score
criterion 1	1	1	1	1	1.0	1
criterion 2	1	1	1	1	1.0	1
criterion 3	1	1	2	3	1.75	2
criterion 4	1	1	2	3	1.75	2
criterion 5	1	2	1	2	1.75	2

Table 4: Y's distances from the voter's ideal point

4 Core conditions and aggregation paradoxes

Perhaps the best-known results on spatial models pertain to the conditions under which a core outcome exists (Banks 1995; McKelvey and Schofield 1987; Saari 1997). Recall: the core is the set of majority undominated outcomes:

$$x \in C \Leftrightarrow xMy, \forall y \in W$$

where M is the weak majority preference relation; i.e., xMy means either that x beats y with a majority of votes or that there is a tie between the two. These core results are based on the assumption that the ideal points, distance measures, and, more generally, utility functions of the individuals are well-defined in the policy space. The results characterize the number of issues for which a "stable core"[1] does, or does not, exist (Saari 1997; see Saari 2004 for an exposition) and explain how the majority rule performs when the core is empty (McKelvey 1976, McKelvey and Schofield 1987).

The definition of a core ensures that if the spatial representation of voter ideal points has a core and if, say, X is in the core, then the Ostrogorski paradox cannot occur. Even stronger:

Theorem 2 *If an Ostrogorski or Anscombe paradox occurs, then any spatial representation of the voters' ideal points has an empty majority vote core.*

Proof: For $\mathbf{p}$ to be a majority vote core point over k issues, a majority of the voters' ideal points cannot be on either side of *any* hyperplane passing through $\mathbf{p}$. (If a majority were on one side, there is a point on the normal to the plane at $\mathbf{p}$ that this majority prefers.) With an odd number of voters, this condition requires $\mathbf{p}$ to coincide with some voter's ideal point; with an even number of voters, either $\mathbf{p}$ coincides with some voter's ideal point, or each ideal point has a companion ideal point where their connecting line contains $\mathbf{p}$ forming a "Plott configuration" (1967).[2] (For instance, four ideal points define a quadrilateral; the core point is the intersection of the two diagonals.)

[1] "Stable" means that if a core does exist, it continues to exist even with any very slight change in preferences. To illustrate instability, a core exists if three ideal points lie on a line in $\mathbb{R}^2$. But this core is unstable because it disappears by ever so slightly moving any point off the line.

[2] If $\mathbf{p}$ is not at an ideal point with an odd number of voters, any hyperplane passing through $\mathbf{p}$ and an ideal point must have less than half of the voter ideal points on either side. Slightly perturbing this plane moves the ideal point to one side and creates a plane with a majority on one side. If n is even and

Because of Thm. 1, we can assume that $\mathbf{X} = (X_1, \ldots, X_k)$ and $\mathbf{Y} = (Y_1, \ldots, Y_k)$ are points selected to ensure that this is an Ostrogorski paradox where the ideal points for majority party members are closer to $\mathbf{X}$ than $\mathbf{Y}$ and each majority party voter is on the losing side of at least one issue. Suppose $\mathbf{p}$ coincides with some voter's ideal point. As a majority of the voters have their ideal points closer to $\mathbf{X}$ than to $\mathbf{Y}$, this voter cannot be in the minority; i.e., $\mathbf{p}$ coincides with the ideal point of a majority party voter. To be a member of the majority party, this voter must agree with $\mathbf{X}$ over a majority of the issues. The Ostrogorski paradox requires this party to lose a majority of the issues, so this voter is on the losing side of, say, the j^{th}, issue. Pass a hyperplane through $\mathbf{p}$ that is orthogonal to the j^{th} coordinate axis; i.e., the j^{th} coordinate of this hyperplane agrees with that of the identified voter, so it is closer to X_j than to Y_j. As a majority of the j^{th} coordinates of voter ideal points are closer to Y_j, a majority of the ideal points are on one side of this hyperplane, so $\mathbf{p}$ cannot be a core point.

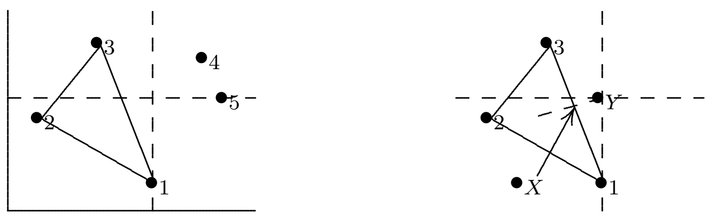
The remaining case is where, with an even number of voters, $\mathbf{p}$ is not at an ideal point. The hyperplane argument shows that for the j^{th} issue, $j = 1, \ldots, k$, a majority of the j^{th} components of the ideal points cannot be larger than, or smaller than, the value of p_j; i.e., p_j is in the median of the j^{th} component of all ideal points. Thus, if Y_j is the winning issue, all ideal points with their j^{th} component on the Y_j side of p_j *and* at least one that is on the X_j side, are in the winning coalition. As $\mathbf{p}$ lies between each companion pair of ideal points p_j lies between their j^{th} components for each j. Thus, at least one of these ideal points is on the winning side at least half of the time. (If k is odd, this is for a majority of the time.) As each ideal point is in a companion pair, at least half of the ideal points are on the winning side at least half of the time, so an Anscombe or Ostrogorski paradox cannot occur. This completes the proof. □

So, with an Anscombe paradox, any spatial representation for the voters' ideal points has an empty core. But even if the core fails to exist, an Ostrogorski or Anscombe paradox need not arise. To see this, slightly modify the above numerical example by replacing each 3.2 value with 3.6; while these changes suffice to avoid the Ostrogorski paradox, the core remains empty. (Generically, the majority vote core is empty with three or more issues (Saari 1997, 2004).) The reason the Ostrogorski paradox need not occur is that the more demanding core concept compares $\mathbf{X}$ with *all possible* $\mathbf{Y}$ locations, but Ostrogorski and Anscombe compare $\mathbf{X}$ only with a fixed $\mathbf{Y}$.

Nevertheless, the core and Ostrogorski paradox remain closely intertwined in that whenever the voters' ideal points define an empty majority vote core, points $\mathbf{X}$ and $\mathbf{Y}$ can be selected to create an Ostrogorski paradox! To exhibit the ideas, which is a spatial voting extension of Thm. 1, this assertion is illustrated in the next theorem for five voters and two issues; the construction clearly extends to all settings.

$\mathbf{p}$ is not an ideal point, select a plane passing through $\mathbf{p}$ than meets no ideal points; i.e., $\frac{n}{2}$ points are on either side. If moving the plane meets one ideal point, but not a companion, slightly change the plane to pass through this ideal point to create a setting with a majority on one side; thus each ideal point is accompanied by a companion on the line passing through $\mathbf{p}$. As $\mathbf{p}$ must be in the convex hull defined by all majority settings of candidates, these companion points must be on opposite sides of $\mathbf{p}$.

Theorem 3 *Whenever five ideal points in* $\mathbb{R}^2$ *have an empty core, points* $\mathbf{X}$ *and* $\mathbf{Y}$ *can be selected to create an Ostrogorski paradox.*



a. General setup　　**b.** Finding $\mathbf{X}$

Figure 1. Core vs. Ostrogorski

Proof: Let $\mathcal{P}_C$ be the convex hull (the triangle in Fig. 1) defined by any three ideal points forming a coalition C. This is the coalition's Pareto set because moving any point in $\mathcal{P}_C$ leads to a poorer outcome for at least one coalition member. Moreover, it follows from the triangle inequality that for any $\mathbf{q}$ not in the triangle, this coalition prefers the point $\tilde{\mathbf{q}}$ that is the closest point in the triangle to $\mathbf{q}$. As such, if a core point exists, it must be in $\mathcal{P}_C$. But as the core is empty, the intersection of the 10 possible triangles is empty.

For each issue (axis) find the median of the voters' ideal points; use these points to construct an axis in $\mathbb{R}^2$ (the dashed lines in Fig. 1a). Place $\mathbf{Y}$ at the center of this axis. Because the core is empty, we can select a triangle that does not include $\mathbf{Y}$. Let $\mathbf{X}$ be the point in this triangle that is closest to $\mathbf{Y}$. It follows from the triangle inequality that $\mathbf{X}$ is closer to each of these three ideal points than is $\mathbf{Y}$; i.e., these three ideal points define the majority party.

To ensure that $\mathbf{Y}$ wins over each issue, it suffices to prove that $\mathbf{X}$ is in the interior of some quadrant (so that $\mathbf{X}$ is farther from each median line). If the leg of the selected triangle that is closest to $\mathbf{Y}$ is not parallel to any axis, then either $\mathbf{X}$ is an ideal point on the triangle or the point of the intersection of this leg and perpendicular line passing through $\mathbf{Y}$ (the slanted dashed line in Fig. 1b). In either case, the geometry forces $\mathbf{X}$ to be in the interior of a quadrant.

If this leg is on, or parallel to some axis, say the vertical one, and if this closest leg is parallel, but not on the axis, then all three points are strictly on the same side of $\mathbf{Y}$ for some issue, which violates the definition of $\mathbf{Y}$. If all three ideal points are on the axis, and as $\mathbf{Y}$ is not in the triangle, all three ideal points are strictly on the same side of $\mathbf{Y}$ on this axis, which again violates the condition that $\mathbf{Y}$ is at the median of each issue. Thus, at least one ideal point is off this axis, and (again, to ensure $\mathbf{Y}$ is at the median) it must either be on, or on the other side of the remaining axis. Thus this leg is not parallel to any axis; this completes the proof. □

The Ostrogorski paradox generated by the Fig. 1 example is in Table 5. This example reflects our concern whether a voter should be modeled as preferring a closest point, or in terms of his votes over issues. After all, the choice of $\mathbf{Y}$, which is determined by the voters' votes over issues, is uniquely defined. In contrast, there are *five* three-voter coalitions in Fig. 1a satisfying the condition that their Pareto set misses $\mathbf{Y}$ (coalitions

issue	*issue 1*	*issue 2*	*the voter votes for*
voter 1	Y	X	X
voter 2	X	X	X
voter 3	X	Y	X
voter 4	Y	Y	Y
voter 5	Y	Y	Y
winner	Y	Y	?

Table 5: Ostrogorski's paradox with two issues

$\{1,2,3\}, \{2,3,4\}$, $\{3,4,5\}, \{1,4,5\}, \{1,2,5\}$); these different majorities create an ambiguity in the choice of the "closest point for a majority of the voters" as each coalition defines a different $\mathbf{X}$ choice ($\mathbf{X}_C$). Even more, it follows from the geometry that there is no Condorcet winner among these five $\mathbf{X}_C$ choices; e.g., all voters in $\{3,4,5\}$ prefer their $\mathbf{X}_{3,4,5}$ choice over the $\mathbf{X}_{1,2,3}$ choice. Instead, these five $\mathbf{X}_C$ define the cycle

$$\mathbf{X}_{1,2,3} \succ \mathbf{X}_{1,4,5} \succ \mathbf{X}_{2,3,4} \succ \mathbf{X}_{3,4,5} \succ \mathbf{X}_{1,2,3} \tag{1}$$

5 Resolving Kelly's conjecture

The conditions under which Ostrogorski's paradox occurs have been studied by several authors (e.g. Rae and Daudt 1976, Deb and Kelsey 1987, Kelly 1989, Laffond and Lainé 2000, Shelley 1984). Rae and Daudt establish a connection between Ostrogorski's paradox and cyclic majorities.

To explain, in the Table 1 Ostrogorski setting, X beats Y with a majority vote. Could a third candidate Z enter the race where Z would beat X? Using a nice argument that improves upon the Rae and Daudt observation, Kelly (1989) shows that, yes, such a Z can be found, but Y would beat Z to create a cycle! Even stronger, Kelly proved for three issues that "every occurrence of the Ostrogorski paradox implies that among the 8 possible candidates there can be no Condorcet winners." In other words, no matter how another candidate Z is selected, she is not a Condorcet winner. Kelly conjectured that this "no Condorcet winner" relationship holds for an odd number of issues. This conjecture is captured by the Eq. 1 cycle where each $\mathbf{X}_C$ represents a candidate.

For our purposes, if Kelly's conjecture is true, it would cast doubt on the standard approach where voters select a candidate based on criteria rather than issues. This is because none of the many possible candidates dominates.

Insights about Kelly's conjecture follow from Thm. 2, which asserts that *any* spatial representation of the voters' ideal points with an Ostrogorski or Anscombe paradox must have an empty majority vote core. With an empty core, McKelvey's (1976) chaos theorem applies; namely, for *any* starting and intermediate points, $\mathbf{x}^s$ and $\mathbf{x}^i$, a majority vote agenda $\{\mathbf{x}_i\}$ can be crafted where the first and last terms are $\mathbf{x}^s$ and an intermediate term agrees with $\mathbf{x}^i$. With this agenda, $\mathbf{x}_{i+1}$ beats $\mathbf{x}_i$ for all i; i.e., if the core does not exist, expect to find cycles among candidates and proposals. Thus

> *McKelvey's spatial voting result verifies Kelly's conjecture for any number of votes and issues; i.e., in a spatial context, an Ostrogorski or Anscombe paradox ensures that there is no Condorcet winner.*

While this observation fully answers Kelly's conjecture in a spatial setting, it does not for the discrete setting. The problem is illustrated with Table 5 where, without having the advantage of distances, it is not clear whether voters 1 and 3 are indifferent (the usual assumption with dichotomous choices), or whether they prefer X or Y. But as the next theorem shows, even with this complication, Kelly's conjecture is true for any number of issues greater than two, not just for an odd number of them.

Theorem 4 *For any number of issues greater than two, the occurrence of an Ostrogorski or Anscombe paradox means that there cannot be a Condorcet winner.*

Proof: To convert preferences over k issues into vertices of a k-dimensional cube $[-1,1]^k$, for each issue, designate the losing choice by a -1 and the winning choice by a 1; e.g., in Table 1, voter A's preferences become $(-1,-1,1)$. Thus, for each j, the sum of the j^{th} entries over all voter vectors is positive and the winning portfolio is the cube vertex $\mathbf{y} = (1,1,\dots,1)$. As a candidate's platform is represented by a cube vertex, there are 2^k possible candidates. To compare voter preferences between two candidates, take the difference between the vectors; e.g., if $\mathbf{x} = (-1,-1,\dots,-1)$ (representing the losing side on each issue), then $\frac{1}{2}(\mathbf{x}-\mathbf{y}) = (-1,-1,\dots,-1)$. Thus $\frac{1}{2}(\mathbf{x}-\mathbf{y})$ is a normal vector for a plane passing through the cube's center; all voters with vectors (vertices) above this plane prefer $\mathbf{x}$, those below prefer $\mathbf{y}$, and those on the plane are indifferent; i.e., a voter's choice is determined by whether the dot product of $\frac{1}{2}(\mathbf{x}-\mathbf{y})$ with the voter's vector is, respectively, positive, negative, or zero. As each $\frac{1}{2}(\mathbf{x}-\mathbf{y})$ entry is a -1, the dot product with a voter's vector equals the difference between the number of issues for which this voter is on the losing and winning sides; e.g., a voter on the losing side of most issues has a positive value. An Anscombe paradox requires a majority of the voters to be on the losing side a majority of the time, so a majority of the voters prefers $\mathbf{x}$ to $\mathbf{y}$.

We now know that $\mathbf{x}$ is preferred to $\mathbf{y}$; we must show that for any other vertex, say $\mathbf{c}$, a vertex can be found that is preferred to $\mathbf{c}$. Other than $\mathbf{y}$, all vertices (candidate's positions) have at least one "-1" in its vector representation. Select any such candidate, say $\mathbf{c}$, where "-1" is in the j^{th} position. Let $\mathbf{d}$ be the vector that agrees with $\mathbf{c}$ in all components except the j^{th} where it has a $+1$. Thus $\frac{1}{2}(\mathbf{d}-\mathbf{c})$ is the vector with 1 in the j^{th} component and zeros elsewhere; its scalar product with a voter's vector has a positive value (supporting $\mathbf{d}$ over $\mathbf{c}$) for a voter who is on the winning side of the j^{th} issue. Thus the sum of j^{th} components over all voter vectors is the difference between the number of voters preferring $\mathbf{d}$ to $\mathbf{c}$. This sum is positive, so $\mathbf{d}$ is preferred to $\mathbf{c}$. As this holds for any $\mathbf{c}$, the theorem is proved. □

This argument can be used to fashion all sorts of cycles. The natural idea is that a candidate can improve her standing by assuming the winning side of an issue. What creates the cycle is the peculiar choice of preferences allowing an Anscombe paradox, as reflected by $\mathbf{x} \succ \mathbf{y}$, where a majority of the voters are on the losing side a majority of the times; e.g., for five issues,

$$(-1,-1,-1,-1,-1) \succ (1,1,1,1,1). \tag{2}$$

A five issue cycle is $(1,1,1,1,1) \succ (1,1,1,1,-1) \succ (1,1,1,-1,-1) \succ (1,1,-1,-1,-1) \succ (1,-1,-1,-1,-1) \succ (-1,-1,-1,-1,-1) \succ (1,1,1,1,1)$. To create a cycle involving all 32 candidates, vary the number of voters of different types.

6 Differences between dichotomous and spatial models

Deb and Kelsey (1987) derive the following necessary and sufficient condition for an Ostrogorski's paradox to occur:

$$kn - 2ny - 2kx - 12xy \geq 0. \tag{3}$$

Here $x = 1$ when n (the number of voters or criteria) is even, and $x = 1/2$ when n is odd. Similarly, $y = 1$ when the number k of issues is even and $y = 1/2$ when k is odd.

To illustrate, the Table 1 example has $n = 5$ and $k = 3$, so $x = y = \frac{1}{2}$. Substituting into Eq. 3 yields $(3)(5) - 2(5)\frac{1}{2} - 2(3)\frac{1}{2} - 12(\frac{1}{2})(\frac{1}{2}) = 15 - 5 - 3 - 3 = 4 > 0$, which means that an example can be constructed. Now try to modify this example to create one with only two issues. Here, $n = 5, k = 2, x = \frac{1}{2}, y = 1$, so Eq. 3, has the negative $(2)(5) - 2(5) - 2(2)\frac{1}{2} - 12(\frac{1}{2})(1) = -2$ value. The Deb and Kelsey condition ensures that no such two-issue example exists.

But, in seeming contraction with the Deb and Kelsey condition, such two-issue examples do exist! One was constructed in Fig. 1 and Table 5. This conflict reflects the difference between dichotomous and spatial voting examples; the former is incapable of handling votes that could be interpreted as being a tie, as with voters 1 and 3 in Table 5. In a spatial voting context, the more refined distances resolve these difficulties. The Deb and Kelsey condition, then, identifies all settings that suffer this ambiguity.

The following statement converts the Deb and Kelsey condition into a format that is easier to understand. As Cor. 1 shows, the main complexities arise with small number of even issues where these problems about whether a voter is a tie vote arise.

Corollary 1 *An Ostrogorski paradox can be created in the following situations:*
Suppose both n and k are odd. If $k \geq 3$, then $n \geq 3$.
Suppose n is odd and k is even. If $k = 4$, then $n \geq 5$. Otherwise, $n \geq 3$.
Suppose both n and k are even. If $k = 4$, then $n \geq 10$; if $k = 6$ or 8 then $n \geq 6$; if $k \geq 10$, then $n \geq 4$.

Proof: For odd values of k (so $y = \frac{1}{2}$), Eq. 3 assumes the $n \geq 2x + \frac{8x}{k-1}$ form. Thus, for $x = \frac{1}{2}$ (or n odd valed), $n \geq 3$; for $x = 1$ (or n even valued), $n \geq 2 + \frac{8}{k-1}$ and the assertion follows. Similarly, for even values of k (so $y = 1$), Eq. 3 assumes the $n \geq 2x + \frac{16x}{k-2}$ form; the remaining conclusions follow. $\square$

issue	*issue 1*	*issue 2*	*issue 3*	*issue 4*
voters 1 & 2	Y	X	X	X
voters 3 & 4	X	Y	X	X
voters 5 & 6	X	X	Y	X
voters 7 - 10	Y	Y	Y	Y
winner	Y	Y	Y	X

Table 6: Ostrogorski's paradox with four issues

Ten voters are required to create a four-issue Ostrogorski paradox! (See Table 6.) If distances could be used, voters in the majority could use (X, X, Y, Y) preferences and be closer to X. But without the aid of distances, such a preference is judged as a tie, which requires using preferences of the (X, X, X, Y) type. This change is what increases the number of necessary voters from three to ten. Thus the mysterious nature of Eq. 3 is to capture the these kinds of situations involving ties that arise with small, even numbers of issues. In contrast, the general spatial voting theorem (using the obvious extension of Thm. 3 and (Saari 1997)) is that

> *a spatial voting Ostrogorski paradox always can be created with any generic*[3] *placement of any odd number (greater or equal to three) of voters' ideal points with two or more issues, and any generic placement of any even number (greater or equal to four) of voter's ideal points with three or more issues.*

The role of Eq. 3 is to identify which small numbers of voters and issues cause problems about who a voter prefers, so there is no reason to believe that larger values of this inequality have any other meaning. Nevertheless, one might wonder whether larger values of this inequality might indicate a larger likelihood of the paradox. Although Kelly's (1989) computer simulations (using an impartial culture assumption) suggest that the paradox's probability increases with voters (or criteria), but decreases with an increase in issues, it is doubtful that Eq. 3 plays a role other than coincidental; likelihood issues must reflect those profile structures that allow the paradox. Our explanation of all such paradoxes sheds light on this and other questions.

7 Relating and explaining the paradoxes

Surprisingly, all of the "Yes-No," "X-Y" voting issues described in this article are caused by the same kinds of profile configurations of voter preferences. As these configurations also explain all paired comparison votes, explanations for paradoxical problems in one setting can be transferred to explain puzzles in others. For instance, the structure causing an Ostrogorski paradox is essentially that required to create a Condorcet cycle.

An important source of these problems (Saari and Sieberg 2001, 2004) is that a voting rule's issue-by-issue outcomes are *not* determined by the actual profile, but rather by a

[3]Namely, either the ideal points, or an arbitrarily small change in them, will create an example.

set of associated profiles. In a real sense, problems arise because the voting rule cannot identify the actual profile; instead, the selected conclusion is an appropriate one for the largest subset of profiles in the associated set!

To illustrate, the Table 1 outcomes occur because, for the first three voters, each issue has the $X \succ Y$ outcome with a 2:1 vote. Now suppose it is only known that the rule respects anonymity and, for each issue, there is the $X \succ Y$ outcome with a 2:1 vote. Armed with this information and by permuting the names of the voters, the following five profiles constitute all supporting choices; they are indistinguishable to the rule:[4]

$$\begin{array}{llll} 1. & (X,X,X),\ (X,X,X),\ (Y,Y,Y) & 2. & (X,X,X),\ (X,X,Y),\ (Y,Y,X) \\ 3. & (X,X,X),\ (Y,X,X,),\ (X,Y,Y) & 4. & (X,X,X),\ (Y,X,X,),\ (X,Y,Y) \\ 5. & (X,X,Y),\ (X,Y,X),\ (Y,X,X) & & \end{array} \quad (4)$$

With each of the first four choices, the last voter's preferences are closer to Y than to X; only with the fifth—the actual choice—are all three voters positioned closer to X than to Y. By adding the remaining two Table 1 voters (who support Y on all issues), it follows from Eq. 4 that for 80% of the possibilities (i.e., where the two Y voters join with each of the first four choices), there is no paradox; Y wins with a majority vote over each issue and Y is the closest point for three of the five voters. Only one choice from this set—the actual profile—creates a conflict.

This example makes it reasonable to believe (as made mathematically precise in Saari and Sieberg 2001) that the majority vote rule handles the ambiguity about the actual profile by selecting an outcome that is appropriate for *most profiles* within the associated set. Thus this approach provides an appropriate, non-paradoxical answer for a majority of the cases; e.g., for the first four cases, Y *is the winner* with either way of computing. This approach can create a paradoxical conflict only for a minority of settings, such as the Table 1 profile. This explanation—where a rule emphasizes the associated set of profiles rather than the actual one—holds for all of the above paradoxes.

As an illustration of this associated set, the Table 7 profile belongs to the associated set of profiles defined by the Table 2 profile. As Table 7 illustrates the Ostrogorski paradox, it follows that the Table 2 profile, which reflects just the Anscombe paradox, has an Ostrogorski paradox in its associated set.[5] The next question is to identify the configurations of profiles for which the associated set includes the foundation for an Ostrogorski and Anscombe paradox.

The kinds of profiles that cause these paradoxical concerns are identified with a coordinate system developed to analyze n-alternative voting problems (Saari 2000, 2008). Certain profile coordinate directions are responsible for all possible problems that affect

[4]The set associated with a given profile is the set of profiles that is obtained permuting each column in all possible ways.

[5]But an Anscombe paradox need not include an Ostrogorski paradox in its associated set. To create an illustrating example, take a strong Ostrogorski paradox with an even number of issues where Y always wins. For the odd numbered issues, exchange the X and Y names to create a strong Anscombe example. The same majority loses over each issue, but as X and Y each win half the issues, no permutation of preferences can create an Ostrogorski paradox. We leave it as an exercise for the reader to determine necessary and sufficient conditions for an Ostrogorski paradox to be in an Anscombe associated set.

issue	*issue 1*	*issue 2*	*issue 3*
voter A	X	Y	Y
voter B	Y	X	Y
voter C	Y	Y	X
voter D	X	Y	X
voter E	X	X	X
winner	X	Y	X

Table 7: Connecting Anscombe and Ostrogorski

paired comparisons. To construct these profile directions, use the Fig. 2 ranking wheel (Saari 2000, 2008). Uniformly near the edge of the wheel place the numbers 1 to n; these are the ranks. On the wall, place the names of the alternatives in any specified order. Read off the ranking; in Fig. 2 it is $A \succ B \succ C \succ D \succ E \succ F$. Rotate the wheel so that the ranking number 1 is by the next name and read off the new ranking. Repeat until each candidate has been in first place precisely once. The Fig. 2 configuration is

$$\begin{array}{ccc} A \succ B \succ C \succ D \succ E \succ F, & B \succ C \succ D \succ E \succ F \succ A, & C \succ D \succ E \succ F \succ A \succ B, \\ D \succ E \succ F \succ A \succ B \succ C, & E \succ F \succ A \succ B \succ C \succ D, & F \succ A \succ B \succ C \succ D \succ E. \end{array} \tag{5}$$

Each ranking defines a unique set and each ranking in a set defines the same set, so there are $\frac{n!}{n} = (n-1)!$ distinct sets. These sets are the building blocks for the *Condorcet profile directions*; each Condorcet coordinate combines, in a particular manner, the set defined by a ranking and the set defined by reversal of this ranking. As such, there are $\frac{(n-1)!}{2}$ orthogonal Condorcet profile directions. (Orthogonality follows because each ranking is in one, and only one, Condorcet direction. But, because of a subtle feature caused by profiles that do not affect the outcomes of any voting rules, there are only $\frac{(n-1)(n-2)}{2}$ directions.) The theorem (Saari 2000) is that all paired comparisons problems are due to a profile's components in the Condorcet directions; e.g., if a profile is orthogonal to all Condorcet directions, then no paired comparison difficulties occur.

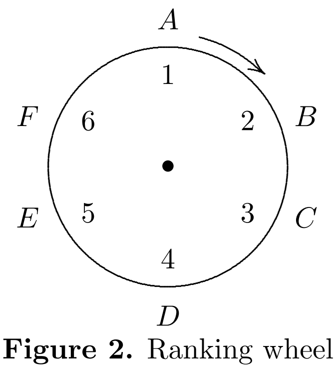


Figure 2. Ranking wheel

To see how the Condorcet directions cause non-transitivity problems, notice that the Eq. 5 profile defines the $A \succ B$, $B \succ C$, $C \succ D$, $D \succ E$, $E \succ F$, $F \succ A$ cycle where the tally for each pair is 5:1. An important point is that, for any number of alternatives, all non-transitive outcomes and other paired comparison problems are caused by the Condorcet profile directions—including the paradoxes of our interest.

To illustrate, Cor. 1 asserts that a six-issue Ostrogorski paradox can be created with three voters, so the majority has two voters. To create an example, select two alternatives from Eq. 5, say A and B, and relabel them with a Y; denote all other alternatives by X. Now, the first and third Eq. 5 choices become (Y, Y, X, X, X, X), (X, X, X, X, Y, Y). By letting the third voter have all Y's, the majority loses over the first, second, fifth, and sixth issues. An examination of all examples in this article shows that the preferences for the majority always have this structure; e.g., Table 1 has three issues, so the majority party's preferences can be constructed from the $A \succ B \succ C, B \succ C \succ A, C \succ A \succ B$ Condorcet direction. By replacing C with Y and replacing the names of the other two alternatives with X, the majority party's preferences are (X, X, Y), (X, Y, X), (Y, X, X) of the table.

This connection is not a coincidence; The next theorem asserts that this "renaming" process captures all Ostrogorski and Anscombe paradoxes.

Theorem 5 *For any Ostrogorski or Anscombe paradox, let $\mathcal{L}$ be the set of k issues over which a majority of the voters lose, but where each voter is on the winning side of at least one issue. The preferences of these voters over $\mathcal{L}$ can be identified with components in the Condorcet profile directions for k alternatives.*

The condition that each voter is on the winning side of at least one issue in $\mathcal{L}$ is for convenience. With the Ostrogorski paradox, for instance, each voter must support X on a majority of issues, so he must be on the winning side of at least one issue in $\mathcal{L}$.

Proof: The proof reverses the renaming manner used to create the above three-voter, six-issue example. List the k issues in the standard column format with X's and Y's. For the j^{th} column, $j = 1, \dots, k$, rename the winning and losing issue, respectively, with $A_j \succ A_{j+1}$ and $A_{j+1} \succ A_j$, where A_{k+1} is identified with A_1. The winner for each issue (column) represents the pairwise tally for the particular pair of alternatives, which, by construction, is a cycle.

We must show that each row (voter's preferences) can be identified with a transitive ranking; what provides flexibility is that the renaming construction specifies only paired comparisons of the $\{A_j, A_{j+1}\}$ type. If these rankings of adjacently listed alternatives do not create a cycle, then the remaining pairs can be ranked to create a transitive ranking. The only way a cycle can be created by these k pairs is if this voter is on the winning side of each issue (which is impossible by being from an Anscombe paradox), or on the losing side of each of the k issues, which again is excluded by assumption.

The renaming process creates a set of transitive preferences that causes a cycle, so it follows from (Saari 2000, 2008) that the cycle is caused by Condorcet direction components in the profile. □

Theorem 5 finally establishes what has been long suspected; the Ostrogorski and Anscombe paradoxes always are intimately connected with non-transitive paired comparison behavior. An added benefit is how this connection explains our quandary about whether voters' preferences should be described by the closest point, or by issues. Namely, it is shown in (Chap. 2 of Saari 2008) how, with the Condorcet profile directions, the pairwise vote drops the assumption that voters have transitive preferences (which then explains phenomena such as Arrow's Impossibility Theorem).

In the context of the Ostrogorski paradox, it means that voting over issues divorces the connection of how a voter relates the issues to a particular party. So, if issues are separate from one another, if they are not connected or related by a voter to determine his support for a party, then the issue-by-issue approach provides the better connection. But if the issues are related, e.g., if they are related in terms of, say, compromises to create a coherent platform, then the selection of the closest point is the appropriate choice. A similar description holds for the voter trying to select a candidate; if individual issues matter more than how their combination characterizes a criterion for a candidate, select the candidate according to issues. But if the combined stance over issues characterize how a candidate should be viewed with respect to specified criteria, then select the candidate whose position is the closest point in issue space.

8 Supermajority voting

To further use Thm. 5, return to the comment about how Ostrogorski's paradox can viewed according the "degree of contradiction" involved in computing election outcomes (i) over issues separately and then determining the overall winner, or (ii) over voters and then determining the election winner. With our voter trying to select between two candidates, for instance, he might use a more severe decision rule than the majority count; e.g., he may continue to support a status quo favorite candidate (or party) unless its competitor is closer to his position on, say, more than two-thirds of the issues.

Indeed, as reflected by the super majority requirements that many countries impose on certain kinds of legislation to ensure some political stability, the status quo may be preferred unless more than, say, $\frac{3}{5}$ of the electorate prefers its competitor. This behavior is illustrated in Table 8 with 3 issues and 5 voters where the status quo party 1's position is 1 on every issue and the competing party 0's position is 0 on every issue. If each voter votes for 0 unless 1 is closer to his position and if it takes a $\frac{4}{5}$ vote to win on each of the issues, no Ostrogorski's paradox emerges. The lower $\frac{3}{5}$ requirement, however, does admit the paradox. (Comparing this example with the core, if the voting rule requires a winning position to have 3 out of 5 voters, the core is generically empty with three or more issues. The stronger rule requiring 4 of the 5 voters to pass a proposal can have a non-empty core (Saari 1997).)

Wagner (1983, 305-306) shows that an Anscombe's paradox cannot arise if, on average, there is a sufficiently strong support over all winning issues:

> If N individuals cast yes-or-no votes on K proposals then, whatever the decision method employed to determine the outcomes of these proposals, if the average fraction of voters, across all proposals, comprising the prevailing coalitions is at least three-fourths, then the set of voters who disagree with

issue	*issue 1*	*issue 2*	*issue 3*
voter A	1	1	0
voter B	1	0	1
voter C	0	1	1
voter D	0	0	0
voter E	0	0	0

Table 8: Ostrogorski's paradox: 0-1 version

a majority of outcomes cannot comprise a majority.

This assertion is about an *average 'yea' vote over all issues*; it does *not* mean (as interpreted by some authors) that an Anscombe paradox can be avoided by using a three-fourths rule. Instead, by using Thm. 5, we prove (Thm. 6) that an Anscombe paradox can occur even with a rule that is one vote shy of requiring unanimity support.

Definition 1 *For integer $n > 2$, let q be an integer satisfying $\frac{n}{2} < q < n$. A "q rule with n voters" is where the winner of a paired comparison must receive q or more votes.*

Theorem 6 *Suppose the majority party wants to pass all issues and each party member agrees with most of these issues. For any q rule, even a q rule that is one vote away from unanimity, the majority party can lose over a majority of the issues. Indeed, the majority party may never win a single issue.*

To indicate why this extreme result does not contradict Wagner's assertion, the example developed below has $2m-1$ voters and issues. As the maximum number of "Yes" votes is $m + (m-1)(2m-1)$, the average of "Yes" votes is bounded by $\frac{m+(m-1)(2m-1)}{(2m-1)^2}$, which is slightly over a half and far from Wagner's three-fourths threshold.

Proof. For $n = 2m - 1$, let the majority party have m voters, a minority party have $m - 1$ voters, and $q = n - 1 = 2m - 2$. For the first m of the $2m - 1$ issues, use the ranking wheel from Thm. 5 to create the set of m rankings

$$A_1 \succ A_2 \succ \cdots \succ A_m, \ldots \quad \ldots, A_m \succ A_1 \succ \ldots A_{m-1}. \tag{6}$$

To assign voters' choices in the majority party, create an $m \times m$ array as in Eq. 7. The k^{th} row represents Eq. 6's k^{th} ranking; the j^{th} column represents A_j. The j^{th} column, k^{th} row entry is the ranking wheel position of A_j in the k^{th} ranking. Thus, the first row's entries are $1, 2, \ldots, m$. For the second row, because A_2 is top ranked and A_1 bottom ranked, the entries are $m, 1, 2, 3, \ldots, m-1$.

$$\begin{array}{cccc} A_1 & A_2 & \ldots & A_m \\ \hline 1 & 2 & \ldots & m \\ m & 1 & \ldots & m-1 \\ \ldots & \ldots & \ldots & \ldots \\ 2 & 3 & \ldots & 1 \end{array} \tag{7}$$

The ranking wheel places each alternative in each position precisely once, so a specified integer in Eq. 7 is in precisely one row for each column and one column for each row.

To create an example that proves the theorem, treat each A_j as an issue to be decided by a "Yes-No" vote. Select an integer between 1 and m, perhaps 1. Everywhere this integer appears, replace it with a "Yes." Replace all other integers with a "No." For each of these m issues, have each of the $m-1$ minority party members vote "No." Over each of these m issues, the "No" side wins with a $q = (m-1) + (m-1) = 2m - 2 = n - 1$ vote, which is one vote shy of unanimity.

For the remaining $m-1$ issues, have each majority party member vote 'Yes." (As required, each majority party member agrees with the party a majority of the time.) Over these issues, assign either a "Yes" or "No" to each minority member. If "No" is assigned to each minority party member for each issue, the outcome is not decided (as neither side obtains the quota). Thus the majority party cannot win anything.

To create an illustrating example where the majority party barely loses on a $q = n-s$ rule, select s different integers from Eq. 7, and, in each column, rename each of these integers with "Yes;" rename the other integers in this column with "No." The rest of the construction is the same. However, to ensure that each voter of the majority party supports a majority of the party's issues, notice that over the m issues, he supports s of them, and disagrees on $m - s$. So, instead of adding $m-1$ additional issues, it suffices to add l issues, where $l + s > m - s$, or to add only $m - 2s$ new issues. □

As Thm. 5 asserts, all illustrating examples must be of this kind. To see why, suppose a different example is created for the $q = n - 1$ case. Each column has precisely one "Yes," and, as the same voter cannot vote "Yes" on more than one of the first m issues (to be a member of the majority party), each row has precisely one "Yes." In the first row, a sole "Yes" appears in some column; name that column A_1. In general, in the j^{th} row, the number 1 appears in only one column; call that column A_j. The result, then, is a permutation of the columns of the original example. In other words, all examples are permutations of the columns and/or rows for Eq. 7.

As Thm. 6 makes no mention about the number of issues, it is reasonable to question whether this kind of super-majority voting result requires a certain number of issues. While we have not explored the answer for "Yea-Nay" voting, the answer is known for spatial voting. According to results in (Saari 1997, 2004) about the existence of a core,

> expect that a spatial voting q-rule Ostrogorski paradox can be created with at least $2q - n + 1$ issues.

So, it is possible to create a spatial voting example with a $q = 5, n = 7$ rule with at least $2(5) - 7 + 1 = 4$ issues where the ideal points do not define a three-dimensional subspace. The qualifying "expect" is added only because the existence of a core for $\frac{q}{n} > \frac{3}{4}$ has a complicated structure; for precise values, consult (Saari 1997, 2004).

The Ostrogorski paradox requires each member of the majority party to support the party over a majority of the issues. To introduce a new, related class of paradoxes, we impose stronger conditions on what issues the majority party can adopt. Namely, it is reasonable to require a certain percentage of majority party members to support a specific proposal before it can be put forth by the party.

Definition 2 *For α satisfying $0 < \alpha < 1$, a party provides α-support for an issue if at least the fraction α of party voters are in favor of it.*

The choice of α may require αn to be just one vote, at least 25% support, a majority vote, or maybe even a particular supermajority. To simplify the arithmetic, replace the q rule with a $\beta > \frac{1}{2}$ rule requiring, for passage, that a rule has at least the fraction β support. (With n voters, the associated q-rule is $q = \beta n$.) The new issue is to determine when Ostrogorski problems can plague the majority party if it has α-support over all issues and the passage of a measure requires a β rule.

Theorem 7 *Suppose a majority party has α-support over all issues. The party can lose in a majority of the issues if the β rule satisfies*

$$\frac{2-\alpha}{2} \geq \beta. \tag{8}$$

If the majority party has a sufficient number of members, then Eq. 8 is a necessary and sufficient condition for this kind of Ostrogorski paradox.

So if α represents the majority (i.e., $\alpha > \frac{1}{2}$), then the specified problem occurs only for $\beta < \frac{3}{4}$ rules, which captures the spirit of Warner's result and strengthens it by imposing a particular rule (i.e., *both* $\alpha > \frac{1}{2}, \beta \geq \frac{3}{4}$). If a β rule determines outcomes *and* a majority party's support for an issue (so $\alpha = \beta$), the problem can exist for any $\beta < \frac{2}{3}$; a way to avoid the problem is to require $\alpha = \beta = \frac{2}{3}$.

Proof. Again let $n = 2m - 1$; the most extreme case is with m voters in the majority party and $m - 1$ in the minority. As αm of voters from the majority party must vote "Yes," the number of possible "No" votes is $(1 - \alpha)m$ from the majority party and $m - 1$ from the minority for a total of $2m - 1 - \alpha m$ votes. This negative vote is victorious if and only if the β rule satisfies $2m - 1 - \alpha m \geq \beta(2m - 1)$, or

$$\beta \leq \frac{2m - 1 - \alpha m}{2m - 1} = \frac{2 - \alpha - \frac{1}{m}}{2 - \frac{1}{m}}.$$

As the derivative of $\frac{2-\alpha-x}{2-x}$ is positive, the function has its minimum value at $x = 0$, so $\frac{2-\alpha-\frac{1}{m}}{2-\frac{1}{m}} > \frac{2-\alpha}{2}$. So, the "No" side is assured victory if $\frac{2-\alpha}{2} \geq \beta$; i.e., examples exist where the majority party loses over each of these issues. For $\beta > \frac{2-\alpha}{2}$, there exist m values where $\beta > \frac{2m-1-\alpha n}{2m-1} = \frac{2-\alpha-\frac{1}{m}}{2-\frac{1}{m}}$, so the "Nea" side will not win.

Now add $m - 1$ other issues where each majority member votes "Yes." (This allows each member to support the majority party on a majority of the issues. But as each majority party voter already supports the party on αm issues, just add k new issues where the k value ensures that each majority party voter supports the party a majority of the time; i.e., a k value where $\frac{\alpha m+k}{m+k} > \frac{1}{2}$, or any k satisfying $k \geq m - 2\alpha m + 1$.) To illustrate with the $q = n-1$ rule of Thm. 6, $\alpha m = 1$, so the smallest $k = m-2+1 = m-1$.) The majority party loses over each of the first m issues—a majority of the outcomes.

To prove that such scenarios exist, select a positive integer value for m so that αm is an integer; e.g., m is a multiple of the denominator of α. In the $m \times m$ array given by Eq. 7 select αm integers; replace each with a "Yes." Rename all other integers with a "No." Each issue (each column) has $(1-\alpha)m$ "No" votes and each of these voters (each row) has "Yes" on αm of the issues. The rest of the construction follows the above. □

9 Simpson's paradox: A shadow over the sure-thing principle

A basic theme of the last two sections is how the troubling paradoxes arise because, rather than the actual profile, the decision rule uses information from an associated set of profiles; the selected outcome is a "reasonable one" for most profiles in this set. The idea appears to extend to other paradoxical settings. For instance, Ostrogorski's paradox is only one of several compound majority paradoxes (e.g., see Nurmi 1999); e.g., Simpson's paradox can arise when dealing with standard problems such as rates of improvement, recovery, growth etc. To quote Blyth (1972, 364; for two general methods to generate these paradoxes, see Saari 1990, 2001):

> ... Savage's sure-thing principle ("if you would definitely prefer g to f, either knowing that event C obtained, or knowing that C did not obtain, then you definitely prefer g to f.") is not applicable to alternatives f and g that involve sequential operations.

Consider the following example where f and g may represent incentive schemes or experimental treatments where the fractions indicate success, efficiency, or quality (e.g. frequencies of exceeding some performance threshold):

	g	f
event C	1/3	1/4
event non-C	2/3	1/2

While this example suggests that g is, indeed, preferable to f, the following table, which raises doubt about this assertion, is consistent with the above data:

	g	f
event C	40 out of 120	10 out of 40
event non-C	10 out of 15	45 out of 90
total	50 out of 135	55 out of 130

The "total" row now makes it natural to accept the opposite conclusion that f, with its higher aggregate success rate, is preferred to g.

Formally, this paradox can be expressed as follows (Blyth 1972): Let A, B and C denote three distinct properties or predicates, such as being a victorious candidate, being a big campaign spender, supporting certain legislation, living in a given neighborhood etc. Let A', B' and C' denote the absence of A, B and C, respectively, and let

$$P(A|B) < P(A|B'); \tag{9}$$

i.e., A is more likely to occur if B does not. Simpson's paradox occurs whenever inequalities 10 and 11 also hold.

$$P(A|B \cap C) \geq P(A|B' \cap C) \tag{10}$$

$$P(A|B \cap C') \geq P(A|B' \cap C') \tag{11}$$

More dramatic examples are where these inequalities (Eqs. 9, 10, 11) involve larger margins. Blyth gives the conditions for extreme forms of Simpson's paradox. To outline of his conditions consider the comparing the effectiveness of using an innovative, rather than the standard approach to combat racial prejudice.

A = property of creating a more positive attitude about racial issues,
B = property of using the innovative educational approach,
B' = property of using the standard approach
C = educational program in district 1
C' = educational program in district 2.

The paradox associated with Eqs. 9-11, then, is that, in general, the standard approach provides better results than the innovative approach (Eq. 9), but in both districts the innovative approach is more successful (Eqs. 10 and 11)! An extreme case of the paradox (which resembles the q-rules of the last section) is where Eqs. 10 and 11 are replaced with Eqs. 12 and 13 for a choice of $\gamma \geq 1$.

$$P(A|B \cap C) \geq \gamma P(A|B' \cap C) \tag{12}$$

$$P(A|B \cap C') \geq \gamma P(A|B' \cap C') \tag{13}$$

As shown below, we could have $P(A|B) \approx 0$ and $P(A|B') \approx \frac{1}{\gamma}$, so with $\gamma = 1$, we have $P(A|B) \approx 0$ and $P(A|B') \approx 1$, which suggests an extremely strong level of success with the standard approach, yet this association is reversed in each district!

To understand Simpson's paradox, consider the decomposition of the conditional probabilities $P(A|B)$ and $P(A|B')$ in terms of C and C':

$$P(A|B) = [P(C|B)]P(A|B \cap C) + [P(C'|B)]P(A|B \cap C') \tag{14}$$

and

$$P(A|B') = [P(C|B')]P(A|B' \cap C) + [P(C'|B')]P(A|B' \cap C') \tag{15}$$

As Eqs. 14 and 15 prove, $P(A|B)$ is a weighted average of $P(A|B \cap C)$ and $P(A|B \cap C')$; $P(A|B')$ is a weighted average of $P(A|B' \cap C)$ and $P(A|B' \cap C')$. The $P(C|B)$ and $P(C'|B)$ weights, then, represent the proportions of people from the two districts who are exposed to a particular educational approach. By varying these weights; all sorts of paradoxical examples follow. Some conditions, of course, avoid the problem; e.g., if B, B', C and C' are independent, no paradox could ensue.

If the ease with which instances of Simpson's paradox can be constructed is an indicator of how often they may occur, then Saari's procedure (1990) suggests that they may be very common. To illustrate with the above example, let x and y represent, respectively, the likelihood of the innovative and standard approach having success. As both values range over $[0, 1]$, the (x, y) outcome lies in the square $\mathcal{S} = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. For instance, $(0, 0) \in \mathcal{S}$ represents where both approaches have a zero success rate, while $(\frac{1}{2}, \frac{1}{3}) \in \mathcal{S}$ is where the innovative approach enjoys a 50% success rate, but the standard one has only a 33.3% chance of success.

To create a Simpson's paradox choose two points, $X = (X_x, X_y) \in \mathcal{S}$ and $Y = (Y_x, Y_y) \in \mathcal{S}$, each of which is located below the $x = y$ line connecting (0,0) and (1,1); i.e., these points indicate that the innovative approach had a greater likelihood of success. These points represent what happens in each district; i.e., expressons 10, 11.

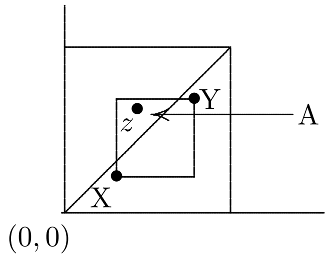


Figure 3. Generating Simpson's paradoxes

Form a rectangle by drawing lines parallel to the coordinate axes through X and Y as in Fig. 3. A portion, denoted by A, of the area of the rectangle spanned by X and Y is located above the $x = y$ line; for these points, the standard approach is more likely to succeed. To create a paradox, select any $z = (z_x, z_y) \in A$ to represent the Eq. 9 outcome of what happens in general.

We now must select appropriate weights for the weighted averages of Eqs. 14, 15. This is immediate; let

$$s = \frac{(z_x - X_x)}{(Y_x - X_x)}, \quad t = \frac{(z_y - X_y)}{(Y_y - X_y)}.$$

To explain these values by using the s term, notice that $X_x \leq z_x \leq Y_x$, so s merely determines the z_x location from X_x with the $X_x - Y_x$ scale. In other words,

$$z_x = sX_x + (1 - s)Y_x, \quad z_y = tX_y + (1 - t)Y_y. \tag{16}$$

Thus the $s, 1 - s, t, 1 - t$ values determine the appropriate Eqs. 10, 11 weights.

This construction holds for any $z \in A$, so, in spirit, Simpson's paradox is related to the Ostrogorski paradox. This is because A is in the associated set of Eq. 9 outcomes defined by specific Eqs. 10, 11 values. Again, notice that most outcomes (i.e., those *not* in A) are consistent.

By selecting different z values, we can explore the extremes of Simpson's paradox. For instance, choosing X near $(0,0)$ and Y near $(1,1)$, both below the $x = y$ line, represents nearly zero success for either approach in one district, but near certainty in the other. The z value now can be selected almost anywhere in $\mathcal{S}$, meaning that anything can happen in general. Selecting z near the $(0,1)$ vertex, for instance, corresponds to the earlier $P(A|B) \approx 0$ and $P(A|B') \approx 1$ assertion.

These are the basic outlines of Saari's procedure. It is based on geometrical properties of cones and, in particular, on the fact that cones can be used to represent a wide class of decision situations (for details, see Saari (1990, 2001)). Saari's paradox machine[6] begins with two sub-population distributions located on the same side of the $x = y$ line. The closer the points representing those sub-populations are to (0,0) and (1,1), respectively, the more freedom one has in generating instances of Simpson's paradox. Comparing this procedure with what was said above in the context of Blyth's analysis of Simpson's paradox, we see that moving the selected point z is tantamount to manipulating the weights $P(C|B)$, $P(C|B')$, $P(C'|B)$ and $P(C'|B')$ weights.

10 Conclusion

By establishing a connection between standard concerns in spatial voting, such as the existence of a core and the McKelvey "chaos theorem," new insights are obtained about long standing issues coming from the Ostrogorski and Anscombe paradoxes. We learn, for instance, that these paradoxes are discrete versions of the chaos theorem, that Kelly's conjecture about no Condorcet winner not only is true, but it is a version of the spatial chaos theorem, that while the Ostrogorski paradox is a special case of an Anscombe paradox, with the same ideal points a change in the parties converts an Anscombe paradox into an Ostrogorski one, and that the reason a standard Ostrogorski paradox cannot arise with certain number of issues is primarily a matter of asserting whether a (X, Y) outcome represents a tie or support for a particular party.

Many distinguished researchers in this area have noted connections between Condorcet's voting problems and Ostrogorski's paradoxes (a partial list includes Bezembinder and Van Acker 1985, Kelly 1989, Laffond and Lainé 2006); we create a stronger connection by proving that they are essentially equivalent; with a renaming of the alternatives, one setting can be converted into the other. Then, by establishing that both the Condorcet cycles and Ostrogorski's problems arise with precisely the same kind of profile configurations, it becomes possible to relate q-rule voting problems to supermajority Ostrogorski voting paradoxes showing, for instance, that the Ostrogorski difficulty can occur even with a rule that is one vote shy of unanimity.

A reason that the identified profiles cause these problems is that with issue-by-issue voting, or paired comparisons, the outcomes are not determined by a particular profile,

[6] Nurmi coined this term in a previous paper.

but rather by an associated set of profiles. The actual outcome is one that is reasonable for a wide selection of profiles in this associated set, but it can be paradoxical for the actual profile. This is not a problem restricted to paired comparisons and Ostrogorski; as shown, it also is the cause of the statistical Simpson paradox.

What makes the Ostrogorski problem intriguing is its path dependency characteristic where reaching a decision in one manner differs from the conclusion when making the decision in a different manner. Which is correct? By understanding the kinds of profiles totally responsible for the Ostrogorski paradox, we now know that the pairwise vote strips away any connections–intended or otherwise–among the issues! (See Chap. 2 of Saari 2008 for more discussion). If the issues are, indeed, divorced from one another, the issue by issue approach probably is better. But should the issues be related, such as where the issue of health care may be modified to adjust the concerns from the issue of governmental deficits, then issues-by-issue voting defeats the purpose. In other words, pragmatics dictates that in the real world, Ostrogorski's paradox most surely will be with us for a long time.

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