Symmetry of Nonparametric Statistical Tests on Three Samples

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Abstract: By identifying symmetry structures of data, properties of nonparametric procedures are identified, which explain why different methods can have different rankings of three samples when analyzing the same data.

1 Introduction

As known, the peculiarities of different nonparametric tests complicate the choice of an appropriate test statistic. To analyze this issue, we explain why different nonparametric *procedures* allow different, even conflicting results to occur with the same data set. Our approach is based on the reality that before noticeable differences in tests can arise, differences emerge among the implicit rankings that are defined by the associated procedures. For instance, before a difference can occur between the Kruskal-Wallis (1952) and some other test, differences arise in the rankings of the k samples as determined by the Kruskal-Wallis procedure and the one associated with the other test. For this reason, we analyze the more sensitive ranking behavior to explain why differences arise.

To do so, we dissect data structures to identify and characterize which kinds of data configurations force different classes of nonparametric rules to have different outcomes. Knowing which kinds of data configurations cause rules to have different outcomes provides a deeper understanding about the behaviors and peculiarities of various nonparametric tests. If, for instance, a certain data structure that is viewed as being unimportant influences the rankings of a specified procedure, then the associated test may not be an appropriate one.

Our work is influenced by that of Haunsperger who, in her thesis (1991) and in a subsequent paper (1992) introduced the concept of a "dictionary" for nonparametric statistical tests over k samples. To explain this term, suppose a data set for the three groups, A, B, C, defines the $A \succ B \succ C$ ranking for the Kruskal-Wallis procedure while the pairwise rankings, as determined by the Mann-Whitney (1947) or Wilcoxon (1945) rules (denoted by MWW), are the conflicting $B \succ A$, $B \succ C$, $A \succ C$. A list of rankings, such as this ($A \succ B \succ C, B \succ A$, $B \succ C$, $A \succ C$), that comes from the same data set is called a *Kruskal-Wallis (KW) word*.

A *dictionary* for specified nonparametric procedures consists of all possible words that could ever be defined in the above sense. Thus, for any number of alternatives, Haunsperger's dictionaries identify *all possible* lists of rankings that can be supported by a data set; these rankings are over all possible subsets of alternatives. By characterizing all dictionaries for a continuum of nonparametric procedures, her results catalogue all possible paradoxes and inconsistencies that these procedures can admit. As an interesting illustration involving the n alternatives $\{A_j\}_{j=1}^n$, her KW dictionary

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establishes that data sets exist whereby the KW ranking of $A_1 \succ A_2 \succ \cdots \succ A_n$ conflicts with MWW pairwise rankings because *every* alternative other than A_n is pairwise ranked above A_1.

To obtain her results, Haunsperger established a connection between nonparametric rules and a class of standard voting rules. By converting data sets into rankings over the alternatives, she proved that these nonparametric rules inherited the basic properties of the associated voting rules. This connection permitted her to transform results about voting rules into results about nonparametric statistical tests. In particular, by transferring conclusions about "dictionaries of voting rules" (Saari 1989), Haunsperger created her "dictionaries for nonparametric statistical rules."

Haunsperger's results are based on conclusions that catalogue what happens with different voting rules as alternatives are dropped or added. After her paper appeared, more refined information was discovered (Saari 1999, 2001) that explains *why* all possible voting paradoxes and differences in election rankings occur and shows how to construct any number of illustrating examples. More precisely, by using mathematical symmetries of data, which are profiles of voter preferences, configurations of voter preferences were discovered over which different voting rules must have different outcomes. These configurations define a coordinate system for the space of voter profiles. By use of this coordinate system it becomes easy to explain why different voting rules have different election rankings with the same data and to construct illustrating examples.

In this current paper, we combine these new structural results about voting theory with Haunsperger's insights about how to transfer conclusions from voting theory to nonparametric statistics. Namely, we characterize why different nonparametric procedures have different outcomes, which data structures cause these differences, and how to construct examples illustrating unexpected behavior. For other results in this direction, see Bargagliotti (2007).

2 Haunsperger's approach and the decomposition of voter profiles

We emphasize three alternatives, or sample groups, denoted by A, B, C, where a data set of n items is collected for each alternative. An item may be, for example, the temperature of a chemical, or the bending strength of a material sample. The information is listed as in the following array of raw, unranked data

A	B	C
r_1	s_1	t_1
r_2	s_2	t_2
$\dots$	$\dots$	$\dots$
r_n	s_n	t_n

Replace these values with integers ranging from 1 to 3n, which indicate how a value ranks across all samples; smaller numbers correspond to lower temperatures, or weaker bending strength. This creates the following table of ranked data (denote the space by $\mathcal{RD}$)

A	B	C
a_1	b_1	c_1
a_2	b_2	c_2
$\dots$	$\dots$	$\dots$
a_n	b_n	c_n

(1)

where the a_j, b_k, c_s terms are the ranking integers that range from 1 to $3n$.

2.1 Data to voting profiles

The profile space for voting rules over three alternatives, denoted by $\mathcal{PS}$, lists the number of voters' preferences that are given by each of the six possible (complete, transitive) ways to strictly rank them. Define the mapping

$$G : \mathcal{RD} \to \mathcal{PS} \tag{2}$$

to be where a ranked data set is converted into a profile as follows: Start with a ranked data set in a Table 1 form and list all n^3 triplets (a_j, b_k, c_m). Replace each triplet with the ranking where larger values are better. The image of G defines a profile that lists how many triplets have each ranking.

As an illustrating example, the ranked data set

$$\mathbf{d} = \begin{array}{c|c|c} A & B & C \\ \hline 6 & 5 & 4 \\ 1 & 2 & 3 \end{array} \tag{3}$$

which resides in $\mathcal{RD}$, define the eight triplets

$$(6,5,4),\ (6,5,3),\ (6,2,4),\ (6,2,3),\ (1,5,4),\ (1,5,3),\ (1,2,4),\ (1,2,3)$$

with, respectively, the associated rankings

$$\begin{array}{cccc} A \succ B \succ C, & A \succ B \succ C, & A \succ C \succ B, & A \succ C \succ B, \\ B \succ C \succ A, & B \succ C \succ A, & C \succ B \succ A, & C \succ B \succ A. \end{array} \tag{4}$$

Thus, $G(\mathbf{d})$ is the profile with two each of $A \succ B \succ C$, $C \succ B \succ A$, $A \succ C \succ B$, and $B \succ C \succ A$.

The profile space, $\mathcal{PS}$, resides in a six-dimensional space; each of the six rankings defines a $\mathbb{R}^6$ coordinate direction. Choosing the coordinate directions in the order

$$(A \succ B \succ C,\ A \succ C \succ B,\ C \succ A \succ B,\ C \succ B \succ A,\ B \succ C \succ A,\ B \succ A \succ C)$$

means that, for the above $\mathbf{d}$, $G(\mathbf{d}) = \mathbf{p} = (2,2,0,2,2,0)$.

2.2 Positional rules and a profile decomposition

A positional voting rule tallies ballots by assigning a specified number of points to each alternative based on its ballot position; e.g., for the positional rule $(3,0,-1)$, three, zero, and minus one points are assigned, respectively, to the top, middle, and bottom positioned alternative on a ballot. Without loss of generality, adjust the weights so that the bottom ranked alternative receives zero points; e.g., the $(3,0,-1)$ choice becomes $(4,1,0)$. Next, scale them so that one point is assigned to the top-ranked alternative; e.g., the $(4,1,0)$ choice becomes $(1,\frac{1}{4},0)$. In this manner, any three alternative positional method can be represented by $(1,s,0)$ for a specific value of $s \in [0,1]$.

Definition 1 *The mapping $P_s : \mathcal{PS} \to \mathbb{R}^3$ lists the candidate's tallies when the positional method $(1,s,0)$, $s \in [0,1]$ is used with a profile. The P_s components, or tallies, are listed in the A, B, C order.*

With the above $\mathbf{p} = (2,2,0,2,2,0)$, then, $P_s(\mathbf{p}) = (4, 2+4s, 2+4s)$ where the tallies are listed in the A, B, C order; e.g., B receives $2+4s$ points. Notice how the choice of s alters the final ranking; e.g., with $s < \frac{1}{2}$, A has a higher tally than B, but with $s > \frac{1}{2}$, B has a higher vote than A.

An important positional method is the Borda Count; this is where, when tallying an n-alternative ballot, $n-j$ points are assigned to the j^{th} positioned alternative. For three alternatives, the Borda Count is given by $(2,1,0)$ with the normalized form of $(1, \frac{1}{2}, 0)$, so $s = \frac{1}{2}$. With the above example, $P_{\frac{1}{2}}(\mathbf{p})$ is a complete tie.

The profile coordinate system is determined by the symmetries of the data structure. The following structures and terminology come from Saari (1999).

- The S_3 orbit of any ranking is where there is precisely one of each of the six possible rankings, or $K = (1,1,1,1,1,1)$. For $c > 0$, cK is where each of the six rankings is supported by the same number of voters. This is called a *kernel profile.*

- The $\mathbb{Z}_3$ orbit of ranking $A \succ B \succ C$ is the set

$$\{A \succ B \succ C,\, B \succ C \succ A,\, C \succ A \succ B\}. \tag{5}$$

  This particular *Condorcet triplet* has the $(1,0,1,0,1,0)$ profile representation. The other Condorcet triplet is generated by $A \succ C \succ B$ and given by $(0,1,0,1,0,1)$. To construct either configuration of preferences, move the top ranked alternative in one ranking to the bottom of the next ranking. This construction ensures that each alternative is in first, second, and third place precisely once in the triplet.

- The $\mathbb{Z}_2$ orbit of the ranking $A \succ B \succ C$ is the set $\{A \succ B \succ C, C \succ B \succ A\}$; such a configuration consists of a ranking and its reversal. For alternative X, $X = A, B, C$, an X-*Reversal profile* consists of the two rankings where X is top ranked in one ranking and the reversal of this ranking; it consists of the two terms in the resulting $\mathbb{Z}_2$ orbit. Using the vector profile representation, the A-Reversal profile is $R_A = (1,1,0,1,1,0)$, while the B-Reversal profile is $R_B = (0,1,1,0,1,1)$. The Eq. 4 profile is $\mathbf{p} = 2R_A$.

The importance of these profile configurations is captured by the following theorem, which identifies when different voting rules have different outcomes.

Theorem 1 *(Saari, 1999) For three alternatives, a simultaneous complete tie for all possible positional rules and all majority votes over pairs occurs if and only if the profile is a kernel profile.*

The situation where two different positional methods have a complete tie and there is at least one non-tied pairwise majority vote occurs if and only if all positional methods have a complete tie, the three pairwise majority votes define a cycle (where the difference in tallies is the same for each pair) and the profile is the sum of a kernel and a multiple of a Condorcet triplet.

The situation where all majority votes over pairs are complete ties, but at least one positional method for $s \neq \frac{1}{2}$ is not a complete tie occurs if and only if the Borda ranking is a complete tie and all non-Borda positional methods are not ties; the profile consists of a kernel profile plus a linear combination of Reversal profiles. In this setting, the $P_s(\mathbf{p})$ ranking for $s < \frac{1}{2}$ must be the reversal of the $P_s(\mathbf{p})$ ranking for $s > \frac{1}{2}$.

Theorem 1 completely identifies which kinds of profiles cause different types of conflict among different voting rules. Clearly, by appropriately combining these data structures, it is possible to create examples where the pairwise and positional methods can differ as desired. What remains is to find profiles whereby all rules agree; for these configurations, there is no conflict of any kind.

Definition 2 *For alternative X, $X = A$, B, C, a X-*Basic profile *is where there are two voters for each of the two rankings where X is top-ranked, one voter for each ranking where X is middle-ranked, and zero voters for each ranking where X is bottom-ranked.*

Using our coordinate representation, the A-Basic profile is given by $B_A = (2,2,1,0,0,1)$, while the B-Basic profile is $B_B = (1,0,0,1,2,2)$. The importance of Basic profiles is captured in the next theorem.

Theorem 2 *(Saari, 1999) For any linear combination of Basic profiles, all positional methods have the same election ranking, the differences between the tallies assigned to alternatives is the same for all P_s rules, and the differences in majority vote tallies between two candidates is a fixed positive multiple of the difference of the positional tally of these candidates. Moreover, the majority votes for pairs satisfy a strong sense of transitivity in that the difference in the A and B majority vote tallies added to the difference in the B and C majority vote tallies equals the difference between the A and C majority vote tallies. (This holds for any change of names of the alternatives.)*

If election outcomes satisfy the above properties, or even if two different positional outcomes and the pairs satisfy the properties, then the profile is kernel profile plus a linear combination of Basic profiles.

With Basic profiles, then, nothing goes wrong in the sense that knowing the outcome for one kind of voting rule identifies what happens with all other positional and majority vote rules. According to Theorems 1, 2, the Borda Count is the sole positional rule where its ranking never is affected by the Condorcet profile terms (because it is a positional method), nor by the Reversal terms (because the Borda Count is explicitly exempt from these effects). Consequently, the Borda outcome, and only the Borda outcome, is completely determined by the Basic profiles. Conversely, the Borda tally determines the tally of all positional methods when computed over the Basic component of the profile. These statements have implications for nonparametric statistical rules.

2.3 Profile coordinates and an example

For three alternatives, Theorems 1 and 2 characterize all possible behavior for positional voting rules and majority votes over pairs. To prove that nothing else can occur, modify these choices to create a coordinate system (Saari 1999), which only requires removing kernel components from each choice. For instance, the Condorcet triplet $(2,0,2,0,2,0)$ is where two voters have each of the Eq. 5 rankings. To make this vector orthogonal to the kernel vector $K = (1,1,1,1,1,1)$, use

$$C = (2,0,2,0,2,0) - (1,1,1,1,1,1) = (1,-1,1,-1,1,-1) \tag{6}$$

where the $+1$ terms define one Condorcet triplet and the -1 terms define the other one. The interpretation of a negative value in a profile, then, is to subtract this number of voters from the specified ranking when the profile is added to another profile.

The resulting coordinate system defined for the six-dimensional $\mathcal{PS}$ consists of the kernel vector K, the Condorcet coordinate C of Eq. 6, which creates all differences in majority votes over pairs but has no effect on positional rules, a two-dimensional space spanned by the modified Reversal profiles $\tilde{R}_a = (1,1,-2,1,1,-2), \tilde{R}_B = (-2,1,1,-2,1,1)$, which causes all differences in positional outcomes but has no effect on pairwise tallies, and the two-dimensional space spanned by the modified Basic profiles $\tilde{B}_A = (1,1,0,-1,-1,0), \tilde{B}_B = (0,-1,-1,0,1,1)$, which is where all rules are in complete agreement. As a technical aside, while $\{\tilde{B}_A, \tilde{B}_B\}$ are not orthogonal, the two-dimensional space they span is orthogonal to the other subspaces. Also $\tilde{B}_C = -[\tilde{B}_A + \tilde{B}_B]$. Similar comments hold for the reversal terms, such as $\tilde{R}_C = -[\tilde{R}_A + \tilde{R}_B]$.

This six-dimensional coordinate system for $\mathbb{R}^6$ ensures that all possible profiles can be described in terms of Theorems 1 and 2. Consequently, all possible differences in outcomes are characterized by these data structures, and the behavior of the rules on these structures is as described in the theorems. In (Saari, 1999), the appropriate linear algebra relationships are developed to decompose a profile into its component parts, etc. The modified choices are also used in the decomposition of $G(\mathbf{d})$ vectors in the Section 4. The only difference between $\tilde{B}_X$ and B_X terms, or $\tilde{R}_X$ and R_X terms is that the latter ones include kernel components.

Part of what we learn from Theorems 1 and 2 is that:

- The Borda Count ranking, $P_{\frac{1}{2}}$, is determined strictly by the profile's Basic components; the other profile components do not matter.

- If $s \neq \frac{1}{2}$, then the P_s outcome is strictly determined by the Basic and the Reversal components of a profile. On the Basic component, the P_s outcome must agree with the Borda outcome. This means that *all possible differences* among positional outcomes are determined by how the different P_s rules react to the Reversal components.

- All majority vote outcomes over pairs are strictly determined by the Basic and the Condorcet components in a profile. Because all outcomes over the Basic component agree with the Borda outcome and because the Borda outcome is not effected by Condorcet terms, it follows that all possible other differences are caused by how the majority vote over pairs reacts to the Condorcet component of a profile.

A way to illustrate these results is to create an example where the Borda ranking is $A \succ B \succ C$ while the plurality ranking is the reversed $C \succ B \succ A$ and the pairwise majority votes define a cycle $B \succ A$, $A \succ C$, $C \succ B$. Because the Borda outcome is strictly determined by the Basic profiles, start with

$$3B_A + 2B_B = (6, 6, 3, 0, 0, 3) + (2, 0, 0, 2, 4, 4) = (8, 6, 3, 2, 4, 7). \tag{7}$$

As Theorem 2 ensures, all positional and majority vote rankings over pairs agree with $A \succ B \succ C$.

To obtain the desired $C \succ B \succ A$ plurality vote, Theorem 1 requires adding Reversal terms, which never affect Borda or pairwise rankings. Adding x units of C - Reversal and y units of B - Reversal, $x > y > 0$, to Eq. 7 leads to the $A : B : C$ plurality tallies of $14 + x + y : 11 + x + 2y : 5 + 2x + y$. A desired outcome satisfies the inequalities $5 + 2x + y > 11 + x + 2y > 14 + x + y$, or $y > 3, x > 6 + y$; one solution is $y = 4, x = 11$.

To create a majority vote cycle, add z units of the Condorcet triplet $A \succ C \succ B, C \succ B \succ A, B \succ A \succ C$ to Eq. 7; this particular Condorcet triplet was selected because it creates a cycle in the desired direction. As a Condorcet term does not affect positional outcomes, a choice for z can be determined by adding this term to Eq. 7 rather than the more complicated expression involving the above x and y terms. With this addition, the A:B, A:C, B:C tallies are, respectively, $17 + z : 13 + 2z$, $21 + 2z : 9 + z$, $19 + z : 11 + 2z$. The desired majority vote rankings are obtained by converting these tallies into inequalities, which leads to $z > 8$. Let $z = 9$.

Adding these terms leads to the profile $(19, 19, 18, 22, 8, 31)$. As adding or subtracting a kernel component does not effect positional or majority vote rankings, subtract a value of eight from each component to create the profile $(11, 11, 10, 14, 0, 23)$ that has the desired properties.

3 Statistical procedures on three samples

Haunsperger proved that many statistical procedures over three samples are subsumed by the following definition.

Definition 3 *(Haunsperger, 1992) For a specified value of s satisfying $0 \le s \le 1$ and ranked data $\mathbf{d}$, $\mathcal{NP}_s$ is the nonparametric procedure defined as*

$$\mathcal{NP}_s(\mathbf{d}) = P_s(G(\mathbf{d})). \tag{8}$$

To compute $\mathcal{NP}_s$ for a specified data set $\mathbf{d}$, first determine the profile $G(\mathbf{d})$ and apply the positional rule $\mathcal{P}_s$. As Haunsperger proves, the $\mathcal{NP}_0$ rule is equivalent to the Bhapkar (1961) V procedure, while $\mathcal{NP}_{\frac{1}{2}}$ is equivalent to the Kruskal-Wallis procedure. This means that the Bhapkar V test inherits the properties and peculiarities of the plurality vote while the KW test inherits those of the Borda Count. Other tests use other choices of s; e.g., the Bhapkar-Deshpande (1968) procedure assigns $+1$ points to an alternative each time it is top-ranked in $G(\mathbf{d})$ and -1 points each time it is bottom-ranked. With three samples, the Bhapkar-Deshpande and the KW tests are equivalent; for four or more alternatives, they are not. (Assigning $(1, 0, -1)$ points is equivalent to assigning $(2, 1, 0)$ points, which is equivalent to $(1, \frac{1}{2}, 0)$, or $s = \frac{1}{2}$.)

Other s values define procedures that use different weights with $G(\mathbf{d})$. For instance, $s = 1$ represents the positional method $(1, 1, 0)$, or even $(0, 0, -1)$; i.e., $\mathcal{NP}_1$ penalizes samples that are bottom ranked within the data. As for pairwise comparisons, carrying out the computations for the MWW rule is equivalent to computing the majority vote over a specified pair of alternatives from the $G(\mathbf{d})$ profile.

3.1 Properties of statistical tests for three samples

Our goal of understanding why different nonparametric procedures have different outcomes is partially achieved by using the above structure. Examples of possible results follow; proofs are in Sect. 6, but the basic intuition comes from Eq. 8 and the assertion that $\mathcal{NP}_s$ rules inherit the properties of P_s rules.

Theorem 3 *(Haunsperger 1992) For any two choices of $s_1, s_2 \in [0, 1], s_1 \ne s_2$, select a ranking for $\mathcal{NP}_{s_1}$ and another one for $\mathcal{NP}_{s_2}$. There exists a data set achieving these two outcomes.*

This theorem asserts, for instance, that there is no reason to expect a ranking from the Bhapkar V procedure to be related, in any manner, to the rankings that are defined by the KW procedure. We improve upon Thm. 3 by explaining why it is true in terms of data structures.

Theorem 4 *If the ranked data structure $\mathbf{d}$ is such that $G(\mathbf{d})$ does not have Reversal components, then all $\mathcal{NP}_s$ methods have the same ranking of the samples. If the ranked data structure consists strictly of Reversal components, then the KW test is a complete tie and the $\mathcal{NP}_s$ tests for $s < \frac{1}{2}$ rank the alternatives in the opposite order from where $s > \frac{1}{2}$. If any two $\mathcal{NP}_s$ rules have different ranking of the alternatives, then the ranked data $\mathbf{d}$ is such that $G(\mathbf{p})$ contains Reversal components. If any two $\mathcal{NP}_s$ rankings differ with both s values on the same side of $\frac{1}{2}$, then $G(\mathbf{p})$ contains Reversal and Basic components.*

This result follows immediately from Theorem 1. Theorem 4, then, completely identifies the particular kind of data structures responsible for all differences in these nonparametric rules. This data structure is implicitly, but completely, described by the fact that $G(\mathbf{d})$ has Reversal components. If the ranked data is such that $G(\mathbf{d})$ has minimal, or no Reversal components, then there will be compatibility among the $\mathcal{NP}_s$ rules. Otherwise, expect conflict and differences.

The next issue is to understand when and why the rankings for a nonparametric test can differ from pairwise rankings. That such behavior can occur follows from another Haunsperger assertion.

Theorem 5 *(Haunsperger 1992) If $s \neq \frac{1}{2}$, independently select any ranking for $\mathcal{NP}_s$ and any rankings for each of the three pairs. There exists a data set whereby all specified rankings occur. In contrast, the ranking defined by the KW test and the pairwise rankings admit relationships; e.g., if the pairwise rankings are transitive, then the KW test always ranks top-ranked alternative from this pairwise rankings above the bottom-ranked one.*

Again, the explanation for these assertions follows from how different $\mathcal{NP}_s$ tests react to different data structures.

Theorem 6 *If the ranked data structure is such that $G(\mathbf{d})$ has no Condorcet components, then the rankings of the pairs forms a complete transitive ranking that agrees with the KW ranking. If the ranked data structure is such that $G(\mathbf{d})$ has no Reversal terms and the pairs do not define a cycle, then there is a relationship between the $\mathcal{NP}_s(\mathbf{d})$ ranking and the ranking of the pairs.*

In other words, all difficulties and complexities caused by pairwise rankings are strictly due to the Condorcet component of $G(\mathbf{d})$. No other term plays a role.

3.2 Summary of the data structure

The above results permit us to completely characterize which data structures affect the different nonparametric tests. This structure provides a complete explanation why different procedures have different outcomes. More specifically:

- The KW procedure is determined strictly by the portion of ranked data that defines the Basic component in $G(\mathbf{d})$. Non-Basic components have no effect on the KW ranking.
- For $s \neq \frac{1}{2}$, the $\mathcal{NP}_s$ outcome is determined strictly by the components of the ranked data that create the Basic and Reversal components of the $G(\mathbf{d})$ profile. For the data portion creating a Basic component, the $\mathcal{NP}_s$ rule agrees with the KW outcome. All possible differences occur by how different $\mathcal{NP}_s$ rules react to the Reversal components in $G(\mathbf{d})$. These $\mathcal{NP}_s$ outcomes define a line in $\mathbb{R}^3$ centered with the KW outcome.
- All MWW rankings of pairs are strictly determined by the data portions that introduce Basic and the Condorcet components in $G(\mathbf{d})$. The outcome over the Basic component agrees with the KW outcome. All other differences are created by the cyclic effect introduced by the Condorcet component of a $G(\mathbf{d})$ profile.

According to this description, cycles of pairwise rankings, different $\mathcal{NP}_s$ rankings, and differences among the pairwise and $\mathcal{NP}_s$ rankings occur because the outcomes for different rules rely on different portions of the data structure.

4 Data structures

Because nonparametric procedures are influenced in different ways by different data structures, the next step is to identify the properties of ranked data that create the various kinds of $G(\mathbf{d})$ data structures. To do so for three alternatives and $n = 2$, where the unranked data does not have ties, just compute each possible $G(\mathbf{d})$ profile and then use the decomposition described in Eq. 21 coming from (Saari 1999). To simplify the task, notice that each column of Eq. 1 can be permuted in any manner without affecting the triplets or G outcome. Thus, assume that each column is ranked from the largest value down to the smallest; e.g., $a_j > a_{j+1}$. As the names of the alternatives can be permuted, further assume that $a_1 > b_1 > c_1$. Using these symmetries, the $n = 2$ setting is reduced from the original 6! possibilities to the fifteen following cases.

Theorem 7 *The following ranked data sets define the associated profile decompositions; each decomposition also includes an* $\frac{4}{3}K$ *term:*

$$\begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} \to \tfrac{4}{3}\tilde{B}_A + \tfrac{2}{3}\tilde{B}_B - \tfrac{1}{3}\tilde{R}_B + \tfrac{2}{3}C,$$

$$\begin{pmatrix} 6 & 5 & 4 \\ 2 & 1 & 3 \end{pmatrix} \to \tfrac{1}{3}[\tilde{B}_A - \tilde{B}_B - \tilde{R}_C] + \tfrac{2}{3}C,$$

$$\begin{pmatrix} 6 & 5 & 4 \\ 2 & 3 & 1 \end{pmatrix} \to \tilde{B}_A + \tilde{B}_B - \tfrac{1}{3}\tilde{R}_B,$$

$$\begin{pmatrix} 6 & 5 & 3 \\ 1 & 4 & 2 \end{pmatrix} \to \tfrac{2}{3}\tilde{B}_A + \tfrac{4}{3}\tilde{B}_B + \tfrac{2}{3}\tilde{R}_A + \tfrac{4}{3}C,$$

$$\begin{pmatrix} 6 & 5 & 3 \\ 2 & 4 & 1 \end{pmatrix} \to \tfrac{4}{3}\tilde{B}_A + \tfrac{5}{3}\tilde{B}_B - \tfrac{1}{3}\tilde{R}_B + \tfrac{2}{3}C,$$

$$\begin{pmatrix} 6 & 5 & 2 \\ 3 & 4 & 1 \end{pmatrix} \to 2\tilde{B}_A + 2\tilde{B}_B - \tfrac{2}{3}\tilde{R}_A - \tfrac{2}{3}\tilde{R}_B,$$

$$\begin{pmatrix} 6 & 4 & 3 \\ 5 & 2 & 1 \end{pmatrix} \to \tfrac{7}{3}\tilde{B}_A + \tfrac{2}{3}\tilde{B}_B + \tfrac{1}{3}\tilde{R}_A - \tfrac{2}{3}\tilde{R}_B + \tfrac{2}{3}C,$$

$$\begin{pmatrix} 6 & 4 & 2 \\ 5 & 3 & 1 \end{pmatrix} \to \tfrac{8}{3}\tilde{B}_A + \tfrac{4}{3}\tilde{B}_B - \tfrac{4}{3}\tilde{R}_B + \tfrac{4}{3}C.$$

$$\begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \end{pmatrix} \to \tfrac{2}{3}\tilde{R}_A,$$

$$\begin{pmatrix} 6 & 5 & 4 \\ 1 & 3 & 2 \end{pmatrix} \to \tfrac{1}{3}[\tilde{B}_B - \tilde{B}_C] + \tfrac{2}{3}\tilde{R}_A + \tfrac{2}{3}C$$

$$\begin{pmatrix} 6 & 5 & 4 \\ 3 & 1 & 2 \end{pmatrix} \to \tilde{B}_A + \tfrac{1}{3}\tilde{R}_A + \tfrac{1}{3}\tilde{R}_B$$

$$\begin{pmatrix} 6 & 5 & 3 \\ 4 & 1 & 2 \end{pmatrix} \to \tilde{B}_A + \tilde{R}_A + \tfrac{2}{3}\tilde{R}_B$$

$$\begin{pmatrix} 6 & 5 & 3 \\ 4 & 2 & 1 \end{pmatrix} \to \tfrac{4}{3}\tilde{B}_A + \tfrac{2}{3}\tilde{B}_B + \tfrac{2}{3}\tilde{R}_A + \tfrac{2}{3}C,$$

$$\begin{pmatrix} 6 & 5 & 2 \\ 4 & 3 & 1 \end{pmatrix} \to \tfrac{7}{3}\tilde{B}_A + \tfrac{5}{3}\tilde{B}_B - \tfrac{1}{3}\tilde{R}_A - \tilde{R}_B + \tfrac{2}{3}C,$$

$$\begin{pmatrix} 6 & 4 & 3 \\ 5 & 1 & 2 \end{pmatrix} \to 2\tilde{B}_A + \tfrac{2}{3}\tilde{R}_A,$$

Notice that six of the fifteen terms have no Condorcet component: this means (Theorem 6) that the KW and the pairwise rankings completely agree over these data sets. All fifteen choices, however, include Reversal terms, which force differences to occur in the associated tallies, and maybe even the rankings, of different $\mathcal{NP}_s$ rules.

The following computational rules help to quickly compute and compare outcomes.

Proposition 8 *The* P_s *tally for* $\tilde{B}_X$ *assigns 2 points to* X *and* -1 *points to each of the other two alternatives. The* P_s *tally for* $\tilde{R}_X$ *assigns* $2 - 4s$ *points to* X *and* $2s - 1$ *points to each of the other alternatives. The* P_s *tally for* K *assigns* $2 + 2s$ *points to each alternative.*

The majority vote tallies for $\tilde{B}_X$ *in an* $\{X, Y\}$ *election assign 2 points to* X *and* -2 *to* Y*. In a pair not including* X*, both alternatives receive zero points. The Majority vote tallies for the Condorcet profile* C *are* $A : B, B : C, C : A$ *each by* $1 : -1$*. The majority votes of a pair for* K *assign three votes to each alternative.*

To illustrate, the $\left(\begin{smallmatrix} 6 & 5 & 3 \\ 1 & 4 & 2 \end{smallmatrix}\right)$ data decomposition of $\frac{2}{3}\tilde{B}_A + \frac{4}{3}\tilde{B}_B + \frac{2}{3}\tilde{R}_A + \frac{4}{3}C + \frac{4}{3}K$ ensures the $B \succ A \succ C$ KW ranking; the KW ranking is our standard basis for comparison. The large Condorcet term raises doubts whether the pairwise rankings will agree with KW ranking; they do not as the above computations prove there is a $A = B, A = C$ tie with $B \succ C$. The A-Reversal term suggests that the Bhapkar V ranking might differ from the KW ranking, and it does with $A = B \succ C$. At the other extreme, the G_1 ranking differs in a different direction; it is $B \succ A = C$. The earlier statement that all $\mathcal{NP}_s(\mathbf{d})$ outcomes define a line with endpoints $\mathcal{NP}_0(\mathbf{d})$ and $\mathcal{NP}_1(\mathbf{d})$, ensures that all other $\mathcal{NP}_s(\mathbf{d})$ rankings are $A \succ B \succ C$, but with different tallies.

On the other hand, the $\left(\begin{smallmatrix} 6 & 5 & 4 \\ 2 & 3 & 1 \end{smallmatrix}\right)$ ranked data has the $\tilde{B}_A + \tilde{B}_B - \frac{1}{3}\tilde{R}_B + \frac{4}{3}K$ decomposition. The Basic terms require the KW ranking of $A = B \succ C$, and the absence of a Condorcet term ensures agreement with the pairwise comparisons of $A = B, A \succ C, B \succ C$. The Reversal term requires the Bhapkar ranking to penalize B, as it does with the ranking $A \succ B \succ C$.

We leave to the reader to decide whether to emphasize

- the KW test because the KW procedure maximizes consistencies over rankings and ignores data causing Reversal and Condorcet terms,
- a G_s rule for $s < \frac{1}{2}$, which interprets data including $A \succ B \succ C, C \succ B \succ A$ pairs of reversals by rewarding the two top ranked alternatives at the expense of inconsistencies with pairs,
- or a G_s rule for $s > \frac{1}{2}$, which rewards the middle ranked alternative with data including $A \succ B \succ C, C \succ B \succ A$ pairs, at the expense of inconsistencies with pairs.

5 Characterizing data sets

Different rules react differently to different kinds of data sets, so the next step is to characterize those data sets that strictly define kernel, Condorcet, and Reversal terms. To create examples exhibiting different kinds of behavior, combine these structures. However, care is needed because of the inherent sense of nonlinearity of $G(\mathbf{d})$; i.e., in general, $G(\mathbf{d}_1 \cup \mathbf{d}_2)$ does not equal $G(\mathbf{d}_1) + G(\mathbf{d}_2)$.

Definition 4 *A ranked data set* $\mathbf{d}$ *is called a kernel, Reversal, Condorcet, Basic data set if and only if* $G(\mathbf{d})$ *is, respectively, a kernel profile, the sum of a kernel and (nonzero) Reversal profiles, the sum of a kernel and (nonzero) Condorcet profile, the sum of a kernel and (nonzero) Basic profiles.*

The characterization of the disruptive Condorcet and Reversal aspects of the data is straightforward. The difficulty is to prove that such data examples exist. In the following characterization, let XYZ be the set of all triplets constructed from the ranked data of the Eq. 1 form that have the ranking $X \succ Y \succ Z$, and let $|XYZ|$ be the number of such triplets. Similarly, XY are all triplets that have X ranked above Y and $|XY|$ is the number of such triplets. Proofs of the following results are in Section 6.

Particular interest is in the Reversal behavior as it forces different $\mathcal{NP}_s$ outcomes.

Theorem 9 *Data set* $\mathbf{d}$ *is a strict Reversal data set if and only if*

$$|ABC| = |CBA|, |ACB| = |BCA|, |CAB| = |BAC| \tag{9}$$

where at least two sets of equalities do not have the same value and

$$|AB| = |BC| = |CA| = |BA| = |CB| = |AC|. \tag{10}$$

Such data sets exist.

Examples seem to require the $\mathbb{Z}_2$ ranking structure whereby rows have opposing rankings. This is the structure of the Eq. 4 example, which is a pure Reversal data set. Also, for the pure Reversal

A	B	C
12	11	10
7	9	8
6	4	5
1	2	3

the first and fourth rows and the second and third rows reverse each other. However, there are examples that do not have the same number of rows with one ranking as with its reversal. A two-sample reversal example is where the A, B ranked information is $(1,3), (2,4), (6,5), (8,7), (10,9), (12,11)$

where the first four rows have $A > B$ while the last two have $B > A$. Three alternatives examples also can be created.

Condorcet data sets are interesting because they create pairwise cycles and differences between pairwise rankings and all $\mathcal{NP}_s$ outcomes.

Theorem 10 *Data set* $\mathbf{d}$ *is a strict Condorcet data set if and only if*

$$|ABC| = |BCA| = |CBA| \neq |ACB| = |CBA| = |BAC| \tag{11}$$

if and only if

$$|AB| = |BC| = |CA| \neq |BA| = |CB| = |AC|, \tag{12}$$

$$|ABC + |ACB| = |BAC| + |BCA| = |CAB| + |CBA| = \frac{n^3}{3} \tag{13}$$

and for any permutation of the letters

$$|ABC| + |ACB| + \frac{1}{2}(|CAB| + |BAC|) = |BAC| + |BCA| + \frac{1}{2}(ABC| + |CBA|) \tag{14}$$

Such data sets exist.

The construction of such data sets captures the spirit of the $\mathbb{Z}_3$ structure, but with complications. To explain, the following pure Condorcet data set is divided into three parts:

$$\begin{array}{c|c|c} A & B & C \\ \hline 27 & 26 & 25 \\ 22 & 24 & 23 \\ 20 & 19 & 21 \end{array}, \quad \begin{array}{c|c|c} A & B & C \\ \hline 16 & 18 & 17 \\ 14 & 13 & 15 \\ 12 & 11 & 10 \end{array}, \quad \begin{array}{c|c|c} A & B & C \\ \hline 8 & 7 & 9 \\ 6 & 5 & 4 \\ 1 & 3 & 2 \end{array} \tag{15}$$

where the first array has the row data arranged in the expected $A > B > C, B > C > A, C > A > B$ order of a $\mathbb{Z}_3$ orbit. The $3^3 = 27$ rankings, however, are $|ABC| = |CAB| = 5, |BCA| = 8$, while $|BAC| = 2, |CBA| = 3$, and $|ACB| = 5$ so, according to Theorem 10, the $G(\mathbf{d})$ ranking is not a pure Condorcet plus kernel term as it slightly favors C. Nevertheless, the outcome is a pairwise cycle, $A > B, B > C$ each by 15:13, and $C > A$ by 16:12.

To create a pure Condorcet term, introduce two more sets of three rows where the top defining rows among the sets have the $\mathbb{Z}_3$ rankings. As the first row of the first set starts with an $A > B > C$ ranking, start the second set with $B > C > A$, where A is slightly favored, and the final set with $C > A > B$, where B is slightly favored. Namely, each set of three rows reflects the $\mathbb{Z}_3$ symmetry, and the three sets are connected with a $\mathbb{Z}_3$ symmetry construction. The resulting Eq. 15 is a Condorcet data set. So far, all Condorcet data sets we have been able to find satisfy this construction, but we expect other structures will be discovered.

Constructing examples with mixed behavior now is immediate. For instance, the first block below is a version of the first three rows of Eq. 15; the second block is the pure reversal Eq. 3.

A	B	C
15	14	13
10	12	11
8	7	9

A	B	C
6	5	4
1	2	3

Combined, the new data set has the anticipated outcome with the KW ranking $A = B = C$, the $\mathcal{NP}_s$ ranking of $A \succ B \succ C$ for $s < \frac{1}{2}$ and $C \succ B \succ A$ for $s > \frac{1}{2}$, and the pairwise rankings for the cycle $A \succ B, B \succ C, C \succ A$.

The kernel data sets are characterized by the following.

Theorem 11 *Data set* **d** *is a kernel data set if and only if*

$$|ABC| = |ACB| = |CB| = |CBA| = |BCA| = |BAC| \tag{16}$$

if and only if

$$|AB| = |BA| = |CB| = |BC| = |AC| = |CA|, \tag{17}$$

$$|ABC + |ACB| = |BAC| + |BCA| = |CAB| + |CBA| = \frac{n^3}{3} \tag{18}$$

and for any permutation of the letters

$$|ABC| + |ACB| + \frac{1}{2}(|CAB| + |BAC|) = |BAC| + |BCA| + \frac{1}{2}(ABC| + |CBA|) \tag{19}$$

Such data sets exist.

An example is

A	B	C
17	16	18
14	15	13
10	12	11
9	7	8
6	5	4
1	2	3

(20)

where the first two rows have the reversal $C > A > B, B > A > C$ assortment, the next two have $B > A > C, C > A > B$ and the last two have $A > B > C, C > B > A$. In other words, a kernel term is created by introducing canceling Reversal components. The same construction can be done with two canceling Condorcet terms.

The final step would be to find a pure Basic data set. We can prove that such data sets do not exist when imposing certain reasonable assumptions, but, as of this writing, we do not know whether this is true in general. We can, however, find conditions whereby the data set has no Basic terms; this is true if and only if the KW ranking is a complete tie.

Theorem 12 *The string of equalities* $|ABC| + |ACB| + \frac{1}{2}(|CAB| + |BAC|) = |BAC| + |BCA| + \frac{1}{2}(|ABC| + |CBA|) = |CAB| + |CBA| + \frac{1}{2}(|ACB| + |BCA|)$ *holds if and only if Kruskal-Wallis technique outputs* $A = B = C$*. This condition also holds if and only if the sums of the Eq. 1 columns are equal (with value* $\frac{n(3n+1)}{2}$*).*

Theorem 13 *The equalities* $|ABC| + |ACB| = |BAC| + |BCA| = |CAB| + |CBA| = \frac{n^3}{3}$ *hold if and only if the V test outputs A=B=C.*

6 Proofs

Proof of Theorem 4. This statement follows from Theorems 1 and 2, and the fact that only Reversal terms cause differences in $\mathcal{NP}_s$ tallies. On Reversal terms, all $\mathcal{NP}_s$ rankings for $s < \frac{1}{2}$ agree and are opposite the common $\mathcal{NP}_s$ ranking for $s > \frac{1}{2}$. Thus, if two $\mathcal{NP}_s$ rankings for s values on the same side of $\frac{1}{2}$ differ, the difference is caused by a Basic term.

Proof of Theorem 6. This theorem follows directly from Theorems 1 and 2.

Proof of Theorem 7. Determining $G(\mathbf{d})$ is an immediate computation. For decomposition, if $\mathbf{p}$ is a standard profile, and if $\hat{\mathbf{p}} = (a_B, b_B, a_R, b_R, c, k)$, meaning $\hat{\mathbf{p}} = a_B\tilde{B}_a + b_B\tilde{B}_B + a_R\tilde{R}_A + b_R\tilde{R}_B + cC + kK$, then (Saari 1999) $\hat{\mathbf{p}}^t = \mathcal{T}(\mathbf{p}^t)$ where

$$\mathcal{T} = \frac{1}{6}\begin{pmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{21}$$

The rest of the proof involves $\mathcal{T}(G(\mathbf{d}))$ matrix computations.

Proof of Theorem 12. The Kruskal-Wallis technique is uniquely identified by $s = \frac{1}{2}$, which means it assigns one point to the first place winner of each triples, $\frac{1}{2}$ to the second place, and zero to the third. Therefore $A = B = C$ if and only if the number of triples that A wins plus the number of triplets in which A is second place equal the number of triplets B wins plus the number of triplets in which B is second place equals the number of triplets C wins plus the number of triplets in which C is second. This is precisely $|ABC| + |ACB| + \frac{1}{2}(|CAB| + |BAC|) = |BAC| + |BCA| + \frac{1}{2}(|ABC| + |CBA|) = |CAB| + |CBA| + \frac{1}{2}(|ACB| + |BCA|)$. An alternative Kruskal-Wallis procedure is to add the Eq. 1 columns. The procedure has a tie if and only if these sums all agree. As the values from 1 to $3n$ sum to $\frac{3n(3n+1)}{2}$, the common value is $\frac{n(3n+1)}{2}$.

Proof of Theorem 13. The V test technique is identified with $s = 0$, which means it assigns one point to the winner of each triplet and zero points to the second and third place alternatives. For the V test to have $A = B = C$, the number of triplets A wins equals the number B wins equals the number C wins, or $|ABC| + |ACB| = |BAC| + |BCA| = |CAB| + |CBA|$. With n^3 triplets when each alternative has n observations, the triplets are divided into three equal parts, so $|ABC| + |ACB| = |BAC| + |BCA| = |CAB| + |CBA| = \frac{n^3}{3}$.

Proofs of Theorems 9, 10, 11. The "if and only if" assertions involving Eqs. 9 and 10, Eqs. 11, 12, 13, and 14, and Eqs. 16 and 17 follow directly from properties $G(\mathbf{d})$ must satisfy for $\mathbf{d}$ to have the designated properties. In practice, we found that one or the other of these conditions, depending on the data set, to be useful when examining data sets.

From a formal perspective, the existence assertion in each of these theorems is verified by finding an example. However, the way in which examples were found and theorems proved (independent of finding examples) was to verify the necessary and sufficient conditions of each of these theorems directly by proving that the $G(\mathbf{d})$ mapping admits the specified properties. The details for Theorems 9 and 11 are given next, details for Theorem 10 are similar and can be found in Bargagliotti (2007).

Lemma 1 *Let the $n \times 3$ data set $\mathbf{d}$ have distinct entries $(a_1, .., a_n, b_1, .., b_n, c_1 .., c_n)$, then $|AB| + |BA| = |BC| + |CB| = |AC| + |CA| = n^2$.*

Proof. Let AB be the set of all elements of the (a_i, b_j) form where $a_i > b_j$, so BA denotes the set of the form (a_i, b_j) where $a_i < b_j$. Thus $AB + BA$ is the set of all elements (a_i, b_j); by counting, there are exactly n^2 of these types of elements. This completes the proof.

This following Lemma highlights the importance of the pairwise relationship in the data, while providing an appropriate structure to capture the Reversal data behavior.

Lemma 2 *The equality* $|ABC| = |CBA|$ *holds if and only if* $|BA| + |CB| = n^2$ *if and only if* $|AB| + |BC| = n^2$.

Proof. By definition $|ABC|$ is the number of triplets of the form $a_i > b_j > c_k$. This number can be expressed in terms of pairwise relationship between A and B and between B and C. In other words, defining $Ab_jC = \{(a_i, b_j, c_k) | 1 \leq i, k \leq n : a_i > b_j > c_k\}$ (i.e. b_j is fixed), we have that $|ABC| = \sum_{j=1}^{n} |Ab_jC|$. By letting $Ab_j = \{(a_i, b_j) | 1 \leq i, j \leq n : a_i > b_j\}$ (i.e. b_j is fixed), this sum can be re-expressed as $\sum_{j=1}^{n} |Ab_j||b_jC|$.

To simplify this expression, notice that with n entries per column, the number of pairs of the $|Ab_j|$ form equals $(n - |b_jA|)$. The reason is that the pairs Ab_j consist of the A's (out of the n total number of A's) that are larger than b_j for a fixed j. Thus the remaining A's (out of n of them) must be smaller than b_j. Consequently, we have that $\sum_{j=1}^{n} |Ab_j||b_jC| = \sum_{j=1}^{n}(n - |b_jA|)(n - |Cb_j|)$.

By factoring, this sum equals $\sum_{j=1}^{n} n^2 - n|Cb_j| - n|b_jA| + |Cb_j||b_jA| = n^3 - n|BA| - n|CB| + [\sum_{j=1}^{n} |Cb_jA|] = n^3 - n|BA| - n|CB| + |CBA|$. In turn, we have that $n^3 - n|BA| - n|CB| + |CBA| = |CBA|$ if and only if $n^3 - n|BA| - n|CB| = 0$ and this only happens if and only if $|BA| + |CB| = n^2$. This completes the proof.

Lemma 3 *The equality* $|ABC| = |CBA|$ *holds if and only if* $|AB| = |CB|$ *if and only if* $|BA| = |BC|$.

Proof. From Lemma 1 we have that $|AB| + |BA| = n^2 = |CB| + |BC|$. According to Lemma 2, $|ABC| = |CBA|$ holds if and only if $|AB| + |BC| = n^2$ if and only if $n^2 = |CB| + |BA|$. So, if $|ABC| = |CBA|$, then the two equations $|AB| + |BA| = n^2$ and $|AB| + |BC| = n^2$ require $|BA| = |CB|$; similarly, $|AB| = |BC|$. Conversely, if $|AB| = |CB|$, then the $|AB| + |BA| = n^2$ expression becomes $|CB| + |BA| = n^2$. According to Lemma 2, this is true if and only if $|ABC|$=$|CBA|$ if and only if $|AB| + |BC| = n^2$. A similar argument holds if $|BA| = |BC|$.

Lemma 4 *The equalities* $|ABC| = |CBA|, |BCA| = |ACB|$, *and* $|CAB| = |BAC|$ *hold if and only if* $|AB| = |BA| = |CA| = |CB| = |BC| = |CB|$.

Proof. By Lemma 3, $|ABC| = |CBA|$ if and only if $|AB| = |CB|$ if and only if $|BA| = |BC|$. This implies that $|ACB| = |BCA|$ if and only if $|AC| = |BC|$ if and only if $|CA| = |CB|$ and $|CAB| = |BAC|$ if and only if $|CA| = |BA|$ if and only if $|AC| = |AB|$. Then $|ABC| = |CBA|, |BCA| = |ACB|$, and $|CAB| = |BAC|$ if and only if $|AC| = |BC|, |CA| = |CB|, |CAB| = |BAC|, |CA| = |BA|, |AC| = |AB|$, and $|CA| = |CB|$ which imply $|AB| = |BA| = |CA| = |CB| = |BC| = |CB|$.

Proof of Theorem 9. This follows from the definition of a kernel data set and Lemma 4. Theorem 11 also follows from the above lemmas.

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