File: 08-09-10-p-add-UofG

Utility of Gambling Under p-Additive Joint Receipt and Segregation or Duplex Decomposition*

C. T. Ng[(a)], R. Duncan Luce[(b)], A. A. J. Marley[(c)]

[(a)] Department of Pure Mathematics, University of Waterloo, Waterloo, Ont., N2L 3G1 Canada
[(b)] Institute for Mathematical Behavioral Sciences, University of California, Irvine, CA, 92697-5100
[(c)] Department of Psychology, University of Victoria, Victoria B.C., V8W 3P5, Canada

Corresponding Author:
R. Duncan Luce
Institute for Mathematical Behavioral Sciences
Social Science Plaza
University of California, Irvine, CA, 92697-5100
Telephone: 949-824-6239
FAX: 949-824-3733
e-mail: rdluce@uci.edu

* The work of Luce and Marley was supported in part by National Science Foundation grant SES-0452756 to the University of California, Irvine, and by the Natural Sciences and Engineering Research Council (NSERC) of Canada Discovery Grant 8124 to the University of Victoria for Marley. That of Ng was supported in part by the NSERC of Canada Discovery Grant 8212. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Abstract: This article continues our exploration of the utility of gambling, but here under the assumption of a non-additive but p(olynomial)-additive representation of joint receipt. That assumption causes the utility of gambling term to have a multiplicative impact, rather than just an additive one, on what amounts to ordinary weighted utility. Assuming the two rational recursions known as branching and upper gamble decomposition, we investigate separately the rational property of segregation and the non-rational one of duplex decomposition. Under segregation, we show that each pair of disjoint events either exhibit weight complementarity or both support no utility of gambling, a property called UofG-singular. We develop representations for both cases. The former representation is simple weighted utility with each term multiplied by a function of the event underlying that branch. The latter representation is ordinary rank-dependent utility with Choquet weights but with no utility of gambling. Under duplex decomposition, we show that weights have the intuitively unacceptable property that there is essentially no dependence upon the events, making duplex decomposition, in this context, of little behavioral interest.

Key words: duplex decomposition, branching, p-additive utility, segregation, rank-dependent utility, gamble decomposition, UofG-singular, utility of gambling

1 The p-Additive Representation

In five articles on the utility of gambling (Luce and Marley, 2000; Luce, Ng, Marley, and Aczél, 2008a,b; Ng, Luce, and Marley, in press a,b) we have worked with a binary operator $\oplus$ over gambles that represents the idea of having or receiving both gambles. Among the properties assumed so far were that $\oplus$ has an additive representation U, i.e., (12) below with $\delta = 0$, and that separability, (19), below also holds. We pointed out that another equally possible separable representation exists, namely the so-called polynomial-additive (somewhat abbreviated as p-additive) representation, (12) below, which is equivalent to a multiplicative representation, (13), when $\delta \neq 0$. Here we begin to investigate that case.

Our results show that when the structure is p-additive and the "rational" property of segregation, (24), holds, and the two recursions, branching, (22), and upper gamble decomposition (UGD), (23), are both satisfied, then either the weights over events exhibit complementarity or an event exists that satisfies a property called UofG-singular (Definition 18). For the former the general representation takes a weighted utility form with each term modified by an event-dependent factor ((36) of Theorem 20). For the latter, the standard rank-dependent representation arises but with no utility of gambling term, (Theorem 21).

If segregation is replaced by the "non-rational" property of duplex decomposition, (45), then under both branching and UGD the weights show essentially no dependence upon the events (Proposition 24), a clearly unacceptable property for behavioral applications.

1.1 General background[1]

Let X denote a set of *pure consequences* — ones for which chance or uncertainty plays no role. The adjective "pure" is used to distinguish such consequences from those that are, in fact, gambles; see below. Let X include a distinguished element e that is interpreted as representing *no change from the status quo.*

Let $\mathfrak{B}$ be a set of events (sets) containing, with any two events, also their union and difference, thus also their intersection and the empty element $\varnothing$. In mathematical terms we say that $\mathfrak{B}$ is a Boolean ring. Let $\mathfrak{B}^* = \mathfrak{B}\backslash\{\varnothing\}$.

Let $\mathbf{E}$ be a finite "experiment " or "chance phenomenon". Let $(C_1, ..., C_n)$, $C_i \in \mathfrak{B}$, $C_i \cap C_j = \varnothing$ if $i \neq j$, denote the partition defining $\mathbf{E}$. The "universal" event for the experiment $\mathbf{E}$ is the union $\Omega = \bigcup_{i=1}^{n} C_i$. We place quotes around "universal" because the event Ω is only universal for the purpose of a particular local chance experiment $\mathbf{E}$, not in the global sense of a state space Savage (1954). Thus, we admit the possibility of many experiments each having its own Ω.

A *first-order gamble* consists of n, $1 \leq n < \infty$, (pure consequence, event) pairs, each of which is called a *branch* of the first-order gamble. Thus, replacing the parentheses of the branches by semicolons, we may write such a gamble as

$$g_{[n]} = (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n). \tag{1}$$

Whenever we use Ω along with $g_{[n]}$, it is implicit that $\Omega = \cup_{i=1}^{n} C_i$ is the universal event underlying the gamble. We sometimes suppress the subscript and write g in place of $g_{[n]}$.

A *second-order gamble* consists of (consequence, event) pairs, also called *branches*, where by a "consequence" we mean either a pure one or a first-order gamble, and at least one consequence is a first-order gamble. From here on, a *gamble* means either a pure consequence, a first-order gamble, or a second-order gamble, and the unmodified term *consequence* means either a pure consequence or a first-order gamble in a second-order gamble.

In addition we assume that a person can have the joint receipt of pure consequences and gambles, which operation is denoted by $\oplus$.

Let $\mathcal{G}$ denote the closure of all gambles under $\oplus$.

Assume that a decision maker has a *preference order* $\succsim$ over $\mathcal{G}$, which we usually denote just as $\succsim$, and that it is a weak order. As usual, $\precsim$ denotes the converse of $\succsim$ and $\sim$ denotes the corresponding indifference relation: $\sim := \succsim \cap \precsim$ and $\succ := \succsim \setminus \sim$.

The gambles $g \succsim e$ are called *gains*, and those with $g \precsim e$ are *losses*.

We will also assume that joint receipt $\oplus$ is *monotonic* in the sense that, for all gambles $f, f', g \in \mathcal{G}$,

$$f \succsim f' \Leftrightarrow f \oplus g \succsim f' \oplus g. \tag{2}$$

[1] The background given here is extracted, with only minor revisions, from Luce, Ng, Marley, and Aczél (2008a, Section 1, pages 3-4), with kind permission of Springer Science+Business Media.

This was empirically explored in Cho and Fisher (2000) and was not rejected. We assume that X is closed under $\oplus$, i.e., for any $x, y \in X$, $x \oplus y \in X$; $\oplus$ is associative and commutative; no change from the status quo, e, is an identity of $\oplus$; and for each $x \in X$, there is an inverse $x^{-1} \in X$ such that $x \oplus x^{-1} \sim e$. With $\oplus$ defined on the equivalences of $\sim$, the structure is a commutative group. Note that

$$(x \oplus y)^{-1} \sim x^{-1} \oplus y^{-1}.$$

We assume that for each gamble $g \in \mathcal{G}$, the set X is sufficiently rich that it contains an element, denoted $CE(g)$, such that $CE(g) \sim g$. This pure consequence is called a *certainty equivalent* of the gamble. Assuming gambles have certainty equivalents, $\oplus$ is an abelian group operation on the equivalent classes of $\mathcal{G}$.

We also assume the property of *certainty*:

$$(x_1, \varnothing; ...; x_{i-1}, \varnothing; x_i, \Omega; x_{i+1}, \varnothing; ...; x_n, \varnothing) \sim x_i; \tag{3}$$

and the property of *expansibility*:

$$\begin{aligned}(x_1, C_1; ...; x_{i-1}, C_{i-1}; x_i, \varnothing; x_{i+1}, C_{i+1}; ...; x_n, C_n) \\ \sim (x_1, C_1; ...; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; ...; x_n, C_n).\end{aligned} \tag{4}$$

Because of this assumption, from now on when we write a gamble, (1), or a partition $(C_1, C_2, ..., C_n)$, we assume that every event C_i is non-null.

A gamble (1) is *ranked* when the consequences are numbered in order of preference, i.e.,

$$x_1 \succsim x_2 \succsim ... \succsim x_n \tag{5}$$

and the corresponding event partition is treated as an ordered n-tuple with that induced order of indices.

We assume that any permutation of the branches yields an indifferent gamble, i.e., for any permutation π,

$$(x_1, C_1; x_2, C_2; ...; x_n, C_n) \sim (x_{\pi(1)}, C_{\pi(1)}; x_{\pi(2)}, C_{\pi(2)}; ...; x_{\pi(n)}, C_{\pi(n)}). \tag{6}$$

Ranking is essential in formulating some properties, such as co-monotonic consequence monotonicity below. When we assume the ranked form, we explicitly say so. Otherwise, we do not assume that the consequences are ranked.

1.2 Several definitions and assumptions

The first definition arises because we are dealing with ranked gambles.

Definition 1 ***Co-monotonic consequence monotonicity*** *is satisfied iff for every* $i \in \{1, 2, .., n\}$*, when* $x_i, x_i' \in X$ *have the same rank position among the other consequences, then*

$$x_i' \succsim x_i \quad \textit{iff} \quad (x_1, C_1; ...; x_i', C_i; ...; x_n, C_n) \succsim (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n) \tag{7}$$

We assume co-monotonic consequence monotonicity holds.

A function U from the domain of pure consequences and gambles to the real numbers is called a *utility function* if it is order preserving and maps the status quo element e into 0, i.e.,

$$g \succsim h \text{ iff } U(g) \geq U(h), \tag{8}$$

$$U(e) = 0. \tag{9}$$

The existence of certainty equivalents, plus the assumption of co-monotonic consequence monotonicity, justifies using the same notation U over both gambles and pure consequences. Given that $\succsim$ has the numerical representation of (8), then $\succsim$ must be a weak order. Whenever a function U occurs in the remainder of the paper it is to be interpreted as such a utility function.

The next definition arises because we *do not* assume that gambles are in general $e-$idempotent:

$$(e, C_1; e, C_2; ...; e, C_n) \sim e. \tag{10}$$

That fact invites the following two concepts:

Definition 2 *An **element of chance** is any gamble for which every consequence is no change from the status quo, e, i.e., gambles of the form $(e, C_1; e, C_2; ...; e, C_n)$.*

Definition 3 *For any gamble $g_{[n]} = (x_1, C_1; x_2, C_2; ...; x_n, C_n)$, its **kernel equivalent**, denoted $KE(g_{[n]})$, is defined to be the pure consequence solution, which we assume exists, to the following indifference*

$$g_{[n]} \sim KE(g_{[n]}) \oplus (e, C_1; e, C_2; ...; e, C_n). \tag{11}$$

1.3 p-Additive gambling structures

We next state several definitions that are close to those found in Luce (2000) but because of non-idempotency are modified slightly.

Definition 4 *A utility function $U^{(1)}$ is **p-additive (over $\oplus$)** if there exists constant δ such that for any gambles f, g*

$$U^{(1)}(f \oplus g) = U^{(1)}(f) + U^{(1)}(g) + \delta U^{(1)}(f) U^{(1)}(g). \tag{12}$$

When $\delta \neq 0$, a p-additive utility function is equivalent to

$$1 + \delta U^{(1)}(f \oplus g) = \left[1 + \delta U^{(1)}(f)\right] \left[1 + \delta U^{(1)}(g)\right]. \tag{13}$$

Thus, for $\delta \neq 0$, (11) with (13) yields

$$1 + \delta U^{(1)}(g_{[n]}) = \left[1 + \delta U^{(1)}(KE(g_{[n]}))\right] \left[1 + \delta U^{(1)}(e, C_1; e, C_2; ...; e, C_n)\right] \tag{14}$$

We introduce the alternative function

$$V^{(1)}(g) := 1 + \delta U^{(1)}(g). \tag{15}$$

Observe that $U^{(1)}$ is p-additive with $\delta \neq 0$ if and only if $V^{(1)} \neq 1$ and

$$V^{(1)}(f \oplus g) = V^{(1)}(f)V^{(1)}(g). \tag{16}$$

Next, we use the following standard terminology: For a function f from X to Y, the *domain* is X, the *codomain* is Y, and the *range* is the set of elements $y \in Y$ for which there exists $x \in X$ such that $f(x) = y$. If the range of f is equal to Y, we say that f maps X onto its codomain Y.

Proposition 5 *Suppose that $U^{(1)}$ is p-additive over $\oplus$.*

1. *If $\delta = 0$, $U^{(1)}$ is additive and the range of $U^{(1)}$ is unbounded from both above and below.*
2. *If $\delta > 0$, the range of $U^{(1)}$ is bounded from below by $-\frac{1}{\delta}$ and unbounded from above.*
3. *If $\delta < 0$, the range of $U^{(1)}$ is unbounded from below and bounded from above by $\frac{1}{|\delta|}$.*
4. *For cases 2 and 3,*

$$\widehat{U}^{(1)}(x) = sgn(\delta) \ln \left[1 + \delta U^{(1)}(x)\right] = sgn(\delta) \ln V^{(1)}(x)$$

is an additive representation of $\oplus$ with codomain $\mathbb{R}$.

All proofs are in Section 6, Appendix: Proofs.

In what follows, we assume the axioms are sufficient that, for the three cases $\delta = 0$, $\delta > 0$ and $\delta < 0$, the range of $U^{(1)}$ is, respectively, $]-\infty, \infty[$, $]-\frac{1}{\delta}, \infty[$, $]-\infty, \frac{1}{|\delta|}[$. Thus, for $\delta > 0$, $\delta U^{(1)}$ maps onto $]-1, \infty[$ and for $\delta < 0$, $\delta U^{(1)}$ maps also onto $]-1, \infty[$. With no loss of generality, we may assume $\delta = 0, 1, -1$.

We use the alternative notations $U^{(1)}$ or $V^{(1)}$ as convenient, referring to them as utility and value, respectively. Note that by value, we do not implicitly mean money, but rather a particular transformation of $U^{(1)}$. Also note that for $\delta = \pm 1$ the range of $V^{(1)}$ is $]0, \infty[$, that $V^{(1)}$ is order preserving when $\delta = 1$, and that it is order reversing when $\delta = -1$

Definition 6 *A gambling structure is* ***joint-receipt decomposable over gains*** *if for each $x \succsim e$ and each event $C \subset \Omega$ there exists an event $D = D(x, C) \subset \Omega$ such that for all $y \succsim e$,*

$$(x \oplus y, C; e, \Omega\backslash C) \oplus (e, D; e, \Omega\backslash D) \sim (x, C; e, \Omega\backslash C) \oplus (y, D; e, \Omega\backslash D). \tag{17}$$

Parallel definitions hold for losses and for all of X.

Note that (17) is equivalent to

$$KE(x \oplus y, C; e, \Omega \backslash C) \sim KE(x, C; e, \Omega \backslash C) \oplus KE(y, D; e, \Omega \backslash D).$$

Also, the decomposition of a gamble into its KE and elements of chance, plus co-monotonic consequence monotonicity, yields

$$\begin{aligned} x \succsim e &\Leftrightarrow (x, C; e, D) \succsim (e, C; e, D) \\ &\Leftrightarrow KE(x, C; e, D) \oplus (e, C; e, D) \succsim e \oplus (e, C; e, D) \\ &\Leftrightarrow KE(x, C; e, D) \succsim e. \end{aligned} \tag{18}$$

Remember, by the term "utility function" we mean an order-preserving function over gains (respectively, losses, all of X) satisfying (8) and (9).

Definition 7 *Binary gambles are* ***separable over gains*** *if for some utility function* $U^{(2)}$ *and some weighting function* $S_{\Omega}^{(2)}$*,*

$$U^{(2)}(KE(x, C; e, D)) = U^{(2)}(x) S_{\Omega}^{(2)}(C), \quad (x \succsim e). \tag{19}$$

In parallel, ***separability over losses****, and* ***over all of*** X*, refers to* (19) *holding for all* $x \precsim e$*, and for all* $x \in X$*, respectively.*

For gains, $x \succsim e$, the ordinary axioms for conjoint measurement justify this separable representation provided that we add the following stipulation[2]: The order $\succsim_{\Omega}$ induced over Ω by $\succsim$ over $KE(x, C_1; e, C_2)$ satisfies *monotonicity of event inclusion* if $C \subseteq E$ implies $C \precsim_{\Omega} E$ and so $S_{\Omega}^{(2)}(C) \leq S_{\Omega}^{(2)}(E)$. This property with separability is, as is easily shown, equivalent to the condition:

$$C \precsim_{\Omega} E \Leftrightarrow KE(x, C; e, D) \precsim KE(x, E; e, F) \quad (E \cup F = C \cup D = \Omega). \tag{20}$$

A parallel condition holds for losses and for all of X.

Observe that the separable representation is invariant under a power transformation in the sense that $(\tilde{U}^{(2)}, \tilde{S}_{\Omega}^{(2)})$, where

$$\begin{aligned} \tilde{U}^{(2)}(f) &= \alpha \, sgn \left(U^{(2)}(f) \right) \left| U^{(2)}(f) \right|^{\beta}, \\ \tilde{S}_{\Omega}^{(2)}(C) &= S_{\Omega}^{(2)}(C)^{\beta}. \end{aligned}$$

is another separable representation. [**CT** I further adjusted the alignment in the items below a bit.]

Definition 8 *A* ***p-additive gambling structure over gains (over losses) [over*** X***]*** *is any gambling structure with the following features:*

(i) a weak order relation $\succsim$;

[2] A formal axiomatization was given by Marley & Luce (2002) for gains (losses).

(ii) a strictly monotonic joint receipt group operation $\oplus$;

(iii) gambles satisfy co-monotonic consequence monotonicity, (1), and permutation invariance, (6);

(iv) there exists a utility function $U^{(1)}$ which is p-additive, (12), with $\delta \neq 0$;

(v) joint-receipt decomposition, (17), is satisfied over gains (over losses) [over X];

(vi) and there is a utility function $U^{(2)}$ with the representation $(U^{(2)}, S^{(2)}_{C\cup D})$ of $KE(x, C; e, D)$ satisfying one of the following conditions that depend, respectively, on joint-receipt decomposition, (17), being over gains (over losses) [over X]

(a) $(U^{(2)}, S^{(2)}_{C\cup D})$ is separable, (19), over gains and the range of $U^{(2)}$ over gains is $[0, 1[$.

(b) $(U^{(2)}, S^{(2)}_{C\cup D})$ is separable, (19), over losses and the range of $U^{(2)}$ over losses is $]-1, 0]$.

(c) $(U^{(2)}, S^{(2)}_{C\cup D})$ is separable, (19), over X and the range of $U^{(2)}$ is either $]-\infty, 1[$ or $]-1, \infty[$.

1.4 Representations satisfying both p-additivity and separability

To arrive at the strong results that we do, we have assumed that the p-additive utility function $U^{(1)}$ and the separable $U^{(2)}$ of Definition 8 are onto specified intervals, but we shall also assume that the weighting function satisfies the event-density restriction that for each non-null Ω, S_Ω is increasing and onto $]0, 1]$.

Theorem 4.4.6 of Luce (2000), which rests upon a functional equation result of Aczél, Maksa, and Páles (1999), yields a restricted version of the following results. That theorem is restricted in two ways: it is proved only for gains and only for the case[3] $\delta = -1$ for $U^{(1)}$. The theorems given below extend that result to losses and to all of X when separability and joint-receipt decomposability hold over all of X without confining δ to -1 for $U^{(1)}$.

Theorem 9 *For any p-additive gambling structure over gains (over losses) [over X], corresponding to the three cases (a), (b), and (c) of Definition 8, there exist a utility function U and a weighting function S_Ω such that, respectively, (a) U is p-additive with $\delta = -1$ and (U, S_Ω) is a separable representation over gains, (b) U is p-additive with $\delta = 1$ and (U, S_Ω) is a separable representation over losses, and (c) U is p-additive with $\delta = -1$ (respectively, $\delta = 1$) and*

[3] δ in the p-additive representation in Theorem 4.4.6 of Luce (2000) corresponds to $-\delta$ in the p-additive representation used in this article.

(U, S_Ω) is a separable representation over X if $U^{(2)}$ of Def. 8 (vi)(c) has range $]-\infty, 1[$ (respectively, $]-1, \infty[$).

2 Branching and upper gamble decomposition

We study two recursive forms that each reduce a gamble of order n to a compound gamble of order less than n and permit us to use the result for binary gambles, Proposition 14. The idea for the first property, called *branching*, is to form a binary sub-gamble consisting of the two branches having the two most preferred consequences. These could be anywhere in the enumeration; however, by permutation invariance, we may always, with no loss of generality, reorder the gamble so that the consequences are ordered from best to worse. Suppose that $g_{[n]}$, $n \geq 2$, is such a ranked gamble, i.e., with $x_1 \succsim x_2 \succsim ... \succsim x_n$, $C_i \in \mathfrak{B}^*$,

$$g_{[n]} := (x_1, C_1; ...; x_i, C_i; ...; x_n, C_n). \tag{21}$$

(See Section 1.1.) Branching then is the recursion in which the first two branches are combined into a single first-order gamble $g_{[2]}$, i.e.,

Definition 10 ***Branching*** *holds if*

$$\begin{aligned} g_{[n]} &\sim (g_{[2]}, C_1 \cup C_2; x_3, C_3; ...; x_n, C_n) \\ &= ((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; ...; x_n, C_n). \end{aligned} \tag{22}$$

We do not presume that the ordering necessarily satisfies $(x_1, C_1; x_2, C_2) \succsim x_3$, which necessitates some care in the statement of certain parts of Lemma 25.

In like manner, upper gamble decomposition (UGD) is the recursion that treats as indifferent a binary gamble consisting of the branch with the best consequence and all of the remaining branches combined as a single first-order gamble, i.e.,

Definition 11 ***Upper gamble decomposition (UGD)*** *holds if*

$$g_{[n]} \sim \left(x_1, C_1; g_{[n],-1}, \cup_{j \neq 1} C_j\right) \tag{23}$$

where $g_{[n],-1} := (x_2, C_2; ...; x_i, C_i; ...; x_n, C_n)$.

Again, we do not presume the ordering between x_1 and $g_{[n],-1}$, which also necessitates some care in the statement of certain parts of Lemma 25.

3 Results Under Segregation

Definition 12 ***Segregation*** *holds if for $x, y \in X$ with $x \succsim y$ and disjoint events C, D,*

$$(x, C; y, D) \sim (x \oplus y^{-1}, C; e, D) \oplus y. \tag{24}$$

Definition 13 *For any partition* $(C_1, C_2, ..., C_n)$, *define*

$$H(C_1, C_2..., C_n) := U(e, C_1; e, C_2...; e, C_n) \quad (25)$$

$$G(C_1, C_2, ..., C_n) := 1 + \delta H(C_1, C_2, ..., C_n) \quad (\delta \neq 0). \quad (26)$$

Note that under our assumptions, for $\delta = 1$, $G = 1 + U$ maps onto $]0, \infty[$. And for $\delta = -1$, G again maps onto $]0, \infty[$. Therefore, G is always positive. When $G \equiv 1$, there is no utility of gambling, in which case, because $\delta \neq 0$, we have $U(e, C_1; e, C_2...; e, C_n) = 0$, i.e., $(e, C_1; e, C_2...; e, C_n) \sim e$.

3.1 Preliminary results

Proposition 14 *Assume a p-additive gambling structure over gains (over losses), and segregation. Then there is a p-additive utility function* U *over all of* X *with* $\delta = -1$ *(* $\delta = 1$*) such that for* $x, y \in X$ *with* $x \succsim y$,

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x) S_{C\cup D}(C) + \delta U(y) (1 - S_{C\cup D}(C))] G(C, D), \quad (27)$$

$$i.e., V(x, C; y, D) = [V(x) S_{C\cup D}(C) + V(y) (1 - S_{C\cup D}(C))] G(C, D). \quad (28)$$

The kernel equivalent is, of course,

$$V(KE(x, C; y, D)) = V(x) S_{C\cup D}(C) + V(y) (1 - S_{C\cup D}(C)). \quad (29)$$

Definition 15 ***Complementarity*** *holds for weighting function* S *if*

$$S_{C\cup D}(C) + S_{C\cup D}(D) = 1 \quad (30)$$

for all binary partitions (C, D).

Notice that if the **choice property** also holds, i.e., if for all $C \subset D \subseteq \Omega$

$$S_\Omega(C) = S_\Omega(D) S_D(C), \quad (31)$$

then multiplying (30) by $S_\Omega(C \cup D)$, it follows immediately that the weights S_Ω are finitely additive.

Corollary 16 *Let* (U, S_Ω) *be the representation obtained in Theorem 9 for a p-additive gambling structure over* X. *Then segregation implies that complementarity holds for* S_Ω *and that for* $x, y \in X$ *with* $x \succsim y$,

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x) S_{C\cup D}(C) + \delta U(y) S_{C\cup D}(D)] G(C, D). \quad (32)$$

Lemma 17 *Assume a p-additive gambling structure over gains (over losses) and segregation. Suppose that branching, (22), and UGD, (23), hold for* $n = 3$. *Let* (C_1, C_2, C_3) *be any trinary partition of* Ω. *Then*

$$G(C_1, C_2) \neq 1 \quad \text{implies} \quad S_\Omega(C_1 \cup C_2) + S_\Omega(C_3) = 1. \quad (33)$$

Definition 18 *A non-null event E is* ***UofG-singular*** *if for each binary partition (E_1, E_2) of E, $(e, E_1; e, E_2) \sim e$, i.e. $G(E_1, E_2) = 1$.*

Because $\delta \neq 0$, also we have $G(E_1, E_2) = 1$ implies $(e, E_1; e, E_2) \sim e$.

Proposition 19 *Assume a p-additive gambling structure over gains (over losses) that satisfies segregation. Suppose that branching, (22), and UGD, (23), hold for $n = 3$. Suppose that no event is UofG-singular, then S satisfies complementarity.*

3.2 Under complementarity

We first explore complementarity and obtain the following representation theorem. Note that Ω occurring in an expression related to a gamble $(x_1, C_1; ...; x_n, C_n)$ stands for $\cup_{i=1}^{n} C_i$.

Theorem 20 *Assume a p-additive gambling structure over gains (over losses) that satisfies segregation, branching, (22), and UGD, (23) for $n = 3$. Suppose that the weights satisfy complementarity, (30). Then:*

1. *There exists a function $\lambda : \mathfrak{B}^* \to]0, \infty[$ such that*

$$S_{C \cup D}(C) = \frac{\lambda(C)}{\lambda(C) + \lambda(D)} \tag{34}$$

and

$$G(C, D) = \frac{\lambda(C) + \lambda(D)}{\lambda(C \cup D)}. \tag{35}$$

Furthermore, the ranked binary and trinary gambles have the representation

$$\begin{aligned} &V(x_1, C_1; x_2, C_2; ...; x_n, C_n) \\ &= \sum_{i=1}^{n} V(x_i) S_\Omega(C_i) G(C_i, \Omega \backslash C_i) \end{aligned} \tag{36}$$

for $n = 2, 3$.

2. *Assume branching, (22), for all $n \geq 3$. Then the representation (36) extends to all ranked gambles of general length $n \geq 2$.*

3. *Gambles with the representation (36) in fact satisfy branching and UGD independent of the rank order of the consequences.*

4. *The representation also holds for a p-additive gambling structure over X that satisfies segregation, branching and UGD.*

Observe, first, that using (34) and (35), we have

$$S_\Omega(C_i)G(C_i,\Omega\backslash C_i)=\frac{\lambda(C_i)}{\lambda(\Omega)}, \tag{37}$$

which may be inserted in (36). The resulting representation is simpler:

$$V(x_1,C_1;x_2,C_2;...;x_n,C_n)=\sum_{i=1}^{n}V(x_i)\frac{\lambda(C_i)}{\lambda(\Omega)}.$$

For the case $\delta=1$, it is a *simple weighted utility representation* (Marley & Luce, 2005, p. 81). However, the former form, (36), makes very clear to decision scientists what the separate roles of S_Ω and G are. For example, an important generalization would be to find axioms that led to either $G_i(C_i,\Omega\backslash C_i)$ or $S_{\Omega,i}(C_i)$ which could account for empirical fact that an event C in the 1^{st} position is evaluated differently from the n^{th} position.

Second, the expression for the utility of gambling, which is found by setting $x_i=e,\ i=1,...,n$, is

$$H(C_1,C_2,...,C_n)=\frac{1}{\delta}\left[\sum_{i=1}^{n}S_\Omega(C_i)G(C,\Omega\backslash C_i)-1\right].$$

Equivalently,

$$\begin{aligned}G(C_1,C_2,...,C_n)&=\sum_{i=1}^{n}S_\Omega(C_i)G(C_i,\Omega\backslash C_i)\\&=\frac{1}{\lambda(\Omega)}\sum_{i=1}^{n}\lambda(C_i).\end{aligned}$$

3.3 When non-maximal events are UofG-singular

We next explore the case of segregation where all non-maximal events are UofG-singular.

Theorem 21 *Assume a p-additive gambling structure over gains (over losses) that satisfies segregation. Suppose that branching, (22), and UGD, (23), hold for $n=3$ and that $G(C_1,C_2)=1$ for all $3-$partitions (C_1,C_2,C_3). Then,*

1.
$$G(C,D)=\begin{cases}1, & \text{if } C\cup D \text{ is not maximal}\\ A,\ A>0, & \text{if } C\cup D \text{ is maximal}\end{cases}.$$

2. *The weights satisfy the choice property, (31), for which there is*

   (a) *A strictly increasing function $\mu:\mathfrak{B}^*\to]0,\infty[$ such that*

$$S_D(C)=\frac{\mu(C)}{\mu(D)}, \tag{38}$$

(b) *And such that* μ *is p-additive in the sense that for some constant* b *with the unit of* $1/\mu$,

$$\mu(C \cup D) = \mu(C) + \mu(D) + b\mu(C)\mu(D), \tag{39}$$

where the choice of b *is restricted by*

$$1 + b\mu(C) > 0. \tag{40}$$

3. *And for* $A = 1$ *with* $C(i)$ *defined by*

$$C(i) = \begin{cases} \varnothing, & i = 0 \\ \bigcup_{j=1}^{i} C_i, & i = 1, ..., n \end{cases}, \tag{41}$$

then, for all $x_1 \succsim x_2 \succsim ... \succsim x_n$,

$$V(g_{[n]}) = \sum_{i=1}^{n} V(x_i) \left[S_\Omega\left(C(i)\right) - S_\Omega(C(i-1)) \right], \tag{42}$$

for $n = 2, 3$. *If we also assume Branching, or UGD, for general higher lengths, then* (42) *holds for general* $n \geq 2$.

Note: Because $V = 1 - U$ or $1 + U$ and because $\sum_{i=1}^{n} \left[S_\Omega\left(C(i)\right) - S_\Omega(C(i-1)) \right] = 1$, (42) is equivalent to

$$U(g_{[n]}) = \sum_{i=1}^{n} U(x_i) \left[S_\Omega\left(C(i)\right) - S_\Omega(C(i-1)) \right], \tag{43}$$

which is the very well known rank-dependent form extended to both gains and losses.

These forms, (42) and (43), have been a preoccupation of utility theorists for the past 25 years (Luce, 2000). In that literature, however, one was working with the form (43) in which U is additive over $\oplus$, whereas our current U is p-additive over $\oplus$, and so is non-additive

It is also well known that if S_Ω is finitely additive, the weights form a probability measure and the representation (43) reduces to

$$U(g_{[n]}) = \sum_{i=1}^{n} U(x_i) S_\Omega(C_i), \tag{44}$$

which is called *subjective expected utility* (SEU). Because in this case event complementarity is satisfied, the representation is also (36) with $G(C, D) = 1$ and $\lambda = \mu$.

4 Results Under Duplex Decomposition

4.1 Preliminary result

Definition 22 *Gambles are said to satisfy* ***duplex decomposition over*** X *if for all* $x, y \in X$*,*

$$(x, C; y, D) \oplus (e, C'; e, D') \sim (x, C; e, D) \oplus (e, C'; y, D'), \tag{45}$$

where (C', D') *is an independent realization of* (C, D)*.*

In terms of the "bottom line," this concept is less rational than is segregation. For example, on the right of (45) two of the possible consequences are $x \oplus y$, which occurs if C and D' both occur and $e \oplus e$, which occurs if D and C' both occur, whereas on the left neither of these consequences can arise.

Proposition 23 *For a p-additive gamble structure over* X*, duplex decomposition implies*

$$1 + \delta U(KE(x, C; y, D)) = [1 + \delta U(x) S_{C\cup D}(C)][1 + \delta U(y) S_{C\cup D}(D)] \tag{46}$$

and

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x) S_{C\cup D}(C)][1 + \delta U(y) S_{C\cup D}(D)] G(C, D). \tag{47}$$

The V notation of (15) does not simplify the duplex decomposition result.

4.2 Results adding branching and UGD

Proposition 24 *Assume a (separable) utility function* U *that is p-additive over* X *that satisfies (47). If both branching and upper gamble decomposition are satisfied, then, for all trinary partitions* (C_1, C_2, C_3)*,*

$$S_\Omega(C_1) = S_{C_1 \cup C_2}(C_1). \tag{48}$$

Taking a decision making perspective, property (48) is intuitively unsatisfactory because changing the event that underlies a gamble should affect the weight attached to any proper subevent.

5 Conclusions

We have worked out results for the p-additive representations over the full domain of consequences, both gains and losses under the following main assumptions: permutation invariance, separability, joint-receipt decomposition, branching, and upper gamble decomposition. The results are: (i) For the rational property of segregation, we develop representations when the weights over events exhibit complementarity. We also develop separately the case of no utility of gambling. In the first case, we found a utility representation of the following form: a simple weighted utility of consequences multiplied by a function of the event underlying each branch. For the case of no utility of gambling, we found in Theorem 21 the usual rank-dependent representation over the full domain X and gambles with the weights satisfying the choice property with a p-additive representation. (ii) Replacing segregation by non-rational property of duplex decomposition leads to the weights showing essentially no dependence upon the events (Proposition 24), a clearly unacceptable property for behavioral applications.

6 Appendix: Proofs

6.1 Proposition 5

Proof. For the proof, we drop the superscript $^{(1)}$ from U.

1. For any x, define

$$x(1) = x, x(m) = x(m-1) \oplus x. \tag{49}$$

   By induction, $U(x(m)) = mU(x)$, and so the range in unbounded.
   In (13) we may, with no loss of generality select $\delta = \pm 1$

2. Select $\delta = 1$. For $x \succ e$, we have from (13) that $1+U(x\oplus x) = [1+U(x)]^2$. Thus

$$U(x \oplus x) = 2U(x) + U(x)^2 > 2U(x).$$

   More generally, referring to (49), we have $U(x(m)) > mU(x)$ for each $m \geq 2$. So U is unbounded from above.

   Let y be any given consequence less than e. Because U is unbounded from above, there exists x such that $x \succ x \oplus y \succ e \succ y$, i.e.,

$$U(x) > U(x \oplus y) > U(e) = 0 > U(y).$$

   By (13), $1 + U(x \oplus y) = [1 + U(x)]\,[1 + U(y)]$. As $1 + U(x \oplus y) > 0$ and $1 + U(x) > 0$, we get $1 + U(y) > 0$. So $U(y) > -1$, proving that U is bounded from below by -1.

3. Select $\delta = -1$. The argument is similar to Case 2.

■

6.2 Theorem 9

Proof. We first make a general observation on the existence of a p-additive utility function $U^{(1)}$ with a $\delta \neq 0$. It was pointed out earlier that there is no loss of generality to assume that $\delta = 1$ or $\delta = -1$. We shall show further that the existence of a $U^{(1)}$ with a $\delta = 1$ is in fact equivalent to the existence of a $U^{(1)}$ with a $\delta = -1$.

Suppose that $U^{(1)}$ is a p-additive utility function with $\delta = 1$ and range $]-1, \infty[$. Consider

$$\tilde{U}^{(1)} := 1 - \frac{1}{1 + U^{(1)}}.$$

It is easy to check that $\tilde{U}^{(1)}$ is a p-additive utility function with $\delta = -1$ and range $]-\infty, 1[$. Conversely, if $\tilde{U}^{(1)}$ is a p-additive utility function with $\delta = -1$ and range $]-\infty, 1[$, then

$$U^{(1)} := \frac{1}{1 - \tilde{U}^{(1)}} - 1.$$

is a p-additive utility function with $\delta = 1$ and range $]-1, \infty[$.

(a). By assumption, and by the above general observation, we may fix a p-additive $U^{(1)}$ having $\delta = -1$ and so $U^{(1)}$ on gains is onto $[0, 1[$. Also, $U^{(2)}$ on gains is onto $[0, 1[$, and so, by Theorem 4.4.6 of Luce (2000), which rests upon the result in Aczél, Maksa, and Páles (1999), there is a $\beta > 0$ such that

$$U_+ := \left(U^{(2)}\right)^{\beta} \tag{50}$$

over gains is p-additive with $\delta = -1$ and $(U_+, S_{C \cup D})$, where

$$S_{C \cup D} := \left(S^{(2)}_{C \cup D}\right)^{\beta}, \tag{51}$$

is separable over $KE(x, C; e, D)$, $x \succ e$.

Thus, over gains, $x \succ e, y \succ e$, we have

$$\begin{aligned} &U_+(x \oplus y) = U_+(x) + U_+(y) - U_+(x)U_+(y), \\ &\Leftrightarrow V_+(x \oplus y) = V_+(x)V_+(y), \end{aligned} \tag{52}$$

where $V_+ := 1 - U_+$ is defined over all gains. From the underlying group structure, for any $x \in X$ which is a loss, $x^{-1} \in X$ is a gain. We extend V_+ on gains to V on all of X by

$$V(x) = \begin{cases} V_+(x), & \forall x \succ e \\ 1, & x \sim e \\ 1/V_+(x^{-1}), & \forall x \prec e \end{cases}. \tag{53}$$

It is straight forward to check that V indeed extends (52) to a multiplicative function on X:

$$V(x \oplus y) = V(x)V(y), \quad \forall x, y \in X. \tag{54}$$

The range of V is $]0, \infty[$. Let U on all of X be defined by

$$U = 1 - V. \tag{55}$$

It is an extension of U_+ and so $(U, S_{C\cup D})$ is separable over gains. The range of U is $]-\infty, 1[$, and (54) is equivalent to the p-additivity of U, with $\delta = -1$. The pair U and V extend (52) to X. It is not difficult to show that they are the unique extensions. Also, given that $U^{(2)}$ is a utility function over X, and using (50) and (53)-(55), it is not difficult to show that U is a utility function over X.

(**b**). Consider a dual ordering $\succsim'$ which is the transpose of $\succsim$, i.e., $x \succsim' y$ if and only if $y \succsim x$. If a utility function $U^{(1)}$ is p-additive for the structure $\succsim$ with δ, then $-U^{(1)}$ is a corresponding utility function for $\succsim'$ with $-\delta$. If $(U^{(2)}, S_\Omega)$ is a separable representation over losses for $\succsim$, then $(-U^{(2)}, S_\Omega)$ is a separable representation over gains for $\succsim'$. If joint-receipt decomposition holds for $\succsim$ losses, then joint-receipt decomposition holds for $\succsim'$ gains. With this correspondence, the **(b)** statement, which is for $\succsim$, is the **(a)** statement for $\succsim'$.

(**c**) **Part 1**. Suppose that there exists a separable representation $(U^{(2)}, S_\Omega)$ over all of X in which the range of $U^{(2)}$ is $]-\infty, 1[$. Fixing such a separable representation, and, by the leading general observation, we may fix a p-additive utility function $U^{(1)}$ having $\delta = -1$. Then $U^{(1)}$ is onto $]-\infty, 1[$. So, $U^{(1)}$ on gains is onto $[0, 1[$. Also, $U^{(2)}$ on gains is onto $[0, 1[$. Then, by **(a)**, there is a utility function U that is p-additive over all of X with $\delta = -1$, and is $(U, S_{C\cup D})$ separable over gains, i.e., for unitary gambles with gains:

$$U(KE(x, C; e, D)) = U(x)S_{C\cup D}(C) \quad (x \succsim e). \tag{56}$$

So the remaining issue is what happens to the unitary gambles with losses.

Let $(x, C; e, D)$, $x \prec e$, be given. We shall proceed to show that

$$U(KE(x, C; e, D)) = U(x)S_{C\cup D}(C). \tag{57}$$

By joint-receipt decomposition over X,

$$KE(x \oplus x^{-1}, C; e, D) = KE(x, C; e, D) \oplus KE(x^{-1}, C'; e, D') \tag{58}$$

for some C' and D', with $C\cup D = C'\cup D' = \Omega$. Observing $KE(x\oplus x^{-1}, C; e, D) = e$ and applying multiplicative $V = 1 - U$, (55), to the above, we get

$$1 = V(KE(x, C; e, D))V(KE(x^{-1}, C'; e, D')).$$

That is,

$$1 = [1 - U(KE(x, C; e, D))][1 - U(KE(x^{-1}, C'; e, D'))]. \tag{59}$$

As x^{-1} is a gain, so is $KE(x^{-1}, C'; e, D')$, and therefore, by the obtained properties on gains, we have

$$U(KE(x^{-1}, C'; e, D')) = U(x^{-1})S_\Omega(C'), \tag{60}$$

$$KE(x^{-1} \oplus y, C'; e, D') = KE(x^{-1}, C'; e, D') \oplus KE(y, C''; e, D''), \tag{61}$$

$$S_\Omega(C'') = S_\Omega(C')\frac{1 - U(x^{-1})}{1 - U(x^{-1})S_\Omega(C')} \tag{62}$$

where the first equation comes from (56), the second holds for some $(C'', D")$, $C'' \cup D'' = \Omega$, and for all $y \in X$ and is due to joint-receipt decomposition over X, whereas the third relation is known (see Luce (2000), p. 170, penultimate line). Comparing the middle equation at $y = x$ with (58) using the symmetry and strict monotonicity of $\oplus$ we get

$$KE(x, C; e, D) = KE(x, C''; e, D'').$$

Applying $U^{(2)}$ to both sides and using separability we get first $U^{(2)}(x)S_\Omega^{(2)}(C) = U^{(2)}(x)S_\Omega^{(2)}(C'')$. Canceling the common factor $U^{(2)}(x)$, raising to the power β, and referring to (51), we get

$$S_\Omega(C) = S_\Omega(C''). \tag{63}$$

Putting that into (62) gives

$$S_\Omega(C) = S_\Omega(C')\frac{1 - U(x^{-1})}{1 - U(x^{-1})S_\Omega(C')}. \tag{64}$$

Putting (60) into (59) gives

$$\begin{aligned} U(KE(x, C; e, D)) &= 1 - \frac{1}{1 - U(x^{-1})S_\Omega(C')} \\ &= \frac{-U(x^{-1})S_\Omega(C')}{1 - U(x^{-1})S_\Omega(C')}. \end{aligned} \tag{65}$$

But

$$U(x^{-1}) = 1 - V(x^{-1}) = 1 - \frac{1}{V(x)} = 1 - \frac{1}{1 - U(x)} = \frac{-U(x)}{1 - U(x)}. \tag{66}$$

Using that to replace $U(x^{-1})$ in (65) we get

$$U(KE(x, C; e, D)) = \frac{U(x)S_\Omega(C')}{1 - U(x) + U(x)S_\Omega(C')}. \tag{67}$$

Putting (66) into (64) gives

$$S_\Omega(C) = S_\Omega(C')\frac{1 - \frac{-U(x)}{1-U(x)}}{1 - \frac{-U(x)}{1-U(x)}S_\Omega(C')} \tag{68}$$

$$= S_\Omega(C')\frac{1}{1 - U(x) + U(x)S_\Omega(C')}. \tag{69}$$

Comparing that with (67) we arrive at (57) as claimed. This proves (U, S_Ω) separability over X.

(c) Part 2. Suppose that there exists a separable representation $(U^{(2)}, S_\Omega)$ over all of X in which the range of $U^{(2)}$ is $]-1, \infty[$.

Consider a dual ordering $\succsim'$ which is the transpose of $\succsim$, i.e., $x \succsim' y$ if and only if $y \succsim x$. If representation $U^{(1)}$ is p-additive for the structure $\succsim$ with δ, then $-U^{(1)}$ is a corresponding representation for $\succsim'$ with $-\delta$. If $(U^{(2)}, S_\Omega)$ is a separable representation over all of X for $\succsim$, then $(-U^{(2)}, S_\Omega)$ is a separable representation over all of X for $\succsim'$. With this correspondence, the **(c) Part 2** statement, which is for $\succsim$, is the **(c) Part 1** statement for $\succsim'$. ■

6.3 Proposition 14

Proof. Part 1. Assume a p-additive gambling structure over gains. In particular, there exists a p-additive $U^{(1)}$ with $\delta = -1$ and a separable representation $(U^{(2)}, S_\Omega^{(2)})$ over gains. Then, by part **(a)** of the proof of Theorem 9, there is a U that is p-additive with $\delta = -1$ over all of X and S_Ω such that (U, S_Ω) is a separable representation over gains.

Using p-additivity with $\delta = -1$ and the decomposition into KEs and elements of chance, and then segregation, gives

$$\begin{aligned}
1 - U(x, C; y, D) &= [1 - U(KE(x, C; y, D))]\,[1 - H(C, D)] \\
&= [1 - U(KE(x, C; y, D))]G(C, D) \\
&= [1 - U(KE(x \oplus y^{-1}, C; e, D) \oplus y)]G(C, D). \qquad (70)
\end{aligned}$$

By p-additivity with $\delta = -1$ and the definition of the inverse y^{-1}, it is a routine calculation to show that

$$U(x \oplus y^{-1}) = \frac{U(x) - U(y)}{1 - U(y)}.$$

Now, since $x \succ y$, we have $x \oplus y^{-1} \succ e$ and so using separability over gains,

$$\begin{aligned}
U(KE(x \oplus y^{-1}, C; e, D) \oplus y) &= U(KE(x \oplus y^{-1}, C; e, D))\,[1 - U(y)] + U(y) \\
&= U(x \oplus y^{-1}) S_{C\cup D}(C))\,[1 - U(y)] + U(y) \\
&= \frac{U(x) - U(y)}{1 - U(y)} S_{C\cup D}(C)\,[1 - U(y)] + U(y) \\
&= U(x) S_{C\cup D}(C) + U(y)\,[1 - S_{C\cup D}(C)]\,,
\end{aligned}$$

which, with (70), gives (27).

Part 2. Assume a p-additive gambling structure over losses. In particular, there exists a p-additive $U^{(1)}$ with $\delta = 1$ over losses and a separable representation $(U^{(2)}, S_\Omega^{(2)})$ over losses. Then, by part **(b)** of the proof of Theorem 9, there is a U that is p-additive with $\delta = 1$ over all of X and S_Ω such that (U, S_Ω) is a separable representation over losses. Then, for $x \succ y$, we have by segregation that

$$\begin{aligned}
(x, C; y, D) &\sim (x \oplus y^{-1}, C; e, D) \oplus y \oplus \left(x^{-1} \oplus x\right) \\
&\sim \left(\left(y \oplus x^{-1}\right)^{-1}, C; e, D\right) \oplus \left(y \oplus x^{-1}\right) \oplus x \\
&\sim (e, C; y \oplus x^{-1}, D) \oplus x.
\end{aligned}$$

Using this equivalence, separability over losses for $e \succ y \oplus x^{-1}$, and with $\delta = -1$ replaced with $\delta = 1$, the proof of (27) parallels that in **Part 1**. ■

6.4 Corollary 16

Proof. By Theorem 9 for a p-additive gambling structure over X, we have (U, S_Ω) where U is p-additive with $\delta = \pm 1$ over all of X and the representation is separable over all of X. In particular, for $y \precsim e$,

$$U(KE(e, C; y, D)) = U(y)S_{C\cup D}(D). \tag{71}$$

A p-additive gambling structure over X is clearly a p-additive structure over gains, so, by Proposition 14, (27) holds and (29) yields, in particular, for $y \precsim e$, that

$$U(KE(e, C; y, D)) = U(y)[1 - S_{C\cup D}(C)].$$

Comparing it with (71) gives $S_{C\cup D}(D) = 1 - S_{C\cup D}(C)$, i.e., complementarity follows. With complementarity, (27) becomes (32). ■

6.5 Two Lemmas

Lemma 25 *Assume a p-additive gambling structure over gains (over losses) and segregation.*

1. *Suppose that branching, (22), holds for $n = 3$. If*

$$(x_1, C_1; x_2, C_2) \succsim x_3, \tag{72}$$

then

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\
&= 1 + S_\Omega(C_1 \cup C_2)\,(G(C_1, C_2) - 1) \\
&\quad + \delta U(x_1)S_{C_1\cup C_2}(C_1)S_\Omega(C_1 \cup C_2)G(C_1, C_2) \\
&\quad + \delta U(x_2)[1 - S_{C_1\cup C_2}(C_1)]S_\Omega(C_1 \cup C_2)G(C_1, C_2) \\
&\quad + \delta U(x_3)[1 - S_\Omega(C_1 \cup C_2)].
\end{aligned} \tag{73}$$

If

$$x_3 \succ (x_1, C_1; x_2, C_2), \tag{74}$$

then

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\
&= 1 + [1 - S_\Omega(C_3)]\,[G(C_1, C_2) - 1] \\
&\quad + \delta U(x_1)S_{C_1\cup C_2}(C_1)G(C_1, C_2)[1 - S_\Omega(C_3)] \\
&\quad + \delta U(x_2)[1 - S_{C_1\cup C_2}(C_1)]G(C_1, C_2)[1 - S_\Omega(C_3)] \\
&\quad + \delta U(x_3)S_\Omega(C_3).
\end{aligned} \tag{75}$$

2. *Suppose that UGD, (23), holds for* $n = 3$. *If*

$$x_1 \succsim (x_2, C_2; x_3, C_3) \tag{76}$$

holds, then

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\
&= 1 + [1 - S_\Omega(C_1)]\,[G(C_2, C_3) - 1] \\
&\quad + \delta U(x_1) S_\Omega(C_1) \\
&\quad + \delta U(x_2) S_{C_2 \cup C_3}(C_2)\,[1 - S_\Omega(C_1)]\, G\,(C_2, C_3) \\
&\quad + \delta U(x_3)\,[1 - S_{C_2 \cup C_3}(C_2)]\,[1 - S_\Omega(C_1)]\, G\,(C_2, C_3)\,.
\end{aligned} \tag{77}$$

If

$$(x_2, C_2; x_3, C_3) \succ x_1 \tag{78}$$

holds, then

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\
&= 1 + S_\Omega(C_2 \cup C_3)\,[G(C_2, C_3) - 1] + \delta U(x_1)[1 - S_\Omega(C_2 \cup C_3)] \\
&\quad + \delta U(x_2) S_{C_2 \cup C_3}(C_2) S_\Omega(C_2 \cup C_3) G\,(C_2, C_3) \\
&\quad + \delta U(x_3)[1 - S_{C_2 \cup C_3}(C_2)] S_\Omega(C_2 \cup C_3) G\,(C_2, C_3)\,.
\end{aligned} \tag{79}$$

Proof. 1. Using branching for $n = 3$ and (27) we get, for all trinary gambles satisfying (72),

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} = \frac{1 + \delta U((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\
&= 1 + \delta U(x_1, C_1; x_2, C_2) S_\Omega(C_1 \cup C_2) + \delta U(x_3)[1 - S_\Omega(C_1 \cup C_2)] \\
&= 1 + \{[1 + \delta U(x_1) S_{C_1 \cup C_2}(C_1) + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)]]G(C_1, C_2) - 1)\} \\
&\quad \times S_\Omega(C_1 \cup C_2) + \delta U(x_3)[1 - S_\Omega(C_1 \cup C_2)] \\
&= 1 + S_\Omega(C_1 \cup C_2)\,[G(C_1, C_2) - 1] \\
&\quad + \delta U(x_1) S_{C_1 \cup C_2}(C_1) G(C_1, C_2) S_\Omega(C_1 \cup C_2) \\
&\quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)] G(C_1, C_2) S_\Omega(C_1 \cup C_2) \\
&\quad + \delta U(x_3)[1 - S_\Omega(C_1 \cup C_2)]
\end{aligned}$$

which proves (73).

In the case of (74), again with (27), we have

$$\frac{1+\delta U(x_1,C_1;x_2,C_2;x_3,C_3)}{G(C_1\cup C_2,C_3)}=\frac{1+\delta U(x_3,C_3;(x_1,C_1;x_2,C_2),C_1\cup C_2)}{G(C_3,C_1\cup C_2)}$$
$$=1+\delta U(x_1,C_1;x_2,C_2)[1-S_\Omega(C_3)]+\delta U(x_3)S_\Omega(C_3)$$
$$=1+\{[1+\delta U(x_1)S_{C_1\cup C_2}(C_1)+\delta U(x_2)[1-S_{C_1\cup C_2}(C_1)]]G(C_1,C_2)-1)\}$$
$$\times[1-S_\Omega(C_3)]+\delta U(x_3)S_\Omega(C_3)$$
$$=1+[1-S_\Omega(C_3)]\,[G(C_1,C_2)-1]$$
$$+\delta U(x_1)S_{C_1\cup C_2}(C_1)G(C_1,C_2)[1-S_\Omega(C_3)]$$
$$+\delta U(x_2)[1-S_{C_1\cup C_2}(C_1)]G(C_1,C_2)[1-S_\Omega(C_3)]$$
$$+\delta U(x_3)S_\Omega(C_3)$$

which proves (75).

2. Now, assuming UGD, (23), for $n=3$ and using (27) and the assumption (76), we have:

$$\frac{1+\delta U(x_1,C_1;x_2,C_2;x_3,C_3)}{G(C_1,C_2\cup C_3)}=\frac{1+\delta U(x_1,C_1;(x_2,C_2;x_3,C_3),C_2\cup C_3)}{G(C_1,C_2\cup C_3)}$$
$$=1+\delta U(x_1)S_\Omega(C_1)+\delta U(x_2,C_2;x_3,C_3)\,[1-S_\Omega(C_1)]$$
$$=1+\delta U(x_1)S_\Omega(C_1)$$
$$+\{[1+\delta U(x_2)S_{C_2\cup C_3}(C_2)+\delta U(x_3)[1-S_{C_2\cup C_3}(C_2)]\,G(C_2,C_3)-1\}$$
$$\times[1-S_\Omega(C_1)]$$
$$=1+[1-S_\Omega(C_1)]\,[G(C_2,C_3)-1]+\delta U(x_1)S_\Omega(C_1)$$
$$+\delta U(x_2)S_{C_2\cup C_3}(C_2)\,[1-S_\Omega(C_1)]\,G\,(C_2,C_3)$$
$$+\delta U(x_3)[1-S_{C_2\cup C_3}(C_2)]\,[1-S_\Omega(C_1)]\,G\,(C_2,C_3)$$

which proves (77).

Assuming (78), we get

$$\frac{1+\delta U(x_1,C_1;x_2,C_2;x_3,C_3)}{G(C_1,C_2\cup C_3)}=\frac{1+\delta U((x_2,C_2;x_3,C_3),C_2\cup C_3;x_1,C_1)}{G(C_2\cup C_3,C_1)}$$
$$=1+\delta U(x_1)[1-S_\Omega(C_2\cup C_3)]+\delta U(x_2,C_2;x_3,C_3)S_\Omega(C_2\cup C_3)$$
$$=1+\delta U(x_1)[1-S_\Omega(C_2\cup C_3)]$$
$$+\{[1+\delta U(x_2)S_{C_2\cup C_3}(C_2)+\delta U(x_3)[1-S_{C_2\cup C_3}(C_2)]\,G(C_2,C_3)-1\}$$
$$\times S_\Omega(C_2\cup C_3)$$
$$=1+S_\Omega(C_2\cup C_3)\,[G(C_2,C_3)-1]+\delta U(x_1)[1-S_\Omega(C_2\cup C_3)]$$
$$+\delta U(x_2)S_{C_2\cup C_3}(C_2)S_\Omega(C_2\cup C_3)G\,(C_2,C_3)$$
$$+\delta U(x_3)[1-S_{C_2\cup C_3}(C_2)]S_\Omega(C_2\cup C_3)G\,(C_2,C_3)$$

which proves (79). ■

Lemma 26 *Assume a p-additive gambling structure over gains (over losses) and segregation. Suppose that branching, (22) and UGD, (23) holds for $n = 3$. Then the following relations necessarily hold*

$$\frac{S_{C_1\cup C_2}(C_1)S_\Omega(C_1\cup C_2)G(C_1,C_2)}{G(C_1,C_2\cup C_3)} = \frac{S_\Omega(C_1)}{G(C_1\cup C_2,C_3)} \tag{80}$$

$$\frac{(1-S_{C_1\cup C_2}(C_1))\,S_\Omega(C_1\cup C_2)G(C_1,C_2)}{G(C_1,C_2\cup C_3)} = \frac{S_{C_2\cup C_3}(C_2)\,(1-S_\Omega(C_1))\,G(C_2,C_3)}{G(C_1\cup C_2,C_3)} \tag{81}$$

$$\frac{1-S_\Omega(C_1\cup C_2)}{G(C_1,C_2\cup C_3)} = \frac{(1-S_{C_2\cup C_3}(C_2))\,(1-S_\Omega(C_1))\,G(C_2,C_3)}{G(C_1\cup C_2,C_3)} \tag{82}$$

$$\frac{1+S_\Omega(C_1\cup C_2)\,[G(C_1,C_2)-1]}{G(C_1,C_2\cup C_3)} = \frac{1+[1-S_\Omega(C_1)]\,[G(C_2,C_3)-1]}{G(C_1\cup C_2,C_3)} \tag{83}$$

for all ordered partition (C_1,C_2,C_3), where $\Omega = C_1\cup C_2\cup C_3$.

Proof. Let ordered partition (C_1,C_2,C_3) be arbitrarily given.

1. We shall first show that there exist ranked consequences $y_1 \succ y_2 \succ y_3$ such that the gamble $(y_1,C_1;y_2,C_2;y_3,C_3)$ satisfies

$$(y_1,C_1;y_2,C_2) \succ y_3 \tag{84}$$

and

$$y_1 \succ (y_2,C_2;y_3,C_3). \tag{85}$$

Let (C_1,C_2,C_3) be given. There are two cases to consider.

Case 1. Suppose that $\delta > 0$. Using (28), for ranked $y_1 \succ y_2 \succ y_3$, (84) is equivalent to

$$[V(y_1)S_{C_1\cup C_2}(C_1)+V(y_2)\,(1-S_{C_1\cup C_2}(C_1))]\,G(C_1,C_2) > V(y_3) \tag{86}$$

and (85) is equivalent to

$$V(y_1) > [V(y_2)S_{C_2\cup C_3}(C_2)+V(y_3)\,(1-S_{C_2\cup C_3}(C_2))]\,G(C_2,C_3). \tag{87}$$

We may arbitrarily fix $y_2 \succ y_3$. We see that inequalities (86) and (87) hold for $V(y_1)$ sufficiently large. We may choose y_1 accordingly.

Case 2. Suppose that $\delta < 0$. For ranked $y_1 \succ y_2 \succ y_3$ and the fact that V is order reversing, (84) is equivalent to

$$[V(y_1)S_{C_1\cup C_2}(C_1)+V(y_2)\,(1-S_{C_1\cup C_2}(C_1))]\,G(C_1,C_2) < V(y_3), \tag{88}$$

and (85) is equivalent to

$$V(y_1) < [V(y_2)S_{C_2\cup C_3}(C_2)+V(y_3)\,(1-S_{C_2\cup C_3}(C_2))]\,G(C_2,C_3). \tag{89}$$

We may first arbitrarily fix y_3. Because V is order reversing with range $]0,\infty[$, we may then choose y_1 and y_2 with $y_1 \succ y_2 \succ y_3$, and so $V(y_1) < V(y_2) < V(y_3)$, with $V(y_1)$ and $V(y_2)$ sufficiently small so that (88) holds and, in turn, $V(y_1)$ sufficiently smaller than $V(y_2)$ that (89) holds.

2. For ranked consequences $x_1 \succsim x_2 \succsim x_3$ with $U(x_i)$ sufficiently near $U(y_i)$, (72) and (76) hold. By Lemma 25, we have both (73) and (77). Then, the coefficients of the $U(x_i)$ must be equal in the two equations, leading to (80)–(83).

■

6.6 Lemma 17

Proof. Because $G(C_1, C_2) = V(e, C_1; e, C_2)$ and $\delta \neq 0$, the assumption that $G(C_1, C_2) \neq 1$ is equivalent to $(e, C_1; e, C_2) \nsim e$. So either $(e, C_1; e, C_2) \prec e$ or $(e, C_1; e, C_2) \succ e$.

1. Suppose that

$$(e, C_1; e, C_2) \prec e.$$

By choosing x_1, x_2 and x_3, with $U(x_1)$, $U(x_2)$ and $U(x_3)$ sufficiently near $U(e) = 0$, we see that there exist consequences x_1, x_2 and x_3, with $x_1 \succ x_2 \succ x_3$, such that

$$(x_1, C_1; x_2, C_2) \prec x_3$$

holds. This is (74) so (75) holds. For such chosen $x_1 \succ x_2 \succ x_3$, we consider two cases:

Case 1.1. Suppose that some gamble satisfies $x_1 \succsim (x_2, C_2; x_3, C_3)$. This is (76) and so (77) holds. Thus, both (75) and (77) hold. Equating the coefficients we obtain the following relations

$$\frac{S_{C_1\cup C_2}(C_1)G(C_1,C_2)[1-S_\Omega(C_3)]}{G(C_1,C_2\cup C_3)} = \frac{S_\Omega(C_1)}{G(C_1\cup C_2,C_3)} \tag{90}$$

$$\frac{[1-S_{C_1\cup C_2}(C_1)]G(C_1,C_2)[1-S_\Omega(C_3)]}{G(C_1,C_2\cup C_3)} \tag{91}$$

$$= \frac{S_{C_2\cup C_3}(C_2)\,[1-S_\Omega(C_1)]\,G(C_2,C_3)}{G(C_1\cup C_2,C_3)} \tag{92}$$

$$\frac{S_\Omega(C_3)}{G(C_1,C_2\cup C_3)} = \frac{[1-S_{C_2\cup C_3}(C_2)]\,[1-S_\Omega(C_1)]\,G(C_2,C_3)}{G(C_1\cup C_2,C_3)} \tag{93}$$

$$\frac{1+[1-S_\Omega(C_3)]\,[G(C_1,C_2)-1]}{G(C_1,C_2\cup C_3)} = \frac{1+[1-S_\Omega(C_1)]\,[G(C_2,C_3)-1]}{G(C_1\cup C_2,C_3)}. \tag{94}$$

Equating (90) to (80) from Lemma 26, yields (33):

$$S_\Omega(C_1\cup C_2)+S_\Omega(C_3)=1. \tag{95}$$

Case 1.2. Suppose that some gamble satisfies $x_1 \prec (x_2, C_2; x_3, C_3)$. This is (78) and so (79) holds. Thus, both (75) and (79) hold. Equating the coefficients we get

$$\frac{S_{C_1 \cup C_2}(C_1)G(C_1, C_2)[1 - S_\Omega(C_3)]}{G(C_1, C_2 \cup C_3)} = \frac{1 - S_\Omega(C_2 \cup C_3)}{G(C_1 \cup C_2, C_3)} \tag{96}$$

$$\frac{[1 - S_{C_1 \cup C_2}(C_1)]G(C_1, C_2)[1 - S_\Omega(C_3)]}{G(C_1, C_2 \cup C_3)} = \frac{S_{C_2 \cup C_3}(C_2)S_\Omega(C_2 \cup C_3)G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{97}$$

$$\frac{S_\Omega(C_3)}{G(C_1, C_2 \cup C_3)} = \frac{[1 - S_{C_2 \cup C_3}(C_2)]S_\Omega(C_2 \cup C_3)G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{98}$$

$$\frac{1 + [1 - S_\Omega(C_3)]\,[G(C_1, C_2) - 1]}{G(C_1, C_2 \cup C_3)} = \frac{1 + S_\Omega(C_2 \cup C_3)\,[G(C_2, C_3) - 1]}{G(C_1 \cup C_2, C_3)}. \tag{99}$$

Comparing these with the basic relations of Lemma 26 leads to

$$\frac{1 - S_\Omega(C_3)}{S_\Omega(C_1 \cup C_2)} = \frac{1 - S_\Omega(C_2 \cup C_3)}{S_\Omega(C_1)} \tag{100}$$

$$\frac{1 - S_\Omega(C_3)}{S_\Omega(C_1 \cup C_2)} = \frac{S_\Omega(C_2 \cup C_3)}{1 - S_\Omega(C_1)} \tag{101}$$

$$\frac{S_\Omega(C_3)}{1 - S_\Omega(C_1 \cup C_2)} = \frac{S_\Omega(C_2 \cup C_3)}{1 - S_\Omega(C_1)} \tag{102}$$

$$\frac{1 + [1 - S_\Omega(C_3)]\,[G(C_1, C_2) - 1]}{1 + S_\Omega(C_1 \cup C_2)\,[G(C_1, C_2) - 1]} = \frac{1 + S_\Omega(C_2 \cup C_3)\,[G(C_2, C_3) - 1]}{1 + [1 - S_\Omega(C_1)]\,[G(C_2, C_3) - 1]}. \tag{103}$$

As (100) and (101) have a common left hand side, equating their right hand sides we get

$$S_\Omega(C_1) + S_\Omega(C_2 \cup C_3) = 1. \tag{104}$$

Putting (104) back into (100) we get

$$S_\Omega(C_1 \cup C_2) + S_\Omega(C_3) = 1. \tag{105}$$

This proves (33). Conversely, it is clear that when (104) and (105) hold, the system (100)–(103) is indeed satisfied.

Case 2. Suppose that

$$(e, C_1; e, C_2) \succ e.$$

Consider the partition $(D_1, D_2, D_3) := (C_3, C_1, C_2)$. Then

$$(e, D_2; e, D_3) \succ e.$$

This implies the existence of consequences x_1, x_2 and x_3, with $x_1 \succ x_2 \succ x_3$, such that

$$(x_2, D_2; x_3, D_3) \succ x_1$$

holds. This corresponds to (78) and so the corresponding (79) holds. For such chosen $x_1 \succ x_2 \succ x_3$, we consider two cases:

Case 2.1. Suppose that some gamble satisfies $(x_1, D_1; x_2, D_2) \succsim x_3$, which corresponds to (72) and so the corresponding (73) holds.

Then, (73) and (79) hold for such gambles. Equating the coefficients we obtain the following relations

$$\frac{S_{D_1\cup D_2}(D_1)S_\Omega(D_1\cup D_2)G(D_1,D_2)}{G(D_1,D_2\cup D_3)} = \frac{1-S_\Omega(D_2\cup D_3)}{G(D_1\cup D_2,D_3)} \tag{106}$$

$$\frac{[1-S_{D_1\cup D_2}(D_1)]S_\Omega(D_1\cup D_2)G(D_1,D_2)}{G(D_1,D_2\cup D_3)} = \frac{S_{D_2\cup D_3}(D_2)S_\Omega(D_2\cup D_3)G(D_2,D_3)}{G(D_1\cup D_2,D_3)} \tag{107}$$

$$\frac{[1-S_\Omega(D_1\cup D_2)]}{G(D_1,D_2\cup D_3)} = \frac{[1-S_{D_2\cup D_3}(D_2)]S_\Omega(D_2\cup D_3)G(D_2,D_3)}{G(D_1\cup D_2,D_3)} \tag{108}$$

$$\frac{1+S_\Omega(D_1\cup D_2)\,(G(D_1,D_2)-1)}{G(D_1,D_2\cup D_3)} = \frac{1+S_\Omega(D_2\cup D_3)\,[G(D_2,D_3)-1]}{G(D_1\cup D_2,D_3)}. \tag{109}$$

Comparing (106) with (80):

$$\frac{S_{D_1\cup D_2}(D_1)S_\Omega(D_1\cup D_2)G(D_1,D_2)}{G(D_1,D_2\cup D_3)} = \frac{S_\Omega(D_1)}{G(D_1\cup D_2,D_3)}$$

we get

$$1-S_\Omega(D_2\cup D_3) = S_\Omega(D_1)$$

which, as we recall $(D_1, D_2, D_3) := (C_3, C_1, C_2)$, proves (33).

Case 2.2. Suppose that some gamble satisfies $(x_1, D_1; x_2, D_2) \prec x_3$. Then we are in the same situation as **Case 1.2**, leading to relations in parallel with (104) and (105):

$$S_\Omega(D_1) + S_\Omega(D_2\cup D_3) = 1 \tag{110}$$

$$S_\Omega(D_1\cup D_2) + S_\Omega(D_3) = 1 \tag{111}$$

where (110) proves (33). ■

6.7 Proposition 19

Proof. Let binary partition (C, D) be given. By assumption, C is not UofG-singular, therefore there exists a partition (C_1, C_2) of C such that $G(C_1, C_2) \neq 1$. Let $C_3 = D$, and consider the trinary partition (C_1, C_2, C_3) of Ω. By Lemma 17 we get

$$S_{C\cup D}(C) + S_{C\cup D}(D) = S_\Omega(C_1\cup C_2) + S_\Omega(C_3) = 1 \tag{112}$$

proving (30). ■

6.8 Theorem 20

Proof. Part 1. The assumptions of Proposition 14 hold, and then, with weight complementarity, (27) is reduced to: for $x, y \in X$ with $x \succsim y$,

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x)S_{C\cup D}(C) + \delta U(y)S_{C\cup D}(D)]\, G(C, D) \quad (113)$$

for ranked gambles. Because of the symmetry of the right hand side, (113) indeed holds for unranked binary gambles. We shall refer to that fact as *the symmetry of* (113).

Because of the symmetry of (113), there is no need to impose (72) in deriving (73) which results in

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\
&\quad = 1 + S_\Omega(C_1 \cup C_2)\,(G(C_1, C_2) - 1) \\
&\qquad + \delta U(x_1)S_{C_1\cup C_2}(C_1)S_\Omega(C_1 \cup C_2)G(C_1, C_2) \\
&\qquad + \delta U(x_2)S_{C_1\cup C_2}(C_2)S_\Omega(C_1 \cup C_2)G(C_1, C_2) \\
&\qquad + \delta U(x_3)S_\Omega(C_3) \qquad (114)
\end{aligned}$$

for all ranked trinary gambles. Similarly, by (113) and taking its symmetry into consideration, (77) gives

$$\begin{aligned}
&\frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\
&= 1 + S_\Omega(C_2 \cup C_3)\,[G(C_2, C_3) - 1] \\
&\quad + \delta U(x_1)S_\Omega(C_1) \\
&\quad + \delta U(x_2)S_{C_2\cup C_3}(C_2)S_\Omega(C_2 \cup C_3)G\,(C_2, C_3) \\
&\quad + \delta U(x_3)S_{C_2\cup C_3}(C_3)S_\Omega(C_2 \cup C_3)G\,(C_2, C_3) \qquad (115)
\end{aligned}$$

for all ranked trinary gambles.

If we equate the trinomial expression in the variables $U(x_1), U(x_2), U(x_s)$ for $U(x_1, C_1; x_2, C_2; x_3, C_3)$ obtained from (114) to that obtained from (115) and compare like terms we arrive at a system of equations:

$$\begin{aligned}
&S_{C_1\cup C_2}(C_1)S_\Omega(C_1 \cup C_2)G(C_1, C_2)G(C_1 \cup C_2, C_3) \\
&= S_\Omega(C_1)G(C_1, C_2 \cup C_3);
\end{aligned} \qquad (116)$$

$$\begin{aligned}
&S_{C_1\cup C_2}(C_2)S_\Omega(C_1 \cup C_2)G(C_1, C_2)G(C_1 \cup C_2, C_3) \\
&= S_{C_2\cup C_3}(C_2)S_\Omega(C_2 \cup C_3)G(C_1, C_2 \cup C_3)G(C_2, C_3);
\end{aligned} \qquad (117)$$

$$\begin{aligned}
&G(C_1 \cup C_2, C_3)[S_\Omega(C_1 \cup C_2)G(C_1, C_2) + S_\Omega(C_3)] \\
&= G(C_1, C_2 \cup C_3)[S_\Omega(C_2 \cup C_3)G(C_2, C_3) + S_\Omega(C_1)].
\end{aligned} \qquad (118)$$

Note that we have omitted the equation obtained from the coefficients of $U(x_3)$ because it is equivalent to (116). It is apparent that (117) is implied by (116) and can be omitted too.

Define

$$S^*_{C\cup D}(C) := S_{C\cup D}(C)G(C,D). \tag{119}$$

So (116) and (118) take, respectively, the forms

$$S^*_{C_1\cup C_2}(C_1)S^*_\Omega(C_1\cup C_2) = S^*_\Omega(C_1); \tag{120}$$

$$\begin{aligned} &G(C_1,C_2)S^*_\Omega(C_1\cup C_2) + S^*_\Omega(C_3) \\ &= G(C_2,C_3)S^*_\Omega(C_2\cup C_3) + S^*_\Omega(C_1). \end{aligned} \tag{121}$$

Multiplying (30) by $G(C_1,C_2)$ yields

$$S^*_{C_1\cup C_2}(C_1) + S^*_{C_1\cup C_2}(C_2) = G(C_1,C_2). \tag{122}$$

Using (122), (121) is equivalent to

$$\begin{aligned} &[S^*_{C_1\cup C_2}(C_1) + S^*_{C_1\cup C_2}(C_2)]S^*_\Omega(C_1\cup C_2) + S^*_\Omega(C_3) \\ &= [S^*_{C_2\cup C_3}(C_2) + S^*_{C_2\cup C_3}(C_3)]S^*_\Omega(C_2\cup C_3) + S^*_\Omega(C_1). \end{aligned} \tag{123}$$

Using (120) on each side of (123) yields the symmetric expression

$$\sum_{i=1}^{3} S^*_\Omega(C_i),$$

and so (120) implies (123). A parallel argument based on the event pair (C_2,C_3) shows that (118) is implied by (120) and so can be dismissed. We are now left with (120) whose solution[4] is given by

$$S^*_{C_1\cup C_2}(C_1) = \frac{\lambda(C_1)}{\lambda(C_1\cup C_2)} \tag{124}$$

for some $\lambda : \mathfrak{B}^* \to]0,\infty[$. Putting (124) into (122) we get

$$G(C_1,C_2) = \frac{\lambda(C_1)+\lambda(C_2)}{\lambda(C_1\cup C_2)}, \tag{125}$$

which gives (35). Using (124) and (125) in (119), for any disjoint C and D,

$$S_{C\cup D}(C) = \frac{S^*_{C\cup D}(C)}{G(C,D)} = \frac{\lambda(C)}{\lambda(C)+\lambda(D)},$$

which is (34). Clearly, (124):

$$S_{C\cup D}(C)G(C,D) = \frac{\lambda(C)}{\lambda(C\cup D)},$$

[4]Luce (1957) gave it for the finitely additive case. Luce et al. (2008), Theorem 6, extended it to the non-additive case using a result in Ebank et al. (1988).

which for $C \cup D = \Omega$ is (37). Putting the resulting forms of S and G given by (34) and (35) respectively into (113) and (114), and using (37), we obtain (36) for $n = 2, 3$.

Part 2 is a straightforward induction. Part 3 is an observation based on the fact that the right hand side of (36), for general n, is symmetric.Finally, Corollary 16 holds under the assumptions of Part 4. Therefore, (32) which is (113), holds and the proof of the representation is as in Part 1. ■

6.9 Theorem 21

Proof. The assumptions include those of Lemma 25, hence (80)-(83) hold.

1. The assumption that $G(C_1, C_2) = 1$ for all 3-partitions (C_1, C_2, C_3) means that

$$G(C, D) = 1, \quad C \cup D \text{ not maximal.} \tag{126}$$

If the algebra has no maximal event, then $G = 1$ follows, which is the ordinary default meaning of "no utility of gambling." So, with (126), (83) is reduced to

$$G(C_1, C_2 \cup C_3) = G(C_1 \cup C_2, C_3). \tag{127}$$

One can show that (127) leads to

$$G(C, D) = A, \quad C \cup D \text{ is maximal.} \tag{128}$$

for some constant $A > 0$. Together (126) and (128) is the full meaning of the terminology "no utility of gambling "in the subsequent discussion, albeit A needs not be 1.

2. Using the results of part 1, with(80), we have

$$S_{C_1 \cup C_2}(C_1) S_\Omega(C_1 \cup C_2) = S_\Omega(C_1),$$

which is the choice property (CP), (31). As noted earlier, the choice property has the representation (38).

With no utility of gambling and the choice property, (81) becomes

$$S_\Omega(C_1 \cup C_2) - S_\Omega(C_1) = S_{C_2 \cup C_3}(C_2) - S_{C_2 \cup C_3}(C_2) S_\Omega(C_1). \tag{129}$$

Substituting (38), (129) becomes

$$\begin{aligned} &\frac{\mu(C_1 \cup C_2)}{\mu(C_1 \cup C_2 \cup C_3)} - \frac{\mu(C_1)}{\mu(C_1 \cup C_2 \cup C_3)} \\ &= \frac{\mu(C_2)}{\mu(C_2 \cup C_3)} - \frac{\mu(C_2)}{\mu(C_2 \cup C_3)} \frac{\mu(C_1)}{\mu(C_1 \cup C_2 \cup C_3)}. \end{aligned}$$

Rearranging,

$$\frac{\mu(C_1 \cup C_2) - \mu(C_1)}{\mu(C_2)} = \frac{\mu(C_1 \cup C_2 \cup C_3) - \mu(C_1)}{\mu(C_2 \cup C_3)}. \tag{130}$$

We next show, for each non-null C and for all non-null D and $\tilde{D}$ which are disjoint from C, that (130) gives

$$\frac{\mu(C \cup D) - \mu(C)}{\mu(D)} = \frac{\mu(C \cup \tilde{D}) - \mu(C)}{\mu(\tilde{D})}. \tag{131}$$

To that end, we examine four cases:

Case 1. Suppose $D = \tilde{D}$. Then (131) holds trivially.

Case 2. Suppose $D \neq \tilde{D}$ and $D \subset \tilde{D}$. Then letting $C_1 = C$, $C_2 = D$ and $C_3 = \tilde{D} \backslash D$ in (130) we arrive at (131).

Case 3. Suppose $D \neq \tilde{D}$ and $\tilde{D} \subset D$. The argument for (131) is similar to that of Case 2.

Case 4. Suppose that $D \not\subset \tilde{D}$ and $\tilde{D} \not\subset D$. First, let $C_1 = C$, $C_2 = D, C_3 = \tilde{D} \backslash D$ in (130), then

$$\frac{\mu(C \cup D) - \mu(C)}{\mu(D)} = \frac{\mu(C \cup D \cup \tilde{D}) - \mu(C)}{\mu(D \cup \tilde{D})}. \tag{132}$$

Second, setting $C_1 = C$, $C_2 = \tilde{D}, C_3 = D \backslash \tilde{D}$ in (130) yields

$$\frac{\mu(C \cup \tilde{D}) - \mu(C)}{\mu(\tilde{D})} = \frac{\mu(C \cup D \cup \tilde{D}) - \mu(C)}{\mu(D \cup \tilde{D})}. \tag{133}$$

By (132) and (133), (131) follows.

Thus, we have (131) in all cases.

Hence, for each given non-null C there exists a constant, denoted by $f(C)$, say, such that

$$\frac{\mu(C \cup D) - \mu(C)}{\mu(D)} = f(C) \tag{134}$$

holds for all non-null D disjoint from C. Because μ is strictly increasing, we have

$$f(C) > 0 \tag{135}$$

for all non-null non-maximal C. We rewrite (134) as

$$\mu(C \cup D) = \mu(C) + \mu(D) f(C). \tag{136}$$

By the symmetry of the left side in C and D we get

$$\mu(D) + \mu(C) f(D) = \mu(C) + \mu(D) f(C).$$

Therefore,

$$\frac{f(C)-1}{\mu(C)} = \frac{f(D)-1}{\mu(D)},$$

for all disjoint non-null C, D. So

$$\frac{f(C)-1}{\mu(C)} = b$$

where b is a constant, leading to $f(C) = 1 + b\mu(C)$. In view of (135), b is restricted by (40).

Substituting f into (136) gives the p-additivity of μ which is holding for all disjoint non-null pairs C, D. Conversely, it is simple to check that μ satisfying (39) is a solution of (130). In turn, we have solved (129).

By no utility of gambling, (82) also reduces to

$$(1 - S_{C_2 \cup C_3}(C_2))\,(1 - S_\Omega(C_1)) = 1 - S_\Omega(C_1 \cup C_2)$$

which coincides with (129).

So, the system of equations (80)-(83) in the case of no utility of gambling is solved by (38) where μ is p-additive, (39).

3. Putting the above special value of G in (73), namely,

$$\begin{aligned}
&\frac{1+\delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\
&= 1 + S_\Omega(C_1 \cup C_2)\,(G(C_1, C_2) - 1) \\
&\quad + \delta U(x_1) S_{C_1 \cup C_2}(C_1) S_\Omega(C_1 \cup C_2) G(C_1, C_2) \\
&\quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)] S_\Omega(C_1 \cup C_2) G(C_1, C_2) \\
&\quad + \delta U(x_3)[1 - S_\Omega(C_1 \cup C_2)]
\end{aligned}$$

and using the properties of S, we get the representation of the no utility of gambling case:

$$\begin{aligned}
&\frac{1+\delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\
&= 1 + \delta U(x_1) S_\Omega(C_1) \\
&\quad + \delta U(x_2)[S_\Omega(C_1 \cup C_2) - S_\Omega(C_1)] \\
&\quad + \delta U(x_3)[1 - S_\Omega(C_1 \cup C_2)]
\end{aligned} \tag{137}$$

where $G(C_1 \cup C_2, C_3) = 1$ except for the case $C_1 \cup C_2 \cup C_3$ is maximal, in which case $G(C_1 \cup C_2, C_3) = A$.

We now restrict out attention to the case $A = 1$ only. Then (137) simplifies to:

$$\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_3, C_3) \\
&= U(x_1) S_\Omega(C_1) \\
&\quad + U(x_2)[S_\Omega(C_1 \cup C_2) - S_\Omega(C_1)] \\
&\quad + U(x_3)[1 - S_\Omega(C_1 \cup C_2)].
\end{aligned} \tag{138}$$

Now we develop the representation for gambles of length 4 using branching, UGD and the choice property (CP), and then develop the representation for higher lengths using induction.
Using branching and CP we get:

$$\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_3, C_3; x_4, C_4) \\
&= U((x_1, C_1; x_2, C_2; x_3, C_3); C_1 \cup C_2 \cup C_3; x_4, C_4) \\
&= U(x_1) S_{C_1 \cup C_2 \cup C_3}(C_1) S_\Omega(C_1 \cup C_2 \cup C_3) \\
&\quad + U(x_2)[S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2) - S_{C_1 \cup C_2 \cup C_3}(C_1)] S_\Omega(C_1 \cup C_2 \cup C_3) \\
&\quad + U(x_3)[1 - S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2)] S_\Omega(C_1 \cup C_2 \cup C_3) \\
&\quad + U(x_4)[1 - S_\Omega(C_1 \cup C_2 \cup C_3)] \\
&= U(x_1) S_\Omega(C_1) \\
&\quad + U(x_2)[S_\Omega(C_1 \cup C_2) - S_\Omega(C_1)] \\
&\quad + U(x_3)[S_\Omega(C_1 \cup C_2 \cup C_3) - S_\Omega(C_1 \cup C_2)] \\
&\quad + U(x_4)[1 - S_\Omega(C_1 \cup C_2 \cup C_3)]. \qquad (139)
\end{aligned}$$

Using UGD and CP we get

$$\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_3, C_3; x_4, C_4) \\
&= U(x_1, C_1; (x_2, C_2; x_3, C_3; x_4, C_4); C_2 \cup C_3 \cup C_4) \\
&= U(x_1) S_\Omega(C_1) \\
&\quad + U(x_2) S_{C_2 \cup C_3 \cup C_4}(C_2)[1 - S_\Omega(C_1)] \\
&\quad + U(x_3)[S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3) - S_{C_2 \cup C_3 \cup C_4}(C_2)][1 - S_\Omega(C_1)] \\
&\quad + U(x_4)[1 - S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3)][1 - S_\Omega(C_1)]. \qquad (140)
\end{aligned}$$

Equating the above two forms yields

$$S_\Omega(C_1 \cup C_2) - S_\Omega(C_1) = S_{C_2 \cup C_3 \cup C_4}(C_2)[1 - S_\Omega(C_1)] \qquad (141)$$

$$\begin{aligned}
&S_\Omega(C_1 \cup C_2 \cup C_3) - S_\Omega(C_1 \cup C_2) \\
&= [S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3) - S_{C_2 \cup C_3 \cup C_4}(C_2)][1 - S_\Omega(C_1)] \qquad (142)
\end{aligned}$$

$$1 - S_\Omega(C_1 \cup C_2 \cup C_3) = [1 - S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3)][1 - S_\Omega(C_1)]. \qquad (143)$$

But adding these three equations yields a trivial identity, and so (143) follows from (141) and (142). Writing out (141), (142) and (129) with explicit $\Omega = C_1 \cup C_2 \cup C_3 \cup C_4$ we have the system

$$\begin{aligned}
&S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2) - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1) \\
&\quad = S_{C_2 \cup C_3 \cup C_4}(C_2)[1 - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1)] \qquad (144)
\end{aligned}$$

$$\begin{aligned}
&S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2 \cup C_3) - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2) \\
&\quad = [S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3) - S_{C_2 \cup C_3 \cup C_4}(C_2)][1 - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1)] \qquad (145)
\end{aligned}$$

$$S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2) - S_{C_1 \cup C_2 \cup C_3}(C_1) = S_{C_2 \cup C_3}(C_2)[1 - S_{C_1 \cup C_2 \cup C_3}(C_1)]. \qquad (146)$$

If we replace C_3 by $C_3 \cup C_4$ in (146), we get (144).

If we replace C_2 by $C_2 \cup C_3$ and C_3 by C_4 in (146) we get

$$S_{C_1\cup C_2\cup C_3\cup C_4}(C_1 \cup C_2 \cup C_3) - S_{C_1\cup C_2\cup C_3\cup C_4}(C_1)$$
$$= S_{C_2\cup C_3\cup C_4}(C_2 \cup C_3)[1 - S_{C_1\cup C_2\cup C_3\cup C_4}(C_1)].$$

Subtracting (144) from (146) we get (145). Therefore, each equation in (141)–(143) is a consequence of (129). So we have shown that when we go from $n = 3$ gambles to $n = 4$ gambles, there are no additional conditions imposed on S_Ω.

By simple induction, for the case $A = 1$, we can show that (138) and (139) yields the representation for general length n, (42), where S satisfies CP and (129). ■

6.10 Proposition 23

Proof. Apply the p-additive form to (45):

$$\begin{aligned}
&1 + \delta U\left[(x, C; y, D) \oplus (e, C'; e, D')\right] = 1 + \delta U\left[(x, C; e, D) \oplus (e, C'; y, D')\right] \\
&\Leftrightarrow [1 + \delta U(x, C; y, D)]G(C', D') = [1 + \delta U(x, C; e, D)][1 + \delta U(e, C'; y, D')] \\
&\Leftrightarrow [1 + \delta U(KE(x, C; y, D))]G(C, D)G(C', D') \\
&\quad = [1 + \delta U(KE(x, C; e, D))]G(C, D)[1 + \delta U(KE(e, C'; y, D'))]G(C', D') \\
&\Leftrightarrow 1 + \delta U(KE(x, C; y, D)) = [1 + \delta U(x)S_{C\cup D}(C)][1 + \delta U(y)S_{C'\cup D'}(D')] \\
&\quad = [1 + \delta U(x)S_{C\cup D}(C)][1 + \delta U(y)S_{C\cup D}(D)],
\end{aligned}$$

which is (46) and (47) follows immediately. ■

6.11 Proposition 24

Proof. Rewrite (47) in the form:

$$\begin{aligned}
U(x, C; y, D) &= U(x)S_{C\cup D}(C)G(C, D) + U(y)S_{C\cup D}(D)G(C, D) \\
&+ \delta U(x)S_{C\cup D}(C)U(y)S_{C\cup D}(D)G(C, D) + H(C, D).
\end{aligned} \tag{147}$$

Because of (6), H and thus G are symmetric functions. We shall be using the symmetry of H and G without further mention.

Recall that we defined

$$S^*_{C\cup D}(C) := S_{C\cup D}(C)G(C, D), \tag{148}$$

and note that, by definition, $\delta H = G - 1$. So (147) becomes:

$$\begin{aligned}
U(x, C; y, D) &= U(x)S^*_{C\cup D}(C) + U(y)S^*_{C\cup D}(D) \\
&+ \frac{\delta U(x)U(y)}{G(C, D)}S^*_{C\cup D}(C)S^*_{C\cup D}(D) + H(C, D).
\end{aligned} \tag{149}$$

Because of the symmetry of (149), there is no need in the following to concern ourselves with the rank order between x_3 and $(x_1, C_1; x_2, C_2)$.

Using branching and then (149), we get

$$\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_3, C_3) \\
&= U(x_1) S^*_{C_1 \cup C_2}(C_1) S^*_\Omega(C_1 \cup C_2) \\
&\quad + U(x_2) S^*_{C_1 \cup C_2}(C_2) S^*_\Omega(C_1 \cup C_2) \\
&\quad + U(x_3)[S^*_\Omega(C_3) + \frac{\delta}{G(C_1 \cup C_2, C_3)} H(C_1, C_2) S^*_\Omega(C_1 \cup C_2) S^*_\Omega(C_3)] \\
&\quad + \frac{\delta U(x_1) U(x_2)}{G(C_1, C_2)} S^*_{C_1 \cup C_2}(C_1) S^*_{C_1 \cup C_2}(C_2) S^*_\Omega(C_1 \cup C_2) \\
&\quad + \frac{\delta U(x_1) U(x_3)}{G(C_1 \cup C_2, C_3)} S^*_{C_1 \cup C_2}(C_1) S^*_\Omega(C_1 \cup C_2) S^*_\Omega(C_3) \\
&\quad + \frac{\delta U(x_2) U(x_3)}{G(C_1 \cup C_2, C_3)} S^*_{C_1 \cup C_2}(C_2) S^*_\Omega(C_1 \cup C_2) S^*_\Omega(C_3) \\
&\quad + \frac{\delta^2 U(x_1) U(x_2) U(x_3)}{G(C_1, C_2) G(C_1 \cup C_2, C_3)} S^*_{C_1 \cup C_2}(C_1) S^*_{C_1 \cup C_2}(C_2) S^*_\Omega(C_1 \cup C_2) S^*_\Omega(C_3) \\
&\quad + H(C_1, C_2) S^*_\Omega(C_1 \cup C_2) + H(C_1 \cup C_2, C_3). \qquad (150)
\end{aligned}$$

Using UGD and then (149), we get

$$\begin{aligned}
&U(x_1, C_1; x_2, C_2; x_3, C_3) \\
&= U(x_1) \left[S^*_\Omega(C_1) + \frac{\delta}{G(C_1, C_2 \cup C_3)} H(C_2, C_3) S^*_\Omega(C_1) S^*_\Omega(C_2 \cup C_3) \right] \\
&\quad + U(x_2) S^*_{C_2 \cup C_3}(C_2) S^*_\Omega(C_2 \cup C_3) \\
&\quad + U(x_3) S^*_{C_2 \cup C_3}(C_3) S^*_\Omega(C_2 \cup C_3) \\
&\quad + \frac{\delta U(x_1) U(x_2)}{G(C_1, C_2 \cup C_3)} S^*_\Omega(C_1) S^*_{C_2 \cup C_3}(C_2) S^*_\Omega(C_2 \cup C_3) \\
&\quad + \frac{\delta U(x_1) U(x_3)}{G(C_1, C_2 \cup C_3)} S^*_\Omega(C_1) S^*_{C_2 \cup C_3}(C_3) S^*_\Omega(C_2 \cup C_3) \\
&\quad + \frac{\delta U(x_2) U(x_3)}{G(C_2, C_3)} S^*_{C_2 \cup C_3}(C_2) S^*_{C_2 \cup C_3}(C_3) S^*_\Omega(C_2 \cup C_3) \\
&\quad + \frac{\delta^2 U(x_1) U(x_2) U(x_3)}{G(C_1, C_2 \cup C_3) G(C_2, C_3)} \\
&\quad \times S^*_\Omega(C_1) S^*_{C_2 \cup C_3}(C_2) S^*_{C_2 \cup C_3}(C_3) S^*_\Omega(C_2 \cup C_3) \\
&\quad + H(C_2, C_3) S^*_\Omega(C_2 \cup C_3) + H(C_1, C_2 \cup C_3). \qquad (151)
\end{aligned}$$

Because we may vary the x_i independently, the coefficients of the trinomials in the equation (150) and (151) must be equal. In particular, two equations hold for all partitions (C_1, C_2, C_3):

$$S^*_{C_2\cup C_3}(C_2)S^*_\Omega(C_2\cup C_3) = S^*_{C_1\cup C_2}(C_2)S^*_\Omega(C_1\cup C_2), \tag{152}$$

and

$$\frac{S^*_\Omega(C_1)S^*_{C_2\cup C_3}(C_2)S^*_\Omega(C_2\cup C_3)}{G(C_1, C_2\cup C_3)} = \frac{S^*_{C_1\cup C_2}(C_1)S^*_{C_1\cup C_2}(C_2)S^*_\Omega(C_1\cup C_2)}{G(C_1,C_2)}. \tag{153}$$

Dividing (153) by (152) side by side we get

$$\frac{1}{G(C_1, C_2\cup C_3)}S^*_\Omega(C_1) = \frac{1}{G(C_1,C_2)}S^*_{C_1\cup C_2}(C_1).$$

By the definition of S^*, it is equivalent to

$$S_\Omega(C_1) = S_{C_1\cup C_2}(C_1).$$

This proves (48). ■

References

Aczél, J., Maksa, G., & Páles, Z. Solution of a functional equation arising from different ways of measuring utility. *Journal of Mathematical Analysis and Applications* **233,** 740-748 (1999).

Cho, Y.-H., & Fisher, G. R. Receiving two consequences: Tests of monotonicity and scale invariance. *Organizational Behavior and Human Decision Processes* **83**, 61-81 (2000)

Ebanks, B., Kannappan, P., & Ng, C. T. (1988) Recursive inset entropies of multiplicative type on open domains, *Aequationes Mathematicae* **36**, 268-293.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971) *Foundations of Measurement. Vol. I.* Academic Press, New York, N. Y.

Luce, R. D. (1957/2005) Individual Choice Behavior. Wiley, New York, N. Y.; reprinted by Dover, Mineola, N. Y.

Luce, R. D. (2000) Utility of Gains and Losses. Mahwah, NJ: Erlbaum

Luce, R. D., Marley, A. A. J. (2000) On elements of chance. *Theory and Decision* **49**, 97-126

Luce, R. D., Ng, C. T., Marley, A. A. J., Aczél, J. (2008a) Utility of gambling I: Entropy-modified linear weighted utility. *Economic Theory* **36**, 1-33. See the errata at http://www.imbs.uci.edu/personnel/luce/luce.html.

Luce, R. D., Ng, C. T., Marley, A. A. J., Aczél, J. (2008b) Utility of gambling II: Risk, Paradoxes, and Data. *Economic Theory* **36**, 165-187. See the errata at http://www.imbs.uci.edu/personnel/luce/luce.html.

Marley, A. A. J., & Luce, R. D. (2002) A simple axiomatization of binary rank-dependent utility of gains (losses). *Journal of Mathematical Psychology* **46**, 40-55.

Marley, A. A. J., & Luce, R. D. (2005) Independence properties vis-à-vis several utility representations. *Theory and Decision* **58**, 77-143.

Ng, C. T., Luce, R. D., Marley, A. A. J. (in press, a) Utility of gambling: Extending the Approach of Meginniss (1976). *Aequationes Mathematicae.*

Ng, C. T., Luce, R. D., Marley, A. A. J. (in press, b) Utility of gambling when Events are Valued: An Application of Inset Entropy. *Theory and Decision.* See the errata at http://www.imbs.uci.edu/personnel/luce/luce.html.

Savage, L. J. (1954) The foundations of probability. Wiley, New York, NY