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Interpersonal Comparisons of Utility for People With Non-trivial p-Additive Joint Receipts*

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Abstract: This article argues that there is a natural solution to carrying out interpersonal comparisons of utility when the theory is supplemented with a group operation of joint receipts. If so, 3 types of people can exist, and the 2 types with multiplicative representations of joint receipt have, in contrast to most utility theories, absolute scales of utility. That makes possible, at least in principle, meaningful interpersonal comparisons of utility with desirable properties, thus resolving a long standing philosophical problem having potentially important implications in economics. Two behavioral criteria are given for the 3 classes of people. At this point their relative sizes are unknown.

Key words: interpersonal comparison of utility, joint receipt, p-additive utility, welfare economics, weighted utility

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Both the economic and philosophic literatures have focussed some discussion on the issue of interpersonal comparisons (briefly, IPC) of utility. With some exceptions, the consensus of many seems to be that we do not have fully satisfactory ways to justify such comparisons. Narens & Luce (1983) showed their impossibility in an ordinal context. Hammond (1991) is especially forceful in the position that, using standard utility theories, no such comparisons are possible, whereas Nozick (1985) has claimed that they can be made by equating kinks in the utility functions. Nonetheless, in practice, welfare (not utility per se) comparisons are attempted both at the individual level, as in discussions of family compromises, and at the social level in welfare economics without directly invoking utility. Detailed discussions may be found in Harsanyi (1977), Elster and Roemer (1991), List (2003), and Binmore (forthcoming) who provides a brief but useful summary. The fundamental point is that for ratio and interval scales, there is no principled way to choose what are "equal" units of utility for different people. These difficulties, of course, raise huge theoretical problems for welfare economics that do not seem to have been effectively resolved.

In this article a new approach is explored that is based on the idea that not only do gambles have representations that involve both addition and multiplication, but also so should the representations of joint receipts which are assumed to interlock with gambles in a fully rational fashion (segregation, (9)). This leads to representations which turn out to have three quite distinct forms. One is additive over joint receipts, which really is the classical case, and the resulting scale is a ratio one for which IPCs are impossible. The other two differ in having, instead, multiplicative representations that are absolute scales. For these two sub-cases, I propose a simple hypothesis as to what IPC means and arrive at a few of its elementary properties.

It should be added that these ideas by no means solve the issues of social welfare comparisons, although they may lay the ground work for a new approach to it. That remains to be seen.

1 p-Additive Representations

Suppose that X is the set of consequences under consideration, that $\succsim$ is a weak (preference) order over X, and that $\oplus$ is a binary operation (of *joint receipt*) on X. We assume that[1] $\langle X, e, \oplus, \sim\rangle$ satisfies the usual axioms of an abelian (weakly commutative) group with identity e and that $\langle X, e, \oplus, \succsim\rangle$ satisfies[2] usual axioms

[1] As usual, $x \sim y$ means both $x \succsim y$ and $y \succsim x$.

[2] Because the indifference relation $\sim$ is an equivalence relation, we are in reality working with equivalence classes. The induced structure on equivalence classes is a solvable Archimedean ordered abelian group with an isomorphism onto the additive real numbers. This, of course, induces a homomorphism of the original structure.

For the equivalence classes, the operation $\oplus$ is closed, has an identity, is commutative and associative, and each element has an inverse.

of a solvable, Archimedean ordered, abelian group (Hölder, 1901; Krantz, Luce, Suppes, & Tversky, 1971, Chapters 2 and 3). Examples of joint receipt are ubiquitous: Choosing two commodities, such as steak and a can of soup, at a store. As we are all aware, steak is an uncertain alternative whereas a can of soup is highly standard and is often treated as without risk. The steak can be imbedded in X by invoking its certainty equivalent. Another familiar example is receiving a check and a bill in the mail. Both examples generalize to any finite size bundles because the operation is associative.

Instead of investigating the representations into the real numbers $\mathbb{R}$ under just addition, $\langle \mathbb{R}, \geq, + \rangle$, as is usually done, let us suppose, as is true of theories for uncertain alternatives,[3] that the representations of $\oplus$ are onto suitable (defined below) subintervals of $\langle \mathbb{R}, \geq, +, \times \rangle$ that are closed under $+$ and $\times$ (which denotes multiplication and is usually suppressed by writing $\alpha \times \beta = \alpha\beta$, where α, β are in $\mathbb{R}$). Under standard assumptions, the only one consistent with Hölder's axioms has the form:[4]

$$U^*(x \oplus y) = U^*(x) + U^*(y) + \delta^* U^*(x) U^*(y) \tag{1}$$

(Luce, 2000, p. 152). Such a representation is called a p-additive (p is brief for polynomial) utility representation because it is the only polynomial form with $U^*(e) = 0$ that can be transformed into an additive representation when $\delta^* \neq 0$ (by the logarithm of (7) below).

Note that for the representation (1) to make dimensional sense when $\delta^* \neq 0$, the unit of δ^* must be the unit of $(U^*)^{-1}$, i.e., that $\delta^* U^*$ is dimensionless in those cases. So we define

$$\delta := sgn(\delta^*), \tag{2}$$

i.e., the sign of δ^*, and for $\delta \neq 0$

$$U := |\delta^*| \, U^*. \tag{3}$$

So U is dimensionless when $\delta = -1$ or 1. We use the notations U and δ from now on.

1.1 Representations for $\delta = 0$ and $\delta \neq 0$

For the case $\delta = 0$, the U representation corresponding to (1) simplifies to the purely additive form

$$U(x \oplus y) = U(x) + U(y), \tag{4}$$

and the utility of both the gains and losses are unbounded because n iterations of x, denoted nx, has $U(nx) = nU(x)$. Moreover, the additive case is of ratio scale type because e is the identity of $\oplus$.

[3] Often, although somewhat misleadingly, called *gambles*. I use that term below.

[4] The notation U^* is unusual, but I want to reserve U for a related function.

For $\delta \neq 0$ we may rewrite (1) in terms of U as

$$1 + \delta U(x \oplus y) = [1 + \delta U(x)]\,[1 + \delta U(y)]\,, \tag{5}$$

where, now, $\delta = -1$ or 1. Ng, Luce, & Marley (in preparation) examine these cases, showing that under the conditions needed to prove results below, for $\delta = 1$, U maps onto $\left]-\frac{1}{\delta}, \infty\right[\; = \left]-1, \infty\right[$ which simply means that the utility for gains ($x \succsim e$) is (subjectively) unbounded, whereas the disutility for losses ($x \precsim e$) is bounded. For $\delta = -1$, U maps onto $\left]-\infty, -\frac{1}{\delta}\right[\; = \left]-\infty, 1\right[$ so the disutility of losses is (subjectively) unbounded whereas that for gains is bounded.

This means that to the degree that the axioms justifying the representation are satisfied, we may expect people to fall into one of three quite distinct categories with inherently different non-linear forms. A major empirical implication of this fact is that it is very unwise to average data from people who have not been pre-screened for at least type. This is because when one averages inherently different functions, that average function may well be unlike any of it components (see Fig. 1 in Section 3). Therefore, one needs, at a minimum, to ascertain to which of the three types each person belongs, which we take up in Section 1.3.

Observe that for $\delta \neq 0$, the transformation

$$V(x) := 1 + \delta U(x), \tag{6}$$

which we refer to as a *value* representation, is an absolute scale because U, and so δU, is absolute. The V scale maps onto $]0, \infty[$ in both non-zero cases. The difference being that V is order preserving for $\delta = 1$ and order reversing for $\delta = -1$. Moreover, we see that together (5) and (6) yield the multiplicative representation

$$V(x \oplus y) = V(x)V(y). \tag{7}$$

Of course, in general, such a multiplicative representation is unique only up to an arbitrary power[5], but as we shall see even that degree of freedom is lost.

1.2 Binary uncertain gambles

Let x and y be consequences from X, and let C and D be disjoint chance events arising in a family of chance "experiments." A binary gamble is based on an experiment whose "universal set" may be partitioned into C and D. Each subevent leads to its own consequence, so that the gamble has two chance branches (x, C) and (y, D). We write it as $(x, C; y, D)$.

For $x \succsim y$, observe that over the equivalence classes

$$x \oplus y^{-1} = z \Leftrightarrow x = z \oplus y. \tag{8}$$

Two key assumptions linking the representations arising from joint receipt and those from the gambling structure are:

[5] Note that no scale factor $\alpha \neq 1$ maintains the multiplicative representation, (7).

Segregation:

$$(x \oplus y^{-1}, C; e, D) \oplus y \sim (x, C; y, D). \tag{9}$$

Note the highly rational aspect of this property: On the left the consequences are $(x \oplus y^{-1}) \oplus y = x$ if C occurs and $e \oplus y = y$ if D occurs, and the right of course yields exactly the same thing.

Separability: The set of so-called unitary gambles of gains, those of the form $(x, C; e, D)$, where $x \succsim e$, satisfies a version of the axioms of conjoint structure on $X \times \mathcal{B}$, where $\mathcal{B}$ is the algebra of events.[6]

For the case $\delta = 0$, Luce (2000, Theorem 4.4.4) shows that under certain density assumptions, segregation, and separability, that there is a ratio scale utility function U over gains, $x \succsim e$, and a subjective weighting function $S_{C\cup D}$ with the following properties. U and $S_{C\cup D}$ are onto intervals and

$$U(x, C; e, D) = U(x)S_{C\cup D}(C) \quad (x \succsim e). \tag{10}$$

By segregation one obtains for any $x, y \in X$ with $x \succ y$,

$$U(x, C; y, D) = U(x)S_{C\cup D}(C) + U(y)\left[1 - S_{C\cup D}(C)\right]. \tag{11}$$

Assuming that there is a p-additive representation U' with $\delta \neq 0$ and a separable representation $(U'', S''_{C\cup D})$, Ng, Luce, and Marley (in preparation), drawing upon Luce (2000, Theorem 4.4.6) for gains, show that under the above assumptions there is a unique U that is p-additive and $(U, S_{C\cup D})$ is separable, that (11) holds and, with the definition (6) of V also yields, for binary gambles with $x \succ y$,

$$V(x, C; y, D) = V(x)S_{C\cup D}(C) + V(y)\left[1 - S_{C\cup D}(C)\right]. \tag{12}$$

Note that if separability (10) holds not just for gains, but for all $x \in X$, then the following stronger property holds also:

Complementarity:

$$S_{C\cup D}(C) + S_{C\cup D}(D) = 1. \tag{13}$$

With this property, the rank dependence in (11) and (12) disappears.

1.3 Behavioral criteria to distinguish $\delta = 1, 0, -1$

An obvious question to be addressed is how to tell whether $\delta = 1$, 0, or -1. As Marley[7] has remarked, Theorem 4.7, p. 157 of Keeney & Raiffa (1993) addresses a related question but in a much more restricted context. Part II of the following Proposition was motivated by their discussion (p. 230) of *additive independence* in risky money gambles defined by

$$(x + x', \frac{1}{2}; y + y', \frac{1}{2}) \sim (x + y', \frac{1}{2}; y + x', \frac{1}{2}).$$

[6] A formal axiomatization was given by Marley & Luce (2002) for gains (and losses), but for present purposes all that is really needed is (10).

[7] I thank A.A.J.Marley for reminding me of this idea. I made slight changes of notation.

That together with comments by Igor Kopylov (personal communication, February, 2008) and Peter Wakker (personal communication, March 2008) led to the following development of Part 2 of Proposition 2.

It is quite easy to verify in (11) and (12) that if we impose segregation and separability without the condition that $x \succ y$, then both lead to complementarity, (13).

Lemma 1 *Suppose that a p-additive representation exists, that both segregation and separability for gains hold,*[8] *that uncertain binary gambles satisfy (11) for* $\delta = 0$ *and (12) for* $\delta \neq 0$*, and that complementarity (13) holds. If there exist consequences* $x \succ y$ *and events* C, D *such that*

$$(x, C; y, D) \sim (x, D; y, C). \tag{14}$$

then

$$S_{C\cup D}(C) = S_{C\cup D}(D) = \frac{1}{2}. \tag{15}$$

Part I of the following Proposition and its proof are due to C. T. Ng, whom I thank.

Proposition 2 *Suppose that a p-additive representation exists, that both segregation and separability for gains hold,*[9] *that uncertain binary gambles satisfy (11) for* $\delta = 0$ *and (12) for* $\delta \neq 0$*.*

Part I. *Then the following two statements are equivalent:*

1.

$$\delta = \left\{ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right\}. \tag{16}$$

2. *For all* $x, y \in X$*, with* $x \succ e, y \succ e$*, and for all non-trivial event partitions* $\{C, D\}$ *with independent replications* $\{C', D'\}$ *and* $\{C'', D''\}$*, then*

$$(x \oplus y, C; e, D) \left\{ \begin{array}{c} \succ \\ \sim \\ \prec \end{array} \right\} (x, C'; e, D') \oplus (y, C''; e, D''). \tag{17}$$

Part II. *Suppose, further, that complementarity, (13), holds for the structure and* C *and* D *satisfy (15) of Lemma 1, then, for all* $x \succ x' \succ y \succ y'$*, the above two conditions are also equivalent to*

$$(x \oplus x', C; y \oplus y', D) \left\{ \begin{array}{c} \succ \\ \sim \\ \prec \end{array} \right\} (x \oplus y, C; x' \oplus y', D). \tag{18}$$

[8] These three assumptions are redundant because, as shown in Luce (2000, Theorem 4.4.4) any two implies the third.

[9] Again, these three assumptions are redundant.

Clearly, the criterion (17) is more general and easier to check empirically than is (18) which rests upon finding events that are subjectively $\frac{1}{2}$.

Despite that disadvantage, the criterion (18) does not appear to be unduly difficult to check empirically when dealing with money lotteries with known probabilities and it has a very simple intuitive interpretation. So, if for money $x \oplus y = x + y$, as seems quite plausible, then we are speaking of 50:50 bets $(x, \frac{1}{2}; y, \frac{1}{2})$ that have the same expected value but with a larger variance on the left than the right. Often these are called, respectively, risky and safe lotteries. To be specific, several examples of Part II of Proposition 2 are

$$\left(160, \frac{1}{2}; 50, \frac{1}{2}\right) \quad \text{versus} \quad \left(140, \frac{1}{2}; 70, \frac{1}{2}\right)$$
$$\left(190, \frac{1}{2}; 30, \frac{1}{2}\right) \quad \text{versus} \quad \left(120, \frac{1}{2}; 100, \frac{1}{2}\right)$$
$$\left(1900, \frac{1}{2}; 60, \frac{1}{2}\right) \quad \text{versus} \quad \left(1050, \frac{1}{2}; 910, \frac{1}{2}\right).$$

Do you feel $\prec, \sim, \succ$ for each? I, for one, am clearly a $\delta = -1$ ($\prec$) type at least for money gambles. My guess is that people who gamble a good deal are of type $\delta = 1$. Are there any who are of type $\delta = 0$? If not, classical utility theory is not descriptive.

It is clearly important to examine one of the criteria carefully for fairly large and varied populations to estimate, in particular, first, whether the $\delta = 0$ class is non-empty and, second, the relative sizes of the $\delta = 1$ and -1 subpopulations.

2 Interpersonal Comparisons of Utility

Next, I propose an hypothesis concerning interpersonal comparisons (IPC) of utility for the cases with $\delta \neq 0$. Until we have carried out the experiments implicit in Proposition 2, there can be no certainty about just how large a class of people this covers.

But first, let us consider what properties we expect a potential concept of IPC to exhibit. We list possible demands.

2.1 Demands on a definition of IPC

Let k, l, and m be three different people and let $x_{(k)} \approx y_{(l)}$ mean that consequence x for person k has the "same utility" as consequence y for person l. The minimal properties the many feel such an IPC relation should satisfy are:

(i) Reflexivity:

$$\forall x \in X,\ x_{(k)} \approx x_{(k)}. \tag{19}$$

(ii) Symmetry:

$$\forall x, y \in X,\ x_{(k)} \approx y_{(l)} \Leftrightarrow y_{(l)} \approx x_{(k)}. \tag{20}$$

(iii) Transitivity: If $\delta_l \neq 0, \delta_k \neq 0$, and $\delta_m \neq 0$, then for $\forall x, y, z \in X$:

$$\text{If } \; x_{(k)} \approx y_{(l)} \; \textit{ and } \; y_{(l)} \approx z_{(m)}, \; \text{ then } \; x_{(k)} \approx z_{(m)}. \tag{21}$$

Transitivity simply asserts that if one knows that (x, y) is a matching pair between k and l and that (y, z) is a match between l and m, then by transitivity one knows that (x, z) is a match between k and m.

Any $\approx$ satisfying (i)-(iii) is, of course, an equivalence relation.

A rather more controversial condition is whether or not we should expect:

(iv) Invariance under joint receipt:

$$\text{If } \; x_{(k)} \approx y_{(l)} \; \text{ and } \; x'_{(k)} \approx y'_{(l)}, \; \text{ then } \; (x \oplus x')_{(k)} \approx (y \oplus y')_{(l)}. \tag{22}$$

We do not subscript $\oplus$ by the individual because joint receipt is a purely objective matter.

2.2 A possible definition of IPC for $\delta \neq 0$

In the non-additive cases, suppose that we define **interpersonal comparability (IPC)** by:

$$x_{(k)} \approx y_{(l)} \Leftrightarrow U_k(x) = U_l(y). \tag{23}$$

This is well defined because with $\delta \neq 0$, U_k and U_l are absolute scales and so no ambiguity arises in equating them. This contrasts with the unit ambiguity in the case of interval or ratio scales. Of course, in practice it will not be easy to establish $\approx$ because it is necessary to estimate U_k and U_l, which, although possible in principle, can often be difficult to do in practice. The key issue is establishing for any utility estimate using standard methods exactly the value of the asymptote, which can then be used to normalize that function.

As will be shown, the equivalence relation property of $\approx$ follows fairly readily. But, in general, we will not be able to satisfy invariance under joint receipt. Intuitively, the reason is that the utility of two things simply need not "add up" in the same way for different types of people, so in general $U_k(x \oplus x') \neq U_l(y \oplus y')$ when $U_k(x) = U_l(y)$ and $U_k(x') = U_l(y')$. For example (see Section 3 below), suppose that the domain X is amounts of money, that $x \oplus y = x + y$, and that there is a constant $\alpha_k > 0$ such that $U_k(x) = \alpha_k x$ and $\alpha_l > 0$ such that $U_l(x) = 1 - e^{\alpha_l x}$. So U_k is additive and U_l is p-additive with $\delta = -1$. Choose x, x', y and y' such that

$$U_k(x) = U_l(y) \Leftrightarrow \alpha_k x = 1 - e^{\alpha_l y}$$
$$U_k(x') = U_l(y') \Leftrightarrow \alpha_k x' = 1 - e^{\alpha_l y'},$$

then

$$U_k(x \oplus x') = U_k(x + x') = \alpha_k(x + x')$$

and

$$U_l(y \oplus y') = U_l(y + y') = 1 - e^{\alpha_l(y+y')}$$
$$\neq \alpha_k(x + x') = U_k(x \oplus x').$$

There is nothing special about the choice of these U functions except that they are not equal and are contained within the families discussed in Proposition 5 and its Corollary below.

Assuming that IPC is given by (23), we explore the above four properties.

2.3 Two $\delta \neq 0$ people of the same type

Suppose that people k and l have either $\delta_k = \delta_l = 1$ or $\delta_k = \delta_l = -1$. In these cases the multiplicative representations V_k and V_l are absolute scales and so (23) is equivalent to:

$$x_{(k)} \approx y_{(l)} \Leftrightarrow V_k(x) = V_l(y). \tag{24}$$

Proposition 3 *Suppose that a p-additive representation exists, that both segregation and separability over gains hold, that uncertain binary gambles satisfy (12).*[10] *If two individuals l and k are both p-additive with $\delta_l = \delta_k = 1$ or -1, then $\approx$ defined by (23) is an equivalence relation and is invariant under joint receipt.*

2.4 Two $\delta \neq 0$ people of different type

Question: What about the case where $\delta_k = 1$ and $\delta_l = -1$? In this case matches can only arise in the common region where both utility functions exist, namely:

$$\mathcal{D}_{-1,1} = \{(x_{(k)}, y_{(l)})\, |U_k(x) = U_l(y) \in]{-1}, 1[\}.$$

Thus, for $(x_{(k)}, y_{(l)}) \in \mathcal{D}_{-1,1}$ we have $U_k(x) = U_l(y)$ and so, using (6) twice,

$$V_k(x) - 1 = U_k(x) = U_l(y) = 1 - V_l(y). \tag{25}$$

So from (23) and (25),

$$x_{(k)} \approx y_{(l)} \Leftrightarrow V_k(x) - 1 = 1 - V_l(y). \tag{26}$$

Proposition 4 *Suppose that the assumptions of Proposition 3 are satisfied. If person k is of type 1, l is of type -1, and m is of either type, then for $(x, y) \in \mathcal{D}_{-1,1}$ the relation $\approx$ is an equivalence one, but it is not necessarily invariant under joint receipt.*

The above formulations have been for pure consequences, but that extends to gambles f via the use of certainty equivalents, i.e., for a gamble f there is a $CE(f) \in X$ such that $f \sim CE(f)$.

[10] See footnote 7.

3 Utility for Money

3.1 General representations

This section is in response to questions raised in person by Igor Kopylov and Stergios Skaperdas: What amount of money must a wealthier person gain or lose so it has equal utility to that of a poorer person for gaining or losing, say, a dollar. The answer is provided at the end of the section. To that end, we explore the special case of the utility of increments of money. This we take to mean that for any $x, y \in \mathbb{R}$, $\oplus$ is a group operator with identity 0, and U is strictly increasing.

Proposition 5 *Suppose that the assumptions of Proposition 3 are satisfied when $X = \mathbb{R}$ is money amounts. Then there exists a strictly increasing function g that is additive over $\oplus$ such that:*
(i) If $\delta = 0$, then U is strictly increasing and onto $\mathbb{R}$ and

$$U(x) = \alpha g(x) \quad (\alpha > 0). \tag{27}$$

(ii) If $\delta = 1$, then $V = 1 + U$ is strictly increasing, onto $\mathbb{R}_+$, multiplicative, (7), and

$$U(x) = e^{\alpha g(x)} - 1 \quad (\alpha > 0). \tag{28}$$

(iii) If $\delta = -1$, then $V = 1 - U$ is strictly decreasing, onto $\mathbb{R}_+$, multiplicative, and

$$U(x) = 1 - e^{-\alpha g(x)} \quad (\alpha > 0). \tag{29}$$

We next consider a special case which many economists seem to believe should hold for money, at least for rational people, namely, that money joint receipt is just addition:

$$x \oplus y = x + y. \tag{30}$$

Corollary 6 *Under the conditions of Proposition 5, for $x \in \mathbb{R}$, (30) is equivalent to*

$$g(x) = x, \tag{31}$$

and so:
(i) If $\delta = 0$, then

$$U(x) = \alpha x \quad (\alpha > 0). \tag{32}$$

(ii) If $\delta = 1$, then

$$U(x) = e^{\alpha x} - 1 \quad (\alpha > 0). \tag{33}$$

(iii) If $\delta = -1$, then

$$U(x) = 1 - e^{-\alpha x} \quad (\alpha > 0). \tag{34}$$

Note that for $\delta = 1$, where we have (33), it is well known that U is strictly increasing and convex, which many identify as corresponding to risk seeking behavior.

And for $\delta = -1$, it follows from the first and second derivative that U is strictly increasing and concave, which corresponds to risk averse behavior.

Assuming the functions (33) and (34) of the Corollary, let us examine what happens when we have a weighted average of proportion a of type $\delta = -1$ and $1 - a$ of type $\delta = 1$, with the same parameter for all people of the same type. Thus, the average is:

$$\overline{U}(x) = a\left(1 - e^{-\alpha x}\right) + (1 - a)\left(e^{\beta x} - 1\right).$$

Note that

$$\begin{aligned} \overline{U}''(x) &= -a\alpha^2 e^{-\alpha x} + (1-a)\beta^2 e^{\beta x} \gtrless 0 \\ \Leftrightarrow x &\gtrless \frac{1}{\alpha+\beta} \ln\left(\frac{a}{1-a}\right)\left(\frac{\alpha}{\beta}\right)^2. \end{aligned}$$

For the special case $\alpha = \beta$ this yields

$$x \gtrless \begin{cases} 0, & a = \frac{1}{2} \\ \frac{1}{2}\ln 9 \approxeq 1.1, & a = 0.9 \end{cases}.$$

Figure 1 shows for $\alpha = \beta = 1$[11] plots for $\delta = -1$ (Panel A) and for $\delta = 1$. The second and third rows are the averages with $a = \frac{1}{2}$ and 0.9 cases on the left and right, respectively. Row 3 focuses in on the regions where the curvature changes.

Insert Fig. 1 about here.

One readily sees just how misleading such average data can be.

A major open problem is how, from the data that we can collect from an individual, do we go about estimating what amounts to the money amount corresponding closely to the asymptote of the $\delta \neq 0$ functions. This means, developing methods to estimate the α in the Corollary or the expression $\alpha g(x)$ in Proposition 5. Inroads on this kind of problem have been developed by Abdellaoui, Bleichrodt, & Paraschiv (2007).

3.2 Interpersonal comparisons for exponential utility functions

Suppose that the comparison is between two people of type $\delta = 1$ with exponential utility and constants α_k, α_l. Then

$$x_{(k)} \approx y_{(l)} \Leftrightarrow e^{\alpha_k x} - 1 = e^{\alpha_l y} - 1 \Leftrightarrow \frac{y}{x} = \frac{\alpha_k}{\alpha_l}. \tag{35}$$

[11] Observe that αx must be dimensionless, and so to have the amounts of money that are likely to be typical, the abscissa's must be multiplied by 10^n, where n is at least 6 and α by the reciprocal factor.

If we conjecture, as seems plausible, that when person l is wealthier than person k, then a larger increment, y, is needed to match in utility the increment x for a poorer person, i.e., $y > x \Leftrightarrow \alpha_l < \alpha_k$. Exactly what "wealthier than" means does not matter beyond that $x_{(k)} \approx y_{(l)}$ holds only for $y > x$, i.e., that it takes a larger money increment y to give the wealthier person the same satisfaction, as measured by U_l, as the increment x give the relatively poorer person, as measured by U_k. By the transitivity of $\approx$, increasing wealth corresponds to decreasing the exponent in the the exponential utility form.

Similarly, for two people of type $\delta = -1$,

$$x_{(k)} \approx y_{(l)} \Leftrightarrow \frac{y}{x} = \frac{\alpha_k}{\alpha_l}. \tag{36}$$

Finally, for a person 1 of type $\delta = 1$ and person 2 of type $\delta = -1$, we obtain for

$$-\infty < x < \frac{\ln 2}{\alpha_k} \quad \text{and} \quad \frac{-\ln 2}{\alpha_l} < y < \infty,$$

which are needed to keep the utilities of the two amounts within the prescribed interval $]-1, 1[$, then

$$x_{(k)} \approx y_{(l)} \Leftrightarrow y = \frac{-\ln\left(2 - e^{\alpha_k x}\right)}{\alpha_l}. \tag{37}$$

4 Conclusions

In principle, one can use the behavioral criteria provided by Proposition 2 to decide, for any person, whether he or she falls in the $\delta = 1$, 0, or -1 categories. The criterion of Part II works easily for experiments with given probabilities and money consequences, i.e., lotteries. For general events and consequences Part I provides a simple criterion to test without having to estimate any subjective probabilities.

It is not clear, to me at least, whether the class of $\delta = 0$ people – those who perceive utility as unbounded for both gains and for losses – is, in fact, non-empty. This bears empirical investigation.

For the $\delta \neq 0$ cases, there are two disjoint classes of people corresponding to $\delta = 1$ or $\delta = -1$. It seems intuitive to me, although this is certainly speculative, that whether a person exhibits unbounded gains and bounded losses ($\delta = 1$) or bounded gains and unbounded losses ($\delta = -1$) correspond, respectively, to the ideas of risk seeking and risk averse people.

Interestingly, within the framework of $\oplus$ having a p-additive representation, the case where utility is both bounded for gains and bounded for losses simply does not arise.

For the $\delta \neq 0$ case, an hypothesis was formulated as to what "interpersonal comparison of utility" might mean, and several of its elementary properties were derived. These are embodied in Propositions 3 and 4. I do not see any comparable solution for the $\delta = 0$ subpopulation, if it exists, because the utility function is a ratio scale, not an absolute one.

These results were detailed for money consequences.

Three major open problems were mentioned: (1) What is a practical method for estimating the asymptote of utility in the $\delta \neq 0$ cases? (2) How well are the money data for individuals fit by functions of Corollary 6, in particular assuming (30) holds separately for gains and for losses. (3) Can a sensible concept of social welfare be formulated for $\delta \neq 0$ using the the absolute scales that arise.

5 Appendix: Proofs

Proof. Suppose that $x \succ y$ and that C, D are such that

$$(x, C; y, D) \sim (x, D; y, C). \tag{38}$$

By segregation, this becomes when $\delta = 0$

$$\begin{aligned} (x \oplus y^{-1}, C; e, D) &\sim (x \oplus y^{-1}, D; e, C) \\ \Leftrightarrow U(x \oplus y^{-1}, C; e, D) &= U(x \oplus y^{-1}, D; e, C) \end{aligned}$$

By separability,

$$\begin{aligned} U(x \oplus y^{-1}) S_{C \cup D}(C) &= U(x \oplus y^{-1}) S_{C \cup D}(D) \\ \Leftrightarrow S_{C \cup D}(C) &= S_{C \cup D}(D), \end{aligned}$$

and adding complementarity (13) yields (15). For $\delta \neq 0$, the parallel argument involves U replaced by V. The conclusion is not altered.
■

5.1 Proposition 2

Proof. Part I. We begin by assuming that **I.1** holds and prove **I.2.** This proof is due to C. T. Ng. First, we consider the case where x, y are either both gains or both losses. Thus, in both cases $U(x)U(y) > 0$.
Case (i). Suppose that $\delta = 1$. Then keeping in mind that $x \succ e, y \succ e$ and that independent replications of $\{C, D\}$ means $S_{C \cup D}(C) = S_{C' \cup D'}(C') =$

$S_{C''\cup D''}(C'')$,

$$\begin{aligned}
&1 + U(x \oplus y, C; e, D)\\
&\quad = 1 + U(x \oplus y)S_{C\cup D}(C)\\
&\quad = 1 + [U(x) + U(y) + U(x)U(y)]S_{C\cup D}(C)\\
&\quad = 1 + U(x)S_{C\cup D}(C) + U(y)S_{C\cup D}(C) + U(x)U(y)S_{C\cup D}(C)\\
&\quad > 1 + U(x)S_{C\cup D}(C) + U(y)S_{C\cup D}(C) + U(x)U(y)S_{C\cup D}(C)S_{C\cup D}(C)\\
&\quad = 1 + U(x)S_{C'\cup D'}(C') + U(y)S_{C''\cup D''}(C'') + U(x)U(y)S_{C'\cup D'}(C')S_{C''\cup D''}(C'')\\
&\quad = [1 + U(x)S_{C'\cup D'}(C')][1 + U(y)S_{C''\cup D''}(C'')]\\
&\quad = [1 + U(x, C'; e, D')][1 + U(y, C''; e, D'')]\\
&\quad = V(x, C'; e, D')V(y, C''; e, D'')\\
&\quad = V((x, C'; e, D') \oplus (y, C''; e, D''))\\
&\quad = 1 + U((x, C'; e, D') \oplus (y, C''; e, D'')).
\end{aligned}$$

So $U(x \oplus y, C; e, D) > U((x, C'; e, D') \oplus (y, C''; e, D''))$ and therefore

$$(x \oplus y, C; e, D) \succ (x, C'; e, D') \oplus (y, C''; e, D'').$$

Case (ii). For $\delta = 0$, then

$$\begin{aligned}
&U(x \oplus y, C; e, D)\\
&\quad = U(x \oplus y)S_{C\cup D}(C)\\
&\quad = [U(x) + U(y)]S_{C\cup D}(C)\\
&\quad = U(x)S_{C'\cup D'}(C') + U(y)S_{C''\cup D''}(C'')\\
&\quad = U(x, C'; e, D') + U(y, C''; e, D'')\\
&\quad = U((x, C'; e, D') \oplus (y, C''; e, D'')).
\end{aligned}$$

This proves

$$(x \oplus y, C; e, D) \sim (x, C'; e, D') \oplus (y, C''; e, D'').$$

Case (iii). Suppose that $\delta = -1$. Then using the fact that $U(x)U(y) > 0$ because both are gains or both are losses,

$$\begin{aligned}
&1 - U(x \oplus y, C; e, D)\\
&\quad = 1 - U(x \oplus y)S_{C\cup D}(C)\\
&\quad = 1 - [U(x) + U(y) - U(x)U(y)]S_{C\cup D}(C)\\
&\quad = 1 - U(x)S_{C\cup D}(C) - U(y)S_{C\cup D}(C) + U(x)U(y)S_{C\cup D}(C)\\
&\quad > 1 - U(x)S_{C\cup D}(C) - U(y)S_{C\cup D}(C) + U(x)U(y)S_{C\cup D}(C)S_{C\cup D}(C)\\
&\quad = 1 - U(x)S_{C'\cup D'}(C') - U(y)S_{C''\cup D''}(C'') + U(x)U(y)S_{C'\cup D'}(C')S_{C''\cup D''}(C'')\\
&\quad = [1 - U(x)S_{C'\cup D'}(C')][1 - U(y)S_{C''\cup D''}(C'')]\\
&\quad = [1 - U(x, C'; e, D')][1 - U(y, C''; e, D'')]\\
&\quad = V(x, C'; e, D')V(y, C''; e, D'')\\
&\quad = V((x, C'; e, D') \oplus (y, C''; e, D''))\\
&\quad = 1 - U((x, C'; e, D') \oplus (y, C''; e, D'')).
\end{aligned}$$

So $U(x \oplus y, C; e, D) < U((x, C; e, D) \oplus (y, C; e, D))$ and therefore

$$(x \oplus y, C; e, D) \prec (x, C'; e, D') \oplus (y, C'; e, D').$$

So Part I.1 implies Part I.2.

Part II. We now assume that the structure satisfies complementarity, (13). Clearly, that does not affect the equivalence of Parts I.1 and I.2. So, assuming I.1, consider any consequences for which $x \succ x' \succ y' \succ y$.

For $\delta = 0$,

$$\begin{aligned}
&(x \oplus x', C; y \oplus y', D) \sim (x \oplus y, C; x' \oplus y', D) \\
&\Leftrightarrow [(U(x) + U(x') - (U(x) + U(y))] \frac{1}{2} \\
&\quad + [(U(y) + U(y') - (U(x') + U(y'))] \frac{1}{2} = 0 \\
&\Leftrightarrow [U(x') - U(y)] \frac{1}{2} + [U(y) - U(x')] \frac{1}{2} = 0 \\
&\Leftrightarrow [U(x') - U(y)]\, 0 = 0.
\end{aligned}$$

For $\delta \neq 0$, then when $\delta = 1$ (-1) V is order preserving (reversing). Then using (15) and $V(e) = 1$

$$\begin{aligned}
&x \succ y' \\
&\Leftrightarrow V(x) >(<)\, V(y').
\end{aligned}$$

Because $x' \succ y$, we may multiply by $V(x') - V(y) > (<)\, 0$ to get ,

$$\begin{aligned}
&V(x)\,[V(x') - V(y)] > V(y')\,[V(x') - V(y)] \\
&\quad \Leftrightarrow V(x)V(x') + V(y)V(y') > V(x)V(y) + V(x')V(y') \\
&\quad \Leftrightarrow V(x \oplus x') + V(y \oplus y') > V(x \oplus y) + V(x' \oplus y') \\
&\quad \Leftrightarrow V(x \oplus x')\frac{1}{2} + V(y \oplus y')\frac{1}{2} > V(x \oplus y)\frac{1}{2} + V(x' \oplus y')\frac{1}{2} \\
&\quad \Leftrightarrow V(x \oplus x', C; y \oplus y', D) > V(x \oplus y, C; x' \oplus y', D) \\
&\quad \Leftrightarrow (x \oplus x', C; y \oplus y', D) \succ (\prec)(x \oplus y, C; x' \oplus y', D),
\end{aligned}$$

which is the criterion (18) for $\delta \neq 0$.

The converses that **Parts I.2** and **II** each imply **Part I.1** are obvious because the cases are exclusive and exhaustive. ■

5.2 Proposition 3

Proof.
(i) Reflexivity is trivial.
(ii) Symmetry is immediate from the definition.

(iii) Transitivity:

$$\begin{aligned} x_{(k)} &\approx y_{(l)} \;\;\&\;\; y_{(l)} \approx z_{(m)} \\ &\Leftrightarrow V_k(x) = V_l(y) \;\;\&\;\; V_l(y) = V_m(z) \\ &\Rightarrow V_k(x) = V_m(z) \\ &\Leftrightarrow x_{(l)} \approx z_{(m)}. \end{aligned}$$

(iv) Invariance under joint receipt:

$$\begin{aligned} x_{(k)} &\approx y_{(l)} \;\;\&\;\; x'_{(k)} \approx y'_{(l)} \Rightarrow V_k(x) = V_l(y) \;\;\&\;\; V_k(x') = V_l(y') \\ &\Rightarrow V_k(x)V_k(x') = V_l(y)V_l(y') \Rightarrow V_k(x \oplus x') = V_l(y \oplus y') \\ &\Rightarrow (x \oplus x')_{(k)} \approx (y \oplus y')_{(l)}\,. \end{aligned}$$

■

5.3 Proposition 4

Proof.
(i) Reflexivity is trivial.
(ii) Symmetry:

$$x_{(k)} \approx y_{(l)} \Leftrightarrow U_k(x) = U_l(y) \Leftrightarrow U_l(y) = U_k(x) \Leftrightarrow y_{(l)} \approx x_{(k)}.$$

(iii) Transitivity: There are two cases to consider:
Case a: $\delta_m = 1$, then

$$\begin{aligned} x_{(k)} &\approx y_{(l)} \Leftrightarrow V_k(x) - 1 = 1 - V_l(y) \;\; \text{and} \\ y_{(l)} &\approx z_{(m)} \Leftrightarrow z_{(m)} \approx y_{(l)} \Leftrightarrow V_m(z) - 1 = 1 - V_l(y) \\ &\Rightarrow V_k(x) - 1 = V_m(z) - 1 \\ &\Leftrightarrow x_{(k)} \approx z_{(m)}, \end{aligned}$$

which is transitivity.
Case b: $\delta_m = -1$, then using (25)

$$\begin{aligned} x_{(k)} &\approx y_{(l)} \Leftrightarrow V_k(x) - 1 = 1 - V_l(y) \;\; \text{and} \\ y_{(l)} &\approx z_{(m)} \Leftrightarrow V_l(y) = V_m(z) \\ &\Rightarrow V_k(x) - 1 = 1 - V_m(z) \\ &\Leftrightarrow x_{(k)} \approx z_{(m)}, \end{aligned}$$

which again is transitivity.
(iv) If

$$x_{(k)} \approx y_{(l)} \;\; \text{and} \;\; x'_{(k)} \approx y'_{(l)},$$

IPC means

$$
\begin{aligned}
&U_k(x) = U_l(y) \text{ and } U_k(x') = U_l(y') \\
&\Leftrightarrow V_k(x) - 1 = 1 - V_l(y) \text{ and } V_k(x') - 1 = 1 - V_l(y') \quad (25) \\
&\Leftrightarrow V_k(x) = 2 - V_l(y) \text{ and } V_k(x') = 2 - V_l(y') \\
&\Rightarrow V_k(x \oplus x') = V_k(x)V_k(x') \\
&\quad = [2 - V_l(y)]\,[2 - V_l(y')] \\
&\quad \Leftrightarrow V_k(x \oplus x') = 4 - 2V_l(y) - 2V_l(y') + V_l(y \oplus y') \\
&\quad \Leftrightarrow V_k(x \oplus x') - 1 = 3 - 2V_l(y) - 2V_l(y') + V_l(y \oplus y') \\
&\quad \neq 1 - V_l(y \oplus y').
\end{aligned}
$$

Thus, invariance under joint receipt, $x_{(k)} \oplus x'_{(k)} \approx y_{(l)} \oplus y'_{(l)}$, does not follow in the mixed case. ■

5.4 Proposition 5

Proof.
(i) $\delta = 0$ means that the representation is additive, (4), which we repeat as

$$U(x \oplus y) = U(x) + U(y), \tag{39}$$

Aczél (1966, Theorem 3, p. 62) yields the general solutions to a somewhat more general equation than (39). Indeed, in his notation that case becomes: $\oplus$ has a representation of the general form

$$x \oplus y = g^{-1}[g(x) + g(y)]$$

where g is strictly increasing and so (27) is obvious.
(ii) $\delta = 1$ means that $V = 1 + U$ which has the multiplicative form (7) which, in turn, is equivalent to

$$\ln V(x \oplus y) = \ln V(x) + \ln V(y). \tag{40}$$

This with (27) yields the result.
(iii) $\delta = -1$ means that $V = 1 - U$ which is equivalent to

$$\Leftrightarrow \ln V(x \oplus y) = \ln V(x) + \ln V(y). \tag{41}$$

Taking into account that V is strictly decreasing we have $\ln V(x) = -\alpha g(x)$. ■

Comment: we have used the fact that (39), (40), and (41) are all the same equation, but in different functions of the same variables.

The proof of the Corollary is immediate because

$$g(x + y) = g(x) + g(y)$$

is the well-known Cauchy equation which, under strict monotonicity and onto $[0, \infty[$, has the solutions $g(x) = cx$. The rest follows from this and the Proposition 5.

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Figure Caption: Panel A: $U = 1 - x^{-\alpha x}$ ($\alpha = 1$); Panel B: $U = e^{\beta x} - 1$ ($\beta = 1$). Panel C: Average of Panels A and B. Panel D: $9:1$ average of Panels A and B. Panel E: Zoom in on region near $x = 0$ of Panel C. Panel F: Zoom in on region near $x = 1.1$, the inflexion point, of Panel D.