Sen's Theorem: Geometric Proof and New Interpretations

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Abstract

Sen's classic social choice result reportedly demonstrates a conflict between standard welfare concepts from economics and liberalism—even forms of liberalism that are very minimal. By providing what appears to be the first direct general proof of this seminal result, we reinforce a very different interpretation: Sen's result occurs because one of his assumptions negates the effects of another. Our proof leads to other interpretations of Sen's conclusion, which identify other kinds of societal conflicts and describes the problems Sen identified in terms of probability. By clarifying the central source of Sen's result, our proof suggests new and practical ways to sidestep these difficulties. But a general resolution remains to be found. [1] [2]

Problems central to decision and social choice theory were aptly characterized with Amartya Sen's comment (in his 1998 Nobel Prize lecture, also see Sen (1999) that

> a camel is a horse designed by a committee [because] a committee that tries to reflect the diverse wishes of its different members in designing a horse could very easily end up with something far less congruous, half a horse and half something else—a mercurial creation combining savagery with confusion.

Expanding on his quote, it should be a basic objective in choice theory to discover societal decision rules that avoid creating camels when horses are intended. But, is this possible? Sen, a leader in identifying subtle but important barriers that prevent achieving this objective, discovered a fundamental difficulty in his 1970 "Impossibility of a Paretian liberal" (Sen, 1970a, b, 1976c). Sen's theorem, which demonstrates that it is impossible to satisfy even a surprisingly minimal aspect of liberalism when combined with the Pareto condition, appears to describe a fundamental conflict between standard welfare concepts and liberalism.

One reason Sen's Theorem has attracted so much attention (as indicated by the extensive literature spawned by his result) is that his result captures central concerns that arise across

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disciplines. We demonstrate this universality with an example as pragmatic and basic as hiring a new employee.

A traditional way to analyze an impossibility result from choice theory, such as Sen's Theorem, is to modify the basic assumptions until a positive conclusion emerges. But our long term objective is to understand what kinds of new assumptions ensure positive conclusions, so we need to go beyond knowing that assumptions can create a conflict to determine *why* they conflict. Namely, we need to understand why Sen's conditions cause his seminal but negative conclusion.

Saari (1998, 2001) provides a surprisingly simple explanation: Sen's conclusion arises strictly because one of his assumptions (minimal liberalism) negates the effect of another crucial assumption (transitivity of voter preferences). By using this explanation, Saari and Petron (2004) found the unexpected result that *all possible* examples illustrating Sen's negative conclusions require *everyone* to be in strong conflict with someone else. This observation makes it arguable that rather than addressing individual liberties, there are settings where Sen's result models a dysfunctional society.

This current paper identifies settings where Sen's conditions appear to address individual liberties and where they appear to model a contentious society. To do this, we provide what appears to be the first general and direct proof of Sen's assertion.[3] In this way, we can explain how and why Sen's minimal liberalism negates the crucial assumption that individuals have transitive preferences. Even more; as our proof catalogues *all possible profiles* that support a particular Sen outcome, it leads to a "probabilistic" interpretation of Sen's conclusion. Finally, in keeping with our goal to obtain positive assertions, we identify certain subtle features that must be part of any successful approach to sidestep the fundamental problems that Sen raises.

0.1 Sen's Theorem

To suggest why the problems Sen identified have implications even in our economic society, consider a "Human Resource" (HR) story where a company plans to hire one of the candidates Amy, Bill, and Cindy. After the candidates are ranked by specified evaluators, the HR supervisor uses these rankings to determine the HR ranking. As part of the evaluation process, the candidates take a number of tests each based on 100 points. The relevant information, which is of the kind one might find on the HR sheet, follows:

Candidate	**IQ**	**Finance test**	**Personal Skills Test**	**Education**
Amy	120	80	60	BA
Bill	110	70	95	MA
Cindy	100	100	80	PhD

(1)

[3] All proofs of Sen's result that we have seen are based on creating examples, so our proof appears to be the first direct general verification of this result.

The rules for assembling the HR ranking are natural:

- **Unrestricted Domain.** Each evaluator can rank the candidates in any desired manner as long as the ranking is complete and transitive.
- **Pareto.** If all evaluators rank a particular pair in the same manner, this common ranking will be the HR ranking.
- **Minimal Liberalism (ML)—or Division of Expertise.** Evaluators are selected because of their expertise. In particular, suppose Garrett is an expert in finance—an area in which both Amy and Bill claim ability. It is natural to rely on Garrett's talents by asserting that the way Garrett ranks Amy and Bill will be their HR ranking. Similarly, Sandy is an expert in managerial and interpersonal practices where both Bill and Cindy claim abilities: Sandy's ranking of Bill and Cindy determines their HR ranking. (ML is defined in Thm. 1.)
- **No Cycles.** To make a decision, the HR ranking must not contain cycles.

By identifying our HR example with Sen's Theorem, it follows that with three or more candidates and two or more evaluators, *no ranking rule will always satisfy these four conditions.*

Theorem 1 *(Sen, 1970), Assume that each voter has a complete, binary, transitive preference ranking with no restrictions on the preferences. There does not exist a decision rule that ranks the alternatives and always satisfies the following conditions:*

1. *(Minimal Liberalism) There are at least two agents each of whom is decisive over at least one assigned pair of alternatives. Their ranking of the assigned pairs of alternatives determine the societal ranking of the pairs.*
2. *(Weak Pareto) If for any pair of alternatives, all voters rank the pair in the same manner, then this unanimous ranking is the societal ranking of the pair.*
3. *The outcome does not have any cycles.*

Sen's assertion mandates that the HR example admits situations where it is impossible to satisfy the specified requirements. To illustrate, according to Table 1, Garrett prefers Amy over Bill because of their relative performances on the finance test. His one disappointment is that Cindy, who had the best grade on the exam, is not interested in this area. Garrett's evaluation ranking is $C \succ A \succ B$. Sandy, on the other hand, is impressed with Bill's superior personal skills, so she ranks Bill above Cindy. As Sandy does not have a positive opinion of Amy, Sandy's evaluation ranking is $B \succ C \succ A$.

Using a dash to represent settings where information from an evaluator is irrelevant (because the HR decision is determined by another evaluator), the information the HR administrator uses to assemble the HR ranking follows:

Evaluator	**Ranking**	$\{A, B\}$	$\{B, C\}$	$\{A, C\}$
Garrett	$C \succ A \succ B$	$A \succ B$	$-$	$C \succ A$
Sandy	$B \succ C \succ A$	$-$	$B \succ C$	$C \succ A$
	HR Ranking	$A \succ B$	$B \succ C$	$C \succ A$

(2)

Sen's assertion is demonstrated by the cyclic HR outcome. Obvious modifications of our example also hold; e.g., each evaluator could be replaced with a team of evaluators, etc.

0.2 Outline

Sen (1970a, b) asserts that his result demonstrates the inconsistency of ML with the Pareto condition. Saari (1998, 2001) claims that very different explanations for this seminal result follow by observing that ML makes it impossible for the decision rule to determine whether the voters have transitive or cyclic preferences. This explanation pinpoints ML as the main culprit; indeed, the Pareto condition plays such a minimal role that it could be ignored or replaced with many other choices and the same difficulties can arise. Rather than the traditional interpretation of a conflict between ML and Pareto, then, the source of the problem is that the rule ignores the individual rationality of voters. These comments are supported by our geometric proof (Sects. 2, 4) of Sen's Theorem.

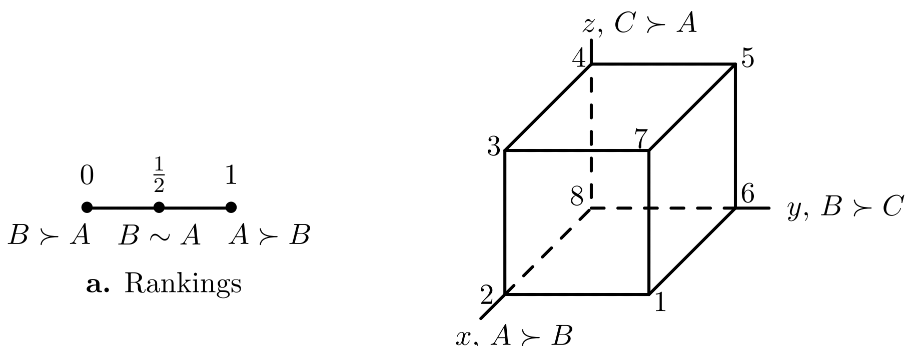
What makes this "cyclic voters" comment surprising is that Sen explicitly requires the agents to have *transitive preferences*. To explain what happens by use of an analogy, notice that the explicitly stated information in the "Education" and "IQ" columns of Table 1 play no role in determining the HR outcome. Consequently, this irrelevant information could be safely removed without affecting the analysis. Applying the same standards, we could safely remove the assumption of "transitive preferences" from the statement of Sen's Theorem as it plays no role in determining the outcome. To prove this statement we consider all possible profiles whether transitive or not. (In doing so, a ranking such as $A \succ B, B \succ C, A \succ C$ can be treated as a transitive ranking, or as a series of binary rankings that accidently satisfy transitivity.) By doing so, the assumption that voters have transitive preferences becomes a restriction of the decision rule mapping to a specific subset of profiles. The analysis, then, requires determining whether and how this restriction plays a role in determining the societal outcome.

Indeed, it is easy to show that a decision rule satisfying Sen's assumptions does permit the decisive agents, or the evaluators in the HR example, to have cyclic preferences. To see this, notice that all of the information used by the above rule to assemble the final HR ranking is

Evaluator	$\{A,B\}$	$\{B,C\}$	$\{A,C\}$
Garrett	$A \succ B$	–	$C \succ A$
Sandy	–	$B \succ C$	$C \succ A$
HR Ranking	$A \succ B$	$B \succ C$	$C \succ A$

(3)

This table underscores the fact that each evaluator's *full* ranking of all three candidates is irrelevant. For instance, we do not know, and it is irrelevant, whether Garrett's ranking of $\{B,C\}$ is $C \succ B$, which would make his ranking transitive, or $B \succ C$, which would make his preferences cyclic. Minimal Liberalism, then, appears to vitiate the assumption of transitive preferences. To examine this comment and understand how Sen's assumptions interact, we allow agents to have all possible preference rankings: cyclic as well as transitive. In this way we can determine whether Sen's assumptions force the decision rule to ignore the individual rationality condition.



a. Rankings

b. Cube

Fig. 1. Cube and rankings

1 Notions of the Geometric Proof

The full geometric proof[4] of Sen's Theorem is in Sect. 4, but we describe a special case with three alternatives $\{A,B,C\}$ to develop intuition about the geometry and the consequences that follow from the proof. For a specified pair, say $\{A,B\}$ (see Fig. 1a), let the three points $\{0,\frac{1}{2},1\}$ on the line interval $[0,1]$ represent, respectively, $B \succ A$ (B is preferred to A), $A \sim B$ (A and B are tied), and $A \succ B$. To represent all three pairs, use the unit cube (Saari 1995) where the x axis represents the $\{A,B\}$ rankings in the described manner, the y direction represents $\{B,C\}$ rankings where $y = 1$ corresponds to $B \succ C$, and the z direction represents the $\{A,C\}$ rankings where $z = 1$ represents $C \succ A$. All of this is displayed in Fig. 1b where, for convenience, the vertices are labeled

[4] A sketch for this proof first appeared as Li's answer for an exam problem when she was a student in a course taught by the other coauthor. Li's outline was motivated by a proof of Arrow's Theorem in Saari 2001.

1 to 8. The rankings associated with the eight vertices are

Vertex	**Ver. No.**	**Ranking**	**Vertex**	**Ver. No.**	**Ranking**	
$(1,1,0)$	1	$A \succ B \succ C$	$(0,0,1)$	4	$C \succ B \succ A$	
$(1,0,0)$	2	$A \succ C \succ B$	$(0,0,1)$	5	$B \succ C \succ A$	(4)
$(1,0,1)$	3	$C \succ A \succ B$	$(0,1,0)$	6	$B \succ A \succ C$	
$(1,1,1)$	7	$B \succ C, C \succ A, A \succ B$	$(0,0,0)$	8	$B \succ A, C \succ B, A \succ C$	

Vertices 7 and 8 correspond to cyclic rankings while the other six represent transitive rankings. The two vertices defining an edge of the cube differ only by the ranking of a single pair: e.g., the rankings assigned to vertices 1 ($A \succ B \succ C$) and 2 ($A \succ C \succ B$) differ only in the $\{B, C\}$ ranking; the difference must involve a $\{B, C\}$ ranking because the connecting edge is parallel to the y axis.

Important for our proof is that the geometry displays connections among transitive and cyclic rankings. For instance, each cyclic vertex is adjacent to three transitive rankings. Consequently, it takes a change in only one particular pair from any of these transitive rankings to create the cyclic one; e.g., as vertices 5 and 7 are on an edge in the x-direction, reversing the $B \succ A$ ranking of vertex 5 ($B \succ C \succ A$) leads to the cyclic ranking of vertex 7. Also, each transitive ranking is "adjacent" to a cyclic one; e.g., each vertex with an "odd" name is adjacent (connected by an edge) to vertex 7 and each vertex with an "even" name is adjacent to vertex 8. Notice the interesting feature: if an edge connects two transitive "vertices," then the change in ranking involves two alternatives that are ranked adjacent to each other; e.g., vertices 1 and 2 differ in $A \succ (B \succ C)$ and $A \succ (C \succ B)$. But an edge connecting a transitive and cyclic ranking changes the ranking of a pair where the alternatives are *not* next to each other in the transitive ranking—they are separated by another alternative; e.g., vertices 3 and 7 also differ in the $\{B, C\}$ ranking where 3 has $C \succ A \succ B$—with A separating C and B—while 7 has $B \succ C$. This "adjacency" geometry, which extends to any number of alternatives, plays a major role in our analysis.

1.1 Creating an example

To illustrate how to use this geometry to prove Sen's Theorem, start with the decisive choices given in the HR example and assume only that each agent ranks each pair. (That is, do not assume that an agent's ranking is transitive.) Garrett is decisive over $\{A, B\}$, so his choice forces the outcome to be on either the front ($x = 1$) face of the cube, with vertices $\{1, 2, 3, 7\}$ for the $A \succ B$ ranking, or on the back ($x = 0$) face with vertices $\{4, 5, 6, 8\}$ for the $B \succ A$ ranking. Similarly, Sandy's choice of $\{B, C\}$ allows her to select either the side face of $y = 1$ representing $B \succ C$, or the $y = 0$ side face representing $C \succ B$. As Garrett's and Sandy's choices determine the societal outcome for these two pairs, the societal outcome must be on both selected faces; i.e., it is one of the vertices on the edge that connects the two faces. There are four possible cases: for each the two selected faces intersect to define a vertical edge that can be identified by its vertices.

Garrett	Sandy	Edge Vertices	Garrett	Sandy	Edge Vertices
$x = 0$	$y = 0$	$4, 8$	$x = 1$	$y = 1$	$1, 7$
$x = 0$	$y = 1$	$5, 6$	$x = 1$	$y = 0$	$2, 3$

(5)

According to Table 5, two of these choices (the bottom row) must have a transitive outcome (represented by either of the two specified vertices) independent of whether the agents have transitive or cyclic preferences and independent of the societal $\{A, C\}$ ranking. The choices from the top row, however, allow a cyclic outcome as defined by either vertex 7 or 8. To ensure the vertex 7 cyclic societal outcome, accompany the $x = 1, y = 1$ choices with *any condition* (this includes any replacement of Pareto to select a ranking for the pair)[5] that places the societal outcome on the $z = 1$ surface. Sen's Theorem specifies two choices: either assign the $\{A, C\}$ pair to a decisive agent, or use the Pareto condition. In the former case, one agent is also decisive over $\{A, C\}$. Here, to have a $z = 1$ societal outcome, let this agent have the $z = 1$ ($C \succ A$) ranking. Alternatively, to ensure a $z = 1$ societal ranking (with the vertex 7 cyclic societal outcome) with the Pareto condition, both Sandy and Garrett must have a $z = 1$ ($C \succ A$) choice. Similarly, to ensure the vertex 8 societal conclusion, supplement the $x = 0, y = 0$ choices from decisive agents with any condition (Pareto or an agent is decisive over two pairs) that forces a $z = 0$ outcome.

It remains to find transitive preferences that yield either the vertex 7 or 8 cyclic outcome. (Notice, the construction does not use the transitivity of individual preferences, so we need to show that one choice could involve transitive preferences.) The analysis is immediate because selecting a ranking for a pair when constructing an example is equivalent to selecting a cube face. The societal outcome is sole vertex on all three selected faces. Each agent's preferences are also identified with cube faces: these faces are the ones the agent has a role in selecting, so the agent's preferences must be on the intersection of the specified faces. A crucial point is that there are two or more decisive agents. So, rather than an agent's individual's preferences being uniquely defined by the intersection of all three faces, these choices are determined by the intersection of at most *two* faces. With Garrett and the vertex 7 outcome, for instance, the choices for his profiles are determined by the intersection of the $x = 1, z = 1$ faces: the intersection generates the edge defined by vertices 3 and 7. The following table lists all possible supporting profiles when the z outcome is determined by the Pareto condition.

Vertex outcome	Garrett	Garrett's edge	Sandy	Sandy's edge
7	$x = 1,\ z = 1$	$3, 7$	$y = 1,\ z = 1$	$5, 7$
8	$x = 0,\ z = 0$	$6, 8$	$y = 0,\ z = 0$	$2, 8$

(6)

[5] For example, the Pareto condition could be replaced with the majority vote over a pair—maybe with a tie breaker, or by a host of other possibilities. The same analysis, showing that the Pareto condition is not important to obtain a Sen-type conclusion, holds for any number of alternatives and agents. *Any rule,* which ranks a pair independent of the choices made by the decisive agents and preferences over other pairs, suffices.

According to the top row of Table 6, the cyclic societal outcome of vertex 7 is supported by four possible profiles where in each pair Garrett's preference is listed first: $\{(3,5), (3,7), (7,5), (7,7)\}$. Only the $(3,5)$ choice endows each agent with a transitive preference to create an example. Similarly, all possible profiles causing a vertex 8 cyclic societal outcome come from the bottom row; they are $\{(6,2), (6,8), (8,2), (8,8)\}$ where $(6,2)$ represents the only transitive preferences.

1.2 Observations

To complete the proof of Sen's result for three alternatives, we need to modify the above to handle all possible choices for which agents are decisive, over what choices of pairs, and whether the Pareto condition is, or is not, needed. The general proof (Sect. 4) for any number of agents and alternatives is similar where the geometry of the cube is replaced with the geometry of hypercubes.

Certain conclusions follow immediately from this geometric construction. For instance, the ranking of a pair, as determined either by a decisive agent or by all agents through the Pareto condition, defines a face of the hypercube. The societal outcome is determined by the intersection of all selected faces. Next, as there are at least two decisive agents, each agent's preferences are restricted by the intersection of *some*, but not all, of these faces. This observation leads to a new geometric proof (Sect. 4) of the following assertion made by Saari.

Theorem 2 *(Saari, 2001) Allow voter preferences to include all ways to rank pairs, rather than just the transitive rankings. Any example of cyclic societal rankings created by choices of decisive agents and the Pareto condition includes the unanimity profile where each voter's preference agrees with the cyclic societal ranking over these pairs.*

Theorem 2 provides a particularly simple way to construct all possible examples illustrating Sen's Theorem (see Saari 2001). Even more: as the theorem holds for all possible examples illustrating Sen's result, it supports the comment that Sen's conditions force a decision rule to ignore the assumption of individual rationality. Instead, the rule tries to service the wishes of non-existent voters with cyclic preferences. (See (Saari 2001) and (Saari and Petron 2004).)

Moreover, our geometric method of proof identifies *all possible profiles* supporting a Sen example, so we can significantly expand on these earlier observations by providing a new statistical interpretation for Sen's Theorem that supports this "non-existent cyclic preferences" comment. We start by determining how often a Sen-type phenomenon can be expected to occur.

2 How often and why do societal cycles occur?

When do Sen's assumptions force a cyclic societal outcome? To address this question, we first determine how often societal cycles occur, and then we analyze the mechanism that causes cycles.

2.1 Likelihoods

Table 5 identifies all profiles that lead to all possible societal conclusions, so it can be used to construct examples where any of the six transitive and the two cyclic rankings could be the societal outcome. Even more: by introducing assumptions about the likelihood of each binary choice, we can determine the likelihood of each societal outcome. In this manner we discover that societal cyclic outcomes are distinctly less likely than transitive ones—even when voters have cyclic preferences.

Theorem 3 *With three alternatives and n-agents ($n \geq 2$), suppose there are two decisive agents where each is decisive over precisely one of two designated pairs. Assume that the voters independently rank each pair, but not necessarily in a transitive manner. Assume that, for each agent, each of the two rankings for a pair is equally likely. Under the condition that the Pareto condition is satisfied for the one pair not determined by the decisive agents, the likelihood of a transitive ranking is $\frac{3}{4}$, and the likelihood of a cyclic outcome is $\frac{1}{4}$. But if each voter's preferences are given by strict transitive rankings, where each is selected independently and with equal likelihood, then, subject to the condition that the Pareto condition is satisfied for the one pair, the likelihood of a transitive societal outcome is $\frac{8}{9}$ while the likelihood of a cyclic outcome is $\frac{1}{9}$.*

If, instead, there are three decisive agents, each decisive over a different pair, then the likelihood of a cyclic outcome is $\frac{1}{4}$; this is true whether the voters can only rank pairs, or have transitive rankings.

Theorem 3 shows that while cyclic behavior is not prominent, it occurs often enough to be bothersome. A similar statement holds for any number of alternatives and decisive voters: as the number of decisive voters increases, even if the agents have transitive preferences, the likelihood of a Sen cycle increases. While the proof is in Sect. 4, a sense of these values follows from the Table 5 list of profiles. The actual proof reduces to a counting argument.

The next natural question is to characterize those settings when a transitive, or cyclic, outcome occurs. Here the structure of a level set (i.e., all profiles that support a specified societal outcome) provides an interesting tool. For instance, if cyclic preferences are not in level set of a particular societal outcome, then cyclic preferences play no role in determining that outcome. Conversely, if cyclic preferences dominate, then they play a major role. Thus a way to understand whether and how the minimal liberalism condition undercuts the assumption of transitive preferences is to analyze when, and how often, cyclic preferences are in the level sets of outcomes.

Proposition 1 *With three alternatives and two agents, where each is decisive over a different pair and they agree on the ranking of the remaining pair, if the societal outcome is transitive, then either all choices of profiles for the two agents must be transitive, or half of the profiles require a transitive preference for one of the agents. No profile supporting a transitive societal outcome consists strictly of cyclic preferences. On the other hand, should the outcome be cyclic, then three-fourths of the possible supporting profiles include a cyclic ranking, and both rankings are cyclic for one choice.*

The proof is in Sect. 4, but a sense of why this result holds comes examining all profiles in the above example that lead to different kinds of outcomes. A similar result holds in general, but the approach developed next provides an easier way to capture this fact.

2.2 Interpretation

As Prop. 1 shows, it is reasonable to associate a transitive societal outcome with transitive profiles. After all, in such settings, all voters have transitive profiles for two-thirds of the supporting profiles. While the remaining one-third allows one voter to have cyclic preferences, the other voter must have transitive preferences.[6] In an undefined sense (even if just accidentally), then, the individual rationality of voters is respected; it affects the transitivity of the societal outcome.

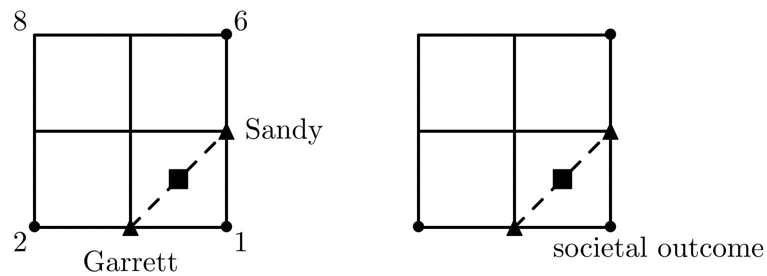
The situation drastically changes with a cyclic societal outcome: here only one-fourth of the possible supporting scenarios endow all agents with transitive preferences. In contrast, half of the profiles have at least one cyclic voter and one-fourth require both voters to have cyclic preferences. These comments suggest that when a decision rule (which satisfies Sen's conditions) yields a cyclic outcome, the difficulty occurs because, by ignoring the assumption that the voters have transitive preferences, the rule appears to capture the views of the non-intended cyclic preferences. In other words, despite the explicit individual rationality requirement, Sen's result arises because the decision rule reflects the properties of the non-existing cyclic voters rather than the actual transitive ones. In this section, we make this notion precise.

To support these comments, we modify the notions Saari and Sieberg (2001) developed to

[6]Four profiles are associated with each pair of edges. With three pairs of edges, there are twelve possible profiles. Eight choices are transitive; four choices have one cyclic preference.

explain pairwise voting paradoxes. By examining all possible profiles that support any specified set of pairwise outcomes, they showed that a societal outcome accurately reflects the *average* of all possible supporting profiles—but not necessarily any specific one. Paradoxes, then, manifest situations where a specified profile deviates from the average of *all* supporting profiles. As an illustration, the "paradox of voting" given by profile $A \succ B \succ C$, $B \succ C \succ A$, $C \succ A \succ B$ is where, by 2:1 votes, the pairwise cyclic outcome $A \succ B$, $B \succ C$, $C \succ A$. There are, however, five profiles supporting the same outcome where the "average" is a cyclic profile with rankings that agree with the cyclic outcome.

To motivate our approach, which must differ from Saari and Seiberg because of the nature of Sen's assumptions, suppose the $A \succ B \succ C$ "vertex 1" outcome arises with Garrett decisive over $\{A, B\}$ and Sandy over $\{B, C\}$. In terms of the geometric construction, Garrett's preference ranking is given either by vertex 1 or 2, and Sandy's by either vertex 1 or 6. These preferences are represented by the bullets in the large square of Fig. 2a, which represents the bottom face of the Fig. 1b cube. The square is subdivided into four smaller squares where the ranking associated with a point is determined by the closest vertex. So all points in the bottom right-hand small square have the $A \succ B, B \succ C, A \succ C$ ranking.



a. A specific example **b.** In general

Fig. 2. Explaining Sen's outcomes

Of the four possible profiles, $(1, 1)$, $(1, 6)$, $(2, 1)$, $(2, 6)$, the societal outcome is conclusively supported by $(1, 1)$ with its unanimous support of the vertex 1 ($A \succ B \succ C$) outcome. Only one voter in next two choices prefers $A \succ B \succ C$ so while these profiles do not provide as strong support, the outcome still makes sense. Only the outlier $(2, 6)$ profile, where Garrett prefers $A \succ C \succ B$ while Sandy prefers $B \succ A \succ C$, might raise questions and doubts about the $A \succ B \succ C$ outcome.

To devise an argument that involves all four profiles and supports the outcome, notice that in an example illustrating Sen's Theorem, flexibility in how a voter ranks a specified pair occurs only when the pair's societal outcome is determined by another agent who is decisive over the pair. A natural way to handle these unknown preferences is to replace them with the average of the two possible choices. In Fig. 2a, then, Garrett's two choices of $B \succ C$ and $C \succ B$ are averaged to obtain the $B \sim C$ ranking: this is indicated by the triangle on the bottom edge of Fig. 2a. Similarly, averaging Sandy's two choices for the $\{A, B\}$ ranking leads to a $A \sim B$ ranking indicated by the triangle on the right edge of Fig. 2a. The "average of the averaged rankings" is represented by the dark square; it is the average of the two "averaged points" represented by triangles. To describe this "averaging of averaged rankings" approach with coordinates, the Fig. 1b coordinates for Garrett are $(1, 0, 0)$ and $(1, 1, 0)$, so the averaged value is $\frac{1}{2}(1, 0, 0) + \frac{1}{2}(1, 1, 0) = (1, \frac{1}{2}, 0)$. Similarly, the averaged coordinates for Sandy are $(\frac{1}{2}, 1, 0)$. In turn, the averaged outcome of the two averaged rankings is $\frac{1}{2}(1, \frac{1}{2}, 0) + \frac{1}{2}(\frac{1}{2}, 1, 0) = (\frac{3}{4}, \frac{3}{4}, 0)$, which represents the $A \succ B$, $B \succ C$, $A \succ C$ rankings of vertex 1. The next definition makes all of this precise.

Definition 1 *All rankings for n specified pairs can be represented in an "orthogonal cube" in R^n defined by the 2^n vertices with entries consisting of zeros and ones. Each coordinate direction designates a particular pair where 0 and 1 represent the pair's two strict rankings. The ranking for each pair that is assigned to a point in the orthogonal cube is determined by its proximity to each vertex where "closer is better."*

Let $\mathcal{K}_j$ be a set of k n-vectors where each coordinate entry consists of zeros and ones; $\mathcal{K}_j$ represents k of the choices the jth agent can make for strict rankings over all n pairs. The agent's "averaged preference ranking" is the average of the k-vectors.

If a vector representing a ranking, or an averaged ranking, is assigned to each of a agents, then the "averaged ranking over the agents" is the average of the a vectors representing the rankings.

As Fig. 2b indicates, the same geometric argument explains any societal outcome—even a cyclic one. Rather than a ML and Pareto conflict, the geometry shows that a Sen outcome reflects an average over the voters' choices. The next result shows that this is always true (proof in Sect. 4); the averaged ranking of the averaged choices for the voters always agrees with the Sen outcome.

Theorem 4 *For any number of alternatives, for any specified number of decisive agents where each is assigned a specified number of (distinct) pairs, consider only the portion of a decision rule's societal outcome that is determined by the choices made by the decisive agents and by the Pareto condition. One of the profiles supporting this outcome is the unanimity profile where each voter's preference (for this portion of the rankings) agrees with the societal outcome. Moreover, this portion of the societal outcome always coincides with the ranking determined by the average of the averaged profile (with no assumption of transitivity) for each agent.*

While is it too much to assume that a decision rule accurately reflects the intentions of each and every profile, the part of the societal outcome determined by decisive agents and the Pareto condition (or a substitute for Pareto) accurately reflects either a particular unanimity profile, or the average of all possibilities for the space of all complete binary rankings (transitivity need not hold). In particular, a cyclic societal outcome, which illustrates Sen's result, does *not* respect the crucial assumption of individual rationality. Instead, by reflecting the average of possible profiles, the cyclic societal outcome more accurately reflects the cyclic preferences of non-existent (by assumption) voters. Rather than playing a central role in the analysis, the actual transitive preferences are statistically treated as unlikely outliers.

2.3 Individual liberties?

Theorem 4 and the geometry of the cube helps us identify when it is arguable that Sen's result captures individual liberties and when it identifies a contentious society. Our argument uses the following definition.

Definition 2 *(Saari 2001, Saari and Petron 2004) In a transitive ranking where alternative X is ranked above Y, let $[X \succ Y, \alpha]$ represent the relative ranking where α designates the number of alternatives in the transitive ranking that separate X and Y.*

A strict ranking for a pair of alternatives, (X, Y), that is determined by a decisive agent imposes a strong negative externality *on another agent if that other agent's transitive preference ranks the pair in the opposite manner, say $[Y \succ X, \alpha]$, and the alternatives are separated by at least one other alternative, that is $\alpha > 0$.*

To illustrate the definition, if Sandy is decisive over $\{B, C\}$ and selects $B \succ C$, this does not create a strong negative externality for Ann's $C \succ B \succ D \succ A$ preference ranking because her $[C \succ B, 0]$ ranking shows that C and B are not separated by an alternative. Jane, with a $C \succ A \succ B \succ D$ ranking, or $[C \succ B, 1]$, does experience a *strong negative externality* by Sandy's choice because the alternatives in her opposing $C \succ B$ ranking are separated by A. The second part of the following theorem is in (Saari 2001) and (Saari, Petron 2004) but a new geometric proof is given. The first part, which we use to interpret transitive societal outcomes, is new.

Theorem 5 *For any number of agents, where each has transitive preferances, assume a specified number (two or more) of them are decisive where each is assigned at least one (distinct) pair of alternatives. If the part of a societal ranking determined by the decisive agents and the Pareto condition is transitive, then each agent has choices of transitive preferences that support the outcome and ensure that the agent does not suffer a strong negative externality with the choices made by the decisive agents. If, however, this part of the societal outcome is cyclic, then, for each cycle,* each *agent suffers a strong negative externality for a choice made by some decisive agent.*

A transitive societal outcome, then, provides no apparent reason to question whether minimal liberalism models individual liberties. It is possible, of course (Thm. 5) for other agents to disagree with the choices made by the decisive agents, but the disagreement need not be severe. Instead, the average of the averages of their possible choices coincides with the transitive outcome (Thm. 4). The situation differs significantly with a cyclic societal outcome: as *everyone* suffers a strong negative externality imposed by someone else, it becomes arguable that some of the decisive agents may be abusing the freedom of minimal liberalism by creating a society that is not well adjusted.

3 Sidestepping Sen-type consequences

The ubiquity of Sen's assumptions suggests that ways to sidestep his negative conclusions must be situation-specific. If, for instance, Sen's cycles occur because decisive agents abuse their power by inflicting strong negative externalities on others, then an obvious solution is to abridge the rights of the decisive agents. Saari and Petron (2004) develop this approach by allowing a decisive agent to be decisive only if the decision does not impose a strong negative externality on someone else. But they did not identify what should be the outcome for this pair because a resolution requires information beyond whether a strong negative externality exists. Before explaining the nature of this added information, we give a partial answer.

Theorem 6 *Limit a decisive voter's rights in the following manner: if a decisive voter's ranking of a pair imposes a strong negative externality on another agent, then replace the decisive voter's choice with a tie ranking for the alternatives. The portion of the societal outcome made by this modified form of decisive voters and the Pareto condition does not create cycles.*

While this modification of minimal liberalism avoids societal outcomes with cycles, the outcome need not be transitive. This feature is captured by the initial HR example whereby Garrett's $A \succ B$ choice creates a strong negative externality for Sandy with her $B \succ C \succ A$ preferences. As such, Garrett's choice is replaced by the tie $A \sim B$. For similar reasons, Sandy's $B \succ C$ choice is replaced by $B \sim C$. As such, the societal ranking is the $C \succ A$, $A \sim B$, $B \sim C$ quasi-transitive outcome leading to the hiring of Cindy.

While Thm. 6 provides one approach, we now want to explore what must be done, in general, to sidestep Sen's problems. Any approach must recognize that the Sen's problems occur because *over the societal rankings determined by decisive agents and the Pareto condition, the decision rule can never use information about the transitivity of individual preferences. Instead, transitivity is an accident or, as true with "single-peakedness," a manifestation of restricted choices of preferences.*

These comments suggest that determining the strict ranking of a pair by blindly using the strong negative externality condition, or any other condition that is independent of other societal events, need not resolve the difficulty. Instead, it could even convert a transitive societal ranking into a cyclic one. To see that this can happen, let Ann, Barb, and Carl be decisive agents over, respectively, $\{A, B\}$, $\{B, C\}$, and $\{A, C\}$ where everyone except David, has the $A \succ B \succ C$ preference ranking; David prefers $C \succ B \succ A$. The transitive societal outcome of $A \succ B \succ C$ imposes negative externalities of $[C \succ B, 0], [B \succ A, 0]$ and $[C \succ A, 1]$ for David. If a rule were imposed that if someone suffered a strong negative externality, then the outcome would reverse, we would allow disgruntled David's choices to convert the transitive $A \succ B \succ C$ into the cyclic $A \succ B, B \succ C, C \succ A$. To avoid cycles, corrections to a pair must be coordinated with other societal events—a pair's societal ranking must be coordinated with what happens with other pairs.

Thus, the "obvious" approach of correcting a cycle by identifying and reversing the ranking of a particular pair is about the only approach. After all, to break a cycle, only one ranking needs to be reversed; but, which one? As there is freedom in the selection of the pair, it may be that some agents continue to suffer strong negative externalities, bu, then, t this may be an integral part of the modeling for settings, say, with political or religious beliefs. To illustrate what can be done with the special setting where decisive agents are abusing their rights, we could select the decisive agent whose action creates the most egregious violation of the rights of others. A natural way to handle this is for each agent suffering a negative externality $[X \succ Y, \alpha]$ to cast a ballot listing the α value. The decisive agent with the largest tally is causing the strongest problem, so this agent's rights are restricted.

Theorem 7 *If a decision rule satisfying Sen's conditions creates a societal cycle with the actions of decisive agents and the Pareto condition, then the actions of at least two decisive agents are involved. For the choice made by each decisive agent in the cycle, compute the following tally: sum the alpha values for all voters with a negative externality $[X \succ Y, \alpha]$. Reverse the ranking of the agent with the largest tally by replacing the agent's choice with indifference. (Break ties in some manner.) The societal outcome will not have a cycle.*

While the Thm. 7 procedure most surely would never be used as specified, it does capture the spirit of what is actually done in practice. Examples include the numerous laws such as sound abatement. To be enforced, someone has to complain. But lodging a complaint involves a cost, Not only must the sound create a negative externality for the complainant, but, because making a complaint involves a cost, it must be a strong negative externality. Now consider a society with many complaints against several individuals. Not all problems can be handled, so realistic budget constraints require establishing priorities for actual enforcement. Theorem 7 provides one way to decide; restrict the choices of an agent who receives the most and loudest complaints.

To illustrate with an example, suppose the information is

Person	**Preferences**	$\{A,B\}$	$\{B,C\}$	$\{C,D\}$	$\{A,D\}$
Anne	$ABCD$	AB	BC	CD	–
Barb	$CDAB$	AB	–	CD	DA
Connie	$CDAB$	AB	–	CD	–
Outcome		AB	BC	CD	DA

(7)

The $\{B,C\}$ choices have both Barb and Connie with $[C \succ B, 2]$, so the score for Anne is 4. For the $\{A,D\}$ choices, Anne has $[A \succ D, 2]$, but Connie agrees with the choice. Thus, only Anne's ranking is replaced to obtain $C \succ D \succ A \succ B$.

4 Proofs

General proof of Thm. 1: three alternatives A general proof of Sen's result requires handling all possible choices for which agents are decisive, over what pairs, and whether the Pareto condition is, or is not, needed. Sen's assumptions for three alternatives and any finite number of agents require each of at least two agents to be decisive over separate pairs. (If they were decisive over the same pair, it would be trivial to create a contradiction.) Start with two agents each decisive over a pair. Using the geometry of the cube, for each assigned pair, the decisive agent selects one of two opposing cube faces. The societal outcome must be on each selected face, so it is one of the two vertices on the edge that connects the two faces selected by the two decisive agents. The four possible pairs of faces define four edges that are parallel to each other. Each edge is defined by two vertices: each vertex represents the outcome depending on how the remaining pair of alternatives is ranked.

In any coordinate direction, the cube has four parallel edges. Each of the cube's eight vertices is on precisely one edge. Thus, each cyclic vertex is on one of the four edges defined by the above intersection process: as the cyclic vertices are diametrically opposite one another, both cannot be on the same edge. So, for each decisive agent, select a ranking of their assigned pair of alternatives (equivalently, for each agent, select the appropriate cube surface) so that the pair of selected faces defines an edge with a cyclic vertex.

The next step is to determine the societal ranking for the remaining pair; that is, we need to select one of the two vertices of the identified edge. If one of the two agents (or a third agent) is decisive over the remaining pair, select the agent's ranking for this pair so that it requires the cyclic vertex. If no agent is decisive over the remaining pair, then use the Pareto condition: select each agent's ranking for the remaining pair to force the cyclic vertex ranking.

It remains to find transitive preferences that support this societal outcome. If an agent is decisive, the societal outcome must be represented by one of the vertices on the cube face she selected. Likewise, if a ranking for the societal outcome is determined by the Pareto condition, then each agent's ranking must be on that face. An important point is that at most two conditions are imposed on any agent's preferences. For instance, if an agent is decisive over two pairs, then the above construction specifies his preferences for two pairs (i.e., for two faces). Thus his preferences must be one of the two vertices along the edge defined by these particular cube faces. Similarly, if an agent is decisive over one pair and shares a common (Pareto) ranking with another pair, these choices define two surfaces, so her preferences are represented by one of the two vertices on the defined cube edge. Finally, for a non-decisive agent, either there are no restrictions on his preferences (when others are decisive over all three pairs), or his preferences are restricted to the four vertices on the cube face mandated by the Pareto condition where everyone agrees on the ranking of a pair. These vertices catalogue all possible supporting profiles.

The societal outcome is the vertex surviving the intersection of the specified cube faces, so one of the choices for each agent is the cyclic vertex representing the societal outcome. But the choices for each agent includes an edge, and, according to the geometry of the cube, each edge emanating from a cyclic vertex has a transitive vertex. Select this vertex. This completes the proof. □

Notice from the above argument that of the four choices of profiles for the two decisive agents, only one is transitive.

Proof for $n \geq 4$ alternatives. The n alternatives define $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of alternatives. For each pair, (X, Y), represent its three rankings $X \succ Y, X \sim Y, Y \succ X$, by the points $0, \frac{1}{2}, 1$ on a particular $\mathbb{R}^{\binom{n}{2}}$ coordinate axis. Over all pairs, the integer points define the vertices of a unit hyper-cube in $\mathbb{R}^{\binom{n}{2}}$: each vertex defines the rankings over all pairs of alternatives. We will use a lower k-dimensional cube.

Assign each of $2 \leq k \leq \binom{n}{2}$ pairs to a decisive agent; there are at least two decisive agents. Take two of the pairs, $\{(X_i, Y_i)\}_{i=1}^{j}$ that are assigned to two different decisive agents, say (X_1, Y_1) and (X_2, Y_2). The pairs cannot be the same, so either at most one alternative is in both pairs, or they are disjoint. In the former case, assume that $Y_1 = X_2$ and add the pair (Y_2, X_1). As this setting reduces to the three alternative setting already analyzed, the proof is completed. (For pairs outside of this triplet, let everyone agree on the ranking.) In the latter case, add the pairs (Y_1, X_2) and (Y_2, X_1). The four pairs define a hyper-cube in the four dimensional space R^4 with $2^4 = 16$ vertices. There are two cyclic vertices: one is given by $X_1 \succ Y_1, Y_1 \succ X_2, X_2 \succ Y_2, Y_2 \succ X_1$ and the other reverses the rankings. As a 0 in a vertex coordinate (i.e., representing the ranking of one pair) for one cycle is replaced by a 1 for the other cycle, these vertices are diametrically opposite each other. By being four-dimensional, four edges emanate from each vertex of this cube. As each edge coming from a cyclic preference is where the ranking of precisely one pair reverses, this change will will make the ranking transitive (with respect to the specified pairs). Therefore, just as in the three-dimensional setting, the cyclic rankings have a special geometric relationship with transitive rankings.

The rest of the construction mimics the three-alternative case. Select a cycle. Let the rankings for the two pairs determined by decisive agents coincide with the selected cycle. For the remaining two pairs, if assigned to a decisive agent, select the ranking according to the chosen cycle. If not, use the Pareto condition by requiring all voters to have the specified ranking. Each selection corresponds to selecting a cube face. By construction, the specified cyclic vertex is on all cube faces, so it defines the societal outcome for these pairs.

To find transitive preferences for the voters, notice that with two (or more) decisive agents, the above construction mandates for each agent the choice of preferences for at most three pairs (at most three cube faces). This means that the selection of an agent's preference is one of the vertices on the faces the agent must select. By construction, the choices of vertices always includes the two vertices on at least one edge coming out of the cyclic vertex. One of the vertices must be transitive: transitive preferences can always be found.

While the above proves the theorem, notice that the argument generalizes to any number of pairs involving any number of cycles. First note that a cycle must involve at least two decisive agents. If this were not the case, then some agent's choice must agree with each ranking in the cycle (through the Pareto condition and by being decisive), so this agent must have cyclic preferences. (This is the only place the transitivity assumption explicitly occurs: it shows the need for at least two decisive agents.) Second, a cycle involving k pairs can be represented by a vertex on a k-dimensional cube: each vertex on an edge with the cyclic vertex is a transitive ranking. □

Proof of Thm. 2. The proof follows immediately from the comments made prior to the statement of the theorem. The societal outcome is the intersection of all faces of a hypercube determined by the decisive agents and the Pareto condition. Each agent's preferences are given by the vertices of the object that result from the intersection of *some* (but not all) faces. Consequently, the societal outcome must be one of these vertices. □

Proof of Thm. 3. With the assumptions on preferences, this is a counting exercise. When the voters can rank each pair independently, then the condition that all voters rank one pair in the same manner means that there are four remaining choices of rankings for each voter. The ranking of the other two pairs, however, are determined by the two decisive voters, so the choices of the $n-2$ voters do not matter. Their choice define four parallel edges; each is equally likely. On each edge, the Pareto condition mandates that only one vertex can be selected. Thus, the outcomes are one of the four vertices on a cube face; each is equally likely. But on each cube face, three of the vertices are transitive and one is cyclic. This completes the proof for the first part.

Now assume that the voters can select only transitive preferences, and that they agree on the ranking for one pair. Thus each voter has three choices for the transitive ranking as determined by whether the remaining alternative is ranked above, below, or between the specified pair. There are nine possible rankings for the two decisive voters. As demonstrated in the proof for three alternatives, for each way the societal ranking of a pair can be determined by the Pareto condition, there is precisely one transitive profile for the two decisive agents that gives rise to the cyclic outcome. Consequently, the likelihood of a cyclic outcome is $\frac{1}{9}$ and of a transitive outcome is $\frac{8}{9}$.

With three decisive agents, each agent must select the societal ranking for one pair. It is immaterial how they rank the other pairs. Therefore, there are eight choices for the selected rankings (cube faces); each is equally likely. Of the eight choices, two define cyclic vertices and six define transitive ones. Thus, the likelihood of a cyclic outcome is $\frac{1}{4}$. $\square$

Proof of Prop. 1. According to the above construction of any example and the hypothesis, each agent's preferences are given by the vertices of the edge that is on both of the agent's specified faces of the cube. Moreover, the edges for the two agents must meet in the vertex that describes the societal outcome, so the edges are orthogonal.

According to the geometry of the cube (Fig. 1b), three edges emanate from each vertex; thus, for each vertex representing a societal outcome, there are three pairs of edges that satisfy the "two-edge" condition. Also according to the geometry of the cube, if the common vertex represents a transitive ranking, then there is one pair of edges where all four vertices represent transitive rankings. For the other two pairs of edges, one vertex on one edge corresponds to a cyclic ranking, while the other three vertices represent transitive rankings. (For example, if 3 is the common vertex (so this vertex represents the societal outcome of $C \succ A \succ B$), then the set of all possible pairs of edges are $\{3-2, 3-4\}$, $\{3-2, 3-7\}$, and $\{3-4, 3-7\}$. The first choice involves only transitive rankings; the other two involve the cyclic vertex 7.) Each pair of edges gives rise to four possible profiles; e.g., the $\{3-2, 3-4\}$ edges define the profiles $(3,3), (3,2), (2,3), (2,4)$. If none of the edges has a cyclic vertex, then all profiles are transitive. If one edge has one cyclic vertex, then half of the possible profiles have a cyclic preference, but no profile consists strictly of cyclic preferences.

If the outcome is cyclic, then each edge for each of the three pairs of edges, has one cyclic vertex. When computing the four resulting profiles, one choice has both rankings cyclic, two others have one cyclic ranking, and the fourth has both transitive rankings. This completes the proof. $\square$

Proof of Thm. 4. The vertices that can be assigned to each agent consist of all vertices on an edge, or a face, or a two-dimensional cube, or ... . Each of these geometric objects have the societal outcome as one of the vertices. As such, each person's averaged outcome is a vertex of the subcube with the societal outcome as a vertex. By the properties of decisive agents with choice over different alternatives, at least two of these averaged outcomes are not on the same edge of the subcube. Thus, the average of the averaged ranking must be in the interior of the cube. This proves the theorem. □

Proof of Thm. 5. To handle the first part of the theorem, we know from Thm. 4 that one choice of preferences is where the preferences of all agents agree with the transitive societal outcome. With agreement, the choices made by the decisive agents do not even register as a negative externality. Then there are $\binom{n}{2}$ edges coming out of this vertex: $n-1$ of them connect to another transitive ranking—these edges represent a pair of alternatives that are adjacently ranked in a ranking representing the vertex. (For instance, for $A \succ B \succ C \succ D$, these would be the edges for $\{A, B\}, \{B, C\}, \{C, D\}$.) By being adjacent, reversing the ranking of a pair does not affect transitivity and, while a change in these rankings would create a negative externality, it would not be a strong negative externality. What remains are $\binom{n}{2} - (n-1) = \binom{(n-1)}{2}$ edges: here a change in the indicated pair leads to a cycle. This is where the pairs are not adjacently ranked in the original ranking. (With $A \succ B \succ C \succ D$, reversing the $\{A, B\}$ ranking leads to a cycle.)

For a cyclic ranking, changing the ranking of any pair defines a transitive ranking. These changed alternatives cannot be ranked adjacent to one another in the transitive ranking—if they were, then reversing them again to return to the original setting would create a transitive rather than a cyclic ranking. Thus, each voter suffers a strong negative externality. □

Proof of Thm. 6. With a cycle, only one ranking needs to be changed to convert it into a transitive ranking. Changing one of these rankings into indifference converts the ranking into a quasi-transitive ranking because, by removing an offending strict ranking, the remaining strict rankings are transitive. □

Proof of Thm. 7. If a cycle created by decisive agents and the Pareto condition involved only tone decisive agent, then the decisive agent's preferences would have to agree with the cyclic outcome. This violates the assumption of transitive preferences. (As asserted earlier, this is about the only place the assumption of transitivity is used.) The remainder of the theorem follows because a way is identified to reverse the ranking of one pair: that is all that is necessary to have a transitive ranking. □

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