Utility of Gambling When Events Are Valued: An Application of Inset Entropy*

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Abstract

The present theory leads to a set of subjective weights such that the utility of an uncertain alternative (gamble) is partitioned into three terms involving those weights – a conventional subjectively weighted utility function over pure consequences, a subjectively weighted value function over events, and a subjectively weighted function of the subjective weights. Under several assumptions, this becomes one of several standard utility representations, plus a weighted value function over events, plus an entropy term of the weights. In the finitely additive case, the latter is the Shannon entropy; in all other cases it is entropy of degree not 1. The primary mathematical tool is the theory of inset entropy.

Keywords: duplex decomposition, functional equation, Shannon entropy, gamble decomposition, inset entropy, segregation, utility of gambling, valued events

JEL classification: C 91, D 46, D 81

Utility theorists have long shunned two issues that were talked about early on: the utility of gambling and the intrinsic value of some chance events, e.g., aspects of weather itself or crashes of vehicles. This is discussed by Luce, Ng, Marley, & Aczél (in press a), but the solution they arrived at did not really make explicit any partition of the two aspects — the utility of gambling and the intrinsic value of events — except in the case of risk where only the utility of gambling term appeared. The goal of this article is to achieve such a partition. Several unanswered problems are cited in context and are summarized in Section 6.

1 General Background

1.1 Additive utility over joint receipts of pure consequences

Assume a decision maker, DM, whose empirically known preference ordering $\succsim_X$ is a weak order over the set X of pure consequences and their joint receipts $\oplus$. The joint receipt $x \oplus y$ means having both x and y (Luce, 2000). Pure consequences are those for which the decision maker believes there is no uncertainty (which, of course, sometimes turns out to be illusory—a rotten egg, a car that is a "lemon," etc.).

Suppose that axioms of additive extensive measurement (see Ch. 3, Krantz et al., 1971) are satisfied that are sufficient to show the existence of a ratio scale utility function U_X over pure consequences that is additive[1]:

$$U_X(x \oplus y) = U_X(x) + U_X(y) \quad (x, y \in X). \tag{1}$$

Among the properties used to arrive at (1) are that $\oplus$ is commutative and associative, both necessary properties.

As is discussed in Luce et al. (in press a), if the pure consequences include money, then (1) implies that the utility of money is proportional to money. We also pointed out that the so-called counter example of the St. Petersburg paradox simply does not realistically rule that out, contrary to what is often claimed.

1.2 Certainty equivalents, kernel equivalents and elements of chance

An uncertain alternative (for short, gamble) is of the form

$$g_{[n]} := (x_1, C_1; x_2, C_2; ...; x_n, C_n), \tag{2}$$

where $x_i \in X$, $x_1 \succsim_X x_2 \succsim_X ... \succsim_X x_n$, and $(C_1, C_2, ..., C_n)$ is a partition of some "universal" event Ω into subevents. Such a gamble $g_{[n]}$ can be viewed as a collection of n branches (x_i, C_i). In constructing a gamble $g_{[n]}$ we assume that all the subevents C_i belong to a Boolean ring $\mathfrak{B}$ with $\varnothing$ the zero and $\mathfrak{B}^* := \mathfrak{B} \backslash \{\varnothing\}$. We assume that $\Omega = \bigcup_{i=1}^{n} C_i$ belongs to $\mathfrak{B}^*$ and, usually, that each C_i belongs to $\mathfrak{B}^*$.

We assume that a person can have the joint receipt of pure consequences and gambles, which operation is also denoted by $\oplus$ and is an extension of the joint receipts of pure consequences. We assume that no change from the status quo, e, is a two-sided identity of $\oplus$.

Let $\mathcal{G}$ denote the closure of all gambles and pure consequences under $\oplus$.

We plan to deal with order extensions of $\succsim_X$ to $\mathcal{G}$. A typical extension is denoted $\succsim$. As usual, $\precsim$ denotes the converse of $\succsim$ and $\sim$ denotes the corresponding indifference relation: $\sim := \succsim \cap \precsim$ and $\succ := \succsim \setminus \sim$.

We assume that for each gamble $g_{[n]}$, the set X is sufficiently rich that it contains an element, denoted $CE(g_{[n]})$, such that $CE(g_{[n]}) \sim g$. This pure consequence is called a *certainty equivalent* (CE) of the gamble.

Because we are considering theories where gambling per se has utility, we do not assume e-idempotence,

$$(e, C_1; e, C_2; ...; e, C_n) \sim e. \tag{3}$$

We speak of $(e, C_1; e, C_2; ...; e, C_n)$ as an *element of chance*, and following Luce et a. (2000), we partition any gamble into its element of chance joint with a pure consequence $KE(g_{[n]})$ that solves the following indifference

$$g_{[n]} \sim KE(g_{[n]}) \oplus (e, C_1; e, C_2; ...; e, C_n). \tag{4}$$

We call $KE(g_{[n]})$ the *kernel equivalent* (KE) of the gamble $g_{[n]}$. In what follows we will be working with many different orderings $\succsim$ in which case we will write $CE_{\succsim}(g_{[n]})$, $KE_{\succsim}(g_{[n]})$, etc., to make explicit their dependence upon $\succsim$.

1.3 Separability and the choice property

We assume that DM satisfies the axioms (very slightly modified but equivalent, as explained in section 2.1 below) of the theory of Luce et al. (in press a) for each extension $\succsim$ of $\succsim_X$. For a given $\succsim$, their theory gives rise to a single utility function $U_{\succsim}$ over gambles that agrees with U_X, and to a special weighting function on Ω denoted $S_{\succsim,\Omega}$ such that $U_X S_{\succsim,\Omega}$ is a *separable* representation in the sense that

$$U_{\succsim}(KE_{\succsim}(x, C; e, \Omega \backslash C)) = U_X(KE_{\succsim}(x, C; e, \Omega \backslash C)) = U_X(x) S_{\succsim,\Omega}(C). \tag{5}$$

Other axioms were shown to imply that any such $S_{\succsim,\Omega}$ must satisfy the *choice property* (Luce, 1959/2005)[2], i.e., for $C \subseteq D \subseteq \Omega$

$$S_{\succsim,D}(C) = \frac{S_{\succsim,\Omega}(C)}{S_{\succsim,\Omega}(D)}. \tag{6}$$

Reflecting on the choice property, Theorem 6 of Luce et al. (in press a) establishes that there exists an increasing function $\mu_{\succsim}$ on $\mathfrak{B}$, positive on $\mathfrak{B}^*$ and $\mu_{\succsim}(\varnothing) = 0$, such that

$$S_{\succsim,\Omega}(C) = \frac{\mu_{\succsim}(C)}{\mu_{\succsim}(\Omega)}. \tag{7}$$

Clearly, both separability and the choice property are invariant under power transformations but, of course, for $\beta \neq 1$, U_X^β is not additive over $\oplus$.

2 Basic assumptions on a family of order extensions

Although the order $\succsim_X$ and the utility function U_X are fixed, the route we are going to pursue involves considering a family $\mathcal{O}$ of weak preference orders $\succsim$ over pure consequences and gambles meeting certain axiomatic assumptions. Our goal is to understand what general class of representations results.

2.1 Basic assumptions

We assume[3] that $\oplus$ is monotonic in the ordering $\succsim$:

$$f \succsim f' \Leftrightarrow f \oplus g \succsim f' \oplus g \Leftrightarrow g \oplus f \succsim g \oplus f'. \tag{8}$$

Then, as we show next, each preference order $\succsim \in \mathcal{O}$, where $\succsim$ agrees with $\succsim_X$ over X and their representations also agree, i.e., $U_{\succsim}(x) = U_X(x)$ for all $x \in X$, has a representation $U_{\succsim}$ which is additive over joint receipts of gambles:

$$U_{\succsim}(f \oplus g) = U_{\succsim}(f) + U_{\succsim}(g). \tag{9}$$

The proof is: because $f \sim CE(f)$, $g \sim CE(g)$,

$$\begin{aligned} U_{\succsim}(f \oplus g) &= U_{\succsim}\left(CE(f) \oplus CE(g)\right) && \text{(2 uses of (8))} \\ &= U_X\left(CE(f) \oplus CE(g)\right) && (U_{\succsim}(CE(f)) = U_X(CE(f)) \\ &= U_X\left(CE(f)\right) + U_X(CE(g)) && (1) \\ &= U_{\succsim}\left(CE(f)) + U_{\succsim}(CE(g)\right) \\ &= U_{\succsim}(f) + U_{\succsim}(g). \end{aligned}$$

Further, we assumed that every preference order $\succsim \in \mathcal{O}$ satisfies the conditions of either Theorem 13 or Theorem 16 of Luce et al. (in press a). Because we draw heavily on the theoremss, we summarize each result and its assumptions in Appendices A and B, respectively. These results differ depending upon whether we assume a property called segregation, (12), or duplex decomposition, (32), both below.

Let $H_{\succsim}$ be defined by

$$H_{\succsim}(C_1, ..., C_n) = U_{\succsim}(e, C_1; ...; e, C_n). \tag{10}$$

We have, by (4) and (9),

$$U_{\succsim}(x_1, C_1; ...; x_n, C_n) = U_X(KE_{\succsim}(x_1, C_1; ...; x_n, C_n)) + H_{\succsim}(C_1, ..., C_n), \tag{11}$$

where various specific forms for $U_X(KE_{\succsim})$ and $H_{\succsim}$ arise depending on specific assumptions (see Appendices A and B). In both the segregation and the duplex decomposition cases it is shown that the elements of chance term has the following representation

$$H_{\succsim}(C_1, ..., C_n) = \frac{1}{\mu_{\succsim}(\Omega)}\left[h_{\succsim}(\Omega) - \sum_{i=1}^{n} h_{\succsim}(C_i)\right]$$

for some function $h_{\succsim}$ with $h_{\succsim}(\varnothing) = 0$.

2.2 Two conceptual interpretations

Two possible interpretations come to mind of the huge families, $\mathcal{O}$, of orders that we will require. The first is that we have a single DM but, lacking any information concerning his or her ordering over gambles, we, as theorists, must be prepared to deal with whatever order DM actually exhibits.

Recall that we are assuming that our current knowledge does not include a specific order $\succsim$ over gambles that extends $\succsim_X$, but we will assume that it belongs to a family of possible orders that satisfy a certain set of assumptions. This is a conceptual assumption that cannot be empirically assessed.

A second interpretation of our formalism is that we can find distinct DMs each of whom agrees with $\succsim_X$ over the pure consequences, X, and a DM corresponding to each extension $\succsim \in \mathcal{O}$.

Because both interpretations are entirely hypothetical, there is no real intellectual distinction between them. In particular, neither is empirical and they are formally identical.

3 Assumptions and representation under segregation

3.1 Definition of segregation

The theory of Luce et al. (in press a) divides into two quite distinct parts, leading to their Theorems 13 and 16, depending on whether we assume segregation or duplex decomposition, both behavioral properties. Here we define the former and the latter will be defined in Section 5.

Segregation:

$$(x, C; e, D) \oplus y \sim (x \oplus y, C'; y, D'), \tag{12}$$

where (C', D') is an independent realization of (C, D). Note that the two sides have the same "bottom line."

Under segregation, Theorem 13 of Luce et al. (in press a) establishes that S_Ω satisfies the choice property, (6), and (7), and that either S_Ω is finitely additive, i.e., for all $C, D \subseteq \Omega$, $C \cap D = \varnothing$,

$$S_\Omega(C \cup D) = S_\Omega(C) + S_\Omega(D) \tag{13}$$

or that S_Ω is (non-trivially) p-additive, i.e., for some constant $\Delta \neq 0$,

$$S_\Omega(C_1 \cup C_2) = S_\Omega(C_1) + S_\Omega(C_2) + \Delta\mu(\Omega)S_\Omega(C_1)S_\Omega(C_2) \quad (C_1 \cap C_2 = \varnothing). \tag{14}$$

3.2 The representation

We call a probability vector $\mathbf{p} = (p_1, ..., p_n)$ *non-trivial* when $p_i \in]0, 1[$, $i = 1, ..., n$, and an ordered event partition $\mathbf{C} = (C_1, C_2, ..., C_n)$ of $\Omega = \cup_{i=1}^n C_i$ *non-trivial* when $C_i \neq \varnothing$, $i = 1, ..., n$. All references to the pair $(\mathbf{p}, \mathbf{C})$ implicitly assume that both are non-trivial. An event C is *maximal* if there does not exist $C' \in \mathfrak{B}^*$ such that $C' \supseteq C$ and $C' \neq C$.

Definition 1 ***Weight solvability*** *holds for a family of orders, $\mathcal{O}$, if for each non-trivial probability vector $\mathbf{p} = (p_1, ..., p_n)$ and for each non-trivial ordered event partition $\mathbf{C} = (C_1, C_2, ..., C_n)$ of $\Omega = \cup_{i=1}^n C_i$, there exists at least one ordering $\succsim \in \mathcal{O}$ such that*

$$S_{\succsim,\Omega}(C_i) = p_i \quad (i = 1, ..., n). \tag{15}$$

If $\mathcal{O}$ denotes a set of orders meeting the conditions of Theorem 13 (Appendix A), then we may partition $\mathcal{O}$ into $(\mathcal{O}_1, \mathcal{O}_2)$ such that for each $\succsim \in \mathcal{O}_1$ the corresponding $S_{\succsim,\Omega}$ is finitely additive and for each $\succsim \in \mathcal{O}_2$ the corresponding $S_{\succsim,\Omega}$ is non-trivially p-additive (cf. (13) and (14)).

Theorem 2 *Let $\mathcal{O}$ be a family of orderings $\succsim$ each satisfying the assumptions of Theorem 13 of Luce et al.(in press a) (see Appendix A). Assume:*

(i) *Weight solvability, Definition 1, is satisfied by $\mathcal{O}$.*

(ii) *Let $(p_1, ..., p_n)$ be a non-trivial probability vector and let $(C_1, C_2, ..., C_n)$ be a non-trivial ordered event partition of Ω. For $\succsim, \succsim' \in O$, suppose that each gives rise to a weighting function with*

$$S_{\succsim,\Omega}(C_i) = S_{\succsim',\Omega}(C_i) = p_i \quad (i = 1, ..., n). \tag{16}$$

Let $y \in X$ be a certainty equivalent (CE) of an element of chance under $\succsim$, i.e.,

$$y \sim (e, C_1; ...; e, C_n). \tag{17}$$

Then y is also a CE under $\succsim'$, i.e.,

$$y \sim (e, C_1; ...; e, C_n) \Leftrightarrow y \sim' (e, C_1; ...; e, C_n). \tag{18}$$

(iii) *Regularity: For a fixed $(e, C_1; e, C_2)$, the set of all values of $U_{\succsim}(e, C_1; e, C_2)$, over all $\succsim \in \mathcal{O}_1$, is bounded.*

Then:

1) *Assumption (i) is satisfied by $\mathcal{O}_1$, where $S_{\succsim,\Omega}$ is finitely additive for each $\succsim \in \mathcal{O}_1$, and there exist a constant A and a function $V : \mathfrak{B} \to \mathbb{R}$, with*

$V(\varnothing)=0$, such that for all gambles $g_{[n]}$ and for each $\succsim\in\mathcal{O}_1$,

$$U_{\succsim}(g_{[n]})=\sum_{i=1}^{n}U_X(x_i)S_{\succsim,\Omega}(C_i)+V(\Omega)$$
$$-\sum_{i=1}^{n}V(C_i)S_{\succsim,\Omega}(C_i)-A\sum_{i=1}^{n}S_{\succsim,\Omega}(C_i)\log_2 S_{\succsim,\Omega}(C_i). \tag{19}$$

2) *For each $\succsim\in\mathcal{O}_2$, $S_{\succsim,\Omega}$ is non-trivially p-additive and there exists a constant $A_{\succsim}$ such that for all gambles $g_{[n]}$,*

$$U_{\succsim}(g_{[n]})=RDU_{\succsim}(g_{[n]})+\left\{\begin{array}{ll}0, & \textit{if } C(n) \textit{ is not maximal}\\ A_{\succsim}, & \textit{if } C(n) \textit{ is maximal}\end{array}\right., \tag{20}$$

where

$$RDU_{\succsim}(g_{[n]})=\sum_{i=1}^{n}U_X(x_i)S_{\succsim,\Omega}(C_i)\left[1+\Delta_{\succsim}\mu_{\succsim}(\Omega)S_{\succsim,\Omega}(C(i-1))\right]$$

and $C(j):=\cup_{k=1}^{j}C_k$.

All proofs are found in Appendix D, but for many of those proofs Appendix C is vital.

Note that (18) is equivalent to

$$CE_{\succsim}(e,C_1;...;e,C_n)\sim_X CE_{\succsim'}(e,C_1;...;e,C_n), \tag{21}$$

which is the form used in the proof of this Theorem and of Theorem 12.

The recursion of Appendix C (Section 7.3.4), which is used in the proof of the above theorem, is that of inset entropy (Aczél & Daróczy, 1978; Ebanks et al., 1988). Under our assumptions, it holds for all non-trivial vectors belonging to $\Omega_n\times\Gamma_n$ and for general $n\geq 2$. Also, it follows from our assumptions that $H_{\succsim}$ is symmetric, i.e., invariant with respect to permutations of the pairs (C_i,p_i). The inset entropy representation is derived under very mild additional regularity assumptions, such as (64).

The roles of Assumptions (i) and (ii) need brief discussion. Basically, the former, weight solvability, postulates that $\mathcal{O}$ is sufficiently rich so as to have an order $\succsim\in\mathcal{O}$ such that its weighting function $S_{\succsim,\Omega}$ maps $\mathbf{C}$ onto $\mathbf{p}$. And Assumption (ii) constrains $\mathcal{O}$ to be not too rich so that two orders with matching weights, (16), end up with the same certainty equivalent for the elements of chance. Although it would be desirable to have behavioral conditions — meaning formulated entirely in terms of the primitives of the structure and can be evaluated by experiment — equivalent to these assumptions, it is unclear how this would be possible for either of the previously mentioned interpretations: A single DM whose ordering is unknown or multiple DMs exhibiting the possible orders but who are otherwise alike in V and A. This issue arises again in testing Definition 10.

Note that the function V is a value function attached to the events and is independent of the order $\succsim \in \mathcal{O}_1$ whereas $S_{\succsim,\Omega}$ is attached also to events but it varies with $\succsim$. The dependence on $S_{\succsim,\Omega}$ is as the Shannon (1948) entropy of these subjective probabilities. One may trivially rewrite (19) as

$$U_{\succsim}(g_{[n]}) = \sum_{i=1}^{n} \left[U_X(x_i) + V(\Omega) - V(C_i) - A \log_2 S_{\succsim,\Omega}(C_i)\right] S_{\succsim,\Omega}(C_i). \quad (22)$$

In the hypothetical interpretation of one DM whose ordering over gambles is unknown to the theorist, it seems plausible that the function V and constant A of Part 1 of Theorem 2 are independent of the order $\succsim$ DM actually reveals because they are properties of the single DM in the same way as is the utility function U_X. In the interpretation of multiple DMs, with at least one corresponding to each extension, the Theorem establishes that the utility representation for every DM has the same V and A, much as if they are, in that respect, all clones of a common DM.

Mark Machina[4] pointed out that the representation (22) is a special case of what economists call "state-dependent SEU with a non-additive measure" (see, e.g., Karni, 1985). Define

$$\widehat{U}_{\succsim,\Omega}(x_i, C_i) := \frac{U_X(x_i)}{-\log_2 S_{\succsim,\Omega}(C_i)} + \frac{V(\Omega) - V(C_i)}{-\log_2 S_{\succsim,\Omega}(C_i)} + A,$$
$$\widehat{S}_{\succsim,\Omega}(C_i) := -S_{\succsim,\Omega}(C_i) \log_2 S_{\succsim,\Omega}(C_i).$$

Then, by dividing and multiplying by $-\log_2 S_{\succsim,\Omega}(C_i)$ in (22) we have the form

$$U_{\succsim,\Omega}(g_{[n]}) = \sum_{i=1}^{n} \widehat{U}_{\succsim,\Omega}(x_i, C_i) \widehat{S}_{\succsim,\Omega}(C_i),$$

which is the general form of state-dependent SEU.

3.3 Mixed Uncertainty and Risk

The following remarks were stimulated by our reading a draft manuscript by Mark Machina, followed by Luce's face-to-face discussion with him on December 30, 2006 about it.

Various decision situations, such as the Ellsberg paradox, involve both events that are quite uncertain (often called ambiguous) and others for which a probability is in some way known to the decision maker. The question is, how do we specialize our representations to such cases. Suppose that C_i and C_j, $i \neq j$, are both uncertain events but that $C_i \cup C_j$ is known to occur with a fixed, given probability $p_{i,j}$. Clearly, the underlying assumption of inset entropy that any event partition can occur with any non-trivial probability vector is not consistent with such a constraint. So our axiomatic approach does not automatically work.

We can take another tack common in the utility literature of simply choosing that ordering $\succsim$ such that some of the $S_{\succsim,\Omega}$ values are prescribed and that

$$\max\left(S_{\succsim,\Omega}(C_i), S_{\succsim,\Omega}(C_j)\right) \leq S_{\succsim,\Omega}(C_i) + S_{\succsim,\Omega}(C_j) = S_{\succsim,\Omega}(C_i \cup C_j) = p_{i,j}.$$

This simply means that under segregation the representation (19) of Theorem 2 some of the $S_{\succsim,\Omega}$ have prescribed values, whereas the others are subjective.

Consider the special case where all the events have known probabilities, i.e., the gamble is risky. Then, with $g_{[n]} = (x_1, p_1; ...; x_n, p_n)$, where $\mathbf{p} = (p_1, ..., p_n)$ is a non-trivial probability vector, if we knew that (22) held with $S_{\succsim,\Omega}(C_i) = p_i$ for all i, then we would have

$$U_{\succeq}(g_{[n]}) = \sum_{i=1}^{n} \left[U_X(x_i) + V(\Omega) - V(C_i) - A\log_2 p_i\right] p_i. \tag{23}$$

In Luce et al. (in press b), we developed a risk specialization of the general theory for a fixed ordering $\succsim$ that gave the representation (23) with $V(\Omega) - V(C_i) = 0$ for all C_i. We now consider how the two results — one with no value of events and the other with valued events — can be reconciled.

The representation with a constant V term was developed under the supposition that the probabilities were all that was given to the respondent, as in many experiments, and no events were specified. We assumed that the respondent had in mind an hypothetical mechanism giving rise to a space of events and that the representation we had arrived for such events was true. Because these events were entirely hypothetical, they were automatically without any value, positive or negative. By contrast, the present representation begins with events, some of which have prescribed probabilities and at the same time may have inherent value. So, the results are not really incompatible; it is simply that the present case of mixed risk and uncertainty is more general, albeit currently unaxiomatized.

3.4 Equivalences to (18) of Theorem 2

Although we do not in fact use the following proposition, we feel that, to some degree, it illuminates Assumption (ii) of Theorem 2.

Proposition 3 *Suppose that for each ordering* $\succsim \in \mathcal{O}$,

$$U_{\succsim}(KE_{\succsim}(g_{[n]})) = \sum_{i=1}^{n} U_X(x_i) S_{\succsim,\Omega}(C_i), \tag{24}$$

and that $\succsim, \succsim' \in \mathcal{O}$ *satisfy (16). Then, the following are equivalent:*

1. $U_{\succsim}(g_{[n]}) = U_{\succsim'}(g_{[n]})$ *holds.*

2. *The elements of chance satisfy:*

$$\begin{aligned} H_{\succsim}(C_1, ..., C_n) &:= U_{\succsim}(e, C_1; ...; e, C_n) \\ &= U_{\succsim'}(e, C_1; ...; e, C_n) \\ &= H_{\succsim'}(C_1, ..., C_n). \end{aligned} \tag{25}$$

3. *Equivalence (18) of Assumption (ii) is satisfied.*

4 Qualitative Conditions Relevant to Finite Additivity

4.1 Finite Additivity of $S_{\succsim,\Omega}$

4.1.1 The Definitions

Because under segregation, and as we shall also see under duplex decomposition, the case of finitely additive $S_{\succsim,\Omega}$ can arise, one may want to verify additivity by direct experimentation. To that end, we formulate two behavioral conditions and then show that each is equivalent to the weights being finitely additive. We follow the convention that one or more primes on an event mean independent realizations of that event. We invoke this without further comment.

Definition 4 ***Qualitative event additivity, I:*** *If*

$$x \oplus (e, C_1; e, C_2) \oplus (e, C_1'; e, C_2') \sim (x, C_1; e, C_2) \oplus (e, C_1'; x, C_2') \tag{26}$$

*holds for every binary event partition, we say that $\succsim$ is x-**qualitative event additive, I**. It is **qualitative event additive, I**, if it is $x-$qualitative event additive, I, for all $x \in X$.*

Definition 5 ***Qualitative event additivity, II:*** *If*

$$\begin{aligned} &(x, C_1 \cup C_2; e, \Omega\backslash C_1 \cup C_2) \oplus (e, C_1'; e, \Omega\backslash C_1') \oplus (e, C_2'; e, \Omega\backslash C_2') \\ &\quad \sim (x, C_1''; e, \Omega\backslash C_1'') \oplus (x, C_2''; e, \Omega\backslash C_2'') \oplus (e, C_1''' \cup C_2'''; e, \Omega\backslash C_1''' \cup C_2''') \end{aligned} \tag{27}$$

*holds for every trinary event partition, we say that $\succsim$ is x-**qualitative event additive, II.** It is **qualitative event additive, II**, if it is $x-$qualitative event additive, II, for all $x \in X$.*

Note that when the property of *certainty,* which for present purposes can be written in the form $(x, \Omega; e, \varnothing) \sim x$**,** holds, qualitative event additivity, II, implies qualitative event additivity, I, because in the special case $\Omega = C_1 \cup C_2$, (27) reduces to (26).

Were the elements of chance idempotent, as is typical of most utility theories, then qualitative event additivity, I, reduces to

$$x \sim (x, C_1; e, C_2) \oplus (e, C_1'; x, C_2'),$$

and qualitative event additivity, II, reduces to

$$(x, C_1 \cup C_2; e, \Omega \backslash C_1 \cup C_2) \sim (x, C_1; e, \Omega \backslash C_1) \oplus (x, C_2; e, \Omega \backslash C_2).$$

But in the non-idempotent case, one must be very careful in formulating appropriate testable properties that take the element of chance into account.

Under segregation and the monotonicity of $\oplus$, qualitative event additivity, I, is equivalent to

$$\begin{aligned}(x, C_1; x, C_2) \oplus (e, C_1'; e, C_2') &\sim x \oplus (e, C_1; e, C_2) \oplus (e, C_1'; e, C_2') \\ &\sim (x, C_1''; e, C_2'') \oplus (e, C_1'; x, C_2'). \end{aligned} \tag{28}$$

4.1.2 The Result

Making use of the decomposition of a gamble into kernel equivalents and elements of chance, (4), the additivity of U_X over $\oplus$, and separability we may prove:

Proposition 6 *Let $\succsim$ be a preference order. Suppose that the following assumptions are satisfied: decomposability into KEs and elements of chance, (4), $U_{\succsim}$ is additive over joint receipt $\oplus$, (1), the kernel equivalents of unitary gambles have a separable representation $U_{\succsim} S_{\succsim,\Omega}$. Let $x \in X$, $x \nsim e$.*

1. *If $S_{\succsim,\Omega}$ has the choice property, then the following are equivalent:*
   - *(a) $S_{\succsim,\Omega}$ is finitely additive, (13).*
   - *(b) $\succsim$ is qualitative event additive, I, Definition 4.*
   - *(c) $\succsim$ is $x-$qualitative event additive, I, Definition 4.*
2. *The following are equivalent:*
   - *(a) $S_{\succsim,\Omega}$ is finitely additive, (13).*
   - *(b) $\succsim$ is qualitative event additive, II, Definition 5.*
   - *(c) $\succsim$ is $x-$qualitative event additive, II, Definition 5.*

4.2 Finite Additivity of $S^{\rho}_{\succsim,\Omega}$

In the next Section 5 we make use of a strong assumption, namely, that for some ρ independent of $\succsim$, $\succsim \in \mathcal{O}$, $S^{\rho}_{\succsim,\Omega}$ is finitely additive. In this section, we formulate behavioral conditions equivalent to this assumption. Toward that end, we define:

Definition 7 ***Weak weight complementarity*** *holds if for all binary partitions (C_1, C_2) and (D_1, D_2),*

$$\begin{aligned} &KE_{\succsim}(x, C_1; e, C_2) \succsim_X KE_{\succsim}(x, D_1; e, D_2) \\ \Leftrightarrow\ &KE_{\succsim}(x, D_2; e, D_1) \succsim_X KE_{\succsim}(x, C_2; e, C_1). \end{aligned} \tag{29}$$

Using this leads us to the first result toward our goal.

Theorem 8 *Assume that the kernel equivalents of unitary gambles have a separable representation $U_{\succsim} S_{\succsim,\Omega}$ and that $S_{\succsim,\Omega}$ satisfies the choice property. For given $x \not\sim e$, the following statements are equivalent:*
(i) *Weak weight complementarity is satisfied.*
(ii) *There exists a constant $\rho_{\succsim} > 0$ such that $S_{\succsim,\Omega}^{\rho_{\succsim}}$ is finitely additive.*

Note that (ii) follows from (i) holding for some $x \not\sim e$; whereas (ii) implies that (i) holds for every $x \in X$.

Proposition 9 *Assume that the kernel equivalents of unitary gambles have a separable representation $U_{\succsim} S_{\succsim,\Omega}$ for an order $\succsim$ and that $x \not\sim e$. Then, for disjoint $C_1, C_2 \in {}^*$, the following statements are equivalent:*

(a) $(x, C_1; e, C_2) \sim (x, C_2; e, C_1)$.

(b) $KE_{\succsim}(x, C_1; e, C_2) \sim_X KE_{\succsim}(x, C_2; e, C_1)$.

(c) $S_{\succsim, C_1 \cup C_2}(C_1) = S_{\succsim, C_1 \cup C_2}(C_2)$.

Such disjoint events (C_1, C_2) are called a $\succsim$**-subjectively equal pair**.

The following definition gives a behavioral property that plays an important role in the following Theorem.

Definition 10 *The **equal event property** holds for a pair of orders $(\succsim, \succsim')$ provided that for all $\succsim$-subjectively equal (C_1, C_2) and $\succsim'$-subjectively equal (D_1, D_2),*

$$KE_{\succsim}(x, C_1; e, C_2) \sim_X KE_{\succsim'}(x, D_1; e, D_2). \tag{30}$$

In the special case $\succsim = \succsim'$, we say that the equal event property holds for $\succsim$.

Theorem 11 *Assume that the choice property and that the kernel equivalents of unitary gambles have a separable representation $U_{\succsim} S_{\succsim,\Omega}$ hold for a pair of orders $(\succsim, \succsim')$. Suppose further that $S_{\succsim,\Omega}^{\rho_{\succsim}}$ and $S_{\succsim',\Omega}^{\rho_{\succsim'}}$ are finitely additive. Then the following statements are equivalent.*

(i) *For some $\succsim$-subjectively equal (C_1, C_2) and $\succsim'$-subjectively equal (D_1, D_2), (30) holds:*

$$KE_{\succsim}(x, C_1; e, C_2) \sim_X KE_{\succsim'}(x, D_1; e, D_2). \tag{31}$$

(ii) $\rho_{\succsim} = \rho_{\succsim'}$.

(iii) *The equal event property holds for $(\succsim, \succsim')$.*

Although the careful reader will realize that Theorem 8 concerns a single ordering $\succsim$ whereas Theorem 11 holds for any pair of orders, we nonetheless make the following observation. The two event partitions (C_1, C_2) and (D_1, D_2) that appear in Theorem 8 are restricted only by (29) and so when we discuss $S^{\rho_{\succsim}}_{\succsim,\Omega}(C_1 \cup C_2)$ and prove finite additivity there is no constraint on C_1 and C_2 beyond $C_1 \cap C_2 = \varnothing$. By contrast, the partitions (C_1, C_2) and (D_1, D_2) discussed in Theorem 11 are constrained to be subjectively equal pairs, which is a tiny subfamily of all binary partitions. But given Theorem 8 that is sufficient to prove $\rho_{\succsim} = \rho_{\succsim'}$.

5 Representation of Elements of Chance Under Duplex Decomposition

5.1 Definition of duplex decomposition

Somewhat parallel to segregation is an alternative, less restrictive assumption:

Duplex decomposition:

$$(x, C; y, D) \oplus (e, C'; e, D') \sim (x, C; e, D) \oplus (e, C'; y, D'), \tag{32}$$

where (C', D') is an independent realization of (C, D). Note that the two sides do not have the same bottom line: on the left either x or y but not both arises whereas on the right there are four possibilities: $e = e \oplus e, x = x \oplus e, y = e \oplus e, x \oplus y$.

5.2 The representation

In obtaining the representation, we invoke the inset entropies of degree κ (see Notes after Theorem 12 and Aczél & Kannappan, 1978, Ebanks et al., 1988) for a general κ. The only way we have seen so far to do so is to impose a very strong assumption about the finite additivity of some weights. Comment 4 following the Theorem discusses the testability of this assumption.

Theorem 12 *Suppose that:*

- **(i)** *for each $\succsim$ belonging to $\mathcal{O}$, the assumptions of Theorem 16 of Luce et al.(see Appendix B) are satisfied,*
- **(ii)** *there exists a positive constant ρ such that $S^{\rho}_{\succsim,\Omega}$ is finitely additive for each $\succsim$ belonging to $\mathcal{O}$,*
- **(iii)** *with the weights $S^{\rho}_{\succsim,\Omega}$ replacing $S_{\succsim,\Omega}$, Assumptions (i)-(ii) of Theorem 2 are satisfied, and when $\rho = 1$ also assume (iii), regularity, of Theorem 2.*

Then, there exist a constant A and a function $V : \mathfrak{B} \to \mathbb{R}$, with $V(\varnothing) = 0$, such that for all gambles $g_{[n]} = (x_1, C_1; ...; x_n, C_n)$ and all $\succsim \in \mathcal{O}$,

$$
\begin{aligned}
U_{\succsim}(g_{[n]}) &= \sum_{i=1}^{n} [U_X(x_i) - V(C_i)]\, S_{\succsim,\Omega}(C_i) + V(\Omega) \\
&\quad - A \begin{cases} \sum_{i=1}^{n} S_{\succsim,\Omega}(C_i) \log S_{\succsim,\Omega}(C_i), & \rho = 1 \\ 1 - \sum_{i=1}^{n} S_{\succsim,\Omega}(C_i), & \rho \neq 1 \end{cases} .
\end{aligned} \tag{33}
$$

A number of observations are in order.

1. Together, the assumptions of the Theorem allow us to define a function which, with our other assumptions, leads to the recursions of the theory of inset entropy (Appendix C, Section 7.3) whose solutions are a basis of the above result.

2. Note that, as was true under segregation, e.g., in (19), the value function V in (33) is attached to the events and is independent of the order $\succsim \in \mathcal{O}$, whereas the weight function $S_{\succsim,\Omega}$ varies with $\succsim$.

3. Because $p_i := S_{\succsim,\Omega}(C_i)^{\rho}$, when $\rho \neq 1$ the term $A\left[1 - \sum_{i=1}^{n} S_{\succsim,\Omega}(C_i)\right]$ takes the form $A \sum_{i=1}^{n}(1 - p_i^{\kappa})$, where $\kappa := 1/\rho$, which is called the entropy of degree κ (Havrda et al., 1967) of these subjective probabilities. It has been quoted lately as "Tsallis entropy" (by Suyari, 2002, among many others), referring to Tsallis (1988) and other works by Tsallis. It was, however, first introduced by Havrda et al. (1967) and examined in depth by Daróczy (1970). See also, among others, the monograph by Aczél et al. (1975).

4. The apparently very strong Assumption (ii) is, by Theorems 8 and 11, equivalent to assuming the following properties: The weak weight complementary property, (29), of Theorem 8, and the equal event property of Definition 10. However, as noted following Theorem 2, it is unclear how to test behaviorally a property that involves two distinct orders, such as the equal event property of Definition 10.

5.3 A Result about Finitely Additive $S_{\succsim,\Omega}$

Theorem 13 *Suppose that any two of the following three conditions are satisfied by the family $\mathcal{O}$:*

(i) *The assumptions of Theorem 2.*

(ii) *The assumptions of Theorem 12.*

(iii) *For each ordering $\succsim \in \mathcal{O}$, qualitative event additivity, I, holds.*
Then all three conditions are satisfied and for each $\succsim \in O$, the representation (19) holds:

$$U_{\succsim}(g_{[n]}) = \sum_{i=1}^{n} U_X(x_i) S_{\succsim,\Omega}(C_i) + V(\Omega)$$
$$- \sum_{i=1}^{n} V(C_i) S_{\succsim,\Omega}(C_i) - A \sum_{i=1}^{n} S_{\succsim,\Omega}(C_i) \log_2 S_{\succsim,\Omega}(C_i).$$

The result follows immediately using part 1 of Proposition 6. Note that (i) and (iii) imply (ii) with $\rho = 1$.

5.4 Condition for constant V

Now we present a condition that allows us to conclude that the value function V is a constant, i.e., the utility representation reduces to one with no valuation of events, per se.

Proposition 14 *Suppose that the assumptions of Theorem 12 hold. Then, the following are equivalent:*

1. *For all $\succsim, \succsim' \in \mathcal{O}$, if $\mathbf{C} = (C_1, C_2)$ and $\mathbf{D} = (D_1, D_2)$ are non-trivial ordered binary partitions of $\Omega = \cup_{i=1}^{2} C_i$ and*

   $$S_{\succsim,\Omega}(C_i) = S_{\succsim',\Omega}(D_i), \; (i = 1, 2), \tag{34}$$

   then $y \in X$ is a certainty equivalent, CE, of the element of chance, $(e, C_1; e, C_2)$, under $\succsim$ iff it is also a CE of $(e, D_1; e, D_2)$ under $\succsim'$, i.e.,

   $$y \sim (e, C_1; e, C_2) \Leftrightarrow y \sim' (e, D_1; e, D_2). \tag{35}$$

2. *There exist constants a and b such that*

   $$V(C) = \begin{cases} a, & \text{if } C \text{ is not maximal and not } \varnothing \\ b, & \text{if } C \text{ is maximal} \end{cases} .$$

6 Conclusion and Open Problems

6.1 Conclusions

The major premise of this article is that given a preference order $\succsim_X$ over pure consequences, a theorist must be prepared to face any extension $\succsim$ of $\succsim_X$ to gambles (uncertain alternatives). Under that premise, the rational property of segregation leads naturally to two cases because, under the assumptions made, the weighting function $S_{\succsim,\Omega}$ is forced to be either finitely additive or non-trivially p-additive. The former leads to SEU plus a weighted value function V over events plus a utility of gambling term that takes the form of the Shannon entropy of the weights. The latter leads to a version of rank-dependent utility plus a constant that must be 0 for non-maximal Ω.

A second decomposition is by the non-rational property of duplex decomposition. For that case, so far we have a result under the strong condition that there is a constant $\rho \neq 1$, independent of $\succsim$, such that for all $\succsim$, $S^{\rho}_{\succsim,\Omega}$ is finitely additive. In that case, our assumptions lead to a representation as a linear weighted utility plus a weighted value function V over events plus an entropy of degree $1/\rho$.

We state next some of the problems that remain to be solved.

6.2 Open Problems

6.2.1 Constraints on the Function V

The statements of Theorem 2 and later of Theorem 12 assert the existence of the value function V over events, but they do not arrive at any of its properties. In particular, they do not impose constraints on V. As a result, it is of interest to seek out properties that do constrain it.

To that end, define $V^*(C, \Omega) := V(\Omega) - V(C)$. Because Ω is held fixed in the following discussion, we simplify $V^*(C, \Omega)$ to $V^*(C)$. Now, for disjoint $C, D \subset \Omega$, suppose that C is valued positively, i.e., $V^*(C) > 0$ and D is valued no better than C, i.e., $V^*(C) \geq V^*(D)$. Then it seems intuitively plausible that the value of the better event is diluted by being placed in an either/or situation with the less good one, i.e., $V^*(C \cup D) \leq V^*(C)$

Similarly, if event C is valued negatively, $V(C) < 0$, and D is not valued any worse, i.e., $V(C) \leq V(D)$, then the negative value of C is lessened by placing it in an either/or situation with D, i.e., $V^*(C \cup D) \geq V^*(C)$. This case is illustrated by air travel: the event of either a crash or any of the other possible disjoint events such as a late arrival certainly is more valued than the crash itself.

What behavioral properties justify these intuitions? And what other properties can be defended using behavioral properties that, at least in principal, can be evaluated experimentally?

6.2.2 Axiomatize Rank-Dependent Values

As discussed by Luce et al. (in press b), many of the anomalies found in the empirical literature are encompassed by utility representations that include a utility of gambling expressions based on entropy, but some of Michael Birnbaum's examples (for a summary, see Marley et al., 2005) accommodated by his un-axiomatized TAX representation are not handled by our current theory. Moreover, adding the V expressions of the present theory deals with some of the issues but again not all. It appears from the data that the value of an event depends not only upon that event but upon whether it is associated with a better or worse consequence, i.e., on its relative ranking. For example, an event

C has a better value if it is playing the role of C_1 rather than the role of C_n. So it may be desirable to characterize a representation of the following form as a replacement for the representation (19): given $x_1 \succsim x_2 \succsim ... \succsim x_n$,

$$
\begin{aligned}
& U(x_1, C_1; x_2, C_2 ...; x_n, C_n) \\
& \quad = \sum_{i=1}^{n} \left[U(x_i) + V_0(\Omega) - V_i(C_i) - A \log_2 S_{\succsim,\Omega}(C_i) \right] S_{\succsim,\Omega}(C_i),
\end{aligned}
$$

i.e., the common function V is replaced by a family of functions V_i, $i = 1, ..., n$. Nothing so far developed in the entropy literature seems to lead to the above form.

7 Appendices

7.1 A. Summary of Theorem 13, Luce et al. (in press a)

The following definitions arise in the statement of the theorem:

Let $g_{[n]} := (x_1, C_1; x_2, C_2; ...; x_n, C_n)$.

Upper gamble decomposition:

$$
g_{[n]} \sim (x_1, C_1; (x_2, C_2; ...; x_n, C_n),\ \Omega \backslash C_1), \tag{36}
$$

Branching:

$$
g_{[n]} \sim ((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3, ...; x_n, C_n). \tag{37}
$$

The statement of the result is virtually verbatim:

Theorem 13. *Suppose that $n \geq 2$ and that the following assumptions are satisfied: decomposability into KEs and elements of chance, (4), U is additive over joint receipt $\oplus$, (1), the kernel equivalents of unitary gambles have a separable representation US_Ω, and segregation, (12). Suppose that both upper gamble decomposition, (36), and branching, (37), are satisfied. Then there exists an increasing function μ, positive on B^*, $\mu(\varnothing) = 0$, such that*

$$
S_\Omega(C) = \mu(C)/\mu(\Omega),
$$

and, for all gambles $g_{[n]}$, either

1. *S_Ω, and thus μ, are finitely additive and there is a function $h : B \to R$, $h(\varnothing) = 0$ with the dimension of μU, for which*

$$
U(g_{[n]}) = SEU(g_{[n]}) + \frac{1}{\mu(\Omega)} \left[h(\Omega) - \sum_{i=1}^{n} h(C_i) \right], \tag{38}
$$

where $SEU(g_{[n]})$ is given by

$$SEU(KE_{\succsim}(g_{[n]})) := \sum_{i=1}^{n} U(x_i)S_\Omega(C_i), \quad \left(\sum_{i=1}^{n} S_\Omega(C_i) = 1\right). \tag{39}$$

or

2. *S_Ω is p-additive (with $\Delta \neq 0$) and there is a constant A with the dimension of U such that*

$$U(g_{[n]}) = RDU(g_{[n]}) + H, \tag{40}$$

where $RDU(g_{[n]})$ is given by

$$RDU(g_{[n]}) = \sum_{i=1}^{n} U(x_i)S_\Omega(C_i)\left[1 + \Delta\mu(\Omega)S_\Omega(C(i-1))\right], \tag{41}$$

where $C(j) = \cup_{k=1}^{j} C_k$ and H is 0 when Ω is not maximal and a constant A when Ω is maximal.

7.2 B. Summary of Theorem 16, Luce et al. (in press a)

Theorem 16 *Suppose that $n \geq 2$ and that the following assumptions are satisfied: decomposability into KEs and elements of chance, (4), U is additive over joint receipt $\oplus$, (1), the kernel equivalents of unitary gambles have a separable representation US_Ω, and duplex decomposition, (32). Suppose that both upper gamble decomposition, (36), and branching, (37), hold. Then there exist an increasing function μ, positive on B^*, $\mu(\varnothing) = 0$, and a function $h : B \to R$, $h(\varnothing) = 0$, and with the dimension of μU such that*

$$S_\Omega(C) = \mu(C)/\mu(\Omega), \tag{42}$$

and

$$U(g_{[n]}) = LWU\left(g_{[n]}\right) + \frac{1}{\mu(\Omega)}[h(\Omega) - \sum_{i=1}^{n} h(C_i)], \tag{43}$$

where LWU is given by

$$LWU\left(g_{[n]}\right) = \sum_{i=1}^{n} U(x_i)S_\Omega(C_i). \tag{44}$$

7.3 C. Inset Entropy

This article draws heavily on a topic called *inset entropy* in the functional equations literature. We summarize the formulation of the theory and the major results will be imbedded in the statements of Theorems 2 and 12 and in the discussion following the latter result.

7.3.1 The Arrays of Inset Entropy

Let $\mathfrak{B}$ be a ring of sets which, in the current interpretation, are events. The basic ingredient of the theory of inset entropy was initially characterized by Aczél and Daróczy (1978) as follows (using our notation for events rather than theirs):

"We call

$$\left(\begin{array}{cccc} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{array} \right) \in \Omega_n \times \Gamma_n \tag{45}$$

a randomized system of events. We use events C_i as names for the elements of $\mathfrak{B}$, while the p_i are probabilities." In addition, it was assumed that each event partition vector $\mathbf{C} = (C_1, C_2, ..., C_n)$, $C_i \in \mathfrak{B}$, and each probability vector $\mathbf{p} = (p_1, p_2, ..., p_n)$ can be selected "independently," i.e., knowing one does not limit the choice of the other. In that formulation, a nonnegative p_i value was allowed to go with any C_i. A related work is Aczél & Kannappan (1978). Later, the variables p_i were restricted to the open interval $]0, 1[$ and C_i were restricted to $\mathfrak{B}^*$. The restrictions and the new formulations made many older derivations inappropriate. Subsequent works under the new formulation were reported in Ng (1980), Aczél (1980, 1981) and Ebanks et al. (1988). Luce et al. (in press a) assumed the conditions of the new formulation. The "later" works quoted above deal with more general entropies of degree κ, see Theorem 12 below and the Notes following it.

As stated in the body of the paper, a probability vector $\mathbf{p} = (p_1, ..., p_n)$ is called *non-trivial* when $p_i \in]0, 1[$, $i = 1, ..., n$, and an ordered event partition $\mathbf{C} = (C_1, C_2, ..., C_n)$ of $\Omega = \cup_{i=1}^{n} C_i$ is called *non-trivial* when $C_i \neq \varnothing$, $i = 1, ..., n$. All references to the pair $(\mathbf{p}, \mathbf{C})$ implicitly assume that both are non-trivial.

In the work that follows, for each pair $(\mathbf{p}, \mathbf{C})$, we will be working with weights that for some $\rho > 0$ satisfy $S^{\rho}_{\succsim,\Omega}(C_i) = p_i$, $\succsim$ belonging to $\mathcal{O}$. We define

$$H_n \left(\begin{array}{cccc} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{array} \right) := H_{\succsim}(C_1, ..., C_n) \tag{46}$$

and we make assumptions in Theorems 2 and 12 sufficient to insure that the function H_n is well defined.

As we shall see, under segregation the case of $\rho = 1$ arises naturally. Whereas, under duplex decomposition, both the cases $\rho = 1$ and $\rho \neq 1$ arise, but not nearly so naturally, especially the latter.

7.3.2 The Branching Recursion for a Single Order

For a fixed ordering $\succsim$, two important recursions arose in Luce et al. (in press a), one of which is *branching,*

$$g_{[n]} \sim ((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; ...; x_n, C_n), \tag{47}$$

which under our assumptions (see Luce et al., in press a) reduces to the elements of chance satisfying the recursion[5]

$$H_{\succsim}(C_1, ..., C_n) = H_{\succsim}(C_1 \cup C_2, C_3, ..., C_n) + H_{\succsim}(C_1, C_2) S_{\succsim,\Omega}(C_1 \cup C_2). \tag{48}$$

The other recursion, *upper gamble decomposition,* which plays no role in inset entropy, was already given in Appendix A, (36).

7.3.3 Arrays, Additivity and Conditional Probabilities

In the case that (46) holds with $S^{\rho}_{\succsim,\Omega}$ **finitely additive**, i.e., *for all* $C, D \subseteq \Omega$, $C \cap D = \varnothing$,

$$S^{\rho}_{\succsim,\Omega}(C \cup D) = S^{\rho}_{\succsim,\Omega}(C) + S^{\rho}_{\succsim,\Omega}(D), \tag{49}$$

we further connect them to sub-partitions and conditional probabilities. Using the choice property, (6), and the finite additivity of $S^{\rho}_{\succsim,\Omega}$, we see that

$$\begin{aligned} S^{\rho}_{\succsim,\Omega}(C_1 \cup C_2) &= S^{\rho}_{\succsim,\Omega}(C_1) + S^{\rho}_{\succsim,\Omega}(C_2) = p_1 + p_2, \\ S^{\rho}_{\succsim,C_1 \cup C_2}(C_i) &= \frac{S^{\rho}_{\succsim,\Omega}(C_i)}{S^{\rho}_{\succsim,\Omega}(C_1 \cup C_2)} = \frac{p_i}{p_1 + p_2} \quad (i = 1, 2). \end{aligned} \tag{50}$$

By (46) and (50), we get

$$H_{n-1}\begin{pmatrix} C_1 \cup C_2, & C_{3,} & ..., & C_n \\ p_1 + p_2, & p_3, & ..., & p_n \end{pmatrix} = H_{\succsim}(C_1 \cup C_2, ..., C_n) \tag{51}$$

and

$$H_2\begin{pmatrix} C_1, & C_2 \\ \frac{p_1}{p_1+p_2}, & \frac{p_2}{p_1+p_2} \end{pmatrix} = H_{\succsim}(C_1, C_2). \tag{52}$$

7.3.4 The Branching Recursion of Inset Entropy

We continue to assume that $S^{\rho}_{\succsim,\Omega}$ is finite additivity, i.e., (49) holds. Substituting (46), (51) and (52) into the recursion (48) and letting $\kappa := 1/\rho$ we get

$$H_n\begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} = H_{n-1}\begin{pmatrix} C_1 \cup C_2, & C_{3,} & ..., & C_n \\ p_1 + p_2, & p_3, & ..., & p_n \end{pmatrix} + H_2\begin{pmatrix} C_1, & C_2 \\ \frac{p_1}{p_1+p_2}, & \frac{p_2}{p_1+p_2} \end{pmatrix}(p_1 + p_2)^{\kappa}. \tag{53}$$

The recursion above is that of inset entropy of degree κ (Aczél & Kannappan, 1978; Ebanks et al. 1988). The case $\kappa = 1$ is invoked in Section 3.

7.4 D. Proofs

7.4.1 Theorem 2

According to Theorem 13 of Luce et al., each $S_{\succsim,\Omega}$ is either finitely additive or non-trivially p-additive. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ denote those orders of $\mathcal{O}$ with additive, and, respectively, non-trivially p-additive, $S_{\succsim,\Omega}$.

We first show that Assumptions (i) and (ii) are irrelevant for orders in $\mathcal{O}_2$.

A non-trivially p-additive $S_{\succsim,\Omega}$ satisfies

$$S_\Omega(C \cup D) = S_\Omega(C) + S_\Omega(D) + \Delta\mu(\Omega)S_\Omega(C)S_\Omega(D), \tag{54}$$

with $\Delta \neq 0$. This implies that either

$$S_\Omega(C \cup D) < S_\Omega(C) + S_\Omega(D) \tag{55}$$

for all disjoint $C, D \in \mathfrak{B}^*$ contained in Ω, or that

$$S_\Omega(C \cup D) > S_\Omega(C) + S_\Omega(D) \tag{56}$$

for all disjoint $C, D \in \mathfrak{B}^*$ contained in Ω, according to the sign of Δ. Simple induction on the length $n \geq 2$ leads either to strict subadditivity

$$1 = S_\Omega(\Omega) < \sum_{i=1}^{n} S_\Omega(C_i), \tag{57}$$

or to strict super-additivity

$$1 = S_\Omega(\Omega) > \sum_{i=1}^{n} S_\Omega(C_i). \tag{58}$$

Hence, for orders belonging to $\mathcal{O}_2$, the matching $S_\Omega(C_i) = p_i$ with $\sum p_i = 1$ is impossible. Therefore, $\mathcal{O}$ satisfies (i) iff $\mathcal{O}_1$ does; and (ii) only constrains $\mathcal{O}_1$, because every order in $\mathcal{O}_2$ trivially satisfies that condition.

There are two cases to consider.

1. *For orders in $\mathcal{O}_1$, i.e., those with* finitely *additive $S_{\succsim,\Omega}$.*

According to Theorem 13, (i) of Luce et al. (in press a), each $S_\succsim$ satisfies the choice property, and combining (38) with (39) we have the representation

$$U_\succsim(g_{[n]}) = \sum_{i=1}^{n} U_X(x_i)S_{\succsim,\Omega}(C_i) + H_\succsim(C_1, ..., C_n) \tag{59}$$

where

$$H_\succsim(C_1, ..., C_n) := U_\succsim(e, C_1; ...; e, C_n) = \frac{1}{\mu_\succsim(\Omega)}[h_\succsim(\Omega) - \sum_{i=1}^{n} h_\succsim(C_i)]$$

are symmetric functions satisfying the branching relation (48), i.e.,

$$\begin{aligned} H_\succsim(C_1, ..., C_n) = {} & H_\succsim(C_1 \cup C_2, C_3, ..., C_n) \\ & + H_\succsim(C_1, C_2)S_{\succsim,\Omega}(C_1 \cup C_2). \end{aligned} \tag{60}$$

Let $\succsim, \succsim' \in \mathcal{O}_1$ be any two orders with equal weights, (16), then it follows from assumption (ii) that

$$\begin{aligned} H_{\succsim}(C_1, ..., C_n) &= U_{\succsim}(e, C_1; ...; e, C_n) \\ &= U_{\succsim}(CE_{\succsim}(e, C_1; ...; e, C_n)) \\ &= U_X(CE_{\succsim}(e, C_1; ...; e, C_n)) \\ &= U_X(CE_{\succsim'}(e, C_1; ...; e, C_n)) \\ &= H_{\succsim'}(C_1, ..., C_n). \end{aligned} \tag{61}$$

For arbitrary

$$\begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix}, \tag{62}$$

where the C_i form a non-trivial partition and the $p_i > 0$ form a non-trivial probability distribution, we define H_n by

$$H_n \begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} = H_{\succsim}(C_1, ..., C_n) \tag{63}$$

where $\succsim \in \mathcal{O}$ is any order satisfying $S_{\succsim,\Omega}(C_i) = p_i$ ($i = 1, ..., n$). By Assumptions (i), such an order exists and by (61) the value of H_n does not depend on the choice of the order. As was shown earlier, $\succsim$ necessarily is in $\mathcal{O}_1$. Hence the H_n ($n = 2, 3, ...$) are well-defined functions and (63) holds for all $\succsim \in \mathcal{O}_1$.

With that, and reasoning as in Section 7.3.1, (60) gives

$$\begin{aligned} H_n \begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} &= H_{n-1} \begin{pmatrix} C_1 \cup C_2, & C_3, & ... & C_n \\ p_1 + p_2, & p_3, & ... & p_n \end{pmatrix} \\ &+ H_2 \begin{pmatrix} C_1, & C_2 \\ \frac{p_1}{p_1+p_2} & \frac{p_2}{p_1+p_2} \end{pmatrix} (p_1 + p_2). \end{aligned}$$

According to the results established in Ebanks et al. (1988), under mild regularity assumptions, such as

$$p \mapsto H_2 \begin{pmatrix} C_1, & C_2 \\ p, & 1 - p \end{pmatrix} \tag{64}$$

is locally bounded, which is implied by Assumption (iii) of Theorem 2, there exists a function $V : \mathfrak{B}^* \to \mathbb{R}$ and a constant A such that

$$H_n \begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} = V(\Omega) - \sum_{i=1}^{n} V(C_i) p_i - A \sum_{i=1}^{n} p_i \log_2 p_i. \tag{65}$$

Putting the obtained forms of H_n into (63) we get the form of $H_{\succsim}$ for all $\succsim \in \mathcal{O}_1$. Putting that further back into (59) we arrive at the representation (19).

2. *For orders in $\mathcal{O}_2$, i.e., those with non-trivially $p-$additive $S_{\succsim,\Omega}$.*

No assumption impacts $\mathcal{O}_2$ apart from those of Theorem 13, and the asserted RDU representation is that of Theorem 13.

(Using expansibility, which means a gamble with a null event is equivalent to the gamble with the corresponding branch omittted, the form (65) could be extended to include the empty event and zero weights under appropriate conventions.)

7.4.2 Proposition 3

Suppose $g_{[n]}$ is a gamble based on the partition $(C_1, ..., C_n)$. We have the two decompositions

$$\begin{aligned} g_{[n]} &\sim KE_{\succsim}(g_{[n]}) \oplus (e, C_1; ...; e, C_n) \\ g_{[n]} &\sim' KE_{\succsim'}(g_{[n]}) \oplus (e, C_1; ...; e, C_n). \end{aligned}$$

So

$$U_{\succsim}(g_{[n]}) = U_X(KE_{\succsim}(g_{[n]})) + U_{\succsim}(e, C_1; ...; e, C_n) \tag{66}$$

$$U_{\succsim'}(g_{[n]}) = U_X(KE_{\succsim'}(g_{[n]})) + U_{\succsim'}(e, C_1; ...; e, C_n). \tag{67}$$

Using (16) and the representation (24), we have

$$\begin{aligned} U_X(KE_{\succsim}(g_{[n]})) &= \sum_{i=1}^{n} U_X(x) S_{\succsim,\Omega}(C_i) \\ &= \sum_{i=1}^{n} U_X(x) S_{\succsim',\Omega}(C_i) \\ &= U_X(KE_{\succsim'}(g_{[n]})). \end{aligned} \tag{68}$$

So, the kernel equivalent terms are identical in (66) and (67), whence we have the equivalences

$$\begin{aligned} U_{\succsim}(g_{[n]}) = U_{\succsim'}(g_{[n]}) &\Leftrightarrow U_{\succsim}(e, C_1; ...; e, C_n) = U_{\succsim'}(e, C_1; ...; e, C_n) \\ &\Leftrightarrow [y \sim (e, C_1; ...; e, C_n) \Leftrightarrow y \sim' (e, C_1; ...; e, C_n)] \,. \end{aligned}$$

7.4.3 Proposition 6

For this proof and the next one, we omit the subscript $\succsim$ on U and S.

1.(a) implies 1.(b). Suppose that S_Ω is finitely additive. Using the fact that primed events mean independent realizations of an event, we have

$$\begin{aligned} &U(x \oplus (e, C_1; e, C_2) \oplus (e, C_1'; e, C_2')) = U(x) + U(e, C_1; e, C_2) + U(e, C_1'; e, C_2') \\ &= U(x) S_{C_1 \cup C_2}(C_1 \cup C_2) + U(e, C_1; e, C_2) + U(e, C_1'; e, C_2') \\ &= U(x) \left[S_{C_1 \cup C_2}(C_1) + S_{C_1 \cup C_2}(C_2) \right] + U(e, C_1; e, C_2) + U(e, C_1'; e, C_2') \\ &= U(x) S_{C_1 \cup C_2}(C_1) + U(e, C_1; e, C_2) + U(x) S_{C_1' \cup C_2'}(C_2') + U(e, C_1'; e, C_2') \\ &= U(x, C_1; e, C_2) + U(e, C_1'; x, C_2') \\ &= U((x, C_1; e, C_2) \oplus (e, C_1'; x, C_2')). \end{aligned}$$

This proves (26) for all $x \in X$, i.e., 1.(b).

1.(b) implies 1.(c). This follows immediately.

1.(c) implies 1.(a). Suppose that (26) holds for some $x \not\sim e$, tracking the above calculations we get that $U(x)[S_{C_1 \cup C_2}(C_1)+S_{C_1 \cup C_2}(C_2)] = U(x)$. Because $U(x) \neq 0$, we get $S_{C_1 \cup C_2}(C_1) + S_{C_1 \cup C_2}(C_2) = 1$. Multiplying both sides by $S_\Omega(C_1 \cup C_2)$ and using the choice property, we get the additivity of S_Ω.

2.(a) implies 2.(b). Suppose that S_Ω is additive. Using the fact that primed events mean independent realizations of an event, we have

$$\begin{aligned}
&U\left[(x, C_1 \cup C_2; e, \Omega \backslash C_1 \cup C_2) \oplus (e, C_1'; e, \Omega \backslash C_1') \oplus (e, C_2'; e, \Omega \backslash C_2')\right] \\
&\quad = U(x) S_\Omega(C_1 \cup C_2) + U(e, C_1 \cup C_2; e, \Omega \backslash C_1 \cup C_2) \\
&\qquad + U(e, C_1'; e, \Omega \backslash C_1') + U(e, C_2'; e, \Omega \backslash C_2') \\
&\quad = U(x) S_\Omega(C_1) + U(x) S_\Omega(C_2) + U(e, C_1 \cup C_2; e, \Omega \backslash C_1 \cup C_2) \\
&\qquad + U(e, C_1'; e, \Omega \backslash C_1') + U(e, C_2'; e, \Omega \backslash C_2') \\
&\quad = U(x) S_\Omega(C_1) + U(e, C_1'; e, \Omega \backslash C_1') + U(x) S_\Omega(C_2) + U(e, C_2'; e, \Omega \backslash C_2') \\
&\qquad + U(e, C_1 \cup C_2; e, \Omega \backslash C_1 \cup C_2) \\
&\quad = U(x) S_\Omega(C_1'') + U(e, C_1''; e, \Omega \backslash C_1'') + U(x) S_\Omega(C_2'') + U(e, C_2''; e, \Omega \backslash C_2'') \\
&\qquad + U(e, C_1''' \cup C_2'''; e, \Omega \backslash C_1''' \cup C_2''') \\
&\quad = U\left[(x, C_1''; e, \Omega \backslash C_1'') \oplus (x, C_2''; e, \Omega \backslash C_2'') \oplus (e, C_1''' \cup C_2'''; e, \Omega \backslash C_1''' \cup C_2''')\right].
\end{aligned}$$

This proves (27) for all $x \in X$, i.e., 2.(b), from which 2.(c) follows.

2.(c) implies 2.(a). Suppose that (27) holds for some $x \not\sim e$, tracking the above calculations we get that

$$U(x) S_\Omega(C_1 \cup C_2) = U(x)[S_\Omega(C_1) + S_\Omega(C_2)]$$

Because $U(x) \neq 0$, we get $S_\Omega(C_1 \cup C_2) = S_\Omega(C_1) + S_\Omega(C_2)$, i.e., the additivity of S_Ω.

7.4.4 Theorem 8

Suppose that (i) holds. By separability it translates into

$$S_{C_1 \cup C_2}(C_1) \geq S_{D_1 \cup D_2}(D_1) \quad \text{iff} \quad S_{C_1 \cup C_2}(C_2) \leq S_{D_1 \cup D_2}(D_2). \tag{69}$$

In functional terms it means that there exists an order reversing function Γ such that for all binary partitions (C_1, C_2)

$$S_{C_1 \cup C_2}(C_2) = \Gamma(S_{C_1 \cup C_2}(C_1)). \tag{70}$$

This "obvious" result is a special case of those in Aczél (1965).

Remember, we have made one general background assumption on all weighting functions: that for fixed non-empty Ω, $\{S_\Omega(E) | E \subset \Omega\} = [0, 1]$. Therefore, Γ is a homeomorphism of $]0, 1[$ onto $]0, 1[$.

By the choice property, for any nontrivial partition (C_1, C_2, C_3) and $\Omega = C_1 \cup C_2 \cup C_3$, we have the relations

$$\frac{S_\Omega(C_3)}{S_\Omega(C_2 \cup C_3)} = S_{C_2 \cup C_3}(C_3), \tag{71}$$

and

$$\frac{S_\Omega(C_1 \cup C_2) S_{C_1 \cup C_2}(C_2)}{S_\Omega(C_2 \cup C_3)} = S_{C_2 \cup C_3}(C_2). \tag{72}$$

By (70) applied to the nontrivial partition (C_2, C_3), $\Gamma(S_{C_2 \cup C_3}(C_3)) = S_{C_2 \cup C_3}(C_2)$, hence (71) and (72) give

$$\Gamma\left(\frac{S_\Omega(C_3)}{S_\Omega(C_2 \cup C_3)}\right) = \frac{S_\Omega(C_1 \cup C_2) S_{C_1 \cup C_2}(C_2)}{S_\Omega(C_2 \cup C_3)}. \tag{73}$$

Using (70) in parallel fashion for various nontrivial binary partitions, we have $\Gamma(S_{C_1 \cup C_2}(C_1)) = S_{C_1 \cup C_2}(C_2)$, $\Gamma(S_\Omega(C_1 \cup C_2)) = S_\Omega(C_3)$ and $\Gamma(S_\Omega(C_1)) = S_\Omega(C_2 \cup C_3)$. So (73) can be rewritten as

$$\Gamma\left(\frac{\Gamma(S_\Omega(C_1 \cup C_2))}{\Gamma(S_\Omega(C_1))}\right) = S_\Omega(C_1 \cup C_2) \frac{\Gamma(S_{C_1 \cup C_2}(C_1))}{\Gamma(S_\Omega(C_1))}. \tag{74}$$

Letting $r := S_\Omega(C_1 \cup C_2)$ and $s := S_{C_1 \cup C_2}(C_1)$, we get from the choice property that $rs = S_\Omega(C_1)$. With that (74) yields the equation

$$\Gamma\left(\frac{\Gamma(r)}{\Gamma(rs)}\right) = r \frac{\Gamma(s)}{\Gamma(rs)} \quad (r, s \in]0,1[\,). \tag{75}$$

Let $\gamma :]0,1[\to]0,\infty[$ be defined by

$$\gamma(r) = \Gamma(r)/r, \quad (r \in]0,1[\,). \tag{76}$$

Because Γ is an order reversing homeomorphism of $]0,1[$ onto $]0,1[$, γ is an order reversing homeomorphism of $]0,1[$ onto $]0,\infty[$. With that, (75) gives

$$\frac{\Gamma(r)}{\Gamma(rs)} = \gamma^{-1}\left(\frac{\Gamma(s)}{\gamma(r)}\right), \quad (r, s \in]0,1[\,). \tag{77}$$

Taking the logarithm on both sides we get

$$(-\ln\Gamma)(r) - (-\ln\Gamma)(rs) = (-\ln\gamma^{-1})\left(\frac{\Gamma(s)}{\gamma(r)}\right), \quad (r, s \in]0,1[\,). \tag{78}$$

Let $u := -\ln(r), v := -\ln(s)$, and let $F :]0,\infty[\to]0,\infty[$ be defined by

$$F(u) = -\ln\Gamma(\exp(-u)), \quad (u \in]0,\infty[\,). \tag{79}$$

Then F is an order reversing homeomorphism and (78) becomes

$$F(u) - F(u+v) = (-\ln\gamma^{-1})\left(\frac{\Gamma(\exp(-v))}{\gamma(\exp(-u))}\right), \quad (u, v \in]0,\infty[\,). \tag{80}$$

Defining $K : \mathbb{R} \to]0, \infty[$ by

$$K(z) = -\ln \gamma^{-1}(\exp(-z)), \quad (z \in \mathbb{R}), \tag{81}$$

we rewrite (80) as

$$F(u) - F(u+v) = K[u - F(u) + F(v)], \quad (u, v \in]0, \infty[\,).$$

Theorem 1 of Aczél et al. (2000) is applicable, and, in particular, the differentiability of F follows. Tracing back through (79) we obtain the differentiability of Γ.

Equation (75) implies

$$\Gamma\left(\frac{\Gamma(r)}{\Gamma(rs)}\right) = r\Gamma\left(s\frac{\Gamma(r)}{\Gamma(rs)}\right), \quad (r, s \in]0, 1[), \tag{82}$$

which is an equation treated in Ng (1998) (cf. equations (25)), with solution (cf. (38))

$$\Gamma(r) = (1 - r^\rho)^{1/\rho} \quad (r \in]0, 1[) \tag{83}$$

where $\rho > 0$ is a constant. With (83), (79) gives

$$S_{C_1 \cup C_2}(C_1)^\rho + S_{C_1 \cup C_2}(C_2)^\rho = 1.$$

Multiplying by $S_\Omega(C_1 \cup C_2)^\rho$ and using the choice property we get

$$S_\Omega(C_1)^\rho + S_\Omega(C_2)^\rho = S_\Omega(C_1 \cup C_2)^\rho. \tag{84}$$

This proves (ii).

The fact that (ii) implies (i) is straight forward.

7.4.5 Proposition 9

The equivalence between (a) and (b) is a simple consequence of the symmetry in the element of chance terms. The equivalence between (b) and (c) is seen from $KE_{\succsim}(x, C_1; e, C_2) \sim_X KE_{\succsim}(x, C_2; e, C_1)$ iff $U_X(x)S_{\succsim, C_1 \cup C_2}(C_1) = U_X(x)S_{\succsim, C_1 \cup C_2}(C_2)$ iff $S_{\succsim, C_1 \cup C_2}(C_1) = S_{\succsim, C_1 \cup C_2}(C_2)$ because $U_X(x) \neq 0$.

7.4.6 Theorem 11

We have made one general background assumption on all weighting functions: that for fixed non-empty Ω, $\{S_\Omega(E) | E \subset \Omega\} = [0, 1]$. Hence by this range condition on the weights, there exist, respectively, (C_1, C_2) and (D_1, D_2) such that

$$S^{\rho_{\succsim}}_{\succsim, C_1 \cup C_2}(C_1) = 1/2 \tag{85}$$

and

$$S^{\rho_{\succsim'}}_{\succsim', D_1 \cup D_2}(D_1) = 1/2. \tag{86}$$

By the additivity of $S^{\rho_{\succsim}}_{\succsim,\Omega}$, and of $S^{\rho_{\succsim'}}_{\succsim',\Omega}$, we get

$$S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_1) = S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_2)$$

and

$$S^{\rho_{\succsim'}}_{\succsim',D_1\cup D_2}(D_1) = S^{\rho_{\succsim'}}_{\succsim',D_1\cup D_2}(D_2).$$

So,

$$S_{\succsim,C_1\cup C_2}(C_1) = S_{\succsim,C_1\cup C_2}(C_2)$$

and

$$S_{\succsim',D_1\cup D_2}(D_1) = S_{\succsim',D_1\cup D_2}(D_2).$$

Hence the existence of $\succsim$-subjectively equal (C_1, C_2) and $\succsim'$-subjectively equal (D_1, D_2) is not an issue.

Suppose that (i) is satisfied. By Proposition 9, we get respectively

$$(x, C_1; e, C_2) \sim (x, C_2; e, C_1)$$

and

$$(x, D_1; e, D_2) \sim' (x, D_2; e, D_1).$$

By Assumption (i),

$$KE_{\succsim}(x, C_1; e, C_2) \sim_X KE_{\succsim'}(x, D_1; e, D_2), \tag{87}$$

and so

$$U_X(KE_{\succsim}(x, C_1; e, C_2)) = U_X(KE_{\succsim'}(x, D_1; e, D_2)).$$

That is,

$$U_X(x)S_{\succsim,C_1\cup C_2}(C_1) = U_X(x)S_{\succsim',D_1\cup D_2}(D_1).$$

Cancelling $U_X(x) \neq 0$ leaves

$$S_{\succsim,C_1\cup C_2}(C_1) = S_{\succsim',D_1\cup D_2}(D_1). \tag{88}$$

On the other hand, by (85) and (86) we have

$$S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_1) = S^{\rho_{\succsim'}}_{\succsim',D_1\cup D_2}(D_1).$$

Comparing it with (88) we get (ii): $\rho_{\succsim} = \rho_{\succsim'}$.

Next, suppose (ii), that $\rho_{\succsim} = \rho_{\succsim'}$.

To prove (iii), suppose that $(x, C_1; e, C_2) \sim (x, C_2; e, C_1)$ and $(x, D_1; e, D_2) \sim' (x, D_2; e, D_1)$ are given. According to Proposition 9, we get respectively

$$S_{\succsim,C_1\cup C_2}(C_1) = S_{\succsim,C_1\cup C_2}(C_2),$$
$$S_{\succsim',D_1\cup D_2}(D_1) = S_{\succsim',D_1\cup D_2}(D_2).$$

Hence

$$S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_1) = S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_2),$$
$$S^{\rho_{\succsim'}}_{\succsim',D_1\cup D_2}(D_1) = S^{\rho_{\succsim'}}_{\succsim,D_1\cup D_2}(D_2).$$

By additivity we get $S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_1) = 1/2$ and $S^{\rho_{\succsim}{}'}_{\succsim',D_1\cup D_2}(D_1) = 1/2$. Thus

$$S^{\rho_{\succsim}}_{\succsim,C_1\cup C_2}(C_1) = S^{\rho_{\succsim}{}'}_{\succsim',D_1\cup D_2}(D_1).$$

Using $\rho_{\succsim} = \rho_{\succsim'}$ we get (88):

$$S_{\succsim,C_1\cup C_2}(C_1) = S_{\succsim',D_1\cup D_2}(D_1).$$

Using that we can back step to (87), proving (iii).

It is clear that (iii) implies (i).

7.4.7 Theorem 12

The following proof, which is stated in full for completeness, is very similar to that of Part 1 of Theorem 2 with $S_{\succsim,\Omega}$ replaced by $S^{\rho}_{\succsim,\Omega}$.

According to Theorem 16 of Luce et al. (in press a), each $S_{\succsim,\Omega}$ satisfies the choice property, and combining (43) with (44), we have the representation

$$U_{\succsim}(g_{[n]}) = \sum_{i=1}^{n} U_X(x_i) S_{\succsim,\Omega}(C_i) + H_{\succsim}(C_1, ..., C_n) \tag{89}$$

where

$$H_{\succsim}(C_1, ..., C_n) := U_{\succsim}(e, C_1; ...; e, C_n) = \frac{1}{\mu_{\succsim}(\Omega)} \left[h_{\succsim}(\Omega) - \sum_{i=1}^{n} h_{\succsim}(C_i) \right]$$

are symmetric functions satisfying the branching relation (48), i.e.,

$$\begin{aligned} H_{\succsim}(C_1, ..., C_n) &= H_{\succsim}(C_1 \cup C_2, C_3, ..., C_n) \\ &\quad + H_{\succsim}(C_1, C_2) S_{\succsim,\Omega}(C_1 \cup C_2). \end{aligned} \tag{90}$$

Let $\succsim, \succsim' \in \mathcal{O}$ be any two orders with equal weights satisfying (16) with $S_{\succsim,\Omega}$ replaced by $S^{\rho}_{\succsim,\Omega}$. Then it follows from (ii) of assumption (iii) that

$$\begin{aligned} H_{\succsim}(C_1, ..., C_n) &= U_{\succsim}(e, C_1; ...; e, C_n) \\ &= U_{\succsim}(CE_{\succsim}(e, C_1; ...; e, C_n)) \\ &= U_X(CE_{\succsim}(e, C_1; ...; e, C_n)) \\ &= U_X(CE_{\succsim'}(e, C_1; ...; e, C_n)) \\ &= H_{\succsim'}(C_1, ..., C_n). \end{aligned} \tag{91}$$

For arbitrary

$$\begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix},$$

where the C_i form a non-trivial partition and the $p_i > 0$ form a non-trivial probability distribution, we define H_n by

$$H_n \begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} = H_{\succsim}(C_1, ..., C_n) \tag{92}$$

where $\succsim \in \mathcal{O}$ is any order satisfying $S^{\rho}_{\succsim,\Omega}(C_i) = p_i$ ($i = 1, ..., n$). By assumptions (ii) and (iii), such an order exists and by (91) the value of H_n does not depend on the choice of the order. Hence the H_n $(n = 2, 3, ...)$ are well-defined functions.

With that, and reasoning as in Section 7.3.1, (90) gives, with $\kappa := 1/\rho$,

$$H_n \begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} = H_{n-1} \begin{pmatrix} C_1 \cup C_2, & C_3, & ... & C_n \\ p_1 + p_2, & p_3, & ... & p_n \end{pmatrix}$$
$$+ H_2 \begin{pmatrix} C_1, & C_2 \\ \frac{p_1}{p_1+p_2} & \frac{p_2}{p_1+p_2} \end{pmatrix} (p_1 + p_2)^{\kappa}.$$

For $\rho = 1$, the solution was encountered in Part 1 of Theorem 2. For $\rho \neq 1$, according to the results established in Ebanks et al. (1988), there exists a function $V : \mathfrak{B}^* \to \mathbb{R}$ and a constant A such that

$$H_n \begin{pmatrix} C_1, & C_2, & ..., & C_n \\ p_1, & p_2, & ..., & p_n \end{pmatrix} = V(\Omega) - \sum_{i=1}^{n} V(C_i) p_i^{\kappa} - A \left[1 - \sum_{i=1}^{n} p_i^{\kappa} \right]. \tag{93}$$

(As in Theorem 2, using expansibility, the form could be extended to include the empty event and zero weights under appropriate conventions.)

Putting the obtained forms of H_n into (92) we get the form of $H_{\succsim}$. Putting that further back into (89) we arrive at the conclusion of the theorem.

7.4.8 Proposition 14

1. implies 2. If $C \in \mathfrak{B}^*$ is maximal, let $b = V(C)$. Now, for each $\Omega \in \mathfrak{B}^*$, select non-trivial ordered binary partitions $\mathbf{C} = (C_1, C_2)$, $\mathbf{D} = (D_1, D_2)$ of Ω. Then $C_i, D_i, i = 1, 2$, are not maximal and, by weight solvability, Def. 1, for each non-trivial binary probability vector $\mathbf{p} = (p, 1-p)$, there exists at least one pair of orderings $\succsim, \succsim' \in \mathcal{O}$ such that

$$S_{\succsim,\Omega}(C_1)^{\rho} = S_{\succsim',\Omega}(D_1)^{\rho} = p.$$

Then, using (34) and (35) with the representation (33), cancelling common terms and rearranging the remaining terms, gives

$$[V(C_1) - V(D_1)]p^{1/\rho} + [V(C_2) - V(D_2)](1-p)^{1/\rho} = 0.$$

However, the above equality holds for all non-trivial binary partitions $\mathbf{C}, \mathbf{D}$ of each $\Omega \in \mathfrak{B}^*$and all $p \in]0, 1[$, and so $V(C) =$ constant for all non-maximal $C \in \mathfrak{B}^*$.

The converse is trivial.

Notes:

1. Another additive representation consistent with exactly the same axioms, and the only other polynomial one, is the bounded p-additive form

$$U(x \oplus y) = U(x) + U(y) + \delta U(x)U(y) \quad \left(\delta \neq 0, U : X \longrightarrow \left]-\frac{1}{|\delta|}, \frac{1}{|\delta|}\right[\right),$$

   which tranforms into an additive representation $\widehat{U}$ under

$$\widehat{U}(x) := sgn(\delta) \log\left[1 + \delta U(x)\right].$$

   The various developments in Luce et al. (in press a,b) are unchanged if U there is replaced by $\widehat{U}$.

2. Luce et al (in press a) did not explicitly subscript functions such as U and S_Ω by $\succsim$ because $\succsim$ was fixed throughout. Here we will be considering a family of orderings that are each extensions of $\succsim_X$ and so we must make $\succsim$ explicit.

3. Luce et al. (in press a) assumed one-sided monotonicity plus the commutativity of $\oplus$; here monotonicity is two-sided and we derive commutativity and associativity.

4. Conversation with Luce on December 30, 2006.

5. There we indexed H with an integer that redundantly showed the number of arguments. Here we suppress it, although we reintroduce it in a later definition.

References:

Aczél, J. (1965), A remark on functional dependence, *Journal of Mathematical Psychology* **2**, 125-127.

Aczél, J. (1980), Information functions on open domain. III, *Comptes rendus mathématiques de l'Académie des sciences, La Société royale du Canada (Mathematical Reports of the Academy of Science, The Royal Society of Canada)* **2**, 281-285.

Aczél, J. (1981), Notes on generalized information functions, *Aequationes Mathematicae* **22**, 97-107.

Aczél, J. and Daróczy, Z. (1978), A mixed theory of information. I. Symmetric, recursive and measurable entropies of randomized systems of events, *R.A.I.R.O. Informatique théorique/Theoretical Computer Science* **12**, 149-155.

Aczél, J. and Kannappan, P. (1978), A mixed theory of information. III. Inset entropies of degree β. *Information and Control* **39**, 315-322.

Aczél, J., Maksa, Gy., Ng, C. T., and Páles, Z. (2000), A functional equation arising from ranked additive and separable utility, *Proceedings of the American Mathematical Society* **129**, 989-998.

Daróczy, Z. (1970), Generalized information functions, *Information and Control* **16**, 36-51.

Ebanks, B., Kannappan, P., and Ng, C. T. (1988), Recursive inset entropies of multiplicative type on open domains, *Aequationes Mathematicae* **36**, 268-293.

Havrda, J. and Charvát, F. (1967), Quantification method of classification processes. Concept of structural α-entropy, *Kybernetika* **3**, 30-35.

Karni, E. (1985), *Decision Making Under Uncertainty: The Case of State-Dependent Preferences*. Cambridge, MA: Harvard University Press.

Krantz, D. H., Luce, R. D., Suppes, P., and Tversky, A. (1971), *Foundations of Measurement, Vol. I.* San Diego, CA: Academic Press. To be reissued in 2007 by Dover Publications, Mineola, NY.

Luce, R. D. (1959/2005), *Individual Choice Behavior: A Theoretical Analysis.* New York: John Wiley and Sons. Reprinted by Dover Publications, Mineola, NY.

Luce, R. D. (2000). *Utility of Gains and Losses: Measurement-Theoretical and Experimental Approaches*, Mahwah, NJ: Erlbaum, Errata: see Luce's web page at http:www.imbs.uci.edu/personnel/luce/luce.html.

Luce, R. D. and Marley, A. A. J. (2000), On elements of chance, *Theory & Decision*, 49, 97-126.

Luce, R. D., Ng, C. T., Marley, A. A. J.,& Aczél, J. Utility of gambling I: Entropy modified linear weighted utility, *Economic Theory,* in press a.

Luce, R. D., Ng, C. T., Marley, A. A. J.,& Aczél, J. Utility of gambling II: Risk, Paradoxes, and Data, *Economic Theory,* in press b.

Marley, A. A. J. and Luce, R. D. (2005), Independence properties vis-à-vis several utility representations, *Theory & Decision* **58**, 77-143.

Ng, C. T. (1980), Information functions on open domain. II, C*omptes rendus mathématiques de l'Académie des sciences, La Société royale du Canada (Mathematical Reports of the Academy of Science, The Royal Society of Canada)* **2**, 155-158.

Ng, C. T. (1998), An application of a uniqueness theorem to a functional equation arising from measuring utility, *Journal of Mathemetical Analysis and Applications* **228**, 66-72.

Shannon, C. E. (1948), A mathematical theory of communication, *Bell System Technical Journal,* **27**, 379-423, 623-656.

Suyari, H. (2002), On the most concise set of axioms and the uniqueness theorem of Tsallis entropy, *Journal of Physics A Mathematical and General* **35**, 10731-10738.

Tsallis, C. (1988), Possible generalization of Boltzmann-Gibbs statistics, *Journal of Statistical Physics* **52**, 479-487.