Condorcet Domains; A Geometric Perspective

Donald G. Saari*
Institute for Mathematical Behavioral Sciences
University of California, Irvine, CA 92697-5100.

1 Introduction

One of several topics in which Peter Fishburn [4, 5] has made basic contributions involves finding maximal Condorcet Domains. In this current paper, I develop a geometric approach that, at least for four and five alternatives, is equivalent to Fishburn's clever alternating scheme (described below), a scheme that has advanced our understanding of the area.

To explain "Condorcet Domains" and why they are of interest, start with the fact that when making decisions by comparing pairs of alternatives with majority votes, the hope is to have decisive outcomes in that one candidate always is victorious when she is compared with any other candidate. When this happens, the candidate is called the *Condorcet winner.* The attractiveness of this notion, where someone can beat anyone else in head-to-head comparisons, is why the Condorcet winner remains a central concept in voting theory. For a comprehensive, modern description of the Condorcet solution concept, see Gehrlein's recent book [7].

But Condorcet also proved that pairwise rankings can lead to cycles, where a Condorcet winner cannot exist. His three voter example [3],[1] now called the *Condorcet triplet,* has the preferences

$$A_1 \succ A_2 \succ A_3, \quad A_2 \succ A_3 \succ A_1, \quad A_3 \succ A_1 \succ A_2 \tag{1}$$

("$A_1 \succ A_2 \succ A_3$" means that the voter prefers A_1 to A_2 and A_3, and A_2 to A_3). The majority vote generates the cycle where A_1 beats A_2, A_2 beats A_3, and A_3 beats A_1 each with a 2:1 tally. The trouble with cycles is that they frustrate society's ability to make a decision; e.g., who should be selected with this example? Not A_1 because a majority prefers A_3. Not A_3 because a majority prefers A_2. Not A_2 because a majority prefers A_1.

The actual complexity associated with this behavior is much worse because the ways in which cyclic behavior can be manifested extend beyond frustrating the majority vote decision process to cause fundamental theoretical concerns. As we now know (Saari [15]), for instance, aspects of cyclic outcomes are what cause Arrow's seminal theorem [1], which purportedly shows that no non-dictatorial voting rule can satisfy seeming innocuous conditions, and Sen's Paretian Liberal Theorem [20], which identifies what is called a fundamental conflict between individual and societal decisions. (For different interpretations of Arrow's and Sen's theorems, see Saari [15]; also see Saari and Petron [17] and Li and Saari [8].)

A standard way to handle these difficulties is to restrict the preferences that voters are permitted to have. (Gaertner [6] describes other restricted domain conditions that arise in choice theory.)

*This research was supported by NSF grants DMI 0640817 and DMS 0631362. My thanks to Bernard Monjardet and two anonymous referees for their several useful suggestions.

[1] Condorcet's example in his Éssai [3] is not as concise; it involves about sixty voters. But, I expect that somewhere in his writings, Condorcet explicitly stated this triplet.

This leads to the Condorcet Domain problem: it is to identify sets of preference rankings whereby, no matter how many voters have each of the specified rankings, the outcome never admits a cycle. A goal is to find or characterize all such domains—all sets of these preference rankings—and find the ones that have a maximum number of rankings.

This Condorcet Domain challenge has captured the attention of influential contributers to this area; for a brief history with references see Fishburn [4], Monjardet [10], and Monjardet's survey [11] that appears in this volume. Indeed, it was Monjardet's clear presentation [9, 10] at an October 2006 DIMACS/LAMSADE conference in Paris that awakened my interest in this issue leading to this paper. As Monjardet explained, Fishburn's paper [4] contains some of the deepest conclusions about this issue. Fishburn credits his discovery of his "alternating scheme" to clever examples that Monjardet created.

Fishburn's and Monjardet's approaches are essentially combinatoric. So, after introducing the basic problem, I will introduce a geometric approach to describe Fishburn's alternating scheme [4]. My expectation is that the symmetries, which become apparent by use of geometry, will lead to other mathematical tools that can be used to analyze this and other pressing questions. Then, after showing how my geometric approach fits into a broader research theme, I generalize the Condorcet Domain problem by replacing sets of "individual rankings" with sets of "specific configurations of individual rankings." Namely, instead of finding specific rankings that avoid cycles, the new goal is to find configuration of rankings whereby if any number of groups of voters adopt any of these configurations, cycles never occur. Although this generalized problem appears to be far more complicated, the complete solution is in Sect. 4. The original problem, however, remains wide open.

2 Early solutions

The *Condorcet Domain* problem is to identify subsets of preference rankings so that, no matter how many voters are assigned to each ranking, the pairwise majority vote outcomes never admit a cycle. As the number of voters with each ranking is not restricted, each Condorcet Domain defines a subspace of profiles with which not only majority pairwise voting, but several other voting issues avoid the difficulties of pairwise comparisons; this includes Arrow's Theorem [1] as well as some decision problems from engineering (Saari and Sieberg [18]). Because of these advantages, it becomes natural to find Condorcet Domains that have the maximal number of rankings; after all, such a domain defines a maximal dimensional profile subspace that has these desired properties.

An early Condorcet Domain solution, which continues to be widely used, is Black's [2] single peaked condition. While his condition is slightly more general than described next, a flavor of it can be obtained by placing each alternative at a distinct point on a line. Next, place an individual's "ideal point" anywhere on the line; this individual's preference ranking is defined by the distance from his ideal point to each alternative where "closer is better." It is not difficult to show how and why this ordering of the voters' preference rankings always results in orderly pairwise outcomes. (See, for instance, Black [2], Saari [15] among many other references.)

To see what happens with three candidates, notice that the alternative in the middle never is bottom-ranked by any voter. For special cases, if all ideal points are on one side of the alternatives, some candidate never is top-ranked; if the voters are split into polarized left-right regimes, some candidate never is middle-ranked. Black's condition probably motivated the Condorcet Domain solution advanced by Ward [21] and later generalized by Sen [19]. Namely, with three candidates at least one of the following conditions is satisfied:

1. There is some candidate who never is bottom-ranked.

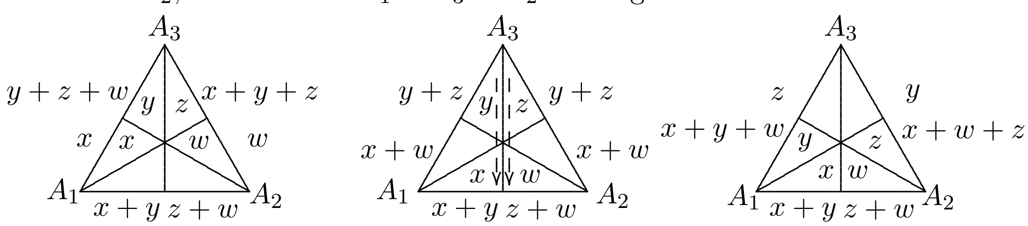
2. There is some candidate who never is middle-ranked.

3. There is some candidate never is top-ranked.

As I indicate next with a geometric representation, when any of these conditions are satisfied, a majority vote pairwise cycle cannot occur.

2.1 Geometry of triangles

My preferred way [16] to represent three-candidate profiles is with an equilateral triangle, where the name of each candidate is assigned to a distinct vertex as illustrated in Fig. 1. The ranking assigned to a point in the triangle is determined by its distance to each vertex, where closer is better. Thus the vertical line represents all $A_1 \sim A_2$ tied rankings; the remaining two indifference lines connect a vertex with the midpoint on the opposite edge. What results is a partitioning of the triangle into what I call "ranking regions;" the open triangles represent strict rankings. For instance, any point in the small Fig. 1a triangle with an x is closest to A_1, next closest to A_3, and farthest from A_2, so it has the $A_1 \succ A_3 \succ A_2$ ranking.



a. A_3 never bottom ranked **b.** A_3 never middle ranked **c.** A_3 never top ranked

Figure 1. Ranking triangles

A way to represent a profile is to place the number of voters with each ranking in the associated ranking region. The z in Fig. 1a, for instance, means that z voters have the $A_3 \succ A_2 \succ A_1$ ranking. A candidate never is bottom-ranked when there are no entries in the two regions farthest from the candidate's vertex; e.g., the Fig. 1a profile never has A_3 bottom-ranked. Figures 1b, c represent the remaining two Ward conditions with respect to A_3.

An indifference line associated with a particular pair divides the six rankings into two regions separating the two possible pairwise rankings; e.g., the vertical line in Fig. 1a separates the three rankings on the left with $A_1 \succ A_2$ from the three rankings on the right with $A_2 \succ A_1$. Thus a quick way to tally majority votes is to project the numbers from the triangle to the appropriate edge and then add them; the sums are listed next to each edge. This projection and summing process is indicated by the dashed arrows in Fig. 1b, which represents all profiles where A_3 is never middle-ranked. Notice that the A_1, A_2 tallies are, respectively, $x+y$ and $z+w$. As the A_1, A_3 and A_2, A_3 tallies agree, A_3 must be either the Condorcet winner or loser; in either case, it follows that cycles cannot occur when some candidate never is middle-ranked. A similar argument holds for the other figures; e.g., in Fig. 1a, if A_1 beats A_2, then $x+y > z+w$, so A_3 beats A_2: as A_2 is the Condorcet loser, cycles cannot occur.

The complementary relationship between Ward conditions and the Eq. 1 Condorcet triplet is illustrated with Fig. 2. There are two possible Condorcet triplets; the Eq. 1 choice is illustrated with stars in the appropriate ranking regions, the other with bullets. To avoid the cycles caused by Condorcet triplets, it is worth examining what happens if one ranking with a star and one with a bullet are prohibited. By symmetry, it does not matter which star choice is selected, so choose the one indicated in Fig. 2. Next, select one of the three bullets; for each choice, the figure indicates which Ward condition is satisfied.

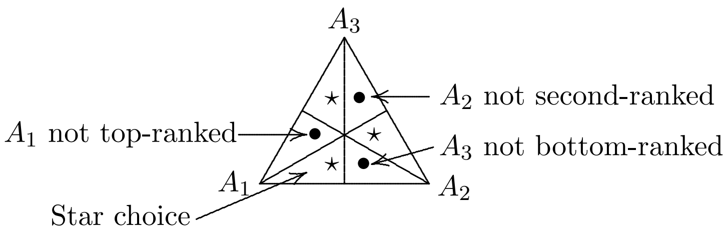


Figure 2. Condorcet triplets

Ward's conditions, then, should be viewed as being the sharpest possible restrictions that avoid a Condorcet triplet. Namely, a way to restate Ward's conditions is that they admit *any* set of rankings from which a Condorcet triplet cannot be created. To complete the complementary connection, each Condorcet triplet has the smallest number of rankings that violate all of Ward's conditions. In summary, a three-candidate Condorcet Domain is any set of rankings from which a Condorcet triplet cannot be created; i.e., it is any set that satisfies one of Ward's conditions. As each candidate defines three different Condorcet Domains, nine different four-dimensional subspaces in the six-dimensional profile space are spared the problems of cyclic behavior.

2.2 More candidates

What happens with more candidates? With four candidates, for instance, can pairwise cycles be avoided whenever some candidate never is bottom-ranked? As illustrated with the Eq. 2 example, where A_3 never is bottom-listed, the answer is no.

$$A_1 \succ A_2 \succ A_3 \succ A_4, \quad A_2 \succ A_3 \succ A_4 \succ A_1, \quad A_3 \succ A_4 \succ A_1 \succ A_2. \tag{2}$$

Here, A_4 beats A_1, A_1 beats A_2, A_2 beats A_3 (each by $2:1$), and A_3 beats A_4 (unanimously) to form a cycle. Notice how this profile defines the $A_1 \succ A_2$, $A_2 \succ A_3$, $A_3 \succ A_1$ cycle with the familiar $2:1$ tallies. Indeed, by focussing attention on the *relative position* of these three candidates, we find that they create a Condorcet triplet so all three of Ward's conditions are violated. This insight leads to the Sen condition [19] that a necessary and sufficient requirement for a set of rankings to define a Condorcet Domain is that when restricting the rankings to any triplet, one of Ward's conditions holds. So for $\{A_1, \dots, A_n\}$, a set of rankings is a Condorcet Domain if and only if when restricting the rankings to any triplet, at least one candidate never is top-ranked, or middle-ranked, or bottom-ranked; i.e, these relative rankings can never be used to create a Condorcet triplet.

By knowing what creates Condorcet Domains, the next step is to find examples and to determine the maximal Condorcet Domains. This is where Fishburn's [4] alternating scheme and "never" conditions play a dominant role. To explain his condition with an example, consider the five candidates $\{A, B, C, D, E\}$. Select a ranking; say $E \succ A \succ D \succ C \succ B$. Assign temporary A_j names according to the ranking's order; e.g., E is called A_1, A is called A_2, ..., B is called A_5.

Fishburn's alternating scheme is as follows:

> List each triplet in the order of their temporary names; e.g., list $\{A_3, A_1, A_4\}$ in the order of their subscripts as $\{A_1, A_3, A_4\}$. If the subscript for the middle alternative is odd, as it is here (it is 3), use the never-top ranked rule for this alternative with the triplet. If it is even, use the never bottom-ranked rule with the alternative. Apply this rule to all triplets. Alternatively, the rule used with all triplets could be that if the subscript for the middle alternative is odd, then use the never-bottom ranked rule; if it is even, use the never-top ranked rule.

The value of this algorithm comes from Fishburn's result stated next; proofs are in his papers:

Theorem 1 *(Fishburn [4, 5]) For $n = 4, 5, 6$ alternatives, a Condorcet Domain has the maximal number of rankings if and only if the set satisfies the alternating scheme. For $n \geq 16$, the alternating scheme does not define the maximal Condorcet Domain.*

What a delightful result! Beyond contributing to a long studied question, his theorem creates a mystery that begs to be investigated. Why does it work? What underlying mathematical structures permit this condition? Is there an intuitive way to appreciate his alternating condition? What is magical about the $n = 16$ cutoff? What happens between 7 and 15? As my objective is to develop insight and intuition, I describe the Ward-Sen and Fishburn conditions in a geometric framework.

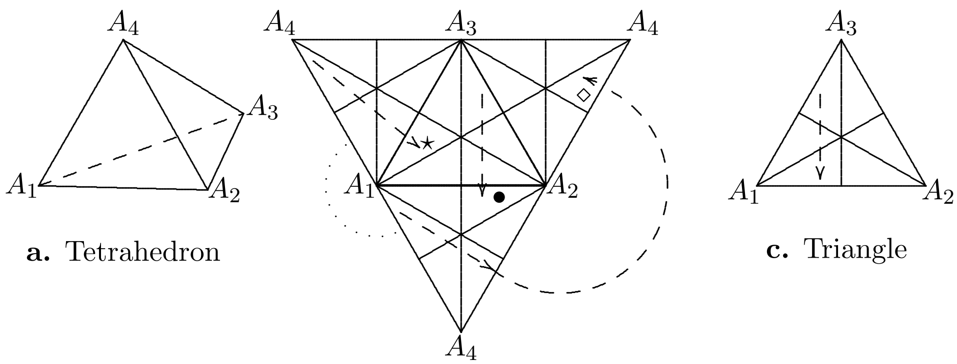


Figure 3. Representation triangle and tetrahedron

3 Geometry

To use geometry to find all four-candidate Condorcet Domains, replace the equilateral triangle with the Fig. 3a equilateral tetrahedron. Again, a ranking is assigned to a point based on its distances to the vertices. To create a two-dimensional representation of the tetrahedron, select a vertex (A_4 in Fig. 3), cut the three tetrahedron edges from the vertex to its base, and open the flaps to create the Fig. 3b object. Each of the 24 small triangles, or ranking regions, represents a particular ranking; e.g., using distances to vertices, the region with the bullet has the $A_2 \succ A_1 \succ A_4 \succ A_3$ ranking, while the one with the diamond has $A_4 \succ A_2 \succ A_3 \succ A_1$. The four large equilateral triangles are the four original tetrahedron faces; alternatively, they represent where one alternative is removed. For instance, the central equilateral triangle with vertices A_1, A_2, A_3 can be used to represent rankings when A_4 is dropped.

To motivate what is done next, recall that to construct a Condorcet Domain we need to find all rankings where after dropping A_4, the remaining triplet satisfies the "A_3 is never middle ranked" or some other Ward-Sen condition. Namely, we need to *avoid* all rankings whereby dropping A_4 leads to either $A_1 \succ A_3 \succ A_2$ or $A_2 \succ A_3 \succ A_1$. More generally, we need to find a way to identify all rankings that have a specified relative ranking after dropping a particular candidate.

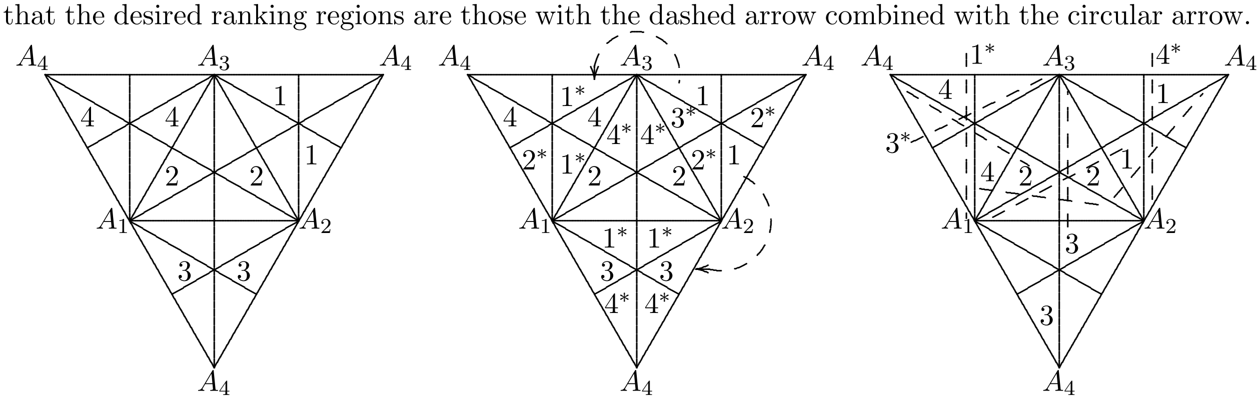
To do this by using geometry, start with the three alternative setting of Fig. 3c. Similar to tallying elections, ignoring A_3 has the effect of projecting the rankings to the A_1-A_2 edge; e.g., the dashed arrow in the triangle represents projecting all three rankings with the $A_1 \succ A_2$ relative ranking to the $A_1 \succ A_2$ portion of the bottom edge. (So, to find all rankings with $A_1 \succ A_2$, just follow that dotted line backwards.) A similar projection occurs with Fig. 3a when an alternative is dropped, but we need help to see the projections. Assistance is provided by Fig. 3b.

Figure 3b easily handles projections when A_4 is ignored and a $\{A_1, A_2, A_3\}$ ranking results. For instance, the starred region has the ranking $A_1 \succ A_3 \succ A_2 \succ A_4$, with the $A_1 \succ A_3 \succ A_2$

relative ranking when ignoring A_4. The four rankings with this $A_1 \succ A_3 \succ A_2$ relative ranking are in the ranking regions with the dashed arrow pointing to the star; i.e., ignoring A_4 effectively projects these four rankings into the starred region. Indeed, "above" (i.e. directly away from the center point of the central triangle) each ranking region in the central equilateral triangle are the four four-candidate rankings with the same relative ranking of the triplet.

Now consider a ranking that is not in the central triangle; e.g., treating the region with a bullet as a triplet, the ranking is $A_2 \succ A_1 \succ A_4$. As A_3 is the missing candidate, one way to handle to geometry is to return to the tetrahedron and open it from the A_3 vertex. Doing so would create four attached equilateral triangles with the A_1, A_2, A_4 triangle in the center; each adjacent triangle has the vertex A_3. But this approach is not satisfactory for our needs as we want to compute the rankings to be removed for all triplets with one diagram. So, an equivalent way to create the same figure that is formed by slicing the tetrahedron open from vertex A_3 is to rotate (the circular dotted line) the A_1, A_3, A_4 triangle about vertex A_1 so that the two A_1-A_4 edges meet, and rotate the A_2, A_3, A_4 triangle about A_2 so that the two A_2-A_4 edges meet. By doing so, it is clear that the ranking regions with the dashed arrow pointing to the bullet are projected to this region. (Here, we did not need to rotate the faces.)

As a final example, the three-candidate ranking for the region with a diamond is $A_4 \succ A_2 \succ A_3$ where A_1 is the ignored alternative. To find all rankings with this relative ranking, rotate the A_1, A_4, A_2 triangle about the A_2 vertex, find the projection, and then rotate back again to show that the desired ranking regions are those with the dashed arrow combined with the circular arrow.



a. Four middle choices

b. Consequences

c. More symmetry

Figure 4. No middle ranked alternatives

3.1 Excluding rankings

To illustrate how to use this geometry, the "never-middle ranked" condition is imposed in Fig. 4a for each triplet. With the A_2, A_3, A_4 triplet, for instance, the 1's indicate those rankings where, when restricted to this triplet, A_4 is middle-ranked; thus, these two rankings are to be excluded and the other four are admitted. Similarly the regions with 4's indicate where A_1 is middle-ranked when restricting admissible rankings to the A_1, A_3, A_4 triplet, so the other four rankings satisfy Ward's condition where A_1 never is middle-ranked.

All rankings that satisfy these conditions, i.e., all ranking regions that project to any of the marked Fig. 4a regions, are marked in Fig. 4b. The top "1" in Fig. 4a, for instance, excludes the three regions indicated by the top circular arrow; one Fig. 4b region already is excluded as it has a 4, the other two regions, marked with 1^*, are excluded because they are projected to a 1. Similarly, the lower circular Fig. 4b arrow identifies the three regions that project to the other 1; one region already is excluded with its 3, and the two with 1^* are excluded by being projected to this 1.

Doing this for all four numbers leaves only four ranking regions without a label; these rankings, $\{A_4 \succ A_3 \succ A_1 \succ A_2,\ A_4 \succ A_3 \succ A_2 \succ A_1,\ A_1 \succ A_2 \succ A_3 \succ A_4,\ A_2 \succ A_1 \succ A_3 \succ A_4\}$ enjoy obvious symmetry relationships made apparent with the figure. (For instance, notice that each ranking is accompanied by its reversal.) They define a "complete Condorcet Domain" in that by adding any other ranking to the set, the new set no longer is a Condorcet Domain.

In general, for each of the four triangles, select a Ward-Sen condition for some alternative. Then, cross off all regions identified by the selected Ward-Sen choices, and all regions that project onto one of these regions. In this manner, all possible four-alternative complete Condorcet Domains can be found. As this approach shows, in profile space (which can be represented by the 24 dimensional Euclidean space $\mathbb{R}^{24}$) the Condorcet Domain is orthogonal to the space of regions that are eliminated by the Ward-Sen conditions.

The geometric challenge, which has the flavor of a Sudoku or crossword puzzle, is to determine which combinations of Ward-Sen structures leave the largest number of blank spaces after the projected regions are crossed off. Thus, finding a Condorcet Domain with a maximal number of rankings requires finding combinations of Ward-Sen conditions with the minimal number of crossed off regions. Clearly, for this to occur, we need to select conditions so that some regions are eliminated by multiple conditions. For instance the regions with a 3 on the right in the A_1, A_2, A_4 triangle is excluded twice; first by being the indicated middle ranking for that triangle and then by being projected to a 1. The goal, then, is to determine which combinations of the Ward-Sen conditions minimize, and which maximize, multiple counting of ranking regions. The answer must involve the geometric structure and its associated symmetries.

An example using symmetry is depicted in Fig. 4c where the four not-middle choices, given by the numbers 1 to 4, are selected in a band. Notice, some numbers are in regions that are projected to other numbers. The projection regions are depicted by dashed lines leading out of regions with a number; three dashed lines are labeled with the donor number n^*. With this choice, five marked ranking regions are used three times, six twice, and only five once. This arrangement leaves eight blank regions that define a complete Condorcet Domain: the first part has A_3 bottom-ranked,

$$\{A_1 \succ A_2 \succ A_4 \succ A_3,\ A_1 \succ A_4 \succ A_2 \succ A_3,\ A_4 \succ A_2 \succ A_1 \succ A_3,\ A_2 \succ A_4 \succ A_1 \succ A_3\}$$

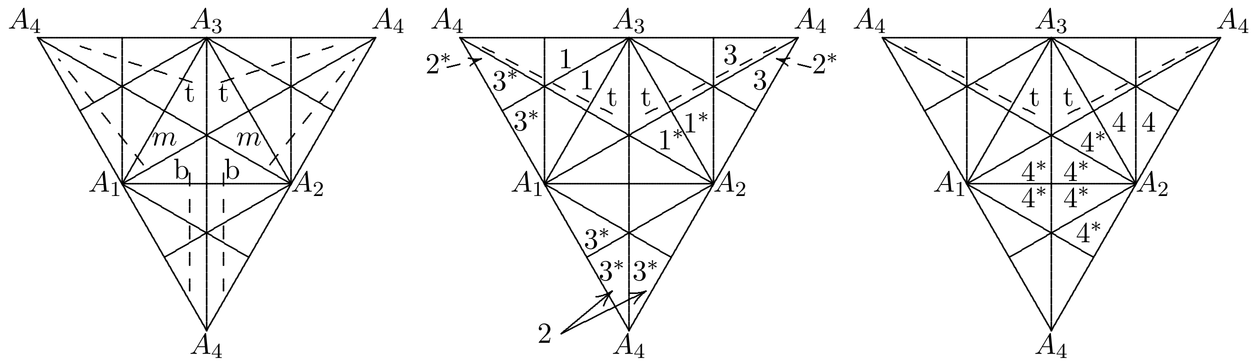
and the second part has A_3 is top-ranked

$$\{A_3 \succ A_1 \succ A_4 \succ A_2,\ A_3 \succ A_1 \succ A_2 \succ A_4,\ A_3 \succ A_2 \succ A_4 \succ A_1,\ A_3 \succ A_4 \succ A_2 \succ A_1\}.$$

Also notice, accompanying each ranking in this Condorcet Domain is its reversal.

3.2 Calculus of Ward-Sen conditions

One of my contributions for this Condorcet Domain problem is to indicate how to create a calculus to determine which ranking regions should be eliminated. To do so, the Ward-Sen conditions are related to the geometry of a tetrahedron. Using the bottom face of Fig. 3a, with vertices $\{A_1, A_2, A_3\}$, which is the central face of Fig. 5a, the never-bottom condition defines an edge; e.g., the b's in Fig. 5a are on the A_1-A_2 edge. Applying this condition to define a Condorcet Domain, it follows from the dashed lines moving out of the "b" regions that it eliminates all rankings in the other face that shares this edge; in Fig. 5a, it is the triangle $\{A_1, A_2, A_4\}$. Thus, a never-bottom condition defines one of the face's edges; it eliminates the two specified never-bottom rankings and *all rankings* in the face sharing the same edge.



a. First steps **b.** Not-top **c.** More not-top

Figure 5. Calculus

A never-top condition defines two regions sharing a vertex; in Fig. 5a, the regions are denoted by t's, and the vertex is A_3. As illustrated by the dashed lines moving out of the two "t" regions, this condition eliminates all six rankings that share the same vertex and two more along the *"designated edge"* that connects the specified vertex with the vertex not in this face; here it is the A_4 vertex. Notice that this A_3-A_4 edge is depicted on two of the faces; the reason is that this is one of the edges along which the tetrahedron was cut open. What is not so obvious is that this edge connects A_3 from the A_1, A_2, A_3 face to A_4 from the A_1, A_2, A_4 face. After all, the same A_4 is on three faces; to see that this is so, just fold up the faces into a tetrahedron.

As indicated by the m's in Fig. 5a, the never-middle condition defines a face base and two adjacent faces; the excluded regions are the two selected rankings and three each in the adjacent faces. These eliminated rankings come in pairs; a ranking and its reversal. Also notice how four of the Fig. 5a rankings are below the $A_1 \sim A_3$ line, the other four are below the $A_2 \sim A_3$ line.

The next step is to identify what rankings disappear by combining these conditions; the ideas can be illustrated by using the same condition with two faces α and β; the remaining two faces (equilateral triangles) are called γ and δ. The easiest case is the never-bottom condition, which emphasizes selected edges.[2] (See Figs. 3a, 5a.) All possible combinations follow:

- If the never-bottom condition used with the α and β faces has the α identified edge bordering face γ and the β identified edge bordering face δ, then there is no overlap of eliminated regions. Thus 16 regions are eliminated; they are all of the γ, δ ranking regions and the four initiating regions. To illustrate with Fig. 5a, let the α face be given by the vertices A_1, A_2, A_3, and the bordering γ face be A_1, A_2, A_4. Then the b's in the α face eliminate all γ face rankings. Let the β face be given by A_2, A_3, A_4 where the two bottom ranked rankings are on the A_3-A_4 edge. These two choices eliminate all rankings in the δ face defined by vertices A_4, A_1, A_3. In total, all rankings from the γ and δ faces, 12 of them, are eliminated along with the four selected rankings for a total of 16.

- If the α edge is on the β face, but the β edge is on face γ, then 14 regions are eliminated— the β face condition eliminates all γ rankings, the α face condition eliminates all β rankings including the two that drop all of the γ rankings, and the two initiating regions in the α face. Again, illustrating with Fig. 5a with the same α face where the β face now is A_1, A_2, A_4, the b's in Fig. 5a satisfy the first condition; all β face rankings are dropped. Now let the γ face be defined by A_2, A_3, A_4. To satisfy the specified conditions, the two bottom ranked rankings

[2]The approach becomes clear and fairly easy with some experience. Therefore I *strongly recommend* that the reader creates versions of the Fig. 5 triangles and carries out the described calculus.

from β must be on the A_2-A_4 edge. These choices eliminate all β and γ rankings (12 of them in total) and the two b rankings in the first face for a total of 14 rankings.

- If both conditions define the same edge, which connects the α and β faces, then both faces, or 12 regions are eliminated. To illustrate, let α and β be as in the last illustration. Let the b be as in Fig. 5a, and let the two choices for β be directly across the A_1-A_2 edge. These choices eliminate all α and β rankings; 12 of them.

- Finally if the α and β edges both border on face γ, only 10 regions are eliminated; each γ region is eliminated by both conditions; the remaining four are the initiating choices in α and β. To illustrate, let α be as above, β the A_2, A_3, A_4 face, and γ the A_1, A_2, A_4 face. Let the b's be as in Fig. 5a, so they eliminate all of the γ rankings. Chose the bottom ranked rankings in β along the A_2-A_4 edge; they, too, eliminate all γ rankings. So, the eliminated rankings are the six in face γ and the four selected ones for a total of ten.

Using the "never-top" conditions with the α, β faces characterizes all combinations of vertices that identify the never-top candidate and the interaction of designated edges.

- If both conditions use the same vertex, there will be overlap in the regions that are eliminated. Here, there are only 10 dropped regions—both "never-top" choices eliminate the six regions around the shared vertex, and each condition eliminates two more regions along the designated edges. This is illustrated in Fig. 5b where face α is given by A_1, A_2, A_3; the two t's eliminate three rankings along the dashed line in the β face of A_1, A_3, A_4 and three rankings along the dashed line in the γ face of A_2, A_3, A_4. The rankings selected in the β face are indicated in Fig. 5b with the 1's. In the γ face, this choice eliminates three rankings, but two of them were already eliminated by t. Similarly, the other 1 eliminates three rankings in the α face, but two of them are t's. Thus this choice eliminates only two additional regions; they are given by the 1^*'s. A total of 10 regions are eliminated.

- If the conditions use two different vertices that share the same designated edge, some overlap occurs meaning that 12 regions are eliminated. In Fig. 5b, the α face is defined by A_1, A_2, A_3 and selected rankings are given by the t's. Thus the designated edge connects vertices A_3 and A_4. To find the other vertex, as the designated edge is to be the same, the face cannot include vertex A_3. Thus this β face must be defined by A_1, A_2, A_4. Moreover, to have the same designated edge, A_4 is the selected vertex, thus the selected regions must be given by the 2's in this face. One 2 eliminates three regions in γ defined by A_1, A_3, A_4, but two of these regions have a dashed line meaning they already were eliminated by the t's. The same behavior occurs in δ defined by A_2, A_3, A_4. Thus the 2^*'s show the two regions not already eliminated by the t's, leading to a total of 12 dropped regions.

- If the designated vertices differ and the designated edges meet only in a single point, then the smaller overlap causes 14 eliminated regions. To illustrate why and what this means, using the same α face and t's, the designated edge connects A_3 with A_4. What we need is to select the β face and its identified never-top rankings in a manner so that the designated line is the A_j-A_4 edge, where A_j is either A_1 or A_2. Suppose it is A_1. Now there is a choice; do we have A_1 or A_4 as the "never-top" ranked candidate? If it is A_1, the face is A_1, A_2, A_3, which is the α face where the never top condition already is specified. Thus, the choice must be A_4 where, as in Fig. 5b, the β face must be given by A_2, A_3, A_4 and the selected never-top rankings must be given by the 3's. The regions newly eliminated are given by the 3^*'s in Fig. 5b. In total, 14 regions are eliminated.

- The remaining condition is for the two vertices differ and the two designated edges not to meet. To see what this means, start with the same α face and the t's. This choice defines the designated line connecting vertices A_3 and A_4. Thus, the other designated line must connect A_1 and A_2; one of these vertices identifies the "never-top ranked" candidate for a particular triplet. If it is A_2, then the choice of the designated line means that the triplet cannot contain A_1; the β face is A_2, A_3, A_4. In Fig. 5c, this situation is given by the 4's. As the eliminated regions do not meet, this last situation drops 16 regions.

The analysis for the never-middle condition is similarly easy. Using the never-middle with faces α and β where both have the same edge as a base, the number of eliminated regions is 12. If the ranking regions for two never-middle choices are adjacent, so they share a portion of an edge of the tetrahedron, the number of excluded regions also is 12. Otherwise, the number of excluded regions is 14. No combination eliminates 16 rankings. Incidentally, for any $n \geq 3$, for each ranking not eliminated by applying the condition to a triplet, its reversal also is not eliminated; i.e., any Condorcet Domain defined strictly with never-middle conditions has an even number of rankings.

Similar straightforward computations hold for other combinations; e.g., when combining a never-bottom with a never-top, emphasize how the never-bottom edge along with the never-top vertex and its designated edge, interact. For instance, using a never-bottom with α where the edge is the designated edge of a never-top condition with face β provides overlap so 11 regions are eliminated. Combining a never-middle with a never-top condition where both designated regions for the never-top already have been eliminated leads to 13 dropped regions.

3.3 Combinations and Fishburn's alternating scheme

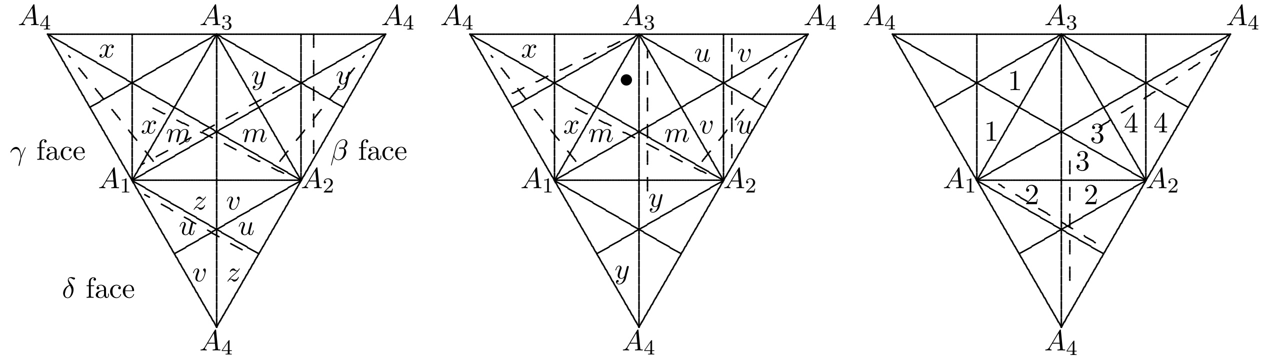
The calculus for three conditions is similar, so, instead of doing so, the above combinatoric rules are now used to obtain insight into what happens with the various combinations of conditions. The first result shows what can be obtained by using the same constraint with each triplet.

Theorem 2 *If the never-top or the never-bottom condition is used with each triplet, then the smallest associated Condorcet Domain is empty; the largest Condorcet Domain has 8 rankings. If the never-middle requirement is used with each triplet, then the resulting Condorcet Domain has either 4 or 8 rankings. The unique arrangement giving 8 rankings is equivalent to Fig. 4c. Each ranking in the never-middle Condorcet Domain has its reversal in the Domain.*

Proof: First consider the never-bottom condition. Use the never-bottom for the α and β faces as defined by the connecting edge; this eliminates all α, β rankings. Doing the same with the γ, δ faces means that all rankings have been eliminated, so the Condorcet Domain is empty. To have a minimum number of eliminated regions, select a face α; each edge of α connects to another face; using each of these edges to define the never-bottom condition for the connecting face means that each of these three conditions eliminate all α regions; in total 12 rankings have been dropped. It remains to use the never-bottom with α; the selected edge will eliminate the remaining four rankings from the connecting face, leaving the specified 8 rankings.

For the never-top condition, to eliminate all rankings, just use all four vertices. About each vertex, the condition eliminates all six rankings where that candidate is top-ranked, so all rankings are eliminated. At the other extreme, select a vertex; it connects three faces. For each face, select the never-top condition defined by that vertex. As the six rankings with that candidate top-ranked are eliminated three times, the total number of eliminated rankings is 12. The choice for the last face must be selected. The three conditions already selected define three designated legs. Select a vertex in this face so that it defines the same designated leg; only four more regions are eliminated.

Hence the associated Condorcet Domain has 8 rankings. That this is best possible follows from the construction and the above combinatoric rules.



a. Face eliminated **b.** Only one ranking **c.** Alternating scheme

Figure 6. Middle-ranked and alternating scheme

The never-middle conditions are left for last as they indicate a general strategy. For instance, to show that the never-middle conditions cannot have an empty Condorcet Domain, assume that it could; thus all rankings from each face must be eliminated. So we try to find what conditions permit this to obtain a contradiction. With the m's in Fig. 6a, the required conditions to eliminate all rankings in this α face defined by A_1, A_2, A_3 follow immediately: There is one "never-middle" condition from the β face of A_2, A_3, A_4 that never eliminates any regions from α; the other two never-middle choices from β leave two blank regions in α. A similar statement holds for any of the three faces bordering on α. Indeed, it is easy to see that the positioning of the x's in γ defined by A_1, A_3, A_4 and and the y's in β, where the x is adjacent to an m, and the y is lifted a region, will eliminate all α face rankings. This choice is unique up to symmetry.

As the never-middle choices for three faces are uniquely specified to drop all α face rankings, it remains to find the never-middle choice for the bottom face δ given by A_1, A_2, A_4. As it is easy to check, each of the three choices of u, v, or z leaves four blank regions, so the associated Condorcet Domain has four rankings. Because this setting describes where all rankings from one face are eliminated, it follows in general that if all rankings from any face are not eliminated, then each face must have at least one blank region; i.e., with the never-middle conditions, the Condorcet Domain must always have at least four rankings.

If we do not want to have all rankings dropped from each face, then the next step is to determine how to select never-middle conditions so that a face has precisely one blank region. The two choices are where the blank region and one of the never-middle rankings are either the bottom two, or the top two, rankings for some alternative. For the first case, which is illustrated in Fig. 6b, the goal is to keep the $A_3 \succ A_1 \succ A_2 \succ A_4$ ranking; this ranking region is identified with the bullet. It is easy to see that the only choice for the x and y never-middle rankings are uniquely determined as illustrated. It remains to find the rankings for the β face. To keep the designated region blank, the only choices are denoted by u and v. If u is selected, all rankings in the side face are eliminated, which returns to the earlier case of four rankings in the Condorcet Domain. Selecting v is the Fig. 4c case of eight rankings in the Condorcet Domain.

The argument for the second case, where m and the blank region are the top two rankings for a candidate is essentially the same. This requirement uniquely defines the choices of never-middle for two faces. There are only two choices for the remaining face; one creates a face with all rankings removed, so it reduces to the earlier case having a Condorcet Domain of four rankings. The other choice leaves one blank ranking for each face; e.g., rankings of the $A_1 \succ A_2 \succ A_3 \succ A_4$, $A_4 \succ A_3 \succ A_2 \succ A_1$, $A_3 \succ A_1 \succ A_4 \succ A_2$, and $A_2 \succ A_4 \succ A_1 \succ A_3$, where each candidate is in each position once, emerge.

Finally, it is easy to show that it is impossible to have three blank rankings in a face. For four blank rankings, it is equally as easy to show that the situation is equivalent to that of Fig. 4c. This completes the proof. □

Before providing a geometric description of Fishburn's alternating scheme, notice how the above approach can be used to answer several other questions. For instance, is the set of rankings $\{A_1 \succ A_2 \succ A_3 \succ A_4,\ A_2 \succ A_1 \succ A_4 \succ A_3,\ A_4 \succ A_2 \succ A_1 \succ A_3\}$ a Condorcet Domain? If so, is it a complete Condorcet Domain? If not, how can it be completed? To find answers, use the above approach used to determine whether a face can have the specified blank regions. In the same way, it is possible to determine the associated Ward-Sen conditions. If such conditions can be found, the set is a Condorcet Domain. If additional blank regions emerge, then the set is not complete and the added regions define a completion.

This approach leads to a geometric description that is equivalent to Fishburn's alternating scheme. A natural choice is to use select face α. For each of the remaining three faces, use the never-bottom condition adjacent to the α edge. In this way, 12 rankings are eliminated. The problem is that whatever choice is made for α, never-top, never-middle, or never-bottom, it will eliminate four more regions from other faces defining a Condorcet Domain of eight rankings. Alternatively, by using the never-top ranked choice with the same vertex, whatever choice is made for the remaining face, four more rankings are excluded.

The next natural approach coming from the calculus is to use two never-bottom conditions, say for faces γ and δ, where both eliminate all α face rankings, and two never-top conditions, for the remaining faces α and β that use the same vertex. In Fig. 6c, the never-bottom choices are illustrated with the 1's and 2's; they eliminate all α rankings. The only two choices for the common vertex of the α and β faces are A_2 and A_3. Either works; I selected A_2 as given by the 3's and 4's. The advantage of this construction is that it creates overlaps with the never-bottom condition, so the Condorcet Domain has nine rankings—the nine blank regions in Fig. 6c outside of the α face.

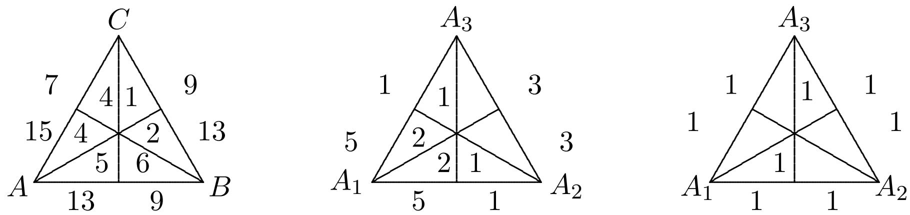
Using the above machinery of computing when all rankings from a face are eliminated, etc., it is not difficult to show that this is the maximum, and it can be attained only in this manner. To recover Fishburn's alternating scheme, select the names of the vertices in an appropriate manner.

3.4 More candidates

The approach for $n \geq 4$ candidates is similar, but assistance coming from concrete geometric objects is missing for $n \geq 6$. (For $n = 5$, the simplex opens into a tetrahedron, which can be opened into a 96 region version of Fig. 3b plus another copy for 24 interior ranking regions.) Any Ward-Sen condition with triplet eliminates $\frac{n!}{3}$ rankings.

The structure remains similar; e.g., the "never-middle ranked" condition eliminates rankings and their reversals; these rankings are along two "indifference ranking" surfaces. If a triplet includes two of the alternatives from the specified triplet, the never-middle condition eliminates half of them; if it has one or none, it eliminates all of the triplet rankings. The never-bottom ranked condition defines an edge and eliminates all rankings in $[\frac{n!}{3} - 3]/6$ triplets. The never-top condition defines a vertex; it eliminates all $(n-1)!$ rankings sharing this vertex (that is, all rankings where the candidate identified with the vertex is top-ranked) and $\frac{(n-1)!}{3}(n-3)$ other rankings which involve rankings on both sides of edges from the designated vertex to the other vertices not on this face. Again, if the triplet includes two alternatives from the specified triplet, the excluded rankings are along an edge; if it includes one or none, the triplet is eliminated.

In this manner, calculus rules for combining Ward-Sen conditions can be determined. The way to do so is to emphasize the interactions among edges, bases, and vertices. For instance, check whether any of the designated edges from the never-top condition coincide with the edge from the never-bottom condition. In this manner, the analysis to determine what happens with $n = 5$ turned out to be straightforward, and it is not overly difficult to find conditions to have a fixed number of blank regions left in a triangle. I have yet to examine what happens with $n \geq 6$.



a. No cycles **b.**Basic **c.** Reversal

Figure 7. A non-cylic example

4 Profile Coordinates

It is widely appreciated that a Condorcet Domain imposes a far too strict constraint to avoid cyclic behavior. This is illustrated with the Fig. 7a profile, which fails *all* of the Ward conditions. Nevertheless, the majority vote rankings are transitive, and, going far beyond what could ever be expected from a Condorcet Domain, the differences in tallies satisfy an extreme "tally consistent transitivity" whereby adding the difference in A, B tallies $(13 - 9 = 4)$ to the difference in B, C tallies $(13-9 = 4)$ equals the difference in the A, C tallies $(15-7 = 8)$! If we embrace the interest of the Condorcet Domain problem, then it becomes necessary to understand why this example, which violates all of Ward conditions, enjoys a much stronger form of majority vote transitivity.

This example was constructed by adding multiples of Figs. 7b, c profiles with appropriate permutations of the A_j names; the component profiles do satisfy Ward conditions. Indeed, the Fig. 7a profile is two units of the Fig. 7b profile where $\{A_1, A_2, A_3\} = \{A, B, C\}$ plus one unit where $\{A_1, A_2, A_3\} = \{B, A, C\}$ plus two units of Fig. 7c where $\{A_1, A_2, A_3\} = \{C, A, B\}$.

Consider the "Generalized Condorcet Domain" problem; it is to determine how to replace "individual rankings" with specific "configurations of rankings" in a way so that any multiples of these configurations never allow cycles. This generalized problem can be completely solved for any n. For three candidates, this generalized problem be solved and the tallies will satisfy the tally-consistent transitivity of majority votes if and only if the profile is a sum of multiples of permutations of the Fig. 7b, c profiles (Saari [12])! Notice, these configurations of profiles define a five-dimensional subspace, which is a dimension larger than possible with the Condorcet Domain. Staying with the theme of the Condorcet Domain, it turns out that with these Fig. 7b. c, configurations, and only with these configurations, any multiple of them be added without ever encountering a cyclic outcome, or without ever violating tally-consistency.

To generalize the discussion, recall that a "positional rule" tallies ballots by assigning specified number of points for candidates based on their position on a ballot. The plurality vote assigns one point to a voter's top-positioned candidate and zero to all others. The Borda Count for n candidates assigns $n - j$ points to a voter's j^{th} positioned candidate.

A recent approach (Saari [12, 13, 14, 16]), which currently is being refined, is to find appropriate profile coordinate systems to handle all possible combinations of positional rules. The goal is similar to that of a Condorcet Domain; it is to find appropriate configurations of profiles—coordinates—so that when adding any multiple of a coordinate to a profile, we know in advance the effect it will have on all possible positional methods. As true with the Condorcet Domain, no restrictions are imposed on how much of a particular profile coordinate is added or subtracted. The difference is that the Condorcet Domain problem concentrates on individual rankings; the profile coordinate system concentrates on specified configurations of preferences.

As an illustration, the Fig. 7b, c configurations denote certain three-alternative coordinate directions; it is easy to show that the Fig. 7b configurations never permit conflict among positional and binary rankings while Fig. 7c configurations, which consist of a ranking and its reversal, has no effect on binary rankings but change positional outcomes. To further illustrate, to understand how and why positional outcomes over triplets differ from positional outcomes over all four candidates and over pairs, we need a coordinate direction that affects the positional election outcomes of triplets without ever affecting binary or four-canidate positional rankings. An example of how this can be done is with the earlier derived Condorcet Domain $\{A_1 \succ A_2 \succ A_3 \succ A_4,\ A_4 \succ A_3 \succ A_2 \succ A_1,\ A_3 \succ A_1 \succ A_4 \succ A_2,\ A_2 \succ A_4 \succ A_1 \succ A_3\}$ where each candidate is in each position precisely once (so all four-candidate positional outcomes end in a tie), and for each pair $\{A_j, A_k\}$, two rankings have $A_j \succ A_k$ while two others have $A_k \succ A_j$; i.e., all pairwise outcomes end in ties. All non-Borda Count positional outcomes for any triplet, however, never are ties. By discovering and using configurations of this type, it becomes possible to explain all differences among all positional elections of all possible subsets of candidates.

Of particular relevance for the current paper is that one part this profile coordinate system identifies those profile configurations that cause cyclic effects in pairwise voting. As these coordinates are closely related to the Condorcet Domain problem, they are described in more detail.

5 The source of all pairwise cycles

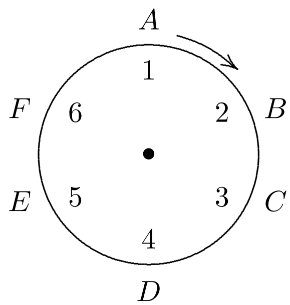
The Condorcet Domain problem searches for the maximal dimensional profile subspaces where cycles never occur. The approach can be described as finding a space of rankings that is orthogonal to cycles by use of the Ward-Sen conditions; the coordinate directions are determined by individual rankings. Precisely the same program is carried out next except that specific coordinate directions replace individual rankings. Namely, the objective is to find appropriate profile coordinates, and the associated profile subspace, so that *any profile orthogonal to this subspace never allows a cycle with the majority vote.* The following theorem states the result.

Theorem 3 *(Saari [13, 12]) For $n \geq 3$ alternatives, in the $n!$ dimensional profile space, there is a $\frac{(n-1)!}{2}$ dimensional subspace with the following property. Any profile that is orthogonal to this subspace can never have a majority vote cycle. Thus, this cycle-free subspace has dimension $n! - \frac{(n-1)!}{2} = \frac{(n-1)!}{2}(2n-1)$. Each triplet $\{A_j, A_k, A_s\}$ has the tally-consistent transitivity property where adding the difference of the majority vote tallies between A_j and A_k to the difference between A_k and A_s equals the difference between A_j and A_s.*

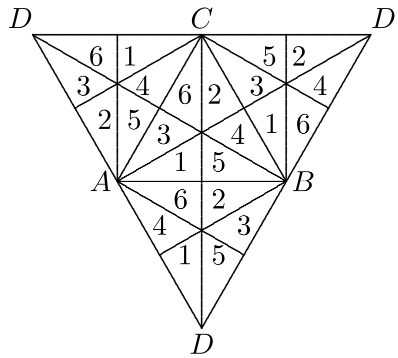
The last statement goes far beyond assuring non-cyclic outcomes to ensure the transitivity of pairwise rankings *and* tally-consistent transitivity. These results, then, are much stronger than possible with the Condorcet Domain. Also, the dimension of the orthogonal space, $\frac{(n-1)!}{2}$, is much smaller than the number of dimensions dismissed by *just one Ward-Sen condition applied to just one triplet,* which is $\frac{n!}{3}$. Thus, the cycle-free subspace ensured by Thm. 3 has a dimension significantly larger than that of any Condorcet Domain. For instance, the largest dimension of a subspace attached to a four-candidate Condorcet Domain is nine, while the subspace from Thm. 3 has dimension $24 - 3 = 21$, so it is more than twice as large. The largest dimension of a subspace attached to a five-candidate Condorcet Domain has dimension 20; the cycle-free subspace guaranteed by Thm. 3 for five candidates is $5! - \frac{4!}{2} = 120 - 12 = 108$, or a five fold increase.

The following theorem illustrates some positive consequences possible from this subspace.

Theorem 4 *(Saari [15]) For any number of candidates, if profiles are restricted to the* $n! - \frac{(n-1)!}{2}$ *dimensional subspace defined in Thm. 3, an admissible rule satisfying Arrow's assumptions [1] is the Borda Count. In the same subspace, there exist rules where Sen's Paretian Liberal impossibility result [20] does not lead to a cycle.*



a. Ranking wheel



b. Six Condorcet four-tuples

Figure 8. Profile coordinates to Condorcet Domains

5.1 Coordinates

To find a profile coordinate system for the orthogonal subspace, use what I call a ranking wheel (Saari [13, 16]), which is a freely rotating wheel attached at its center to a wall. With n candidates, list the numbers from 1 to n in a uniform manner near the wheel's edge. In Fig. 8a, this is illustrated with $n = 6$. Next, select a ranking and list the names of the candidates on the wall next to the appropriate "ranking number." In Fig. 1, the generating ranking is $A \succ B \succ C \succ D \succ E \succ F$.

The first ranking is as given; for Fig. 8a it is the specified $A \succ B \succ C \succ D \succ E \succ F$. Next, rotate the ranking wheel so that the ranking number "1" is positioned next to the second candidate and read off the new ranking. Illustrating with Fig. 8a, the rotated ranking wheel now has "1" next to B, so the new ranking is $B \succ C \succ D \succ E \succ F \succ A$. Continue in this fashion until the ranking number "1" has been next to each candidate precisely once. I call this the "Condorcet n-tuple" generated by the starting ranking. With the Fig. 8a example, the "Condorcet six-tuple generated by $A \succ B \succ C \succ D \succ E \succ F$" is

$$\begin{array}{lll} A \succ B \succ C \succ D \succ E \succ F, & B \succ C \succ D \succ E \succ F \succ A, & C \succ D \succ E \succ F \succ A \succ B, \\ D \succ E \succ F \succ A \succ B \succ C, & E \succ F \succ A \succ B \succ C \succ D & F \succ A \succ B \succ C \succ D \succ E \end{array} \tag{3}$$

A Condorcet n-tuple can be generated by any ranking, and each ranking is in precisely one Condorcet n-tuple. There are $n!$ possible rankings, so there are precisely $\frac{n!}{n} = (n-1)!$ Condorcet n-tuples. To illustrate with $n = 4$, the six Condorcet triplets are generated by

Name	Ranking	Name	Ranking
1	$A \succ B \succ C \succ D$	2	$D \succ C \succ B \succ A$
3	$A \succ C \succ B \succ D$	4	$D \succ B \succ C \succ A$
5	$B \succ A \succ C \succ D$	6	$D \succ C \succ A \succ B$

(4)

Each Condorcet four-tuple has four rankings; by using the Eq. 4 assigned names, the positioning of these rankings are located in Fig. 8b. Each face of the tetrahedron has precisely one representative from each Condorcet four-tuple. For n candidates, each of the n faces of the corresponding equilateral object has precisely one representative from each of the Condorcet n-tuples.

On each row of Eq. 4, each ranking is the reverse of the other. The same effect occurs for any n, a Condorcet n-tuple generated by a ranking can be associated with a Condorcet n-tuple generated by the reverse of the original ranking. Indeed, a coordinate direction in profile space is given by one unit of one of these Condorcet n-tuples and -1 units of the other. (To see the role of negative numbers in profiles, see Saari [12]. It just means that when adding such a profile to another profile, subtract voters from the specified rankings.) This defines the $\frac{(n-1)!}{2}$ orthogonal coordinate directions for the Thm. 3 subspace.

With three candidates, place a 1 in each Fig. 2 starred region, and a -1 in each of the bulleted regions. Listing the Fig. 7a profile coordinates in a counterclockwise manner starting from the lower left corner defines the vector $(5, 4, 4, 1, 2, 6)$ while the Condorcet profile vector is $(1, -1, 1, -1, 1, -1)$. It now is trivial to show that the two vectors are orthogonal, as required by Thm. 3. However, using Fig. 1a, with the associated vector $(0, x, y, z, w, 0)$, it follows that Ward's never-bottom condition satisfies the tally-consistent property if and only if the coordinates satisfy the added restriction $x + z = y + w$. A similar assertion holds for the other two Ward conditions. Namely, profiles associated with Condorcet Domains still have vestiges of the Condorcet n-tuples that the Ward-Sen approach tries to eliminate.

5.2 Condorcet Domains in Condorcet n-tuples

Central to the Ward-Sen condition is that any three rankings from a Condorcet n-tuple creates a cycle. (For a concrete illustrating example, notice that selecting *any* three rankings from the six choices in Eq. 3 creates a cycle.) Consequently, to construct a Condorcet Domain, we cannot have three rankings from any n-tuple, so at least $n-2$ of the rankings from each Condorcet n-tuple must be dropped. Thus, a Condorcet Domain has at most $2(n-1)!$ terms; this number is where there are two rankings from each Condorcet n-tuple. But the actual value is more severe. The reason is that, as as illustrated in Fig. 8b with projections, the rankings of the different Condorcet n-tuples are intimately intertwined. For instance, using a Ward-Sen condition with any triplet has the effect of dropping rankings from *each* Condorcet four-tuple; i.e., the choice of what rankings to eliminate from each four-tuple are interrelated. For instance, to have a blank in the $A \succ B \succ C \succ D$ region in Fig. 8b, two rankings from the number 1 Condorcet four-tuple, and specific number 4 and 6 rankings must also be eliminated. Indeed, with the use of Fig. 8b and some combinatorics, it can be shown that the maximal number of rankings in a four-alternative Condorcet Domain is less than ten. This interesting connection between the Condorcet four-tuples and the Ward-Sen conditions extends to any $n \geq 3$; i.e., the projection approach captures the weaving interactions needed to eliminate rankings from among the Condorcet n-tuples.

6 Summary

for a practical issue of understanding and avoiding majority vote cycles, the Thm. 3 approach is more useful. Nevertheless, the Condorcet Domain problem remains an intriguing question that uncovers valued structures about voting that should be more carefully examined. The projection approach introduced here is a new way to do so. It would be interesting to carry out this projection approach to $n \geq 6$, which would involve determining the calculus conditions for the different Ward-Sen conditions.

Even more, the symmetries disclosed by analyzing this issue most surely have other applications in understanding the complex problems that arise in social choice theory. Fishburn was blessed with an intuitive insight about how to handle the associated and complex combinatorics that are characteristic of this area. But for those of us who are not gifted with such insight, it is important to create a systematic approach to uncover the source of fundamental problems in this area. My sense is that the appropriate tools involve mathematical symmetries, and a way to uncover the appropriate symmetries of social choice is to appeal to the underlying geometry.

References

[1] Arrow, K., 1951 (2nd ed. 1963), *Social Choice and Individual Values*, Wiley, New York.

[2] Black, D., 1948, On the rationale of group decision-making, *Journal of Political Economy,* **56**, 23-34.

[3] Condorcet, M., 1785, *Éssai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Paris.

[4] Fishburn, P., 1997. Acyclic sets of linear orders, *Social Choice and Welfare* **14**, 113-124.

[5] Fishburn, P., 2002. Acyclic sets of linear orders: A progress report, *Social Choice and Welfare* **19**, 431-447.

[6] Gaertner, W., 2001, *Domain Conditions in Social Choice Theory*, Cambridge University Press, New York.

[7] Gehrlein, W., 2006. *Condorcet's Paradox*, Springer, Berlin.

[8] Li, L. and D. G. Saari, 2004, Sen's theorem: Geometric Proof, New Interpretations, IMBS working paper series.

[9] Monjardet, B., 2006, Invited talk, Condorcet domains and distributive lattices, *Workshop on Voting Theory and Preference Modeling*, DIMACS-LAMSADE, Paris, October 26, 2006.

[10] Monjardet, B., 2006, Condorcet domains and distributive lattices, pp. 285-301, *Annales du LAMSADE*, Université Paris, Daupine

[11] Monjardet, B., Acyclic domains of linear orders:a survey, in this volume.

[12] Saari, D. G., 1999, Explaining all three-alternative voting outcomes, *Jour. Econ. Theory* **87**, 313-355.

[13] Saari, D. G., 2000, Mathematical structure of voting paradoxes 1: pairwise vote. *Econ. Theory* **15**, 1-53.

[14] Saari, D. G., 2000, Mathematical structure of voting paradoxes 2: positional voting. *Econ. Theory* **15**, 55-101.

[15] Saari, D. G., 2001, *Decisions and Elections, Explaining the Unexpected,* Cambridge University Press, New York.

[16] Saari, D. G., 2001, *Chaotic Elections! A Mathematician Looks at Voting*, American Mathematical Society, Providence, RI.

[17] Saari, D. G., and A. Petron, 2006, Negative Externalities and Sen's Liberalism Theorem, *Economic Theory* **28**, 265-281.

[18] Saari, D. G., and K. Sieberg, 2004. Are part wise comparisons reliable? *Research in Engineering Design* **15**, 62-71.

[19] Sen, A. K., 1966, A possibility theorem on majority decisions, *Econometrica* **34**, 491-499.

[20] Sen, A. K., 1970, The impossibility of a Paretian Liberal, *The Journal of Political Economy* **78**, 152-157.

[21] Ward, B., 1965, Majority voting and the alternative forms of public enterprise, in Margolis, J. ed *the public economy of urban communities,* 112-126, Johns Hopkins University Press, Baltimore.