Equal Protection Implies Proportional Representation

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Abstract

This paper shows that for a single-vote electoral system for a representative body to treat all voters and all parties equally, it must produce results essentially identical to list proportional representation (PR). Democratic theory has often been agnostic concerning representative institutions. Different institutions have been compared in terms of behavioral outcomes rather than axiomatic properties. Building on van der Hout et al.'s (2002) result, we show that for an electoral system to completely respect the principle of political equality, its results must be equivalent to those of list PR.

Democratic theory has often been agnostic concerning representative institutions. Different institutions for choosing representative bodies have been compared in terms of behavioral outcomes rather then in terms of their properties. This paper, instead, provides a justification for list proportional representation on the basis of its axiomatic properties. For this purpose, we model electoral systems as seat share allocation rules. These are rules that translate the votes for the parties into a seat share for these parties. Our proofs are limited to single-vote electoral systems, although we discuss the relevance of our results to multiple-vote electoral systems like STV. Building on van der Hout et al.'s (2002) result, we show that for a single-vote seat share allocation rule to treat all voters and all parties equally, it must produce results essentially identical to pure list proportional representation (PR).

Following Dahl (1956, 1988), we can define a process as being democratic only if it respects the values of political equality and popular sovereignty. Popular sovereignty implies that the political process has the final say on all matters, if it should choose to exercise it. Political equality is operationalized as the anonymity and neutrality of the decision-making process – all voters are treated equally and all candidates / alternatives are treated equally. It can certainly be argued that further conditions are required for a decision making process to be democratic in substantive sense; however, we can define popular sovereignty and political equality as minimum formal conditions for a process being democratic.

May (1952) shows that the only binary decision-making process that satisfies anonymity, neutrality, decisiveness and positive responsiveness is majority rule.[1] This argument is accepted by Dahl,[2] and forms the basis of what Dahl calls the "populist" theory of democracy, which argues that political equality implies majority rule.[3] However, political decisions are typically not made directly, but by representative bodies. For this reason Dahl (1956) argues that axiomatic theory cannot say much about the

[1] By binary decision making process we mean a procedure that considers only two alternatives at a time, such as an amendment procedure. If we allow procedure that consider more than two alternatives at a time, there are other procedures that satisfy anonymity and neutrality, such as the Borda Count.
[2] Dahl (1988) refers to May (1952). Dahl (1956) does not cite May. However, the logic of Dahl's argument is identical to May's.
[3] Technically this argument does not follow from May's result, as there are nonbinary procedures considering more than two alternatives at a time that respect neutrality and anonymity, such as the Borda Count (see Young 1974, Saari 2003). However, to our knowledge, these procedures have never been used for legislation. Thus Dahl is correct that the requirements of anonymity and neutrality eliminate all of the commonly used alternatives to majority rule.

institutions that translate the democratic principle into reality. This leads Dahl to dismiss axiomatic theory (and with it the populist theory of democracy) as being useless as a guide to action. Instead he develops a behavioral theory of democracy, based on description of existing systems of government.

We argue, however, that we can make clear axiomatic statements about the institutions of representative government. We can conceive representative democratic decision-making as proceeding in two stages: Firstly elections choose representatives; and secondly those representatives make binding decisions. We know from May (1952) that the only binary decision-making rule that satisfies anonymity and neutrality in the second stage is majority rule. We show that for the first stage any single-vote seat share allocation rule that satisfies anonymity and neutrality must produce a result that is to all intents and purposes identical to that produced by (pure) list proportional representation. Thus to satisfy political equality at the first stage, the seat allocation rule must be equivalent to proportional representation, and at the second stage the decision rule must be majority rule, or some anonymous and neutral rank-ordering procedure (such as the Borda Count). Far from being utopian, the institutional configuration of proportional representation and majority rule is observed in a significant number of countries.

Evaluating Electoral Systems

This paper analyses the axiomatic properties of single-vote seat allocation rules, showing that the axioms of political equality at the individual level entail an election system that produces proportionality in electoral results. This approach can be compared to that taken by both the electoral systems literature and the literature on social choice, two literatures with fairly minimal overlap.[4] The electoral systems literatures studies seat allocation rules, but focuses on their effects rather than their axiomatic properties. That is, electoral systems are compared in terms of how they produce outcomes such as proportionality or stability, or in terms of how many parties they tend to generate, not whether they intrinsically satisfy principles such as political equality. The social choice literature deals with axiomatic properties, but has concentrated on social decision rules, rather than seat allocation rules. That is, most formal work has been carried out on

[4] Dummett (1984) remarks that when he compared the bibliography of two notable books, one on social choice, the other on electoral systems, the only author referenced by both was John Stuart Mill.

rules for aggregating preferences into a single decision, rather than on rules for translating preferences into a distribution of representatives who then deliberate to reach policy decisions.

Whereas our analysis starts with basic normative principles, such as political equality, and sees what this logically requires in an electoral system, the electoral system literature tends to work empirically, starting with existing electoral systems and studying their effects. Much of the electoral system literature has focused on the effect of electoral rules on party systems. Duverger (1954) found that first-past-the-post elections tended to produce two party systems, while proportional representation produces mutipartism. Rae (1967) systematically compared district magnitude (the number of candidates elected from each district) and electoral rules to explain cross-national differences in proportionality, large party advantage and the number of parties. More recent works in this tradition include Taagepera & Shugart (1989) and Lijphart (1994).

When it deals with normative questions of democracy, the electoral systems literature tends to operate in instrumental terms, as typified by the title of Powell's (2000) book "Elections as Instruments of Democracy". Various conceptions of democracy are set out, and different electoral systems are evaluated in terms of how far they produce results compatible with these conceptions. Thus in Powell's account majoritarian conceptions of democracy stress the direct accountability of government to the electorate, as operationalized by how likely a change in popular support is to produce a change in government; while proportional conceptions of democracy see democracy as a multi-stage process requiring "authorized representation", measured in terms of what proportion of the voters voted for a government party, and the degree to which policy outcomes match the preferences of the median voter. Plurality systems do well on the first set of criteria, while proportional systems do well on the second. Similarly Lijphart (1994) contrasts the value of proportionality maximized by proportional systems with the accountability provided by plurality elections. Katz (1997) goes even further, providing a long list of conceptions of democracy (including less credible variants such as "guided democracy", socialist "people's democracy" and Calhounian veto-group "democracy") and tracing the type of election systems required by each. For the most part there is a studied impartiality between plurality and proportional election systems, although there are exceptions. (Elsewhere Lijphart 1999 links proportional election systems with favorable outcomes in

terms of factors such as economic equality, quality of life and environmental protection, while providing similar outcomes in terms of economic growth and stability. Dummett (1997) while being very critical of first-past-the-post and single transferable vote allows that the choice of replacement depends on competing principles, although he does propose a new system based on a modified Borda procedure. Farrell (2001), while accepting that there is a trade-off between the accountability provided by plurality systems and the accurate representation provided by PR, argues that PR is preferable because the main arguments against PR – that it produces unstable government – is empirically untrue.)

The social choice literature studies the axiomatic properties of voting procedures. However, it has paid little attention to electoral systems, focusing instead on social decision rules. Electoral systems, or more specifically seat share allocation rules, translate voters' indicated preferences into a distribution of seats amongst parties or candidates. Social decision rules, on the other hand, translate preferences into a policy decision or some other outcome. Representative government involves a seat allocation rule by which representatives are chosen (or rules if there are multiple chambers); and a social decision rule by which these representatives make collective decisions.

As noted, May (1952) shows that for two alternatives the only binary social decision rule that satisfies anonymity, neutrality, decisiveness and positive responsiveness is majority rule. In a similar vein, Rae (1969) and Taylor (1969) show that majority rule is the decision rule that minimizes the probability that an agent votes for something that is not enacted or votes against something that is. Straffin (1977) shows that majority rule is the decision rule that maximizes responsiveness to individual preferences.. There is a considerable literature on the logical limits of majority rule. Condorcet (1790) showed that majority rule with more than two candidates may produce cycles (situations where a is socially preferred to b, b is preferred to c, but c is preferred to a). Arrow (1952) generalized this finding showing that under some preference profiles all non-dictatorial, Pareto optimal social decision rules were vulnerable to either cycles or having the outcome changed by the addition of an irrelevant alternative. Plott (1967), McKelvey (1976, 1979) and Schofield (1978) showed that under majority rule cycling would occur unless extremely stringent symmetry conditions for preferences were met. Furthermore, this cycling would include the entire set of alternatives.

As electoral systems are seat allocation rules and not social decision rules, the impossibility results of Arrow, McKelvey and Schofield are not immediately relevant to them. Of course, Van Deemen (1993) shows that if we treat list proportional representation as a social decision rule (saying that if party A receives more seats than party B, then party A is socially preferred over party B), then list proportional representation is susceptible to the same voting paradoxes as plurality voting, resulting from the fact that only first preferences are considered. However, the outcome of seat allocation rules cannot immediately be interpreted as a social welfare ordering. For example in list systems of proportional representation, the fact that party A gets more seats then party B does not unambiguously mean that A's policies are preferred to B's.

Nevertheless, a strong reading of the impossibility results might appear to undermine the relevance of the axiomatic properties of seat allocation rules. If the decisions taken by the representatives chosen by our seat allocation rule are arbitrary, indeterminate or simply the result of agenda manipulation, it would matter little that the means for choosing them was procedurally fair. An argument of this type is made by Riker (1982) in "Liberalism Against Populism". Essentially, Riker interprets the global cycling results to mean that the outcome of the democratic process is meaningless – a different agenda would have produced a different result, and an agenda could have been found to produce any result. Therefore it makes no sense to talk about democracy in populist terms of public control of government. Democracy can only serve as a liberal safeguard – a means by which it is possible to remove oppressive governments.

However, there are two problems with Riker's argument. Firstly there is the question of logical consistency pointed out by Coleman and Ferejohn (1986) – global cycling is just as fatal to a procedural defense of liberalism as it is to populism.[5] Secondly, and for us more importantly, Riker's interpretation of the global cycling results is exaggerated. While it is technically true that an agenda could be constructed to get to any conceivable outcome, the probability of this happening under realistic institutions is extremely

[5] Coleman and Ferejohn (1986) argue that if global cycling means that the results of elections are arbitrary and unstable, then elections cannot provide the defense against tyranny that liberalism demands. If election results are random, they will affect the behavior of government no more than the threat of removal from office by being struck by lightening. To discipline governments, "bad" governments need to be removed from office more often than "good" governments. However, if this is so, then elections provide some information about whether the government is good or popular, which allow a weaker form of populism

low. Miller (1980) shows that under a variety of institutional regimes (open amendment, two party competition, fixed agenda with strategic voting) the outcome will be in the "uncovered set". McKelvey (1986) and Schofield (1999) show that the uncovered set is typically a small centrally located set of alternatives. (We say point a covers point b if a beats b and also beats everything that b beats. A point is uncovered if there is no alternative that covers it. The intuition here is that if an alternative is covered, there is no point in proposing it. The alternative that covers it will do at least as well – and maybe sometimes better – against any other alternative.) The significance of these results to us is that the location of the uncovered set (and thus the eventual outcome) will depend on the preferences of the representatives.[6] Therefore it does matter who the representatives are. Questions of procedural fairness in the seat allocation rule that chooses representatives are not moot.

Since our results only consider single-vote seat allocation rules, they do not serve to defend list PR against procedures that are not single-vote. However, some significant procedures are not single-vote. Mixed-member systems (which combine single-member and proportional elections) became quite common in the 1990s (see Shugart and Wattennberg 2001) and, of course, such a system has been in use in Germany since 1949. However, we will argue that these systems can be easily accommodated within the framework of our analysis, since they essentially function as a combination of two single vote seat share allocation rules. More complex are multi-vote electoral rules that consider more than the voter's first preference. Single transferable vote and the Borda count are such procedures, and both satisfy our criteria of neutrality and anonymity. These procedures are considered in a section below.

We take as a starting point of our analysis the principle of political equality. As stated above, Dahl essentially defines democracy in terms of political equality. Of course, it is possible to define democracy is many ways, and as Katz (1997) argues, this often results in the concept of democracy getting loaded with everything that we view as desirable. It can certainly be argued that conditions beyond political equality are

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back in. Coleman and Ferejohn refer to this as "epistemic populism". Elections do not define the popular will, but merely provide evidence about it.

[6] Note that the eventual outcome is not fully dependent on the preferences of the representatives. So, the democratic process is indeed nondeterminate in the sense that the outcome is not only dependent on the initial profile, but also on the method of choice.

required for a decision making process to be democratic in substantive sense.[7] However, we can define political equality as minimum formal conditions for a process being democratic. It is hard to imagine us calling a process democratic if it explicitly and deliberately gave more power to some people than to others (for example by giving some people extra votes, as opposed to using some complex constitutional feature that might obfuscate the inequity). Certainly the idea of political equality has been central not only the populist conception of democracy in the tradition of Rousseau, but also to the liberal tradition, as evidenced by the US Declaration of Independence and Mill's (1859) assertion that the only way for a person's rights to be secure is for them to be fairly represented. However, even if it is not accepted that political equality is a necessary condition for a democratic decision, the main argument of this paper – that political equality implies proportionality – still holds.

We operationalize political equality in terms of the axioms of neutrality (all parties are treated equally) and anonymity (all voters are treated equally). Neutrality is a minimal criterion for fairness between alternatives or parties. It essentially says that the names of the alternatives or parties should not difference to the result – if all the supporters of party a switch to party b and vice versa, party a should get party b's seats. Note that this does not by itself entail proportionality – an electoral rule that gives the largest party two-thirds of the seats is neutral because it gives two-thirds to *whichever* party happens to have the largest vote. Anonymity requires that the identities of the voters do not affect the result. If two voters (or any number of pairs) were to trade their preferences, this does not affect the result.

It should be noted that the standards of political equality we use are considerably less demanding than those usually used to advocate proportional representation. In particular we do not depend upon any notion of group representation[8] (Still 1981, Lijphart 1977, 1994, Farrell 2001) or what Pitkin (1967) called "descriptive" representation" (the idea that a representative body should be a microcosm of those represented). Neither do we rely on the idea of fair representation for political parties, which has frequently

[7] For example, social training in democratic norms (Dahl 1956); a body of basic rights necessary for the functioning of democratic decision making (Dahl 1988); a participatory environment so that people have the political and cognitive skills to act politically (Pateman 1970).

[8] For example, Still (1981) argues that there are six levels of political equality – universal equal suffrage, equal shares (same number of voters for each representative), equal probability of being decisive, anonymity, majority rule and equal group representation. Still argues that each level implies the previous level, but Grofman (1981) shows that this is not so in all cases.

been the angle taken in the popular discourse on PR. In the UK, for example, the debate is often framed in terms of the "injustice" of the Liberal Democrats receiving a handful of seats when they win between 15% and 20% of the vote. Our results, however, only require an individualist conception of political equality – whether all voters and all parties are treated similarly. Rogowski (1981) argues similarly to us that proportional representation is the only electoral system that satisfies political equality because district systems treat voters unequally.

We show that political equality, as operationalized as the axioms of anonymity and neutrality, logically requires a single-vote electoral system to produce a result essentially identical to pure list PR. This result draws heavily on van der Hout et al. (2002), particularly in the relationship between anonymity, neutrality and the plurality ranking property (if a party receives more votes than another, it must receive more seats). The concept of pure list PR is of course an abstraction. It is defined as an electoral system where the legislative voting weight of each party is precisely proportional to its vote share. No existent electoral system meets this criterion, although some come close. The fact that seats are not infinitely divisible means that there is always some divergence from proportionality, although in principle this could be overcome.[9] Other features of many existing PR systems also reduce proportionality. The division of the electorate into electoral districts has this effect, especially if the districts are small and are unevenly sized (for example, in a three seat district, 50% of the vote would win two seats out of three, whereas in a ten seat district it would probably win 5 seats out of 10).[10] However, in many systems this is mitigated by national compensatory seats that are distributed to restore proportionality. Electoral thresholds also reduce proportionality, in that parties that win fewer votes than the threshold get no representation. Thus the systems that come closest to pure proportionality are national list systems with very low thresholds, such as the systems in the Netherlands and Israel. Nevertheless, although there are no empirical examples of pure list PR, it serves a purpose here as a counterfactual ideal, and there are existing systems that approximate it quite well.

[9] For example, it would be possible to implement pure list PR by giving parties "seat" with fractional voting weight, or more realistically by distributing some seats by a lottery where each party's chance of getting the seat is proportional to the difference between its vote share and the seat share it has received.
[10] This assumes a greatest remainder rule with Droop quota.

Of course, it is not argued here that political equality is the only worthwhile political value, or even that it should be paramount. Certainly we may value the protection of rights and minorities, political stability and political accountability, amongst other things. However, political equality is a core democratic value. It is hard to argue that a system is democratic in any meaningful way if it privileges some voters over others by, say, giving them extra votes. We can show that political equality implies certain kinds of institutions. If political equality is compromised in favor of some other value a case has to be made that the loss of equality increases this other value, and that the trade-off is necessary and worthwhile. This case may be harder to make than is sometimes supposed. We will return to this matter in the conclusion.

Results

We show that political equality (operationalized as the axioms of anonymity and neutrality) implies a single-vote seat allocation rule essentially equivalent to pure list proportional representation. Firstly, we show that any single-vote seat share allocation rule that is positively responsive, neutral and anonymous satisfies the strong plurality ranking property (parties that win more votes get more seats). Then we show that this result applies to coalitions as well as parties (if the seat allocation rule is anonymous and neutral, coalitions whose members win more votes must get more seats in aggregate). In parliaments, governments are typically chosen by majority rule, the vote of investiture usually requiring a coalition of parties. Therefore the outcome depends on the coalition formation game defined by the election and the seat share allocation rule. We show that any seat share allocation rule that is anonymous and neutral (and thus satisfies the strong plurality ranking property) defines a coalition game identical to that defined by pure list proportional representation. We show similar results when the seat share allocation rule is assumed to be nonnegatively instead of positively responsive.

First we show that political equality implies the strong plurality ranking property – if party A wins more votes than party B, then it must receive a greater seat share . Political equality is here defined in terms of the axioms of neutrality (treating every alternative equally) and anonymity (treating every voter equally). We only consider single-vote electoral systems, and assume that the electoral system is positively responsive (if party A gains votes – either from other parties or from people who previously abstained – while everyone else votes in the same way, then the seat share of party A relative to every other party must

increase). This is a strong assumption, as it assumes that the seat share allocation function is perfectly responsive – if a party wins one more vote, it must get a greater seat share as a result, a property no empirical seat allocation rule has.[11] However, we can prove very similar results only assuming nonnegative responsiveness – the common sense assumption that if a party wins more votes and everything else remains equal, this party does not lose seat share.

Proposition 1a: Neutrality, anonymity and positive responsiveness imply the strong plurality ranking property. (Proof in appendix.)

Proposition 1b: Neutrality, anonymity and nonnegative responsiveness imply the weak plurality ranking property. (Proof in appendix.)

The intuition here is straightforward. If two parties have the same number of votes, then by anonymity and neutrality, they must have the same seat share. If in this case one party receives more seats, then either the vote allocation system is inherently biased in its favor (violating neutrality) or some voters' votes count for more than others (violating anonymity). If one party then increases its vote at the expense of the other or by gaining the votes of people who previously abstained, by positive responsiveness it must receive a greater seat share than the other party. Therefore, if one party receives more votes than another it must receive more seats (the strong plurality ranking property). If we only assume nonnegative responsiveness, then when a party gains votes, it most at least not lose seats. This implies that a party that wins more votes than another must get at least an equal seat share (the weak plurality ranking property).

We can extend propositions 1a and 1b to apply not just to individual parties, but to coalitions of parties. In multi-party systems, governments are typically not single-party, but are composed of coalitions. Therefore the relative size of various coalitions is as important as the relative size of parties.

Proposition 2a: Anonymity, neutrality and positive responsiveness imply the strong plurality ranking property for coalitions. (Proof in appendix.)

Proposition 2b: Anonymity, neutrality and nonnegative responsiveness imply the weak plurality ranking property for coalitions. (Proof in appendix.)

The proofs of these propositions are essentially identical to that of proposition 1a and 1b. If one coalition of parties receives more votes than another, it must receive a greater total seat share if we assume

[11] However, it would be theoretically possible to implement seat allocation functions that satisfy positive responsiveness, using either fractionally weighted seat, or assigning some seats probabilistically, so that the expected seat share of each party is positively responsive.

positive responsiveness, and at least an equal total seat share if we only assume nonnegative responsiveness.

In representative bodies, governments are typically formed by a process of majority-rule coalition formation. Which coalition forms is a result of a bargaining process. However, the bargaining situation is defined in terms of which coalitions have sufficient seats to win a majority-rule vote of investiture (or confidence) and form a government. We can show that any seat allocation function that satisfies the coalitional strong plurality ranking property defines a coalition formation game identical to that defined by pure list proportional representation. (Pure list proportion representation is defined as a seat share allocation function that allocates a seat share to each party, which is equal to the share of the total vote that the party won.) Therefore anonymity, neutrality and positive responsiveness imply a single-vote seat share allocation that produces a parliamentary outcome to all intents and purposes identical to that produced by pure list proportional representation.

Proposition 3: Anonymity, neutrality and positive responsiveness imply a seat share allocation function that defines a coalition game identical to that defined by a seat share allocation by pure list proportional representation. (Proof in appendix)

The intuition of the proof comes from the fact that under majority rule a coalition is winning if it has a greater seat share than all the parties excluded from it. By proposition 2a, anonymity, neutrality and positive responsiveness require that if a coalition has more votes than another coalition, it must receive a greater seat share. Therefore the set of winning coalitions must be the set of coalitions whose members have more votes than all the parties excluded by them. This is exactly the same as the set of winning coalitions under pure list proportional representation.

If we only assume nonnegative responsiveness, we get slightly different results depending on whether the number of legislative seats is even or odd. If the number of seats is odd, we get the same result as with positive responsiveness (anonymity and neutrality imply a coalition games identical to that defined by pure list proportional representation). If the number of seats is even, then the set of wining coalitions is a subset of the set of winning coalitions under pure list proportional representation. It is possible for a coalition to win a majority of the vote, but only to receive exactly half the seats, and thus be a blocking, but not winning coalition. However, it is impossible to have a "manufactured majority" (a situation where a

party or coalition with a minority of the vote gets a majority of the seats) without violating anonymity or neutrality.

Proposition 4a: In a legislature with an odd number of seats, anonymity, neutrality and nonnegative responsiveness imply a seat share allocation function that defines a coalition game identical to that defined by seat share allocation by pure list proportional representation. (Proof in appendix.)

Proposition 4b: In a legislature with an even number of seats, anonymity, neutrality and nonnegative responsiveness imply a seat share allocation function that defines a majority rule coalition game with a set of winning coalition that is a subset of that defined by seat share allocation by pure list proportional representation. (Proof in appendix.)

The intuition behind these propositions is similar to the case assuming positive responsiveness. By proposition 2b, anonymity, neutrality and nonnegative responsiveness require that if a coalition has more votes than another coalition, it must receive at least an equal seat share. However, if the number of seats is odd, a coalition cannot have an equal number of seats to the parties excluded from it.[12] Therefore the coalition with more votes must have more seats, and any winning coalition under pure list proportional representation must also be a winning coalition under our seat share allocation rule. If the number of seats is even, however, it is possible for a coalition to have an equal number of seats to its complement. Therefore it is possible for a coalition to win a majority of the vote, but to only receive exactly half the seats. Any winning coalition under our seat share allocation rule is a winning coalition under pure list proportional representation, but not vice versa.

Multiple-Vote Electoral Systems

Our results apply to single-vote electoral systems. However, electoral systems that ask voters for several choices are quite common. We can classify these multiple-vote electoral systems into two groups. Firstly there are so called "mixed-member" systems (Shugart & Wattenberg 2001). These typically give voters a vote for a candidate to represent their district and a vote for a party, but only ask for the voter's first choice in each category. We will argue that these systems can easily be accommodated within our framework. Secondly there are rules that ask voters to rank candidates or parties. Notable examples of these

[12] Note the special case where one party or coalition wins precisely (not one vote more or less) 50% of the vote. In this case it is impossible to satisfy anonymity and neutrality with an odd number of seats. Anonymity and neutrality imply that a coalition that wins the same number of votes as its complement wins the same number of seats, but this is impossible if the number of seats is odd.

are single transferable vote (STV) and the Borda procedure. These systems cannot be accommodated within our framework. We will argue that List PR can be defended against these systems on other grounds.

Mixed member systems can be divided into two groups – mixed member plurality and mixed member proportional (Shugart &Wattenberg 2001). An ideal type mixed member plurality system typically allocates a certain number of seats to district elections and a certain number to proportional election, with no compensation between the two. So, this system essentially functions as a combination of two single vote seat share allocation rules. The proportional part of the election largely respects the principles of anonymity and neutrality, while the district election typically does not. Consequently, the overall result will violate the principles of political equality we have defined. An ideal-type mixed member proportional system, on the other hand, distributes seats from the proportional part of the election in a compensatory manner, so that overall seat totals of each party from both stages approximates proportionality. As a result, these systems essentially function as List PR and thus respect political equality.[13]

Our results do not apply to systems that ask the voters to rank alternatives. Furthermore, at least two notable examples of such systems – single transferable vote and the Borda procedure – satisfy both anonymity and neutrality. However, we can defend List PR against these systems on other grounds. First, we can make a normative case for considering only the first preference of voters in allocating seats in legislatures. Under a pure proportional system, everyone gets a representative of the party they chose, however small. In a sense there is no need to consider second place preferences because everyone gets their first preference. Furthermore, the fact that my preferred representative is your very least preferred is irrelevant – their job is to represent me, not you. While this argument is plausible, it requires a stronger theory of representation, such as Powell's (2000) concept of authorized representation. The assumptions required are far more demanding than the minimal requirement of political equality we required in the last section.

[13] Subject to other institutional features. For example, Germany has a mixed member system of this type. The results are approximately proportional, except that there is a 5% electoral threshold, and that parties who win more district seats than their overall vote share would dictate are allowed to keep the "excess" seats.

Secondly, we can discuss the use of these mechanisms as seat allocation rules. They are both anonymous only if applied to a single district. This is particularly significant for single transferable vote, which is usually applied to many, rather small districts (in Eire, the size varies between 3 and 5), rather than a single national district. Farrell (2001) finds that single transferable vote in Ireland does not produce strict proportionality, but that it is far closer to it than plurality elections. It is possible to apply single transferable vote to national elections, but this would result in ballots with hundreds, if not thousands of candidates. One solution to the resulting problem of unwieldiness would be to allow voters to vote a straight party ticket. However, if most voters act in this way, the results will be virtually identical to list PR.

The Borda count[14] satisfies the requirements of anonymity and neutrality, but has some other features that make it extremely problematic as a seat allocation rule. This is not surprising, as it was originally proposed as a rule for ranking candidates, not for distributing representatives. Suppose we have two parties a and b, each of which has two voters who favor it:

Voter	1	2	3	4
	a	a	b	b
	b	b	a	a

Parties a and b both receive a Borda count of 2, and thus receive an equal allocation of seats. However, now let us assume that a faction breaks away from party a to form party a`, giving us the following preference distribution:

Voter	1	2	3	4
	a	a`	b	b
	a`	a	a`	a
	b	b	a	a`

Party a, party a` and party b now all get a Borda count of 4, so all get equal representation. However, this means that the combined representation of parties a and a` is now double that of b. By dividing in two, the original party a has increased its representation at the expense of b. This property of the Borda procedure makes sense when we are ranking candidates – if a new candidate enters the race who is almost identical to a, that candidate should score almost identically to a. However, it is not a desirable quality when distributing seats, because it does not take into account the similarity of candidates or parties. This leads to

some potentially undesirable consequences, such as encouraging party fragmentation and possibly excluding minority representation.[15] Apart from these consequences, the results will be arbitrary as they depend as much on the number of candidates of each type running as on the preferences of the voters.

So, although our results only apply to single vote seat allocation rules, List PR can easily be defended against multiple vote rules too. Single transferable vote satisfies anonymity and neutrality, but only when applied to a large, single district. In this case, either elections would either be unwieldy or results would be produced that are very similar to those of list PR. The Borda procedure also satisfies anonymity and neutrality, but is problematic as a seat allocation rule because it ranks candidates relative to each other and not relative to an overall distribution of representation. We would also take issue with the view that rules that consider a voter's entire preference profile are normatively superior to rules that only consider first preferences, as argued by various authors, including Dummett (1997). The case for this point of view is far from obvious. If we are picking a single candidate, then it would indeed seem arbitrary to count only first choice preferences. However, if we are selecting legislators as representatives of voters, and a system is sufficiently proportional that all voters can have their "first choice" of representative, a strong case can be made that lower-order preferences should be ignored.

Discussion

We have shown that any single-vote seat share allocation rule that satisfies anonymity, neutrality and positive responsiveness must produce results that are to all intents and purposes identical to those of pure list proportional representation. We obtain similar results assuming nonnegative instead of positive responsiveness. This has notable consequences for the theory of democracy. Following Dahl (1956) we can define democracy in terms of popular sovereignty and political equality. Political equality implies that the decision making system should be neutral (all parties are treated equally) and anonymous (all voters are treated equally). In a representative system, the decision making process is divided into two stages: Firstly, elections choose representatives; secondly, representatives make decisions. To satisfy political equality of

[14] If there are n candidates, the Borda procedure gives n-1 points to each voters first choice, n-2 to their second, etc. The scores are then totaled.

the first stage, we have shown that the seat allocation function has to produce results essentially identical to list proportional representation. At the second stage, we already know from May (1952), that the only binary decision-making rule that satisfies anonymity and neutrality is majority rule. Therefore if the goal is to maximize democracy-as-political-equality, axiomatic democratic theory gives us a clear institutional prescription.

This brings us to the limits of axiomatic theory. Few people would argue that democracy-as-political-equality is the only value to be considered in producing a "good" political system. Indeed, Schumpeter (1942) argued that it was not a value worth considering at all, and that democracy should be considered instrumental in terms of producing other values. Certainly we would have to balance the value of political equality against values such as the protection of minority rights, stability and accountability. The question of how such values and others can be balanced is not pursued here. However, it is far from clear that the trade-off between democracy-as-political-equality and these other values is a severe one, if indeed there is a trade-off at all.

There does not appear to be a trade-off between political equality and minority protection in terms of representation. Indeed, proportional representation would appear at first glance to be the electoral system that is most favorable to minorities, in that minorities will receive more representation than in plurality systems, which tend to give a considerable advantage to the largest two parties. There is a tradition going back to Mill (1859) arguing that the only way a group can be secure in its rights is for it to be fully and completely represented. Similarly, Lijphart (1977, 1984, 1999) argues that proportionality is central to what he terms the consensual model of democracy, which he argues is appropriate for plural societies where minority protection is a concern. Katz (1997, 142-3), on the other hand, argues that plurality elections are more suited to a liberal conception of democracy that emphasizes rights protection, on the grounds that it is

[15] To mitigate this, Dummett (1997) suggests a hybrid "quota Borda system" that combines a Borda procedure with a provision that candidates who receive a certain quota of first-place votes are automatically elected.

more possible for a minority to create a short-lived blackmail party that can hold the balance of power. However, Katz presents no evidence for this assertion, and the opposite appears more likely to be case.[16]

The evidence that proportional electoral systems lead to greater political instability is mixed. It is true that average cabinet duration in multi-party governments (typical of proportional representation systems) is shorter than that for single-party governments (typical of single-member district systems) (Lijphart 1984b). However, most of the difference is accounted for by a small number of countries that contribute a large number of short duration cabinets, such as Italy, Israel and Fourth Republic France (see Strom 1990). Powell (2001) finds that policy stability is actually greater in countries with proportional systems, as countries with majoritarian systems have a tendency to oscillate between left and right. In terms of economic stability, Lijphart (1999) finds no difference in the performance of countries with proportional and majoritarian electoral systems.

Various authors (Lijphart 1994, Powell 2000) argue that there is a trade-off between the accurate representation provided by PR and the accountability provided by plurality elections. However, even this trade-off is questionable, especially if plurality elections fail to reduce the number of parties to two and the opposition is divided. It is true that plurality elections are more responsive to changes in support, but only in limited circumstances. As Bingham Powell (2000) argues, what plurality elections do is concentrate responsiveness into a very narrow range – the result is extremely sensitive to changes in support as one party goes from being the larger party to being the smaller. Proportional elections, on the other hand, are more responsive to small changes in support in cases where the parties maintain their rank order. One place where there is likely to be a trade-off between national list PR and district-based systems (whether plurality or PR) is with local accountability. District systems are likely to provide geographically more accurate representation, whereas a national lists tend to produce more accurate representation in terms of other factors, such as gender, ethnicity and, of course, party allegiance (Farrell 2001).

[16] Not only is it far easier for a new party to enter in a proportional system, but small parties may have greater influence on the viability of coalitions. For example, in the Netherlands the entry of the List Pim Fortuyn in 2002 and the two pensioners parties in 1994 prompted changes of governments. Similar stories could be told in the case of Denmark and Israel. It is hard to find similar example in the recent history of the UK or USA.

It is notable that single-member district plurality ("first-past-the-post") elections violate the political equality of voters, not of parties. The axiom that is violated is anonymity, not neutrality. Popular debate about proportional representation has tended to focus on arguments that proportional representation is "fairer" because it treats all parties equally, while first-past-the-post discriminates against small and geographically dispersed parties. However, first-past-the-post is neutral, in that if all the voters for party A were to vote for party B and vice versa, the parties seat totals would be reversed. The inequality in first-past-the-post elections comes from the fact that some people's votes count for more than others. However, the degree to which a vote counts does not depend on the individual characteristics of the voter,[17] but on the distribution of opinion in the constituency in which that voter lives. If the distribution of opinion was random and unpredictable, this might be unproblematic; but this is clearly not the case. Thus the debate on proportional representation could be more appropriately framed in terms of the rights of individual voters rather than in terms of fairness to political parties.

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[17] Unless the electoral districts are of unequal size. However, it is important to note that anonymity is violated even if district size is equal.

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Appendix: Proofs

Let us define the set of eligible voters as N, with voters numbered 1…n, and the set of parties P, numbered 1…p.

The voting correspondence V is defined over the Cartesian product N*P, with $_{i\in N}V_{j\in P}$ reading, "i votes for party j." Assigning the value 1 for true and 0 for false: $\forall i \in N \sum_{j\in P} {}_iV_j \leq 1$. (Each individual either votes for one party or does not vote.)

The function T maps the voting correspondence into the total vote for each party: $T : V \rightarrow [0,n]^p$.

The function E maps the voting correspondence into the seat share for each party: $E : V \rightarrow [0,1]^p$. We will assume that seats are infinitely divisible, to abstract from rounding problems.

We can define the following properties of the seat share function E.

Anonymity:
Let σ be a function that permutes N. Then E is anonymous if $E(V) = E(\sigma V)$.

Neutrality:
Let π be a function that permutes P. Then E is neutral if $\pi E(V) = E(\pi V)$.

Cancellation property:
If $T_j = T_k$, then $E_j = E_k$.
(If the vote share for two parties are the same, then their seat shares must be the same.)

Nonnegative responsiveness:
Let V` be a vote pattern over N*P.
Let V`` = V`, except that some voters or abstainers have switched to party j
$({}_iV`_j \Rightarrow {}_iV``_j; \exists i \in N : {}_iV``_j \wedge \sim {}_iV`_j; \forall i \in N : \sim {}_iV``_j, \forall k \in P, {}_iV`_k \Leftrightarrow {}_iV``_k)$

E is nonnegatively responsive iff $\forall k \in P \frac{E_j(V``)}{E_k(V``)} \geq \frac{E_j(V`)}{E_k(V`)}$.

Positive responsiveness:

E is positive responsive iff $\forall k \in P \frac{E_j(V``)}{E_k(V``)} > \frac{E_j(V`)}{E_k(V`)}$

Weak plurality ranking property:
$T_{j\in P} > T_{k\in P} \Rightarrow E_j \geq E_k$
(If party j wins more votes than party k, party j receives a greater or equal seat share to party k.)

Strong plurality ranking property:
$T_{j\in P} > T_{k\in P} \Rightarrow E_j > E_k$
(If party j wins more votes than party k, party j receives a greater seat share than party k.)

Lemma 1: Anonymity and neutrality imply the cancellation property

Proof:
The proof is isomorphic to the proof of lemma 2 in Van der Hout et al. (2002).
We need to show that
$E_{j\in P} = E_{k\in P}$ for all i, k ∈ P when $T_j = T_k$ and anonymity and neutrality hold.

If $T_j = T_k$, we can permute all voters so that the voters of parties j and k change places. We will call the permutation function σ`.
By anonymity, the seat shares stay the same:
$E_j(\sigma`V) = E_j(V), E_k(\sigma`V) = E_k(V)$.
We can permute the parties so that parties j and k change places. We call this permutation function π`.
By neutrality, the seat share again remains the same:
$E_j(\pi`\sigma`V) = E_k(\sigma`V)$.

However, by construction $\pi`\sigma`V = V$. (We have swapped the supporters of parties j and k, and then swapped the names of the parties, so we are back to the original situation.)

Therefore, $E_j(\pi`\sigma`V) = E_j(V) = E_k(\sigma`V) = E_k(V)$. QED.

Proposition 1a: Neutrality, anonymity and positive responsiveness imply the strong plurality ranking property.

Proof:
The proof is isomorphic to the proof of theorem 2 in Van der Hout et al. (2002).We need to show that neutrality, anonymity and positive responsiveness imply $T_{j\in P} > T_{k\in P} \Rightarrow E_j > E_k$.
By lemma 1, anonymity and neutrality imply the cancellation property:
That is, $T_j = T_k \Rightarrow E_j = E_k$.
Let V`` be a vote pattern where $T_j > T_k$.
Then we can derive a vote pattern V` where $T_j = T_k$ by having an appropriate number of voters for party j abstain, so that the vote for parties j and k are equal.
By the cancellation property: $E_j(V`) = E_k(V`)$.
By positive responsiveness, $\frac{E_j(V``)}{E_k(V``)} > \frac{E_j(V`)}{E_k(V`)}$.
$\therefore E_j(V``) > E_k(V``)$. QED.

Proposition 1b: Neutrality, anonymity and nonnegative responsiveness imply the weak plurality ranking property.

Proof: We need to show that neutrality, anonymity and nonnegative responsiveness imply
$T_{j\in P} > T_{k\in P} \Rightarrow E_j \geq E_k$.
Define vote patterns V` and V`` as in the proof of proposition 1a.
By the cancellation property: $E_j(V`) = E_k(V`)$.
By nonnegative responsiveness, $\frac{E_j(V``)}{E_k(V``)} \geq \frac{E_j(V`)}{E_k(V`)}$.
$\therefore E_j(V``) \geq E_k(V``)$. QED.

Extension to Coalitions

Let us define the set of coalitions $C = Yc : c \subseteq P$.

Let us define the correspondence K on N*C, where ${}_{i\in N}K_{c\in C}$ reads i votes for a party in coalition C.

(Note that while an individual can only vote for one party, that individual may contribute to the vote total of multiple potential coalitions.)

Let us define the coalition vote total function $U : K \to [0,n]^{2^p}\left(= \sum_{i\in N} {}_i K_c\right)$,

and the coalition seat share function $F : K \to [0,1]^{2^p}$.

We can redefine the properties used in proposition 1 for use with coalitions:

Anonymity: Let σ be a function that permutes N. Then F is anonymous if $F(K) = F(\sigma K)$.

Neutrality:

Let π be a function that permutes P. then F is neutral if $\pi F(K) = F(\pi K)$.

Cancellation property:

If $U_j = U_k$, then $F_j = F_k$.

(If the vote totals for two coalitions are the same, then their seat shares must be the same.)

Nonnegative responsiveness:

Let K` be a vote pattern over N*C.

Let K`` = K`, except that some voters have switched to parties in coalition j:

$({}_i K`_j \Rightarrow {}_i K``_j; \exists i \in N : {}_i K``_j \wedge \sim {}_i K`_j; \forall i \in N : \sim {}_i K``_j, \forall k \in C, {}_i K`_k \Leftrightarrow {}_i K``_k)$

E is nonnegatively responsive iff $\forall k \in C \frac{F_j(K``)}{F_k(K``)} \geq \frac{F_j(K`)}{F_k(K`)}$.

Positive responsiveness:

E is positively responsive iff $\forall k \in C \frac{F_j(K``)}{F_k(K``)} > \frac{F_j(K`)}{F_k(K`)}$

Weak plurality ranking property:

$U_{j\in P} > U_{k\in P} \Rightarrow F_j \geq F_k$

(If the parties in coalition j win more votes than the parties in coalition k, coalition j receives a greater or equal aggregate seat share to coalition k.)

Strong plurality ranking property:

$U_{j\in P} > U_{k\in P} \Rightarrow F_j > F_k$

(If the parties in coalition j win more votes than the parties in coalition k, coalition j receives a greater aggregate seat share than coalition k.)

Lemma 2: Anonymity and neutrality imply the cancellation property for coalitions.

Proof: Isomorphic to lemma 1.

Proposition 2a: Anonymity, neutrality and positive responsiveness imply the strong plurality ranking property for coalitions.
Proof: Isomorphic to proposition 1a.

Proposition 2b: Anonymity, neutrality and nonnegative responsiveness imply the weak plurality ranking property for coalitions.
Proof: Isomorphic to proposition 1b.

Majority rule coalition formation

Let us define pure list proportion representation as a seat share allocation function that allocates a seat share (a real number in [0, 1]) to each party, which is equal to the share of the total vote that party won.

Coalition games can be defined in terms of the set of winning coalitions $W \subseteq C$.
Under majority rule, a coalition is winning iff it has a majority of seat share: $c \in W \text{ iff } F_c > F_{P-c}$.
Under pure list proportional representation, a coalition wins a majority of seat share iff it has more than 50% of the vote: $c \in W \text{ iff } U_c > U_{P-c}$.

Proposition 3: Anonymity, neutrality and positive responsiveness imply a seat share allocation function that defines a coalition game identical to that defined by a seat share allocation by pure list proportional representation.

Proof:
Given that the set of majority rule winning coalitions is defined as $\{c \in W : F_c > F_{P-c}\}$, and under pure list proportional representation the set of coalitions with a majority of seat share is $\{c \in W : U_c > U_{P-c}\}$, we need to show that if anonymity, neutrality and positive responsiveness hold, $\forall c \in C : U_c > U_{P-c} \Leftrightarrow F_c > F_{P-c}$.
Anonymity, neutrality and positive responsiveness imply the strong plurality ranking property: $U_c > U_{P-c} \Rightarrow F_c > F_{P-c}$.
We can also show $\forall c \in C : U_c > U_{P-c} \Leftarrow F_c > F_{P-c}$:
Suppose $F_c > F_{P-c}$, but $U_c \leq U_{P-c}$. If $U_c < U_{P-c}$ then by the strong plurality ranking property, $F_c < F_{P-c}$. Contradiction. Anonymity and neutrality also imply the cancellation property. If $U_c = U_{P-c}$, then by the cancellation property $F_c = F_{P-c}$. Contradiction. QED.

Proposition 4a: In a legislature with an odd number of seats, anonymity, neutrality and nonnegative responsiveness imply a seat share allocation function that defines a coalition game identical to that defined by seat share allocation by pure list proportional representation.

Proof:
We need to show that if anonymity, neutrality and nonnegative responsiveness hold, $\forall c \in C : U_c > U_{P-c} \Leftrightarrow F_c > F_{P-c}$.
(1) we show $\forall c \in C : U_c > U_{P-c} \Rightarrow F_c > F_{P-c}$:
Anonymity, neutrality and nonnegative responsiveness imply the weak plurality ranking property.
By the weak plurality ranking property, $U_c > U_{P-c} \Rightarrow F_c \geq F_{P-c}$.
However the fact that there is an odd number of seats means that a coalition and its complement cannot have an equal number of seats, so the inequality is strict and $F_c > F_{P-c}$. QED.

(2) we show $\forall c \in C : U_c > U_{P-c} \Leftarrow F_c > F_{P-c}$:

Suppose $F_c > F_{P-c}$, but $U_c \leq U_{P-c}$. If $U_c < U_{P-c}$ then by the weak plurality ranking property, $F_c \leq F_{P-c}$. Contradiction. Anonymity and neutrality also imply the cancellation property. If $U_c = U_{P-c}$, then by the cancellation property $F_c = F_{P-c}$. Contradiction. QED.

Note: In the special case where a coalition wins precisely 50% of the vote, anonymity and neutrality are incompatible with there being an odd number of seats. Anonymity and neutrality imply the cancellation property. The cancellation property states demands that if a coalition and its complement get equal votes they must receive equal seat shares (as does pure list PR). This, however, is not possible with an odd number of seats. As (anonymity ^ neutrality ^ nonnegative responsiveness) is necessarily false in this case, proposition 3 is necessarily true.

Proposition 4b: In a legislature with an even number of seats, anonymity, neutrality and nonnegative responsiveness imply a seat share allocation function that defines a majority rule coalition game with a set of winning coalition that is a subset of that defined by seat share allocation by pure list proportional representation.

Proof:

Given that the set of majority rule winning coalitions is defined as $\{c \in W : F_c > F_{P-c}\}$, and under pure list proportional representation the set of coalitions with a majority of seat share is $\{c \in W : U_c > U_{P-c}\}$, we need to show that anonymity, neutrality and nonnegative responsiveness imply: $\forall c \in C : U_c > U_{P-c} \Leftarrow F_c > F_{P-c}$. This is shown in part (2) of the proof of proposition 4a.