Finessing a point; augmenting the core *

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Abstract The "finesse point" introduced here extends the notion of a core point; it can be viewed as an optimal location for a candidate in that it requires a minimal reaction to counter moves made by an opponent. The definition is motivated by the dynamics of positive and negative political campaigning, and it includes the "finagle point" (when defined) as a special case. It is used to define a "malicious point," which is an optimal location for a candidate to engage in "negative campaigning."

Introduction

A core point, which is a position that cannot be beaten with the specified voting rule, leads to valued concepts such as the "median voter theorem" that are used to understand elections and competitive interactions. But the core fails to exist for many natural settings, and this has inspired a search for extensions. Suggested notions include the uncovered set, yolk, and so forth. (Miller 2005 provides a nice, informative survey.)

The basic idea of the "finesse point," which we introduce here, is that since a core point cannot be beaten, the finesse point should minimize what it takes to keep from being beaten: it minimizes what is needed to disrupt any minimal winning coalition (i.e., a coalition with just enough voters to ensure victory) supporting an opponent. We then relate the finesse point to the uncovered set, yolk, etc. In terms of elections, the finesse point minimizes what a candidate must do to encourage some of her opponent's supporters to change their vote. Indeed, the development of the finesse point borrows from apparent election dynamics such as negative campaigning.

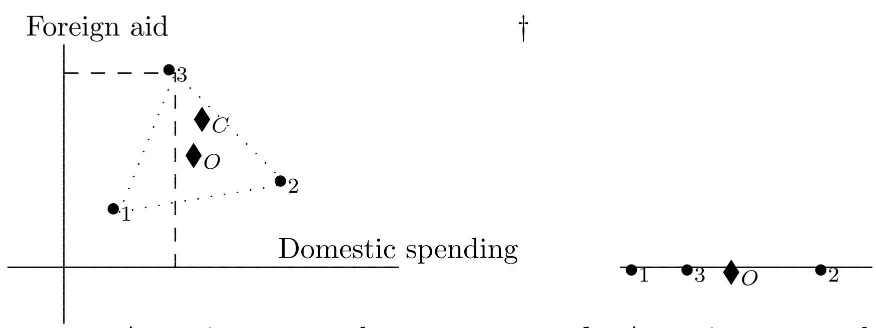
Our finesse point is influenced by a candidate's need to preserve a sense of credibility; e.g., changes in a candidate's position, or charges about her opponent, cannot vary wildly. A candidate, for instance, may wish to limit accusations leveled at an opponent to what is necessary to win. After all, if voters view negative campaigning as demonstrating malice toward an opponent, then being excessive could be counterproductive. To capture this sense, we define a "malicious point" as a stance from which charges cast by a candidate against her opponent can be minimal yet effective in that the opponent does not have a winning position. We then relate the finesse and malicious points. Similarly with "positive campaigning," to preserve credibility a candidate would try to keep relatively consistent positions over issues.

"Minimizing" is a standard objective in analyzing voting rules and solution concepts. For instance, even drastically different methods, such as the plurality vote, Borda Count, Kemeny, and Dodgson rules, can each be characterized as minimizing the differences between voter preferences and the election outcome. Differences, then, depend upon formal definitions and what is being

*Saari's research was supported in part by NSF grant DMI 0233798. We have benefitted from comments made by several people, including participants at an Institute for Mathematical Behavioral Sciences conference on Spatial Voting held at the University of California, Irvine, December 2005; e.g., here we learned about the "strong point" when G. Owen raised the question, which we answer, whether the "strong point" is the "finesse point."

minimized. Similarly, to minimize an opponent's impact in spatial voting, Owen and Shapley (1989) created the appealing notion of a *strong point*, which minimizes the volume of the winning set; i.e., all positions where an opponent can beat a candidate. Wuffle, Feld, Owen, and Grofman (1989) (WFOG in what follows) addressed the same question with their clever "finagle point," which uses a different property of the winning set with the objective of minimizing what it takes to win. But due to the complexity associated with these concepts, both approaches appear to be restricted to limited number of voters in highly symmetric settings. Our approach holds for any number of issues, voters and their positions, and with other voting rules.

With three voters, the finesse and finagle points agree because minimizing what it takes to avoid being beaten and to win turn out to be the same. With more candidates and issues, however, it normally requires a greater change to ensure that a candidate will win than to ensure she will not lose. With eight voters, for instance, a minimal majority vote winning coalition consists of five voters. To avoid being beaten, which is the finesse point, only one voter must be persuaded to change his mind; to ensure winning, the finagle point, two of them must change their voting allegence, which could involve more complicated and extensive changes in stance. Also, WFOG's approach is restricted to highly symmetric settings that include two issues and three voters with the majority vote; e.g., it is not clear how to extend their approach to more realistic settings involving many voters with general views and more issues, and their technical construction, which defines their finagle point, clouds intuition about its merits. While our finesse point uses a different definition, construction and even objective, when they agree, we like to think of the finesse point as extending at least the intent of the finagle point to any number of voters, issues, a wider class of voting rules, with a more transparent construction. We determine how much "finagling," finessing," and "malicousness" is required of a candidate, and we relate the finagle, malicious, and finesse points.



a. A two issue example **b.** A one issue example

Fig. 1. Voter and candidate positions

Basic ideas

We use the standard notions of spatial voting where the "issue space" is represented by an Euclidean space $\mathbb{R}^k$: each of the k coordinate axes represents a separate issue. The components of a voter's *ideal point*, $\mathbf{p} = (p_1, \dots, p_k)$, identify the voter's "ideal" or "most preferred" outcome over these issues. As illustrated in Fig. 1a, where the bullets designate voters' ideal points, the dashed lines show that voter three's ideal position is to spend a moderate amount on domestic issues and a considerable amount on foreign aid. While our results extend to where voter preferences are represented with a smooth, strictly convex utility functions, it suffices for our current purposes to assume that the closer a position is to a voter's ideal point, the more he likes it. These preferences are called "Euclidean;" as is standard, assume that no two ideal points agree.

A candidate's position is also represented by a point in issue space; they are given by the diamonds in Fig. 1a where the subscripts identify their names of "Candidate" and "Opponent." A voter prefers the candidate whose position is closer to his ideal point; e.g., in Fig. 1a, voter 1 prefers Opponent while voter 3 prefers Candidate. Traditionally a candidate's point is interpreted as her actual position. But because negative campaigning tries to change the voters' views of an opponent, it is convenient to treat a candidate's point as the voters' *perception* of her stance rather than her actual stance. As our finesse point supports either interpretation, the main purpose of this comment is to add flexibility in our discussion.

This "perception" interpretation is a realistic consequence of the ever present problem of incomplete information as well as the objectives of negative campaigning. An illustration is the 1988 presidential campaign where the "Americans for Bush" launched a "Willie Horton" ad. This ad, which featured a criminal who committed horrendous crimes while furloughed, was intended to create the perception that the Democratic candidate Dukakis was "soft on crime." For purposes of the 1988 election, Dukakis' actual stance was not as consequential as the voters' perception. As this example also illustrates, there can be multiple perceptions of a candidate's position; e.g., members of one political party may perceive a her stand quite differently than members from another party. We could incorporate such differences into our discussion, but we do not because it would detract from our emphasis of introducing the finesse point. At each instant of time, then, assume there is a unique perception of each candidate.

Independent of how a candidate's position point is interpreted, the role of campaigning, whether positive or negative, is to change voters' perception of some candidate. For purposes of this paper, *positive campaigning* is where a candidate's campaign tactics are directed toward changing her perception, or stance, with the voters. *Negative campaigning* is where her campaign tactics are directed toward changing the voters' perception of her opponent.

Our main contribution extends the game theoretic concept of a "core point." In terms of an election, a candidate's position is a core point if it cannot be defeated by *any* possible position taken by her opponent. To illustrate with the majority vote and a single issue, the core is at the median voter's ideal point; e.g., in Fig. 1b, this is voter three's ideal point. To appreciate this well-known "Median Voter Theorem" (Downs 1957, Hotelling 1929), notice that if Opponent's position is anywhere else, as indicated by the diamond in Fig. 1b, Candidate can win with a position closer to voter three's ideal point to attract the votes of voters one and three.

Obviously, a candidate would like to be positioned in the core. Conversely, if an opponent's position is at a core point, a losing candidate must change this status. If nobody is at a core, or if the core does not exist, then a trailing candidate presumably responds by identifying "weak spots" in an opponent's supporting coalition to encourage these voters to change their vote. To motivate the finesse point, we first show how a trailing candidate can react to *any* winning position of an opponent when a core does not exist, and how a winning candidate can make the campaign more difficult for her opponent. Then we identify what must be done when a core does exist; e.g., how a trailing candidate can move or destroy the core. This leads to the "selective core."

An empty core

Our results are for *q-voting rules* where a winning candidate must receive q, the *quota*, or more of the n votes where $\frac{n}{2} < q < n$. Let $\mathbb{C}(q, n)$ denote the q-rule core. For the simple majority vote, q is the first integer greater than $\frac{n}{2}$, while for the two-thirds rule q is the first integer greater than $\frac{2n}{3}$; q-rules often arise in politics (Nurmi 2002) and even to elect a pope (Saari 1997). We ignore the unanimity $q = n$ rule only because $\mathbb{C}(n, n)$ always exists. The dynamics allowed by q-rules can be troublingly chaotic.

To explain, McKelvey's (1979) so-called chaos result describes surprising consequences for an empty core. Namely, choose *any* two positions in issue space represented by an initial $\mathbf{p}_i$ and a final $\mathbf{p}_f$. If a core for the majority rule does not exist, then there exists a sequence of positions, $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m$ where $\mathbf{p}_1 = \mathbf{p}_i$ and $\mathbf{p}_m = \mathbf{p}_f$ and each $\mathbf{p}_j$ is preferred by a majority of the voters over $\mathbf{p}_{j-1}$. In words, with an empty core, it is possible to start anywhere and end up anywhere else via a carefully selected sequence of majority votes. Tataru (1996, 1999) extended McKelvey's seminal theorem to all q-rules: call this collection of results the "chaos theorem." Richards (1994) made this terminology appropriate by establishing a connection between the voting "chaos theorem" and "chaotic dynamics."

In simpler terms, the chaos theorem asserts that with an empty core and with sufficient time (m steps are needed where m may be a large integer) and skill in packaging successive positions, it is possible to lead the voters from any initial position to any desired final position. This conclusion holds even if *all* voters view the final position as more undesirable than an earlier proposal! With Fig. 1a, for instance, there is a sequence of positions starting at Opponent's position—the diamond near the center of the triangle—and, with a succession of majority vote victories, ending at the dagger located in the far upper right-hand corner of the figure. (For an election campaign, replace the majority vote "victories" with majority vote "opinions" about the candidates' positions.) Clearly, *all voters* in Fig. 1a prefer the original position to the final one.

While the chaos result guarantees that anything can happen, the finesse point is intended to counter this wild dynamic by introducing a centrally located position from which, with a candidate's thoughtful action, a sense of stability can be introduced. The chaos theorem ensures that voters can be led from one position to any other; the finesse point is a stance from which minimal modifications can respond to any proposed change.

To develop intuition to define the finesse point, a natural first question is to identify when a candidate can use positive and/or negative campaigning. As asserted next, under fairly general conditions either approach can *always* be successfully used at any moment, so the choice is determined by other factors. But if both strategies always are viable, we must anticipate a mixture of positive and negative campaigning, and this is what we observe. Incidentally, three-voter results of this type often are easy to prove, but the complexity increases with the number of voters because of the added number of winning coalitions. What makes the proof of Thm. 1 immediate is the chaos theorem; without it the proof would be difficult.

Theorem 1 *For n voters and any number of issues in a two candidate q-rule election, suppose that $\mathbb{C}(q, n)$ is empty. For any current position of an opponent who currently is winning, the losing candidate can adopt a winning position by using either positive or negative campaigning.*

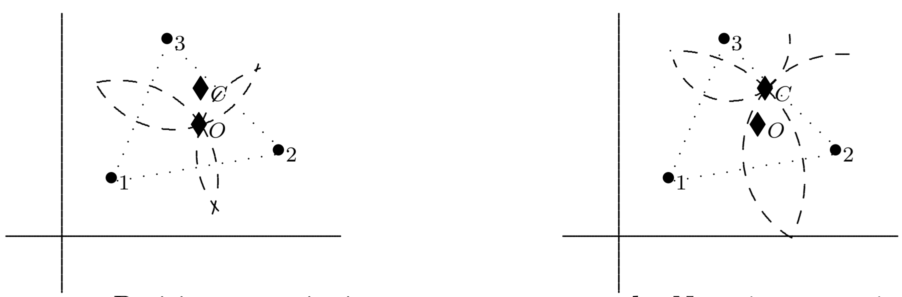
The chaos theorem ensures a sequence of positions $\mathbf{p}_i, \dots, \mathbf{p}_f$ starting at an initial $\mathbf{p}_i$, which we select to be the opponent's current position, and ending at a final $\mathbf{p}_f$, where each position is preferred by q of the voters over the preceding one. A candidate's positive winning strategy is to adopt a position that beats $\mathbf{p}_i$, the opponent's current position. The chaos theorem ensures that, for any q-rule, any number of voters, and any number of issues, such a position always exists. By changing the candidate's own perceived position, this strategy is positive campaigning.

To prove that a negative strategy always exists, choose the initial position $\mathbf{p}_i$ as the opponent's current position, and let the final position $\mathbf{p}_f$ be a specific candidate's current position. According to the chaos theorem, there exists a position that makes $\mathbf{p}_f$ a winning position. Consequently, the candidate's strategy is to campaign negatively to change her opponent's perceived stance to one that the candidate can beat.

An illustrating example

Before illustrating this theorem with the three-voter Fig. 1a preferences, notice that the *Pareto set* (positions that if changed in any manner will hurt some voter) consists of the points inside the convex hull defined by the voters' ideal points; i.e., the triangle defined by the ideal points. In Fig. 1a, both candidates have their positions inside the Pareto set while the dagger is not.

Using Euclidean preferences, voters one and two will vote for Opponent to make her the majority vote winner. All options for Candidate to react and gain a lead are given by Candidate's *winning set* (Shepsle and Weingast 1984). As depicted in Fig. 2a, draw a circle about each ideal point that passes through Opponent's position;[1] as all points inside this circle are closer to the voter's ideal point, he prefers them to Opponent's position. The trefoil defined by these circles is Candidate's *winning set* in that each point in a trefoil leaf is preferred by a majority of the voters over Opponent's position. The lower leaf, for instance, identifies all positions that voters one and two prefer to Opponent. As Candidate's position is not in the Fig. 2a trefoil, Opponent is winning, so Candidate's positive campaigning strategies must move her new stance inside the trefoil. Presumably the easiest approach (i.e., with the least change) is to select a position directly toward voter two's ideal point; this change moves Candidate's newly perceived stance into the leaf on the upper right-hand side and solidifies her support from the winning coalition of $\{2, 3\}$. So, relative to Opponent's position, $\{2, 3\}$ is the closest winning coalition for Candidate.



a. Positive campaigning **b.** Negative campaigning

Fig. 2. Strategic options for Candidate

To describe Candidate's choices in terms of Opponent's winning set depicted in Fig. 2b, pass circles through Candidate's perceived position: as Opponent is positioned in the trefoil's lower leaf, she currently is winning. Candidate could "redefine" her own position to drop Opponent's position out of the new winning set where the optimal strategy is to move directly toward voter two's ideal point (i.e., emphasize these issues); this change reduces the radius of the circle centered at voter two's ideal point and moves the winning set away from Opponent's position.

To illustrate Candidate's choices for negative campaigning with Opponent's winning set in Fig. 2b, her strategies must move Opponent's perceived position outside of this set. By using the closest leaf edge of this example, this objective is most easily accomplished by painting Opponent's views as being more to the left—directly *away from* voter two's ideal point.

Options for the winning candidate

The leading candidate, Opponent in Fig. 2, also has negative and positive campaigning opportunities, which motivates our "selective finesse point." By currently winning, Opponent's position

[1] This set, which denotes all positions that beat Opponent, often is denoted by $\mathcal{W}(O)$. With two candidates, it is more intutive to call this "Candidate's winning set" as it identifies positions where *Candidate* can win. When discussing legislative actions, the traditional notion is preferred.

is in her winning set depicted in Fig. 2b. A way to make Candidate's task more difficult is for Opponent to increase the size of her winning set relative to Candidate's position. As dictated by the geometry, a negative strategy requires altering Candidate's perceived position by increasing the radius defined by voter two's ideal point. This change increases the distance of Candidate's perceived position from the ideal points of Opponent's majority coalition ($\{1, 2\}$ in Fig. 1a). Thus this negative campaigning assumes the flavor of "Not representing the views of the majority consisting of voters one and two," or "Candidate's views are more representative of voter three than any other voter." Contrary to the impression occasionally suggested in the literature, negative campaigning *is* a viable option for a winning candidate, and it is observed in actual elections.

Options for positive campaigning are determined by Fig. 2a where Opponent should change her position to move Candidate's winning set farther from Candidate's current perceived position. According to the geometry, this objective requires shortening the distance from Opponent's position to the ideal points of voter's one and two (with a slight emphasis on voter two): she should change her position to better reflect the preferences of the members of her winning coalition. Using Fig. 2b, her positive strategy moves her position into a wider section in her winning set where she is better identified with her winning coalition. These are commonly observed strategies.

Costs of negative and positive campaigning

With an empty core and both candidate's positions being in the Fig. 2 Pareto set, a negative or a positive approach could change Candidate's fortunes. But which approach involves a smaller change in perception? The result follows.

Theorem 2 *With three voters and two candidates in a majority vote election with an empty core where voters have Euclidean preferences, if campaign costs are measured by the shortest distance to a winning position and if both candidates currently are in the Pareto set, then a positive and a negative strategy are equally expensive for the trailing candidate.*

In other words, when measured by the "distance" required to change voters' perceptions, neither positive nor negative campaigning has an advantage over the other. This equality adds practical significance to Thm. 2. After all, campaign costs (measured in terms of money, public opinion. etc.) per unit change in perception most surely differ for positive and negative strategies. As the *amount of needed change* is the same, the differing costs per unit change determine the least expensive way to attract supporting voters; e.g., if negative campaigning attracts press coverage or inexpensive publicity, then it is cheaper to campaign negatively.

The intuition behind the proof (Appendix) of Thm. 2, which is central for our definition of the finesse point, captures the objective of persuading "swing voters" to create a new minimal winning coalition. As illustrated in Fig. 2, voter two is the swing voter for both of Candidate's "nearest strategies" and voter three is the target for both of Opponent's strategies. In Fig. 2a, a positive campaign by Candidate must be directed toward voter two to encourage him to leave the winning $\{1, 2\}$ coalition and create the winning $\{2, 3\}$ coalition; in Fig. 2b, a negative campaign also is targeted toward voter two to destroy Opponent's winning $\{1, 2\}$ coalition. The amount of change for each strategy to reach a winning coalition is the shortest distance from the targeted perceived position to the leaf edge defined by the swing voter's ideal point. As the edges are circles centered at ideal points, the distances are the radii of the different circles. The proof shows that the nearest distance is the same for a positive or negative approach.

If Candidate successfully changes voters' perception, then Opponent must respond with a positive or negative campaigning strategy. The phrase "if Candidate successfully changes " suggests a candidate should immediately counter an opponent's ability to change this perception. In practice, this is accomplished by the "rapid response" teams; in our terms, it is captured by adjustments to the finesse point.

Finesse Points

A candidate wants to attract a winning coalition. As a winning coalition contains minimal ones, we emphasize these structures. To be precise, a *minimal winning coalition* for a q-rule consists of precisely q voters where the convex hull defined by their ideal points does not contain any other ideal points. By being "minimal," the "winning status" changes should even a single voter leave the coalition; as such, the analysis concentrates on individual preferences.

With any number of voters and issues, we want to find a position that minimizes the amount of change (for positive and/or negative campaigning) to *any* minimal winning coalition. Namely, this position should minimize the change needed to convert *some* member of *any* minimal winning coalition supporting an opponent to join a coalition that supports the candidate's new position. By moving a voter out of the opponent's minimal winning coalition, the coalition no longer is winning, so the candidate avoids being beaten. The many possible ways to do this depend on what a candidate knows about her opponent. If, for instance, her opponent must respect certain beliefs (e.g., she must adopt a specific position on social security or abortion), then certain coalitions are irrelevant while others gain in importance.

Start with the worse case scenario where a candidate has no prior knowledge about her opponent. (With legislative agenda models, this corresponds to having no prior knowledge about possible modifications for a specified plan.) With no information, the approach must involve all possible minimal winning coalitions. Then we describe what modifications occur with more information.

As suggested by Fig. 2, the worse case scenario for positive campaigning is if an opponent's position is precisely at the middle of the widest portion of the biggest leaf of the winning set: to entice some voter to join a different coalition, a candidate's response is half this width. This widest portion is where a leaf intersects the boundary of the Pareto set of a winning coalition. (The "Pareto set of a coalition" are those points where any change would cause some voter in this coalition to have a poorer outcome. With Euclidean preferences, it is the convex hull of the ideal points of a coalition; i.e., with two voters it is a line connecting the two ideal points.) By assuming this position, the opponent identifies herself with this particular coalition, so a candidate must woo at least one of these voters. Actual examples are easy to find; e.g., during the 1980 presidential election R. Reagan courted members of labor unions and the "Reagan Democrats."

Let d_C be the widest width of the winning set defined by the winning coalition C. In Fig. 2b with coalition $\{1,2\}$, $d_{\{1,2\}}$ is the length of the interval on the triangle's lower leg that also is in the lower leaf of the winning set. As the figure shows,

> the $d_{\{1,2\}}$ length is [the distance from 2's ideal point to Candidate's position] plus [the distance from 1's ideal point to Candidate's position] minus [the distance between 1's and 2's ideal points]; i.e., it is the sum of the two radii minus the edge length of the triangle.

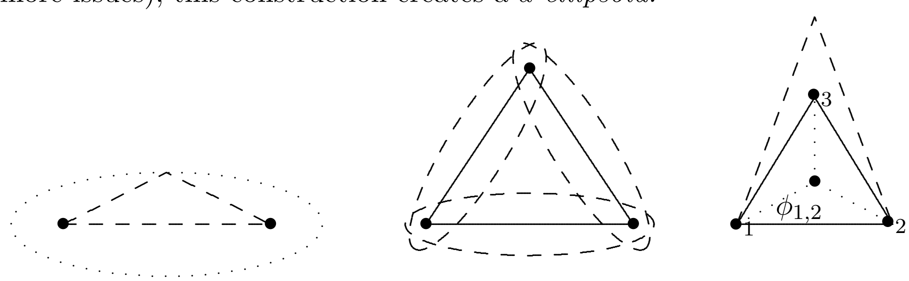
The $d_{\{2,3\}} < d_{\{1,3\}} < d_{\{1,2\}}$ values of Fig. 2 benefit Candidate if Opponent courts the $\{2,3\}$ coalition, but it would create problems if she courted the $\{1,2\}$ group. By assuming that a candidate has no information about an opponent, we must minimize the maximum d_C value *over all possible minimal winning coalitions.* This is accomplished with the finesse point.

A d-finesse point

To introduce a "d-finesse point" for three voters, recall that an ellipse can be created as indicated in Fig. 3a. About two pegs on a board place a string (given by the dashed line) tied in a loop with length that is larger than twice the distance between pegs. Next, put a pencil inside the string and trace it around creating the dotted curve: this curve is an ellipse. Call it a "d-ellipse" where $2d$ is the extra length of the string; i.e., $2d$ equals the length of the string minus twice the length between the pegs. Equivalently (because the part of the string connecting the pegs is taut),

> $2d$ is the sum of the lengths of the two dashed slanted lines (the two radii from the pegs to the point) minus the length between the pegs.

By comparing this definition with the description of d_C, it follows that all points on the d-ellipse defined by two ideal points have $d_C = 2d$ for this two-voter coalition. In higher dimensions (i.e., with more issues), this construction creates a *d-ellipsoid.*



a. Creating an ellipse **b.** Intersecting ellipses **c.** Moving the finesse point

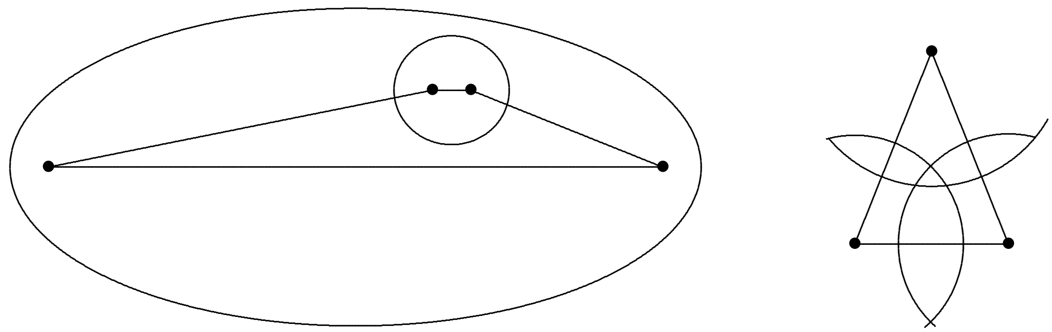
Fig. 3. Finding the d-finesse point

For the majority vote and three voters a minimal winning coalition consists of any two ideal points. About each coalition, construct a d-ellipse. For any two coalitions, these two d-ellipses intersect at a point in the Pareto set from which, with either coalition, the widest portion of an opponent's winning set leaf is $2d$. If this point is not in the d-ellipse for the third coalition, then the widest portion of the opponent's winning set for the third coalition exceeds $2d$. The "d-finesse point" is where all three ellipses intersect in a unique point: this point has the smallest d value over all winning coalitions.

To extend this construction to any number of voters and any specified q-rule, start with a minimal coalition C and let $E^2_d(C)$ be the union of C's Pareto set and all d-ellipsoids (interior and surface) defined by each pair of ideal points in C. (See Fig. 4a.) A point $\mathbf{p}$ on the boundary of $E^2_d(C)$, but not in C's Pareto set, is on the d-ellipsoid boundary for a pair of ideal points; the width of the lens defined by $\mathbf{p}$ and these ideal points is $2d$. For all other pairs of ideal points from C, where its d-ellipsoid does not include $\mathbf{p}$, the lens width from $\mathbf{p}$ is wider than $2d$. The winning set is in the intersection of all lens, so $2d$ is an upper bound on the width of the winning set. This means that if an opponent is in the winning set with C as a supporting coalition, then it is possible to appropriately alter $\mathbf{p}$ by no more than distance d to create a new proposal where C no longer is the opponent's winning set. The following definition mimics the ellipse construction.

Definition 1 *For two issues, a* q-rule d-finesse point *is a point that is in* $E^2_d(C)$ *for all minimal winning coalitions* C *where* d *is the smallest possible value where such a point exists. The* adjustment region*, denoted by* $\mathbb{S}^2_d(q)$*, is a ball centered about the d-finesse point with radius d.*

As shown in the Appendix, the $E^2_d(C)$ structures describe the winning set's width with two issues. But, it turns out that for more issues and $q \geq 3$, the $E^2_d(C)$ construction is too crude of an approximation for the winning set. In Fig. 4b, for instance, the winning set for a proposal $\mathbf{p}$ that happens to be above the triangular Pareto set for C (where $q = 3$) has the winning set of the small triangular shaped region in the interior: here, the width of any lens is much larger. In the worse case scenario, the opponent is positioned in the center of this triangular region (the center of the largest ball that fits in this region); her position is α-units from each of the three curves. We need to replace $E_d^2(C)$ with a geometric structure that captures this α value.



a. Two of the six $E_d^2(C)$ ellipses **b.** Winning set for thee issues

Fig. 4. Structure of sets

To review why ellipses suffice for $q = 2$, if r_j is the distance from the j^{th} ideal point to proposal $\mathbf{p}$, then the lens' width for two ideal points is $2d = r_i + r_j - z_{i,j}$. Thus if $\rho_j = r_j - d$, then $\rho_j + \rho_i = z_{i,j}$ and the circles with radii ρ_i and ρ_j meet in a unique point, which is the lens' center, on the $\{i, j\}$ Pareto set. For k issues and k ideal points that define a $(k-1)$-dimensional simplex, define *the k-fold d-ellipsoid* to be all points where r_j is the distance from the j^{th} ideal point and where the spheres of radius $\rho_j = r_j - d$ meet in a unique point in the convex hull of the k ideal points. The ρ_j's define the center point of the curvilinear region defined by the r_j's, so any point in this curvilinear region is at most d-distance from the nearest surface.

In Fig. 4b, for instance, point $\mathbf{p}$ that defines the arcs is directly above the simplex; a three-fold d-ellipsoid meets $\mathbf{p}$ with a smaller d-values (it equals α) than possible for any of the ellipsoids defined by pairs of points. After all, the three-fold d-value is determined by the distance directly above the center point of the curvilinear triangle, while, for an ellipsoid, part of the d value comes from getting the ellipse to the center point of the curvilinear triangle before moving above it. Incidentally, if, rather than finding the finesse point to avoid losing, we want a point to minimize what it takes to win, the argument involves extending E_C^2; this is because an opponent positioned in one of the wide leaves of Fig. 4b is not in the winning set, but she can block a candidate's victory.

Definition 2 *For a q rule, where C is a minimal winning coalition, let $E_d(C)$ be the union of C's Pareto set, the d-ellipsoid for each pair of ideal points in C, and the k-fold d-ellipsoid for each set of $k \geq 3$ ideal points in C that define a $(k-1)$ dimensional simplex. A* q-rule d-finesse point *is a point that is in $E_d(C)$ for all minimal winning coalitions C such that d is the smallest possible value where such a point exists. The* adjustment region, *denoted by $\mathbb{S}_d(q)$, is a ball centered about the d-finesse point with radius d.*

One of our main results follows (all proofs are in the Appendix).

Theorem 3 *For a q-rule with any finite number of voters and issues, the following are true.*

1. *The q-rule d-finesse point always exists.*
2. *A point is a q-rule d-finesse point for $d = 0$ if and only if it is a q-rule core point.*
3. *A q-rule d-finesse point is the q-rule malicious point. If q is the majority vote and there are an odd number of voters, the finesse point is the finagle point (when defined). The maximum*

amount of change in a position needed to respond to the opponent, with either positive or negative campaigning, is d; it is a change in the adjustment region $\mathbb{S}_d(q)$.

The d-finesse point achieves our goal by being a natural generalization of the core (part 2). Indeed, the d value measures how far coalitions are from defining a core point. To prove part 2, recall that the core is the intersection of the Pareto sets of all minimal winning coalitions. Thus, for all C, a core point is in $E_d(C)$ for all $d \geq 0$, so part of the conclusion follows. The other direction, asserting that a d-finesse point for $d = 0$ is a core point, follows because it requires the point to be in the Pareto sets of all minimal winning coalitions.

This connection with the core yields other conclusions. For instance, as a core point need not be unique, (Thm. 3) the d-finesse points, or the finagle and malicious points, need not be unique. A four-voter illustration is where all ideal points are on a line, so the core is the subinterval between the second and third voters ideal points. Points in this interval cannot beat others, but they cannot be beaten. Similarly when the finesse point is applied to q-rules, the change within distance d creates a setting that cannot be beaten by the opponent.

As part c proves, there are settings where the d-finesse point, finagle point (when defined), and malicious point all agree. At this common point the necessary change is no more than d for negative or positive campaigning, so it is in $\mathbb{S}_d(q)$. Thus Thm. 3 extends Thm. 2 to any number of voters, issues, and any q-rule. Because d measures the amount of change required to counter any action, d provides a measure of the inherent stability, or instability, of the system. Namely, "small" d-values suggest a system requiring only minor changes while "large" d-values suggest that extreme changes might be required to avoid the chaos theorem consequences.

The simpler geometry accompanying three voters helps develop intuition about how changes in voter preferences alter the position of the d-finesse point. Again let r_j be the distance from the j^{th} voter's ideal point to the finesse point and $z_{j,k}$ the distance between the j^{th} and k^{th} voters' ideal points. Changes in preferences change the shape of the triangle, or the $z_{j,k}$ values, and, presumably, the r_j values. An interesting result (Eq. 1) is that the change in, say, r_1 and r_3 is governed by the positioning of the *second voter's ideal point.*

Theorem 4 *Let the three voters be denoted by i, j, k. The position of the finesse point and the value of d satisfy the following expressions.*

$$r_j - r_k = z_{j,i} - z_{k,i}, \tag{1}$$

$$d = r_j - \frac{1}{2}[z_{j,k} + z_{j,i} - z_{i,k}] = \frac{1}{3}\sum_j r_j - \frac{1}{6}\sum_{j<k} z_{j,k}. \tag{2}$$

It follows from Eq. 1, for instance, that if the ideal points define an equilateral triangle (the solid lines of Fig. 3c) where $z_{1,2} = z_{2,3} = z_{1,3}$, then $r_1 = r_2 = r_2$ and the d-finesse point (the bullet) is at the center of the triangle. The d value is surprisingly small; e.g., elementary trigonometry proves that $r_1 = \frac{\sqrt{3}}{3} z_{1,2}$, and, by using Eq. 2, d is only about $0.0773 z_{1,2}$, or less than 8% of the common distance between ideal points.

Moving the top ideal point, voter 3, directly upwards (dashed lines in Fig. 3c) creates an isosceles triangle, where $z_{1,3} = z_{2,3}$, so $r_1 = r_2$ while $r_3 = r_1 + (z_{2,3} - z_{1,2})$. (From Eq. 1, differences in leg lengths transfer to differences in r_j lengths.) Direct trigonometric computations (Appendix) prove that $r_1 = r_2$ will increase; thus the finesse point moves upwards. But no matter how far the ideal point moves upwards, the inequality $\frac{z_{1,2}}{2} < r_1 < z_{1,2}$ always is satisfied; i.e., the finesse point remains near the shorter edge. Also, from, Eq. 2 we have that

$$d = \frac{1}{3}[3r_1 + (z_{1,3} - z_{1,2})] - \frac{1}{6}[3z_{1,2} + 2(z_{1,3} - z_{1,2})] = r_1 - \frac{1}{2} z_{1,2},$$

which, because $r_1 < z_{1,2}$, means that $d < \frac{z_{1,2}}{2}$, or less than half the shortest leg. (It is smaller.)

The r_j values and finesse point position[2] can be determined from Eq. 1, but the solution is sufficiently messy to block intuition. A more informative approach is to indicate how changes in the ideal points change the finesse point. To do so, let $\phi_{k,j}$ be the angle with vertex at the k^{th} voter's ideal point defined by the legs r_k and $z_{k,j}$; $\phi_{1,3}$ is depicted in Fig. 3c. (So, $\phi_{k,j}$ and $\phi_{j,k}$ are different angles.)

Theorem 5 *Assume that $z_{1,2} < z_{1,3} < z_{2,3}$; i.e., the two shorter legs of the triangle are attached to voter one's ideal point, and the two longer legs are attached to voter three's ideal points. Then*

$$r_1 < r_2 < r_3, \quad \textit{and} \quad \phi_{1,2} < \phi_{1,3}, \quad \phi_{2,1} < \phi_{2,3}, \quad \phi_{3,1} < \phi_{3,2}. \tag{3}$$

Thus the finesse point always is closest to the ideal point with the shorter legs and farthest from the ideal point with the longer legs; the point is closer to a coalition with more coherent views than one with separated views. (Indeed, r_j is smaller than the shorter leg connecting the j^{th} voter's ideal point and d is strictly less than half the length of the shortest leg. Should some $z_{j,k}$ leg lengths agree, appropriate r_j and $\phi_{j,k}$ inequalities become equalities.[3]) As $\phi_{j,k} + \phi_{j,i}$ defines the triangle's angle at vertex j, it follows from Eq. 3 that about each ideal point, the finesse point's position must be skewed toward the shorter leg.

$E_d(C)$ captures what change is needed to alter some voter's opinion in coalition C to create a different minimal winning coalition. But with more voters, the $E_d(C)$ structure has surprises because a d-ellipse for a longer $z_{j,k}$ is fatter. (This "fatness" explains the Thm. 5 consequences requiring the finesse point to be farther from the most distant ideal point, and the relatively small values of d for the finesse point.) Indeed, using the equation for an ellipse, it follows that

$$\text{the widest portion of a } d\text{-ellipse for } z_{j,k} \text{ is } 2d\sqrt{1 + \frac{z_{j,k}}{d}}. \tag{4}$$

For small $z_{j,k}$ values, which represents closely positioned ideal points, the ellipse resembles a circle where the widest portion is slightly larger than the circle's diameter, or $2d$. But if $z_{j,k} = 3d$, the width is $4d$, and if $z_{j,k} = 24d$, the width grows to $10d$. So, the d-ellipse of a pair of C points that are sufficiently far apart may be the $E_d(C)$ portion that defines the finesse point' this is true even if this pair is the most distant of all C points from the finesse point! Thus a finesse point will tend to be closer to a minimal winning coalition that is compact, which suggests cohesion in beliefs, than one where the ideal points are spread apart, which indicates diversity in beliefs.

This assertion is dramatically illustrated in Fig. 4b with the four-voter minimal coalition. (C is a minimal winning coalition for a $q = 4$ rule, so n could be $5, 6$, or 7.) With four ideal points, $E_d(C)$ has six ellipses; the two extreme ones are displayed. The d value is the distance between the two closest ideal points. While these two ideal points may be closer to the eventual position of the finesse point, the properties of $E_d(C)$, and the finesse point, can be governed by the d-ellipse defined by the two most separated ideal points. Which C points have the stronger impact on the choice of the finesse point depends on the positions of all other minimal winning coalitions.

Comparison with the yolk, uncovered set, etc.

For all examples we investigated, the finesse point and $\mathbb{S}_d(q)$ refine the yolk. Ferejohn, McKelvey, and Packel (1984) defined the *yolk* to be the sphere of minimal radius that intersects all median

[2]If $\mathbf{p}_j = (p_1^j, p_2^j)$ and $\mathbf{x} = (x_1, x_2)$, then $r_j = \sqrt{[\mathbf{x} - \mathbf{p}_j]^2} = \sqrt{(x_1 - p_1^j)^2 + (x_2 - p_2^j)^2}$. Substituting into any two of the Eq. 1 expressions leads to two equations in the two x_1, x_2 unknowns.

[3]Certain Eq. 3 inequalities reverse with changes in leg lengths; they are equalities when leg lengths agree.

lines. A median line extends the median point: it is a line (hyperplane in higher dimensions) with no more than half of the voters on either side. We extend the yolk to the *"q-rule yolk."*

Definition 3 *For $\frac{n}{2} < q < n$, a q-rule yolk, $\mathbb{Y}(q)$, is the circle (or sphere) of minimal radius that intersects all q-lines where a q-line is a line (or hyperplane) with less than q of the voters' ideal points on either side.*

Just as the median point divides the voters into two non-winning coalitions, a median line divides the space into two parts where no winning coalitions are strictly on either side. A winning coalition must include ideal points from the median line. As a core point must be on all median lines, the yolk is an extension of the core. Similarly, the q-hyperplane divides the space so that no q-vote winning coalition is strictly in either portion. Our next result identifies situations where the d-finesse point and its adjustment region $\mathbb{S}_d(q)$ sharply refine the yolk.

We expect, but have not proved, that $\mathbb{S}_d(q) \subset \mathbb{Y}(q)$ always holds. It is true, for instance, when points in a minimal winning coalition that define the largest distance always are on a limiting q-line (e.g., when all ideal points are on the boundary of the Pareto set), but a general result has eluded us in part due to the complexity of the yolk as demonstrated by Stone and Tovey (1992). (Also see Miller, 2005.) In any case, $\mathbb{S}_d(q)$ is a more stable indicator of the potential election behavior than $\mathbb{Y}(q)$ when adding or removing voters. For instance, with five voters located at the vertices of a square and the fifth at the center, both $\mathbb{Y}(3)$ and $\mathbb{S}_0(3)$ collapse to the center point, which is a core point. While $\mathbb{S}_0(3)$ remains the same should this last voter be removed, $\mathbb{Y}(3)$ now explodes to a ball touching all four edges of the square.

Miller(1980) introduced and McKelvey (1986) further developed the uncovered set for the majority vote. The extension to q-rules is immediate. Namely, position $\mathbf{p}$ is covered by $\mathbf{p}_1$ with the q-rule if $\mathbf{p}_1$ beats $\mathbf{p}$ and every point that beats $\mathbf{p}_1$ also beats $\mathbf{p}$. The *q-rule uncovered set* consists of all points that are not covered by any other point. As a covered proposal probably would not be a legislative outcome, the q-vote uncovered set identifies plausible q-vote legislative outcomes. Thus another favorable property of the q-rule d-finesse point is that it is in the q-rule uncovered set.

The strong point, introduced by Owen and Shapley (1989), minimizes the area of the win set. If θ_j is the angle (in radians), with vertex at the j^{th} voters ideal point and defined by the Pareto triangle, then modifying the usual πr^2 formula for area shows that $\sum_{j=1}^{3} \frac{\theta_j}{2}(\mathbf{x} - \mathbf{p}_j)^2$ is the sum of the portions of the areas of the three circles, with centers at the ideal points and passing through $\mathbf{x}$, that is in the Pareto set. The regions in the win set are double counted by being in two disks, so subtracting the area of the Pareto set yields half of the area of the win set: the total area is $2[\sum_{j=1}^{3} \frac{\theta_j}{2}(\mathbf{x} - \mathbf{p}_j)^2 - \text{Triangle area}]$. By minimizing and using $\sum_j \theta_j = \pi$, the strong point is at

$$\mathbf{x} = \sum_{j=1}^{3} \frac{\theta_j}{\pi} \mathbf{p}_j. \tag{5}$$

A similar argument extends to a $q = n - 1$ rule with $n - 1$ issues: the $(n - 1)$-rule strong point minimizes the volume of the winning set, so it is the minimizing value of $[\sum_{j=1}^{n} \theta_j(\mathbf{x} - \mathbf{p}_j)^{n-1}]$ where the θ_j's are the solid angles at each ideal point.

In the spirit of the strong point and for $n = 3$, it is reasonable to introduce the "minimal width point," which we define as the point that minimizes the sum of the widths of the three lens of a winning set: $\sum_{j<k} d_{j,k}$. As r_j is the distance from a point to the j^{th} ideal point, we have that $2d_{j,k} = r_j + r_k - z_{j,k}$, or $2\sum d_{j,k} = 2\sum r_j - \sum_{j<k} z_{j,k}$. Thus, the minimal width point is located at the point that minimizes $\sum r_j$, which is the sum of the distances from the vertices. This is known as the "Fermat point" (Courant and Robbins 1996).

Theorem 6 *For $n \geq 3$ voters, we have the following.*

1. *For $n = 3$, the finesse point agrees with the finagle point. With $n-1$ issues and n ideal points that define a $(n-1)$-dimensional simplex, $\mathbb{S}_d(q)$ is a proper subset of $\mathbb{Y}(q)$ and the center of the yolk is in $\mathbb{S}_d(q)$.*

2. *A q-rule finesse point generally is not at the center of $\mathbb{Y}(q)$. Indeed, for $n = 3$, the d-finesse point is at the center of $\mathbb{Y}(2)$ iff the ideal points define an equilateral triangle. In general, the strong point, minimal width point, yolk center, and the finesse point do not agree.*

3. *If $\frac{n}{2} < q_1 < q_2 < n$ and the q_1-core is empty, then either the q_2-core exists, or $\mathbb{Y}(q_2) \subsetneqq \mathbb{Y}(q_1)$. Similarly, if $d(q)$ is the d-value for a q-rule finesse point, then $d(q_2) \leq d(q_1)$.*

4. *A q-rule d-finesse point is a q-rule uncovered point.*

As part 1 shows, the adjustment set can be a (sharp) refinement of the yolk. The special case of three voters answers a WFOG's conjecture whether their finagle point is in the majority vote yolk; it is. (Part 1 can be significantly extended, but the proof provides a clean way to envision the d-finesse point construction.) The yolk's center, finesse, strong, and minimal width points tend to differ (part 2), so they measure different attributes of the ideal points.

As for part 3, an example where q_1-core, $\mathbb{C}(q_1, n)$, is empty but $\mathbf{C}(q_2, n)$ exists is Fig. 5b given below. As described below, $\mathbb{C}(3, 5)$ is empty, but $\mathbb{C}(4, 5)$ is voter five's ideal point. We know (e.g., Saari 1997) that if $q_1 < q_2$ and $\mathbb{C}(q_1, n)$ exists then $\mathbb{C}(q_1, n) \subset \mathbb{C}(q_2, n)$, i.e., increasing the quota increases the stability by having more stable (core) points. The $\mathbb{Y}(q_2) \subsetneqq \mathbb{Y}(q_1)$ and $d(q_2) \leq d(q_1)$ assertions has the same spirit; increasing the quota refines the yolk and reduces the adjustment level. While $\mathbb{S}_d(q_2) \subsetneqq \mathbb{S}_d(q_1)$ is true for simple settings we have not proved this in general.

To envision the $\mathbb{Y}(q_2) \subsetneqq \mathbb{Y}(q_1)$ result, consider four points that define an equilateral tetrahedron. If $q = 3$, $\mathbb{Y}(3)$ touches each tetrahedron face but none of the edges. But if $q = 2$, the yolk must touch each edge resulting in a larger ball. (While $q = 2 = \frac{n}{2}$ is not permitted, the geometry captures what happens with five voters in a four-dimensional space when comparing $\mathbb{Y}(3)$ with $\mathbb{Y}(4)$; intuition about four-dimensional objects is difficult to convey.)

Selective finesse points and a selective core

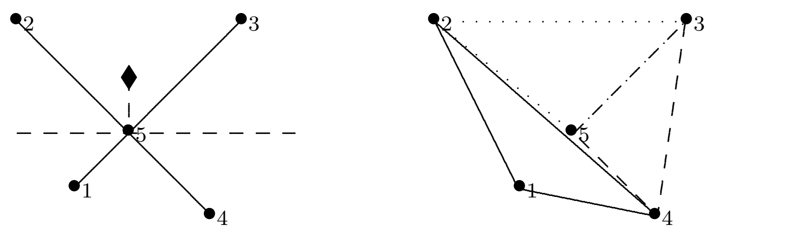
There are settings where certain minimal winning coalitions can be ignored; e.g., when political parties are polarized, certain coalitions are not politically relevant. Here, the d-finesse point is inappropriate as it unnecessarily guards against changes that an opponent will not, or cannot, take. This situation occurred with Fig. 2 when Opponent improved her chances to win by using information about Candidate to adopt a stance moving Candidate's winning set away from Candidate's position. In other words, the more we know about an opponent the better we know which winning coalitions are relevant. Rather than defining the d-finesse point over *all* minimal winning coalitions, the *selective d-finesse point is defined over a specified set of minimal winning coalitions.*

To motivate this definition and one for a *selective core point*, suppose Opponent is unbeatable because her position is in the core. As it is unlikely to replace Opponent's position with Candidate's, Candidate must change the status quo; i.e., she must try to destroy the core.

Plott (1967) appears to have been the first to recognize the delicate relationship between the existence of a majority vote core and the number of issues. Using what are now called "Plott diagrams,"he showed that in any dimensional space and for any number of voters, there are ideal points where a core exists. While Plott used the majority vote, his arguments extend to all q-rules.

But as Plott also observed, with enough issues and even the slightest change in a single voter's ideal point, the core could vanish.

To illustrate this changeable situation, in Fig. 5a the core is at the fifth voter's ideal point where the two lines cross. To verify this, select another proposal as depicted by the diamond. Draw a line from the new proposal to the fifth voter's ideal point, and draw the perpendicular line as illustrated by the dashed lines in Fig. 5a. A majority consisting of the fifth voter and the two below the horizontal dashed line prefers the fifth voter's ideal point to the new proposal. Figure 5b depicts what happens by moving voter two's ideal point slightly to the left. The connecting lines do not cross voter five's ideal point, so the core (with an odd number of voters) vanishes. To prove this, the Pareto set for winning coalition $\{2, 3, 5\}$ (given by the dotted lines) intersects that of winning coalition $\{3, 4, 5\}$ on the line connecting voters' 3 and 5 ideal points, thus, if a core exists, it is on this line. But as the Pareto set for coalition $\{2, 3, 4\}$ misses this line, the core is empty.



a. Core exists **b.** Empty core

Fig. 5. Plott diagram and sensitivity of the core

A message from Plott's argument is that while a core can exist for any q-rule in any dimensional issue space, its existence can be tenuous. The natural question is to determine the maximum number of issues that allows a q-rule core to persist even with small changes in voter preferences. Among others, contributions addressing this issue for general utility functions were made by Schofield (1983), McKelvey and Schofield (1986, 1987), and Banks (1995); the issue was finally resolved by Saari (1997). (An exposition is in Saari 2004.) The actual conclusion is more general than stated next, but a useful informal description (Saari 2004, p. 388) is that "the bound on k identifies the number of issues, one per voter, that need to be raised to change the election outcome." With a $q = 91$ rule where $n = 135$, in order for the $n - q = 44$ voters on the losing side to reverse the outcome, they must convince $91 - 44 = 47$ voters to change their minds. According to Saari's result, this q-rule allows cores to persist for k issues where $k \leq 47$.

For us, the interesting issue is that the core can be destroyed by introducing new campaign issues. This effect is depicted in Fig. 1 for the $q = 2, n = 3$ rule. As a single voter can change a $2 : 1$ majority vote outcome, we know that the core is stable only for a single dimensional issue space: this is the Fig. 1b setting. That this core can be destroyed by introducing another issue is illustrated with Fig. 1a where the Fig. 1b ideal points have the same x-direction spacing as in Fig. 1a, but the y direction differences in opinions destroys the previous core.

Introducing issues and the selective core

A candidate should introduce issues to place her new position either at the core in the newly defined issue space, or, if the core is empty, at a d-finesse point for a small d value; i.e., selecting a new issue is a strategic variable. To indicate how to change the location of an existing core to the advantage of a candidate, notice that the Fig. 1b core exists not because there is a single issue, but because the ideal points (for an odd number of voters) define a straight line. Therefore, if a new issue could rigidly raise the configuration of ideal points, the core persists with a raised location. For instance, suppose the two candidates are positioned at the Fig. 1b core point. If Candidate introduces an issue that lifts each voter's ideal point the same amount in the y-direction, then the ideal points still define a line so voter three's ideal point remains the core. If only Candidate's position also moved in this y-direction, then she solely occupies the core's new position.

For an illustrating example, recall Sen. Edwards campaigning approach during the 2004 presidential primaries. At a stage of the primaries, many of the candidates had similar positions clustered about the presumed core. What temporarily changed this status was when Edwards emphasized that he campaigned positively. The point is *not* that he campaigned positively, but that he converted his campaigning approach into an issue that enjoyed universal acceptance: this new issue "lifted" Edwards' stance above the others. An example that has the same effect with negative campaigning is where an opponent's character is impugned.

Generically, however, adding issues causes the core to shrink or vanish. But while introducing issues cannot be expected to position a candidate at the core, it may position her at a *selective core.* To define this term, notice that with Euclidean preferences, the *core* is the intersection of the Pareto sets of all winning coalitions. But if certain winning coalitions cannot be formed, maybe because of an opponent's position, then they are not of interest.

Definition 4 *A* selective core point *is one that cannot be beaten with specified winning coalitions. Similarly, with Euclidean preferences,* a selective d-finesse point *is a point that is in $E_d(C)$ for all specified minimal winning coalitions C and d is the smallest value allowing such a point to exist.*

With Euclidean preferences, the selective core is determined by the intersection of all specified winning coalitions. While the core is empty for Fig. 2a, if coalition {1, 3} could not be formed, then the "selective core point" is determined by coalitions {1, 2} and {2, 3}. The intersection of the Pareto sets for these coalitions is voter two's ideal point, so it is the *selective core point.*

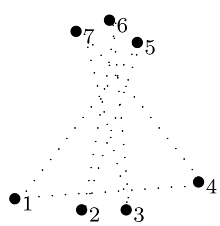


Fig. 6. Polarized setting

Adding an issue, maybe a divisive one, could separate old coalitions or create new ones to strategically replace the original core with a selective core or finesse point. This dynamic is depicted in Fig. 6 where, had there been one issue on the x-axis, the core would be voter six's ideal point with voters one and four widely separated and never together in a minimal winning coalition. But the new issue, depicted by the y-axis, is sufficiently divisive that $\{1, 2, 3, 4\}$ becomes a natural minimal winning coalition. The standard core does not exist, but with a sufficiently polarized setting, where $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$ are the only realistic coalitions, the Pareto set of $\{1, 2, 3, 4\}$ is the selective core. Even if the realistic minimal winning coalitions expand to involve these four voters and, say, voter five, a selective core exists and is near the median of the dominating majority party. But if the winning coalitions consist of $\{1, 2, 3, 4\}$, $\{2, 5, 6, 7\}$ and $\{3, 5, 6, 7\}$, then a selective core vanishes to be replaced by the selective d-finesse point; it is close to the median of the Pareto set of $\{1, 2, 3, 4\}$ with a smaller d value than the d-finesse point.

The following result is immediate.

Theorem 7 *For q-rule and a specified selection of minimal winning coalitions, with any number* *of voters and issues, the following are true.*

1. *The q-rule selective d-finesse point exists. In general, the d value for a selective d-finesse point is smaller than that for a d-finesse point.*

2. *A point is a q-rule selective d-finesse point for $d = 0$ if and only if the point is a selective q-rule core point.*

3. *A selective q-rule d-finesse point is the selective q-rule finagle point and the q-rule malicious point. The maximum amount of change in a position needed to respond to the opponent, with either positive or negative campaigning, is d.*

Of course, certain settings allow a "selective yolk" to be defined relative to specified winning coalitions. When this is possible, the earlier results relating the *d*-finesse point to the yolk extend.

Conclusions

It is informative to compare voting and general dynamics. Voting dynamics linked to a core, or a selective core, force each winning proposal closer to the core, much like "asymptotic stability" from physics where a particle continually moves toward an equilibrium. Physics also uses "orbital stability" where points may not be attracted to a stable point but they remain nearby. Our *d*-finesse point corresponds to a controlled system approach to orbital stability in voting.

The *d*-finesse point, which extends the core while refining the yolk, is a partial response to the negativity of the chaos theorem. The chaos theorem asserts that anything can happen, while the *d*-finesse point identifies a centrally located position that ensures a level of stability by exercising appropriate responses. While adding issues tends to generate instability, it is interesting how introducing appropriate issues can create a sense of stability by making certain winning coalitions ineffective. The associated stability is captured with the selective core and *d*-finesse points.

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Appendix

Proof of Thm. 2: With each candidate in the Pareto set, extending the line connecting a candidate's position with the nearest point on a winning set passes through the swing voter's ideal point. Let $r_{i,j}$ be the radius of the circle centered at voter j's ideal point and passing through candidate i's position. The circle geometry of Euclidean preferences requires that, for positive campaigning, $|r_{C,j} - r_{O,j}|$ is the shortest distance to move Candidate's position to the leaf edge of the circle defined by Opponent's position and voter j's ideal point. So the closest strategy is the smallest value of $\{|r_{C,1} - r_{O,1}|, |r_{C,2} - r_{O,2}|, |r_{C,3} - r_{O,3}|\}$. A similar argument for negative campaigning shows that $|r_{O,j} - r_{C,j}|$ is the shortest distance to move Opponent's position to the leaf edge defined by Candidate's position and voter j's ideal point. As the smallest distance for both possibilities uses the same set of numbers, the conclusion follows. This completes the proof.

Proof of Thm. 3. Part 1. With a finite number of voters, the Pareto set (defined by their ideal points) is bounded. Thus for a sufficiently large value of d and any winning coalition C, set $E_d(C)$ includes the Pareto set. For this d value, all $E_d(C)$ sets intersect. It now follows from standard analysis and the compactness of the $E_d(C)$ sets that a minimal d value can be found where all $E_d(C)$ sets still meet. Such an intersection point is a d-finesse point.

Part 2. Point $\mathbf{p}$ is a core point iff $\mathbf{p}$ is in each winning coalition's Pareto Set iff $\mathbf{p} \in E_0(C)$ for each minimal winning coalition C iff $\mathbf{p}$ is a d-finesse point for $d = 0$.

Part 3. At a d-finesse point, the largest possible distance from an opponent's position to the boundary of the winning set is d. That boundary is created by a circle (or sphere) passing through a candidate's current position with center at a voter's ideal point. By moving a candidate's position d units directly toward that voter's ideal point, the new circle, defining the new winning set, would pass precisely through the opponent's position. According to the definition of a d-finesse point, if a candidate's position differs from a d-finesse point, an opponent could find position requiring the candidate to change by more than d units. Thus, a d-finesse point is a finagle point.

Similarly, with negative campaigning, a candidate must move the perceived position of an opponent outside of the winning set. If a candidate is located at the d-finesse point, then the largest possible distance is d—the largest possible distance in a winning set. However, if a candidate's position is not at the d-finesse point, then, from the definition of the d-finesse point, an opponent can find positions that would require a larger change in her position to move out of the winning set. Thus a d-finesse point is a malicious point.

Proof of Thm. 4. By construction, at the finesse point we have $r_i + r_k = z_{i,k} + 2d$ and $r_i + r_j = z_{i,j} + 2d$. Subtracting the first from the second yields Eq. 1. To derive Eq. 2, adding $r_1 + r_2 = z_{1,2} + 2d$ to the Eq. 1 $r_1 - r_2 = z_{1,2} - z_{1,3}$ leads to $2r_1 = z_{1,2} + z_{1,3} - z_{2,3} + 2d$. Solving for d and expressing the equation in a general form yields $d = r_j - \frac{1}{2}[z_{j,k} + z_{j,i} - z_{k,i}]$; the first part of Eq. 2. Adding the three equation gives $3d = \sum r_j - \frac{1}{2}\sum_{j<k} z_{j,k}$, or the second part of Eq. 2.

Proof of Thm. 5. The leg length inequalities of Eq. 3 follow from Eq. 1. From the law of cosines, $\cos(\phi_{j,k}) = \frac{r_j^2 + z_{j,k}^2 - r_k^2}{2r_j z_{j,k}}$. As larger cosine values correspond to smaller angles, the Eq. 3 angle inequalities follow from the inequalities on leg lengths and the r_j values.

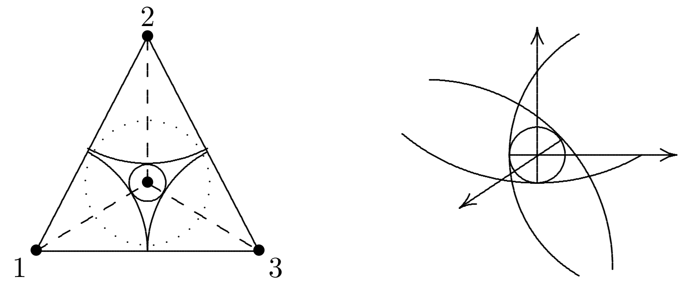
Proof of Eq. 4. An ellipse with both foci (the nails in the description) on the x-axis, z distance apart, and $x = 0$ in the center, has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for some $a > b > 0$ values. When $y = 0$, the d-ellipse is at point $x = \frac{z}{2} + d$; this is the a value. When $x = 0$, the string defining a d-ellipse forms an isosceles triangle, where the height above the $x = 0$ point is

$$\sqrt{(\frac{z}{2} + d)^2 - (\frac{z}{2})^2} = \sqrt{zd + d^2} = d\sqrt{1 + \frac{z}{d}};$$

this is the b value. The width of $2b$ yields Eq. 4. Other properties follow from the d-ellipse's equation, $\frac{x^2}{((z/2)+d)^2} + \frac{y^2}{zd+d^2} = 1$. For instance, a circle of diameter $2d$ with center anywhere between the two ideal points is in the d-ellipse; e.g., the ellipse width at an ideal point is $2d[2 - \frac{1}{1+\frac{z}{2d}}] > 2d$.

Proof of Thm. 6. (Part 1) To show for $n = 3$ that the finagle and finesse points agree, we outline WFOG's construction of the finagle point. Find three circles (Fig. 7a) where each is centered at a voter's ideal point and tangent to the other two. Such circles exist and are uniquely determined. (As such tangencies do *not* generally exist, this limits WFOG's approach.) In the curvilinear triangle defined by the three circles, find the largest inscribed circle: the finagle point is the circle's center.

To prove that this point is our majority vote d-finesse point, let r be the radius of the circle centered at the finagle point. The dashed lines connecting any two ideal points is the distance between the two ideal points plus $2r$, so it is on a r-ellipse. As this is true for all pairs of ideal points, the finagle point is a d-finesse point where $r = d$. That this is the minimum d value follows from the fact that in the Pareto set, the three d-ellipses have a unique point of intersection. While the finagle point is our $d = r$ finesse point for $n = 3$, our construction does not rely on this special Fig. 7a behavior, which holds primarily for triangles and special symmetric figures: this permits our d-finesse point to hold for any number of voters, issues, and q-rules.



a. WFOG's finagle point **b.** $E_d^2(C)$ suffices for two issues

Fig. 7. The yolk vs. the finesse point

To show for $n = 3$ that $\mathbb{S}_d(2)$ is inside $\mathbb{Y}(2)$, notice that $\mathbb{Y}(2)$ touches all three edges of the triangular Pareto Set. (In Fig. 7a, $\mathbb{Y}(2)$ is given by the dotted circle.) Each ideal point is on two edges and $\mathbb{Y}(2)$ is tangent to each triangle edge, so the two distances from an ideal point to the two tangent points are equal. If ρ_j is the common distance from ideal point i, then $\rho_1 + \rho_2 = z_{1,2}$, $\rho_1 + \rho_3 = z_{1,3}$, $\rho_2 + \rho_3 = z_{2,3}$. The locations of the tangent points follow from these three equations with three unknowns. These tangent points also define the three Fig. 7a circles, so $\mathbb{S}_d(2)$ is in the $\mathbb{Y}(2)$ interior. Moreover, the lines from the $\mathbb{Y}(2)$ center to these tangent points are perpendicular to the edges and tangent to the two ρ_j circles. As these perpendicular lines cross at the center of the yolk and also intersect (by construction) $\mathbb{S}_d(2)$, the yolk's center is in $\mathbb{S}_d(2)$.

To show that $\mathbb{S}_d(n-1) \subset \mathbb{Y}(n-1)$ with $(n-1)$ issues where the n-ideal points define a nondegenerate n-simplex, note that $\mathbb{Y}(n-1)$ is tangent to the n planes defined by $(n-1)$ ideal points; each ideal point is in $n-1$ hyperplanes. As $\mathbb{Y}(n-1)$ is a sphere, the line in each hyperplane from the j^{th} ideal point to the $\mathbb{Y}(n-1)$ tangent point has the same length ρ_j. Thus, in any hyperplane, the ρ_j values define a unique point—the $\mathbb{Y}(n-1)$ tangent point. About the j^{th} ideal point, construct a sphere of radius ρ_j. As there are precisely n of these spheres, a unique sphere can be constructed in Pareto set that is tangent to each of n constructed spheres. This is $\mathbb{S}_d(n-1)$, by construction it is in $\mathbb{Y}(n-1)$. Also by construction, the line perpendicular to each hyperplane meets the center of $\mathbb{Y}(n-1)$ and is tangent to each of the $(n-1)$ constructed spheres, so it is in $\mathbb{S}_d(n-1)$. As this is true for all such lines, the yolk center is in the adjustment set.

The proof for $\mathbb{S}_d(q)$ and $\mathbb{Y}(q)$, $\frac{n}{2} < q < n$, involves only minor changes; e.g., each ideal point is in $\binom{n-1}{q-1}$ minimal coalitions and each coalition's Pareto set has dimension $q-1$. At least $n-1$ of them are tangent to $\mathbb{Y}(q)$, while others may intersect this yolk. The construction using these tangent points and spaces is the same as for $q = (n-1)$ except instead of $(n-1)$ choices of ρ_j in each $(q-1)$-dimensional hyperplane, there are q of them, and the orthogonal line from each of these hyperplanes is replaced with a $(n-1)-(q-1)$ dimensional orthogonal hyperplane.

(Part 2.) To prove for $n = 3$ that the d-finesse point is at the $\mathbb{Y}(2)$ center iff the Pareto set is an equilateral triangle, notice that the $\mathbb{Y}(2)$ center is at the intersection of the angle bisectors of the Pareto set's angles. Thus, according to Eq. 3, the d-finesse point is at the $\mathbb{Y}(2)$ center iff $\phi_{j,k} = \phi_{j,i}$ for all j, which (Thm. 5) requires all leg lengths of the Pareto set to agree—it is an equilateral triangle. As the minimal width, strong, and finesse point are based on expressions that are analytic in the location of the ideal points, if they do not agree at one location, they do not agree in general. Thus, we only need show they disagree at a point.

If the Pareto triangle's largest angle exceeds 120^o, the strong and finesse point are in the interior but the Fermat (minimal width) point is at the largest angle's vertex. To show that the strong point can disagree with the finesse point, consider the equilateral triangle with ideal points, 1, 2, 3, respectively at $(0,0), (2,0)$, and $(1, \sqrt{3})$; the strong and finesse points are at the triangle's center where $r_j = 2\sqrt{3}/3$. Now move point 3 to $(1, \sqrt{3}+u)$; we must show that the points change with different rates at $u = 0$. Here, $z_{1,3} = z_{2,3} = \sqrt{1 + (\sqrt{3} + u)^2}$ while $z_{1,2} = 2$. The finesse point is at $(1, \sqrt{r_1^2 - 1})$ where (triangle height) $\sqrt{r_1^2 - 1} + r_3 = \sqrt{3} + u$, or $\sqrt{r_1^2 - 1} + r_1 + z_{1,3} - 2 = \sqrt{3} + u$ (as (Eq. 1) $r_1 - r_3 = z_{1,2} - z_{2,3} = 2 - z_{1,3}$). By implicit differentiation of the height equation, $\frac{dr_1}{du} = r_1' = \frac{(1 - z_{1,2}')\sqrt{r_1^2 - 1}}{r_1 + \sqrt{r_1^2 - 1}}$, so at $u = 0$ the y' value is $\frac{r_1 r_1'}{\sqrt{r_1^2 - 1}}|_{u=0} = (2 - \sqrt{3})/6$. The strong point (Eq. 5) is at $\frac{\theta_2}{\pi}(2, 0) + \frac{\theta_3}{\pi}(1, \sqrt{3} + u)$ where (law of cosines) $\theta_2 = \arccos(\frac{1}{z_{1,3}})$ and $\theta_3 = \arccos(1 - \frac{2}{z_{1,3}^2})$. Thus y' (at $u = 0$) is $(2\pi - 3\sqrt{3})/6\pi$: the derivatives disagree.

To show that the q-rule finesse point generally is not the $\mathbf{Y}(q)$ center, with k-issues (where k may be smaller than the number of issues) and by placing ideal points on top of one another if necessary, the Pareto set can be made into a k-dimensional simplex with the q-hyperplanes the faces of the simplex. With the above construction, construct the finesse point with the tangent points of the yolk on the hyperplanes. If the simplex is not equilateral, some radii are longer than others, so the inscribed sphere cannot have its center at the $\mathbb{Y}(q)$ center.

(Part 3.) For $\frac{n}{2} < q_1 < q_2$, a q_1-line has less than q_1 voter ideal points on either side, so it also is a q_2-line; i.e., $\mathbb{Y}(q_2) \subseteqq \mathbb{Y}(q_1)$. The yolks need not agree because a q_2-line need not be a q_1-line. Similarly, to show that $d(q_2) \leq d(q_1)$, notice that any minimal winning coalition for a q_1 rule, C_1, is contained in some minimal winning coalition for a q_2 rule, say C_2. Thus $E_d(C_1) \subset E_d(C_2)$, so the $E_d(C)$ sets for the q_2 rule cannot meet at a larger d value than those for the q_1 rule.

(Part 4.) To show that a q-rule finesse point $\mathbf{p}$ is a q-rule uncovered point, suppose $\mathbf{p}_1$ beats $\mathbf{p}$; i.e., $\mathbf{p}_1$ is in the winning set defined by $\mathbf{p}$. But as $\mathbf{p}_1$ is not a finesse point, it follows that the winning set it defines for some coalition is larger than that defined by $\mathbf{p}$. Consequently, there are points that beat $\mathbf{p}_1$ that do not beat $\mathbf{p}$, or $\mathbf{p}$ is an uncovered point.

Width of winning set. In defining the finesse point, we claimed that $E_d^2(C)$ suffices for two issues. First, place the largest possible ball in the winning set (Fig. 7b). The ball is tangent to circles about ideal points, so a vector orthogonal to the ball is orthogonal to the circle: it points toward the center of the ball and the particular ideal point. If the ball is not at the center of a lens defined by two circles, then some circle on one side holds it above the lens' center. The two orthogonal vectors for the lens have components toward the lens' center, while the third one points away. Thus, with two dimensions, the convex hull of the three ideal points must include the ball. As the ideal points are in coalition C, for the winning set to have a ball, the point $\mathbf{p}$ must be in a third dimension; i.e., there are more than two issues. Thus the winning set's width for two issues is given by the lens width of two circles. The first ellipse from $E_d^2(C)$ that meets $\mathbf{p}$ captures the smallest width of lens defined by $\mathbf{p}$, so the d value captures the winning set's width.

A similar argument shows that with k issues, the largest ball in a winning set is bounded by at most k circles; the width of this structure is captured by the k-fold d-ellipsoid. Some $\mathbf{p}$ choices may allow the lens of j spheres where $j < k$, to define the winning set's width. To handle such situations, $E_d(C)$ is defined as the union of the $E_d^j(C)$ sets for $j = 2, \ldots, k$.

Proof of Thm. 7. The proofs follow that of the earlier theorems but over a subset of the winning coalitions. Thus, for instance, the minimum values for d are equal or smaller.