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Behavioral Assumptions for a Class of Utility Theories: A Program of Experiments

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Abstract:

An outline is given of the behavioral properties (axioms) that have been proposed, and to some extent empirically evaluated, that involve uncertain (often risky) alternatives, the joint receipt of alternatives, and possible linking properties. Recent theoretical work has established the existence of three distinct types of people, and so evaluations of theories must take respondent type into account. A program of experiments is sketched making clear exactly which empirical studies need to be repeated with respondents partitioned by type.

Key words: gambles, joint receipt, p-additive representation, risk types, uncertain alternatives

JEL classification: C 91, D 46, D 81

The term "behavioral economics" or "experimental economics" seems to have been mostly used for experimental studies of various models of economic interactions. But in my view it should also include theories of individual behavior that have been evolving for more or less realistic economic situations – the type of axiomatic theorizing that leads to forms of utility representations.

During the past 20 years, I have been fairly deeply involved in trying to understand how two distinct economic structures inter-relate and the sorts of numerical representations that, as a result, can arise. The one structure has to do with consequences attached to uncertain events (often called "gambles") or to risky (i.e., with known probability of events occurring) gambles (often called "lotteries"). This is certainly the mainstay of anything that purports to be a theory of utility for uncertain or risky situations. The second structure has to do with the concatenation of consequences which is called the joint receipt of valued items. This riskless aspect of utility seems to have been functionally declared to be outside the scope of utility theory. Nonetheless, a few of us think that it is inherent to economic situations and that by including it we gain considerable richness of structure that can be effectively exploited.

1 The Plan of the Article

My purpose here is to formulate in one place the key behavioral assumptions (axioms) that theorists have proposed and that might be evaluated empirically by others in, first, the (binary) case $n = 2$, and then in the general case of n which is approached recursively starting with the binary results. The proposed experimental program is quite large. Because I am no longer an active teacher and no longer supervise graduate students, this means that I also do not supervise a suitable laboratory. Thus, I am unable to supervise the execution of the needed experiments. I hope this article may stimulate others to do so.

In many ways, the behavioral science that I have mostly pursued is intellectually far closer to the types of physics from the 16-19 centuries – the study of macroscopic phenomena and the discovery of the laws of mechanics, motion, thermodynamics, hydro-dynamics, electromagnetism, relativity, etc. – than it is to the "opening of black boxes" – atomic structure, quantum physics, much of planetary theory, plate tectonics, geology, etc. typical of the 20th and first decade of the 21st centuries. What we currently seem able to do is attempt to discover the behavioral axioms that form compact summaries of behavioral regularities, some of which are invariances. And sometimes these axioms formulate enough structure to be able to derive numerical representations of them. A well known example concerns the axioms giving rise to the subjective expected utility (SEU) representation (originated by Savage, 1954). Of course, I fully realize that huge, very expensive efforts are being made, often involving computational brain models that are loosely tied to imaging data and intended to open the black box that is the human brain, but those "internal networks" never seem to become firmly agreed upon. Important though it may be, such reverse engineering is inherently very, very difficult, as modern physicists and engineers are very much aware.

It is important to realize that utility representations, such as SEU, no more reside in the mind of the decision maker (DM) nor is used by the DM to make choices than the partial differential equations and their solutions of classical physics can be found expllicitly lurking within the objects whose behavior they characterize. Both representation are creatures of scientific attempts to summarize compactly the behavior, thereby making it more convenient to derive predictions from the collection of behavioral "laws."

That is exactly why we bother to work out the representations: to discover behavioral implications of the underlying assumptions (ultimately called "laws," should they withstand empirical testing). We will see a vivid example of this below in §§ 2.2.3-2.2.4 Further, a great deal of the empirical research that the psychologist Michael H. Birnbaum has used to attack the class of rank-dependent models (including the famed cumulative prospect theory (CPT) of Tversky & Kahneman, 1992) is of that character. Of course, aficionados of CPT simply dismiss or ignore Birnbaum's findings. For a summary of his results until 2004 with detailed references to his relevant publications, see Marley & Luce (2005).

My aim here is to propose an empirical program having two distinct parts. The first is designed to evaluate several interrelated putative *binary* "laws" in a fashion that should pinpoint which, if any, assumptions appear to be wrong. Once the binary case is clarified, then we turn to some possible generalizations to gambles with $n > 2$ branches. These are based on inductive axioms that simply have not yet been evaluated empirically. One quite novel feature of the program is that the binary results lead to a classification of people into 3 types, and the inductions must take into account which type it is attempting to model. This, of course, complicates the experimental program considerably.

Neither formal theorems nor proofs are stated here.

2 Underlying Empirical Structures

2.1 Primitives

2.1.1 Consequences and preference order

Suppose that X is the (rich) set of valued, pure consequences under consideration. By "pure" I mean viewed as certain = riskless. A special, but important, example is money. But X can include far more substantive consequences than that, e.g., goods at stores when they are viewed as riskless, etc.

We postulate that the DM exhibits preferences over consequences that, for pairs, can be summarized as a *weak (preference) order* $\succsim$ over X. As usual, $x \sim y$ means that both $x \succsim y$ and $y \succsim x$ hold. Because the indifference relation $\sim$ is an equivalence relation, we are in reality working with equivalence classes. The key property of $\succsim$ is transitivity:

$$\text{If } x \succsim y \text{ and } y \succsim z, \text{ then } x \succsim z. \tag{1}$$

In principle transitivity can be studied empirically, but in practice some fairly troublesome statistical issues have been encountered in making this evaluation. Some data analyses proved to be in error, but the current consensus is we need not reject it. The details until 1999 are found in Luce (2000, pp. 37-39). Crucially important were analyses of Iverson and Falmagne (see, e.g., 1985), Regenwetter, & Davis-Stober (2008), and Regenwetter, Dana, Davis-Stober (2009).

2.2 Joint receipts (JR)

Suppose that $x, y \in X$, then let $x \oplus y \in X$ mean having both x and y. Thus $\oplus$ is a binary operation (of *joint receipt* often abbreviated JR) on X. Examples of joint receipt are ubiquitous – anytime one shops for two (or more) things, that purchase results in a joint receipt of goods. We assume that $\langle X, e, \oplus, \sim \rangle$ satisfies the usual axioms of what is technically called an abelian (weakly commutative) group with identity e. In particular, in addition to $\sim$ being an equivalence relation, the assumptions are that for all $x, y, z \in X$, there is an element $e \in X$ that is an *identity*:

$$x \oplus e \sim x, \tag{2}$$

that *commutativity* holds:

$$x \oplus y \sim y \oplus x \tag{3}$$

and that *associativity* holds:

$$(x \oplus y) \oplus z \sim x \oplus (y \oplus z). \tag{4}$$

The binary definition of $\oplus$ extends to any finite number of goods because we have assumed that $\oplus$ is associative. In the psychophysical context and for some interpretations of $\oplus$, failures of (3) have been found (e.g., the time-order error in audition). I know of nothing comparable in utility theory.

If $x \in X$ and $x \succsim e$, it is called a *gain*. If $x \precsim e$, it is called a *loss*. Clearly joint receipt of two gains is a gain and of two losses is a loss. Joint receipt of a gain and a loss can be seen either as a gain or a loss. As we shall see, the mixed case is an ever present complication in the theory.

In principal, the above axioms can be experimentally evaluated although I am unaware of such a study. For associativity, presumably one would determine for (4) certainty equivalents, i.e., the pure consequence indifferent to the more complex object, for each side

$$u \sim (x \oplus y) \oplus z, \quad u' \sim x \oplus (y \oplus z), \tag{5}$$

and then ask whether (statistically[1])

$$u \approx u'. \tag{6}$$

[1] This is quite a tricky issue that has received some attention in psychophysics, but hardly an accepted solution.

A similar test of commutativity is also possible. Two psychophysical examples are auditory pure tones of different intensities to the two ears and a similar visual one with light patches of different intensities to the two eyes. Such tests have been conducted by Steingrimsson and Luce (2005) and by Steingrimsson (2009). If we denote the respective stimuli (x, y) and (y, x) and match them to (u, u) and (u', u') the question is whether or not (6) holds within the accuracy of the data. References in either of these articles gives the theoretical background.

This operation $\oplus$ will play a role analogous to the concatenation operations of elementary physics, e.g., two masses on a pan balance, two rods abutted, etc.

2.2.1 Hölder's axioms

We assume that $\langle X, e, \oplus, \succsim \rangle$ on equivalence classes is a solvable, Archimedean ordered, abelian group with an isomorphism onto the additive real numbers. For the equivalence classes, the operation $\oplus$ is closed, has an identity, is commutative and associative, and each element has an inverse satisfying the usual axioms of a solvable, Archimedean ordered, abelian group (Hölder, 1901; Krantz, Luce, Suppes, & Tversky, 1971, Chapters 2 and 3).

The key testable assumption beyond transitivity is *monotonicity:* For all $x, y, z \in X$,

$$x \succsim y \Longleftrightarrow x \oplus z \succsim y \oplus z. \tag{7}$$

The literature up to 1999 is discussed in Luce (2000, pp. 137-139, 237-238) and it seems favorable toward monotonicity of joint receipt. For a later and very general discussion of testing many of the axioms I will mention, see the important article by Karabatsos (2005).

2.2.2 p-Additive representations

Classically, and certainly in Hölder's theorem as usually formulated, the only representations that are studied are mappings into the real numbers denoted $\mathbb{R}$ (and sometime into the non-negative real numbers denoted $\mathbb{R}^+$) under just addition, i.e., into the structure $\langle \mathbb{R}, \geq, + \rangle$. But recall that the typical theories for uncertain alternatives, such as SEU and CPT, have representations that involved both addition $+$ and multiplication $\times$. So, why not admit the possibility that the representations of $\oplus$ are onto suitable (defined below) subintervals of $\langle \mathbb{R}, \geq, +, \times \rangle$ that are closed under both addition and multiplication?

Admitting that possibility, under the usual Hölder assumptions, the possible polynomial representations are of the form:

$$U(x \oplus y) = U(x) + U(y) + \delta U(x)U(y), \quad \delta = -1, 0, 1, \tag{8}$$

where U is order preserving and $U(e) = 0$. These are called p-additive because they are the only polynomial forms with $U(e) = 0$ that transform into addition. We do not yet have a proof that, under plausible smoothness conditions, (8) are the only representations.

2.2.3 Three types of people

Corresponding to the value of δ, there are 3 classes or types of people. As we shall see, these types are strikingly different and so I believe that our proposed experiments should be partitioned accordingly.

When $\delta = 0$, the representation is purely additive

$$U(x \oplus y) = U(x) + U(y), \tag{9}$$

which is certainly the type that has been studied the most for 60 years. As Karabatsos (2005) shows, additivity is not well sustained in general.

When $\delta \neq 0$, we may rewrite (8) in terms of U as

$$1 + \delta U(x \oplus y) = [1 + \delta U(x)]\,[1 + \delta U(y)]\,, \tag{10}$$

which means there is a representation in terms of the transformation

$$V(x) := 1 + \delta U(x), \tag{11}$$

which satisfies the multiplicative property

$$V(x \oplus y) = V(x)V(y). \tag{12}$$

Note that no scale factor $\alpha \neq 1$ maintains this multiplicative representation, (12). In general, however, such a multiplicative representation is unique only up to an arbitrary power, $V \to V^{\beta}$, $\beta > 1$, but as we shall see even that degree of freedom is lost. Of course, from (12) it is immediate that $\ln V$ is an additive ratio scale.

2.2.4 Two scale types: ratio and absolute

For those people satisfying (9), the utility function is a ratio scale, i.e., it is unique up to its unit. But for the other two types, $\delta \neq 0$, the fact that in (11) U is either added to or subtracted from 1, means that U must be an absolute scale. Of course V itself is unique up to positive powers and so $\ln V$ is an additive ratio scale. The V scale maps onto $]0, \infty[$ in both non-zero cases. The difference being that V is order preserving for $\delta = 1$ and order reversing for $\delta = -1$. Put in words, U in the case $\delta = 0$ has a free unit, as say with mass, whereas for the cases with $\delta \neq 0$ there is no freedom in the choice of unit, as with probability. These differences are very significant as we see below.

2.2.5 An experimental-procedural implication

The fact that there are 3 types of people corresponding to $\delta = -1, 0, 1$, which as we shall see are qualitatively quite different, should be significant for the experimenter. It means that each participant in the study must be evaluated for type before any further data are examined. A criterion for deciding that is provided in Section 3 below. Moreover, that fact certainly means that group averages, including medians and comparisons of distributions, over an un-screened population are *totally* meaningless. Although averaging is a bit more justified over people of the same type, the fact that utility functions have the same (non-linear) form but with different parameters (see § 3.4) means that here too averaging of raw data is not really justified. This has received some attention in other areas, such as psychophysics, and upshot is that only averages of linear functions are meaningful with the average slope parameter equal to the average of the individual parameters. Thus, in the present case, one must transform the data to linear form as outlined in § 3.4.

Although the Hölder axioms clearly have interesting consequences, there is troubling empirical evidence (e.g., Luce, 2000; Sneddon and Luce, 2001) that some properties seem to hold for gains ($x \succsim e$) and, separately, for losses ($x \precsim e$) but not very well for gambles involving mixed gains and losses. Schneider and Lopes (1986) clearly ran into the same issues with gambling behavior. Whether or not this reflects the fact that a substantial number of respondents are of types $\delta \neq 0$ has not yet been carefully explored; it seems to me to be very important to do so.

2.3 Uncertain alternatives – gambles

2.3.1 Uncertain alternatives

We assume there are a great many families of chance "experiments," i.e., sources of uncertainty. We denote by C, D, etc. typical disjoint chance events arising within a family of chance "experiments." A general uncertain alternative – often, although somewhat misleadingly, called a *gamble*, which I use – begins with a partition of the underlying chance event into n subevents, and to each is assigned a consequence, either an element of X or a first-order gamble.

A *binary* uncertain alternative is a gamble with $n = 2$. We may think of it as an experiment whose "universal set" Ω may be partitioned into C and $\overline{C} := \Omega \backslash C$. Each sub-event leads to its own consequence, so that the gamble has two chance branches (x, C) and $(y, \overline{C})$. We write it as $(x, C; y, \overline{C})$.

Two properties of binary gambles that have received some attention are *event commutativity*

$$\left((x, C; y, \overline{C}), D; y, \overline{D}\right) \sim \left((x, D; y, \overline{D}), C; y, \overline{C}\right), \tag{13}$$

and *right autodistributivity*

$$\left((x, C'; y, \overline{C'}), C; z, \overline{C}\right) \sim \left((x, C'; z, \overline{C'}), C; (y, C''; z, \overline{C'}), \overline{C}\right), \tag{14}$$

where the primes refer to independent realizations of the underlying chance experiment. The reason they are of some interest is made clear in Section 2.5.

2.3.2 Unitary gambles and separable representations

One important subclass of binary gambles has $y = e$; these are called *unitary*. One axiomatic issue turns out to be whether such gambles have a multiplicative conjoint representation:

$$U^*(x, C; e, \overline{C}) = U^*(x) S^*_\Omega(C), \tag{15}$$

which is called a *separable representation.*

The necessary axioms are well known (Krantz, Luce, Suppes, & Tversky, 1971, Ch. 6) to be: transitivity of $\succsim$, *monotonicity*[2], i.e., if for non-null events $C, D, E \in \Omega$,

$$(x, C; e, \overline{C}) \succsim (y, C; e, \overline{C}) \Longleftrightarrow (x, D; e, \overline{D}) \succsim (y, D; e, \overline{D}), \tag{16}$$

and the *Thomsen condition*:

$$\begin{aligned}(x, E; e, \overline{E}) \quad &\sim \quad (z, D; e, \overline{D}) \;\; \& \;\; (z, C; e, \overline{C}) \sim (y, E; e, \overline{E}) \\ &\Longrightarrow \quad (x, C; e, \overline{C}) \sim (y, D; e, \overline{D}).\end{aligned} \tag{17}$$

The monotonicity assumption has received some empirical study. For example, von Winterfeldt, Chung, Luce, & Cho (1997) ran a test of it based on medians and concluded that the evidence for it was mixed. However, Ho, Regenwetter, Neiderée, and Heyer (2005) carried out a far more complete statistical analysis, which was based on the entire distribution of responses, which provided strong support for monotonicity. The second axiom for separable representations, the Thomsen condition, (17), Karabatsos (2005) reanalyzed the then unpublished memory data of W. H. Batchelder and Jarad Smith using Bayesian methods, and the Thomsen condition was well sustained. There is no question, however, that data on unitary gambles, as such, need to be collected and analyzed in a comparable way.

2.4 Two possible linking laws

At this point, we have two distinct measures of utility: the function U from the JR concatenation structure and U^* from the separable representation of unitary gambles. One hopes that some sort of behavioral law exists that forces them to be the same measure in the sense that there is a unique $\gamma > 0$ such that $U = (U^*)^\gamma$. Luce (2000, Theorem 4.4.6) proved that the following behavioral condition suffices. The key mathematical result was proved by efforts of Luce (1996) and Aczél, Luce, and Maksa (1996).

2.4.1 Segregation

Segregation holds if for $x, y \in X$ with $x \succsim y$ and non-null event C,

$$(x \oplus y, C; y, \overline{C}) \sim (x, C; e, \overline{C}) \oplus y. \tag{18}$$

Clearly this is empirically testable, and it has been (see discussion below) because of its crucial theoretical importance. To date, the empirical results have been ambiguous (Karabatsos, 2005). Whether this would be clarified by taking respondent type into account remains to be seen.

[2] Actually called *independence* in the measurement literature.

2.4.2 Duplex decomposition

An alternative linking law has been proposed and studied quite a bit both theoretically and experimentally (more details below). As we will soon see, much more needs to be done.

Gambles are said to satisfy *duplex decomposition* (DD) *over* X if for all $x, y \in X$,

$$(x, C; y, \overline{C}) \oplus (e, C'; e, \overline{C'}) \sim (x, C; e, \overline{C}) \oplus (e, C'; y, \overline{C'}), \tag{19}$$

where $(C', \overline{C'})$ is an independent realization of $(C, \overline{C})$. Duplex decomposition is formulated in a way that that may seem unusual but for the fact that it has a certain symmetry in that the chance experiments invovled are the same on both sides of $\sim$. That has proved to be critical in some work on the utility of gambling (Luce et al., 2008a,b; Ng et al.,2009a,b) where we have avoided assuming *idempotence*, $(e, C'; e, \overline{C}') \sim e$, and have taken $(e, C'; e, \overline{C}')$ to be the qualitative structure that underlies the utility of gambling. Of course, we should attempt to verify, in the context of various judgments about broader class of gambles, whether or not idempotence fails.

Unlike segregation, DD is rather non-rational: on the left one gets either x or y but not both whereas on the right there a four possible consequences: $e \sim e \oplus e$, $x \sim x \oplus e$, $y \sim e \oplus y$, $x \oplus y$.

Although DD has been closely looked at empirically (see Luce, 2000, § 6.2), the most recent empirical study of it and segregation is Cho, Luce, & Truong (2002). Although Karabatsos (2005) claims good support for DD, in the light of the theory motivating this article, those results are inherently ambiguous, Most important, the respondents should be classified as to type, as described in § 3, when evaluating fits to a property. Also, all tests to date assumed idempotence, which property, as I said earlier, needs to be checked.

2.5 The binary representation

If $x \succsim y$, $\overline{C} = \Omega/C$, and the above assumptions hold under segregation, the binary utility representation is

$$U(x, C; y, \overline{C}) = U(x)S_\Omega(C) + U(y)S_\Omega(\overline{C}), \tag{20}$$

(for $\delta = 0$, Luce et al., 2008a, Eq. (28); for $\delta \neq 0$, Ng et al., 2009b, Eq. (39)). Under duplex decomposition and idempotence

$$U(x, C; y, \overline{C}) = U(x)S_\Omega(C) + U(y)\left[1 - S_\Omega(C)\right]. \tag{21}$$

(Luce et al., 2008a, Eq. (60)), whereas for $\delta \neq 0$

$$U(x, C; y, \overline{C}) = U(x)S_\Omega(C) + U(y)S_\Omega(\overline{C}) + \delta U(x)U(y)S_\Omega(C)S_\Omega(\overline{C}). \tag{22a}$$

(Ng et al., 2009b, Eq. (47)).

These representations feed into the general theory via recursive assumptions, § 4.

It is a simple calculation to show that event commutativity, (13), follows from the rank-dependent form (21) but not from (20) unless the weights are finitely additive:

$$S_\Omega(\overline{C}) = 1 - S_\Omega(C),$$

which is the classic *subjective expected utility* representation. Further, right autodistributivity, (14), holds only in that case.

Chung, von Winterfeldt, and Luce (1994) evaluated event commutativity and concluded that for the most part it was sustained. Brothers (1990) in his dissertation explored right autodistributivity and found it not consistent with the data, which is consistent with the many studies that showed the inadequacy of subjective expected utility.

3 Empirical Classification of Individuals

3.1 Events having subjective probability 1/2

If there exist consequences $x \succ y$ and non-empty event C such that

$$(x, C; y, \overline{C}) \sim (x, \overline{C}; y, C). \tag{23}$$

then from either (20) or (21),

$$S_\Omega(C) = S_\Omega(\overline{C}) = \frac{1}{2}. \tag{24}$$

Such event partitions are called "equally likely."

3.2 A criterion

For such events, the following criterion is established by Luce (2009): For all $x, y \in X$, with $x \succ x' \succ y \succ y'$, and for all non-trivial $C \subset \Omega$,

$$\delta = \left\{ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right\} \Longleftrightarrow (x \oplus x', C; y \oplus y', \overline{C}) \left\{ \begin{array}{c} \succ \\ \sim \\ \prec \end{array} \right\} (x \oplus y, C; x' \oplus y', \overline{C}). \tag{25}$$

This is less formidable than it may seem. Luce (2009) mentioned the following examples:

$$\left(160, \frac{1}{2}; 50, \frac{1}{2}\right) \quad \text{versus} \quad \left(140, \frac{1}{2}; 70, \frac{1}{2}\right)$$
$$\left(190, \frac{1}{2}; 30, \frac{1}{2}\right) \quad \text{versus} \quad \left(120, \frac{1}{2}; 100, \frac{1}{2}\right)$$
$$\left(1900, \frac{1}{2}; 60, \frac{1}{2}\right) \quad \text{versus} \quad \left(1050, \frac{1}{2}; 910, \frac{1}{2}\right).$$

For example, the first case arises when $x = 100$, $x' = 60$, $y = 40$, $y' = 10$. Notice each gamble of a row has the same (subjective) expectation but differ in their (subjective) variance, the one on the left being more risky than the safer one on the right.

Ng, Luce, Marley (2009b) have shown that the co-domains (images) of U are

$$\left\{ \begin{array}{c}]-1,\infty[\\ ]-\infty,\infty[\\ ]-\infty,1[\end{array} \right\} \Longleftrightarrow \delta = \left\{ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right\}. \tag{26}$$

This fact together with the criterion makes clear that the $\delta = 0$ people are oblivious to the (subjective) variance difference and so may be called *risk neutral*; those for whom $\delta = -1$ are by contrast like the safe gamble and so are called *risk averse*; and those for whom $\delta = 1$ prefer the riskier gambles and so are called *risk seeking*. Such concepts have repeatedly appeared informally in discussions of risk taking behavior (e.g., Tversky & Kahneman, 1992; Schneider & Lopes, 1986) with criteria that are special cases of (25), which itself is an "if and only if" form. They typically pit a gamble against a pure consequence equivalent to the expected value as the gamble. This criterion is the special case of (25) where $x \oplus y \sim x' \oplus y'$ and idempotence holds.

3.3 Arrow-Pratt measure of local risk

Other authors, mostly economists, have characterized the type of risk attitude as embodied in properties of utility functions for money, not as a property of people themselves. Unlike my and some other psychologists work, the forms for utility functions are usually ad hoc but with apparently desired properties. Moreover, the domain is usually assumed to represent total wealth, not increments in it. So the independent variable can never be negative. That is not true of my work.

The most famous measure, now called the Arrow-Pratt measure of *local risk aversion*, is

$$r(x) = -\frac{u''(x)}{u'(x)}. \tag{27}$$

(Pratt, 1964; Arrow, 1965, 1974). An excellent summary up to 1975 of both theory and applications centered on (27) is Keeney & Raiffa (1976/1993)[3]. Later

[3] A. A. J. Marley reminded me of this reference. The 1993 edition is mainly an update of applications.

discussions of this and related measures are extensive and include Holt & Laury, 2002; Levy & Levi, 2002; Ross, 1981; and references cited in these articles.

In Sec. 3.4 the utility forms derived from my theory and the resulting $r(x)$ for them are stated.

As mentioned, Karabatsos (2005) reanalysis of other authors' data makes clear that $\delta = 0$ is not sustained over the sample of people studied. So, a major empirical problem is to collect data from a "sample" of people not restricted to students and academics to see, first, if each person is consistent in choosing either the risky over the safe, the safe over the risky, or is indifferent between the two. Assuming that people are consistent, then what proportions of each type seem to be found? And is there any substantial correlation with social role. Judging by informal "data" collected at several of my lectures, such academics tended to be type $\delta = -1$, i.e., risk averse with a much smaller fraction of risk takers.

Presumably, type may vary over different domains of activity. A person may have one risk attitude toward financial matters and a different one about such risky sports as mountain climbing and skiing.

A more general criterion that does not depend upon finding an event satisfying (23), due to C. T. Ng, is also reported in Luce (2009). Although more general, it is not nearly as transparent as is (25).

An important empirical issue is whether there are any people who are of type $\delta = 0$? If not, then all the earlier theories of utility theory simply cannot be descriptive. This is true no matter the form of the utility function which, as we see next, is linear with money when (30) holds. This would be a pretty shocking discovery, as is made clear in Regenwetter et al. (2008, 2009)

3.4 Utility of money

Under his assumptions, Luce (2009) showed that there must be an increasing function $g : X \longrightarrow \mathbb{R}$ and a constant $\alpha > 0$ such that

$$U(x) = \left\{ \begin{array}{l} e^{\alpha g(x)} - 1 \\ \alpha g(x) \\ 1 - e^{-\alpha g(x)} \end{array} \right\} \Longleftrightarrow \delta = \left\{ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right\}. \tag{28}$$

So the Arrow-Pratt measure (27) is easily calculated to be

$$r(x) = \left\{ \begin{array}{l} -g''(x) - \alpha g'(x) \\ -\frac{g''(x)}{g'(x)} \\ -g''(x) + \alpha g'(x) \end{array} \right\} \Longleftrightarrow \delta = \left\{ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right\} \tag{29}$$

For money amounts, it seems most plausible that

$$x \oplus y \sim x + y, \tag{30}$$

from which it follows that $g(x) = x$. If so, then from (29)

$$r(x) = \left\{ \begin{array}{l} -\alpha \\ 0 \\ \alpha \end{array} \right\} \Longleftrightarrow \delta = \left\{ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right\}, \tag{31}$$

which, of course, is constant risk aversion or, for $\delta = 1$, risk seeking (Keeney & Raiffa, 1976, explored (28) with $g(x) = x$ and without the ± 1 term, which does not affect $r(x)$).

Notice that

1. For $\delta = 1$ as $x \to -\infty$, then $U(x) = e^{\alpha x} - 1 \to -1$, and
2. For $\delta = -1$ as $x \to \infty$, then $U(x) = 1 - e^{-\alpha x} \to 1$.

The empirical issues of estimating these asymptotes and the parameter α have yet to be tackled systematically.

Given estimates of U, one acceptable way[4] to average over people of the same type is, for $\delta = 1$ to average $\ln(1 + U(x))$ and for $\delta = -1$ to average $-\ln(1 - U(x))$ vs. x.

4 General Gambles

So far we have discussed only binary gambles, but clearly a satisfactory utility theory must deal with gambles having $n > 2$ branches as well. So far, this seems to have best been done via recursive forms.

Let a general gamble, where the consequences are ranked from best to worst, be denoted

$$g_{[n]} := (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n). \tag{32}$$

Branching is the recursion in which the first two branches are combined into a single first-order gamble $g_{[2]} := (x_1, C_1; x_2, C_2)$, i.e.,

$$\begin{aligned} g_{[n]} &\sim (g_{[2]}, C_1 \cup C_2; x_3, C_3; ...; x_n, C_n) \\ &= ((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; ...; x_n, C_n). \end{aligned} \tag{33}$$

Upper gamble decomposition (UGD) is the recursion that treats as indifferent a binary gamble consisting of the branch with the best consequence and all of the remaining branches combined as a single first-order gamble, i.e.,

$$g_{[n]} \sim \left(x_1, C_1; g_{[n],-1}, \Omega \backslash C_1\right) \tag{34}$$

where $g_{[n],-1} := (x_2, C_2; ...; x_i, C_i; ...; x_n, C_n)$.

Some of the theoretical implications of the representations resulting from these properties are explored in Luce et al. (2008a,b) and Ng et al. (2009a) for the additive case $\delta = 0$ and in Ng et al. (2009b) for the cases $\delta \neq 0$. Idempotence was not assumed, but if it is the results described here follow readily. To my knowledge, no experiments have been run on either branching or UGD. There are many opportunities for research here, both empirical and, if these two recursions are shown to be incorrect, then theorist much seek alternative recursions.

[4] It is well known that for linear functions, and only for them, the average function is also linear. Further, the slope of the average is the average of the individual slopes.

5 Experimental Program

For the experiments listed in the first subsection, type does not matter and so these studies need not be rerun. The several experiments discussed in the remainder of the sections were run without any awareness of the classification of people into 3 types. So, if that distinction is important to a property, it must be rerun taking that into account.

5.1 Tests independent of type

Of course, even when type does not directly matter, we must be most cautious about averaging the data from several individuals because there are usually parametric differences as, e.g., in the utility of functions.

- Transitivity of $\succsim$, (1), appears well supported empirically at this point and is not in urgent need of further evaluation.
- Commutativity, (3), and Associativity, (4), of JR have not been directly evaluated, but few doubt that they hold. Indeed, if money satisfies (30), then they must hold by virtue of properties of arithmetic.
- Monotonicity (or independence), (7), of JR was explored by Cho and Fisher (2000) and was mostly sustained.
- The Thomsen condition, (17), has not, to my knowledge, been directly evaluated using unitary gambles. Because practically every theory that has been proposed implicitly or explicitly assumes it to be correct, it is almost certainly worthy of some empirical study.
- Consequence monotonicity, (16), of binary gambles seems first to have been explored empirically by von Winterfeldt, Chung, Luce, and Cho (1997) and reanalyzed by Ho, Regenwetter, Neiderée, and Heyer (2005), as described in § 2.3.2. They concluded that the von Winterfeldt et al. (1997) was not sufficiently thorough and that consequence monotonicity was actually very well sustained.
- Event commutativity, (13), and right autodistributivity, (14), were both discussed in Section 2.5, and there seems no need for further data collection.

5.2 Classification by type

Given the p-additive representation, (8), it is clearly critical to know the types experimental respondent are and to analyze the data separately. To do this, one should find an event with binary symmetry, (23), and determine type by the criterion (25).

- One obvious empirical question is whether or not a person is consistent in adhering to the criterion. At this time, we simply do not know – it seems to be an empirically virgin topic.

- A second thorny issue is whether the criterion is just for gains and losses separately or whether it also works for mixed gains and losses. This topic needs extensive exploration in order to guide future theory construction.

5.3 Linking for risk neutral people

The properties of segregation, (18), and duplex decomposition, (19), have proved important theoretically, and they have received a fair amount of empirical investigation (Sneddon and Luce, 2001; Cho, Luce, and Truong, 2002; Karabatsos, 2005) with somewhat mixed and confusing results. However, at the time, no one recognized the type distinction of the p-additive form, and so the data were not so partitioned. This should be done.

- Those people exhibiting $\delta = 0$ should have, according to the theory (see Table 1 of both Ng et al. 2008a,b articles) a rank-dependent representation under segregation and a linear weighted one under duplex decomposition.

- Once that is done, the recursive properties of branching, (33), and UGD, (34), need direct empirical investigation.

5.4 Linking for risk averse or risk seeking people

Ng et al. (2009b) have studied the representations that follow from segregation, (18), and duplex decomposition, (19).

- For $\delta \neq 0$, segregation leads to the rank-dependent form, which according to some data (of course, not partitioned by type), was rejected because it implies coalescing, which seems empirically wrong, at least when not partitioned. Again, this needs to be restudied with respondent type playing a role.

- With $\delta \neq 0$, duplex decomposition implies that, in essence, the weights do not vary with events, which is absurd. So, DD should fail for these 2 types.

- Should the evidence, when partitioned by types and carefully checked for experimental flaws, show systematic failures of segregation and/or duplex decomposition, then theorists will be forced to devise alternatives and work out their consequences.

5.5 Inductive conditions of branching and UGD

To my knowledge, no empirical work has ever been attempted to check directly either branching, (33), or upper gamble decomposition, (34). Given their current theoretical importance, it is important to do so with respondents partitioned by type.

5.6 Fitting utility and weighting functions to data

Considerable work has been done on fitting functions for both utility and weighting functions to appropriate data (see Luce, 2000, §§ 3.3 and 3.4 for a summary to 1999).

- Mostly this has been done under the implicit assumption of additive joint receipts ($\delta = 0$), and no attempt has been made to fit the form (28) which has been recently derived and the special case (31).

- To do so, we need to develop ways to estimate from finite sets of data the upper bound for $\delta = -1$ and lower bound for $\delta = 1$ types.

- Of the weighting functions, the generally most successful class of functions is the Prelec one for events with known probabilities

$$W(p) = \exp\left[-\beta(-\ln p)^{\alpha}\right],$$

  which includes power functions as the special case $\alpha = 1$ (e.g., Sneddon & Luce, 2001[5]). Luce (2001) presents a fairly simple axiomatic condition equivalent to a Prelec function that should be explored with some care, again partitioning by type.

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[5] The expression for the Prelec function in this article, Eq. (26), was incorrectly stated using $-p$ where it should have been p.

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