

GEOMETRY OF CHAOTIC AND STABLE DISCUSSIONS

DONALD G. SAARI

1. INTRODUCTION

It always seems to be the case. No matter how hard you might work on a proposal, no matter how polished and complete the final product may be, when it is presented to a group for approval, there always seems to be some majority who wants to “improve it.” Is this just an annoyance or is there a reason? The mathematical modeling provides an immediate explanation in terms of some interesting and unexpected mathematics. Even more; the mathematics describing this behavior underscores the reality that it can be surprising easy even for a group sincerely striving for excellence to make inferior decisions. Indeed, these difficulties are so pervasive and can arise in such unexpected ways that it is realistic to worry whether groups you belong to have been inadvertently victimized by these mathematical subtleties based on the orbits of symmetry groups. These problems can occur even if all decisions are reached by consensus during discussions, such as a committee discussing the selection of a new calculus book.

This paper addresses deliberations by discussing a branch of voting theory where Euclidean geometry models an “issue space.” When describing how it is possible to unintentionally make inferior choices, we will encounter mathematical behaviors remarkably similar to “attractors” and “chaotic dynamics” from dynamical systems. Since the coexistence of chaotic and stable behavior is common in the Newtonian N -body problem and dynamical systems, it is interesting that this combination also coexists in the “dynamics of discussions.” Another connection arises when configurations central to the N -body problem play a suggestive role in the analysis; at another step we use singularity theory. What adds to the delight of this topic is that while the mathematics can be intricate, the issues can be described at a classroom level where some even lead to student level research projects.

2. SYMMETRY AND SOME OF ITS CONSEQUENCES

A convenient way to introduce the mathematical structures which cause problems is with an example (Saari [14, 15]) explicitly designed to underscore the reality that an election outcome need not reflect the views of the voters. To emphasize that outcomes can drastically change with the choice of an election procedure, I often joke that

“For a price, I will come to your organization before your next election. You tell me who you want to win. After talking with members of your group, I will design a voting procedure that involves all candidates where your designated choice will be the sincere winner.”

This is a written version of an invited 2003 Mathfest talk in Boulder Colorado. The research reported in this paper was supported by various NSF grants including DMI-9971794.

To illustrate, suppose in a department of 30, the voter preferences for candidates $\{A, B, C, D, E, F\}$ (where “ $A \succ B$ ” means that A is strictly preferred to B) are

Number	Ranking	
10	$A \succ B \succ C \succ D \succ E \succ F$	(2.1)
10	$B \succ C \succ D \succ E \succ F \succ A$	
10	$C \succ D \succ E \succ F \succ A \succ B$	

It is trivial to find a way to elect C . The real challenge is to elect F ; this is because *everyone* prefers C , D and E to F , so this group clearly views F as the inferior choice. But, notice what happens should candidates be compared pairwise with the loser eliminated and the winner advanced to be compared with the next candidate. If we start by comparing D and E , D wins unanimously; comparing D with C , C wins unanimously; comparing C with B , B wins with a two-thirds vote; comparing B with A , A wins with a two-thirds vote; and in the final comparison of A with F , F is the final winner with a two-thirds vote. Nobody likes F , but each comparison is decided with a vote of landslide proportions so who would dare argue with the final outcome?

Such behavior can, and probably often does occur in discussions where the goal is to reach a consensus. Imagine trying to select a calculus book. Let’s see; book E is more expensive than D ; C does a better job of describing limits than D ; B has a better selection of problems than C ; ... The rest of the story is apparent; after selecting book F *everyone* leaves the meeting disappointed not only with the selection but with the tastes and standards of his or her colleagues.

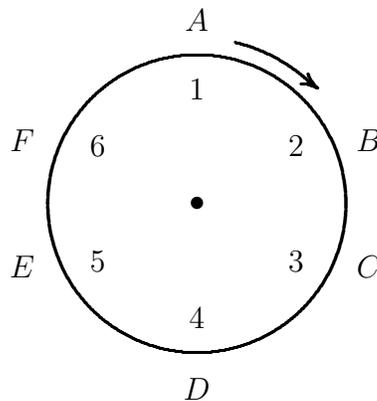


Fig. 1. Ranking wheel and Z_6 orbits

To understand this phenomenon, notice that Eq. 2.1 consists of three of the six terms from the Z_6 orbit of the ranking $A \succ B \succ C \succ D \succ E \succ F$. A convenient way to introduce this structure to non-mathematicians and students is with what I call a *ranking wheel*. Equally spaced, place the numbers 1, 2, up to the number of candidates (six in our case) along the edge of a freely rotating wheel attached to a wall. Then, as illustrated in Fig. 1, write on the wall the names of the candidates as given in an initial ranking. Next, rotate the wheel to move the “1” under the next candidate, B , and read off the second ranking of $B \succ C \succ D \succ E \succ F \succ A$. Continue creating rankings in this manner until each candidate is in first place precisely once. While the construction does not favor any candidate, any three or more rankings from this orbit

create a pairwise voting cycle. With a little experimentation, it becomes obvious how to compare candidates to make any designated choice the “final winner.”

This structure not only provides a way to generate cyclic outcomes; it is the only way. It was recently proved (Saari [13]) that all possible pairwise comparison anomalies with n alternatives, whether used to describe surprising outcomes from tournaments, agendas, pairwise cycles, comparison of pairwise outcomes with other procedures and so forth, arise because the data includes components of Z_n orbits of the alternatives.¹ An extension of these comments explains all those mysteries that occur with other aggregation methods such as the well known difficulties of pairwise comparisons in probability and statistics. More generally, we now know (Saari [13, 14]) that all of those mysterious voting paradoxes, which have been described in many delightful papers but which can cause serious problems with actual elections, occur because embedded within the data (voter preferences) are components of orbits of a wide variety of symmetry groups. Moreover, by imposing a coordinate system on the space of data where some coordinate directions correspond to these symmetry configurations, it follows that almost all data sets must be tainted by these symmetry structures.

If so many unexpected difficulties are caused by the symmetries defined by a finite number of objects, imagine what might occur with a continuum of alternatives constrained only by residing in R^n . An aspect of this issue is explored for the remainder of this paper.

3. ISSUE SPACE

What complicates a selection process are the competing issues. Even when selecting a calculus book we worry about the book size, the cost, the graphics, the exposition, the exercises, and so forth. In national legislation, the issues might involve balancing the amount of money dedicated toward foreign aid and domestic issues including NSF sponsored research. A department’s graduate committee may worry about the level of a TA’s stipend combined with the expected number of hours of work.

Following the lead of Hotelling [4], as extended by others including Enelow and Hinich [2], Kramer [5], McKelvey [7], and Plott [10], the obvious way to model n issues is assign each issue to an axis of R^n . Designate a voter’s level of support for the various issues by a point in R^n called the voter’s “ideal point.” In the graduate student TA example, an ideal point in R^2 represents a voter’s desired level of (stipend, hours of work). Similarly, since a proposal, or an item of legislation, describes a particular combination of the issues, it also is represented as a R^n point. As for voter preferences, it is reasonable to assume that the closer a proposed alternative is to a voter’s ideal point, the more the voter likes it. The first goal is to determine which alternatives will be adopted by a majority vote for a specified set of voters’ ideal points.

To illustrate with a simple example, consider three voters and two issues where the ideal points define the vertices of a triangle in R^2 . Which alternative should these voters adopt? The baricenter? How about favoring a particular voter by selecting her ideal

¹By using this fact, a wide selection of interesting problems can be designed for students. By experimenting with terms from Z_n orbits they can create several examples using pairwise comparisons with counterintuitive outcomes.

point? The surprising fact is that whatever point is selected, a majority of the voters can successfully offer a competing counterproposal! Rephrasing this assertion in terms that many of us have experienced, no matter how refined and complete a proposal may be, during a meeting some majority can propose an “improving amendment” which will pass. At almost any MAA business meeting when bylaws and other legislation are introduced, for instance, expect amendments.

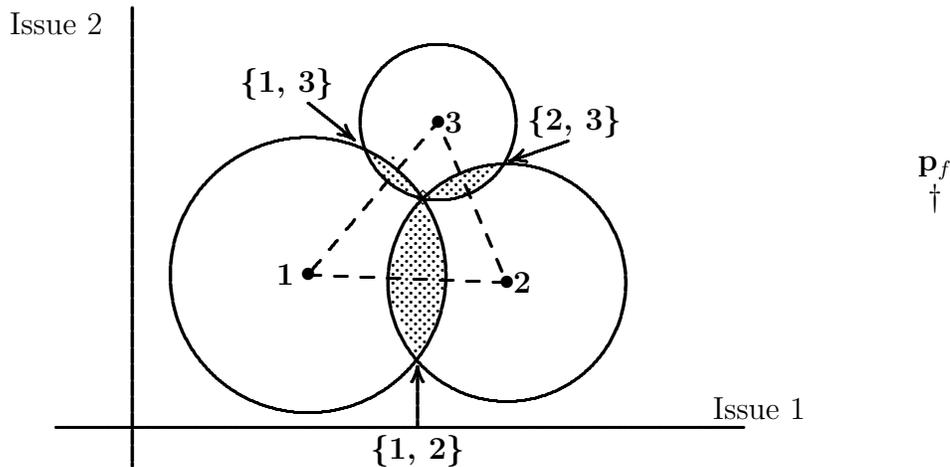


Fig. 2. Forming coalitions

The geometry explaining this situation is demonstrated in Fig. 2 where the ideal points for the three voters, given by bullets, define the vertices of a triangle and the proposal is depicted by the diamond in the interior of the triangle; it is hidden behind the intersection of the three circles. Each circle has its center at a particular voter’s ideal point and passes through the diamond proposal. As all points inside a particular circle are closer to the voter’s ideal point, this voter prefers any of these points to the diamond proposal.

The intersections of the Fig. 2 circles define the trefoil shaded regions. Each shaded leaf identifies points that a majority of the voters prefers to the diamond proposal. The largest region, for instance, consists of alternatives that coalition $\{1, 2\}$ strictly prefers to the proposed diamond, while the upper-right leaf identifies all options strictly preferred by coalition $\{2, 3\}$. In other words, with a surprisingly wide variety of possibilities, any majority can force an “improvement” over the original diamond proposal.

The discouraging observation is that unless the ideal points lie along a line (where an appropriate combination of the issues defines a single issue), circles constructed as above intersect for *any proposal*. Consequently, whatever the new proposal, there always is a majority coalition that can offer an “improved alternative.” The dynamic continues; another majority can be found to propose an even better “improvement” to the just approved “improvement,” and for this proposal . . .

This dynamic forces us to wonder whether, similar to the selection of F in the earlier Z_6 example, a group might adopt an outcome everyone dislikes more than the original proposal. This is the case. Notice, for instance, the dagger hiding in the far right of Fig. 2; as described later there is a sequence of “improvements,” each approved by a majority

vote, starting at the diamond and ending at the dagger. As mathematicians, we need to understand how bad the situation can be and whether other voting procedures offer help. But before addressing these concerns, let me identify what it takes for a proposal to be “durable” in the sense that there are no successful counterproposals.

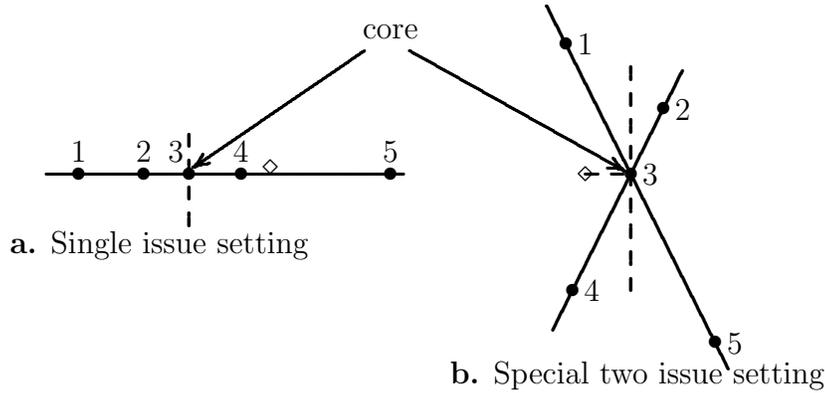


Fig. 3. Finding the core

3.1. Core and attractors. Should the ideal points lie on a line, as in Fig. 3a, certain proposals can never be undone by any majority. As simple experimentation using the “circle geometry” proves, the only durable proposal in the Fig. 3a five voter setting is voter three’s ideal point — the views of the *median voter*. For instance, select any proposal to the right of this ideal point; say the diamond. While voters four and five support the new proposal, all voters with ideal points on or to the left of the dashed vertical line, a majority coalition of $\{1, 2, 3\}$, prefer voter three’s ideal point. More generally, the definition of “median” ensures that any point on one side or the other of the median voter’s ideal point must be supported by less than a majority of the voters; it cannot replace the median voter’s ideal point. This geometry makes it easy to understand why this “median voter” phenomenon is often used to explain the similarity of the political platforms for the major political parties.

In game theory, these durable configurations are called “core points;” this term is used in what follows.

Definition 1. For a specified decision rule which compares two points, if \mathbf{p} is such that no other point is preferred by the decision rule to \mathbf{p} , then \mathbf{p} is called a core point. The core is the set of all core points. If a voter’s ideal point is a core point, call it a “bliss point.”

In Fig. 3a, voter three’s ideal point is the bliss point. Figure 3b depicts a two-dimensional setting that also has a core point; this symmetric five voter arrangement pairs ideal points on opposite sides of straight lines that pass through the third voter’s ideal point. Treat this “Plott configuration” (Plott [10]) as rotating pairs of points off the line where the median voter’s ideal point separates the pair and is the center for the rotation. As true with Fig. 3a, voter three’s ideal point is the Fig. 3b bliss point. To see this, suppose the diamond in Fig. 3b is a counterproposal. Let the line from the diamond to voter three’s ideal point (the horizontal dashed line in the figure) be

the normal vector for the vertical dashed line passing through this point. All voters with ideal points on and to the right of this vertical dashed line, a majority coalition of $\{2, 3, 5\}$ prefers three's ideal point to the diamond.

As Plott's construction can be used in any dimension, for any number of issues there always exists a core point if there are an odd number of voters. The same statement holds for an even number of voters; to see this, add a fictitious "median voter" to create a Plott construction; while this point is not a true ideal point, it is core point for the even number of voters. (In Fig. 3b, for instance, a core point for voters 1, 2, 4, 5 is where voter three — who now is fictitious — had an ideal point.)

The core — in Figs. 3a, b this is a bliss point; voter three's ideal point — enjoys properties reminiscent of an attractor from dynamical systems. To explain, notice that the diamond proposal in Fig. 3a can be beaten by any alternative closer to voter three's ideal point. In turn, the new proposal can be beaten by any proposal even closer to the median voter's ideal point. Similarly, proposals in Fig. 3b that are closer to voter three's ideal point on the horizontal dashed line will beat the diamond proposal. It is this succession of successful proposals and counterproposals converging to the core that resembles an attractor in dynamics.

A natural generalization of the majority rule requires a specified super-majority, such as a two-thirds or four-fifths vote, for victory. This is called a " q -rule."

Definition 2. *For n voters, a q -rule is where an alternative wins if and only if it receives at least q -votes.*

Let $[x]$ be the greatest integer function which rounds a mixed number up to the next integer; the majority rule is where $q = \lceil \frac{n+1}{2} \rceil$. The decision procedure currently used to select a pope for the Catholic Church requires over a two-thirds vote of the cardinals; it is the $q = \lceil \frac{2n+1}{3} \rceil$ rule. Unanimity is where $q = n$. An obvious relationship follows.

Proposition 1. *If \mathbf{p} is a core point for a q_1 rule, then it is a core point for any q -rule where $q > q_1$.*

3.2. Implications of an empty core. If a core ensures stability and even serves as an attractor for the dynamic of proposals and counterproposals, what happens when the core is empty as with the ideal points in Fig. 2? Problems must arise because, by definition, an empty core means that any proposal can be beaten by some other proposal. But even without the stability ensured by a core, it is reasonable to expect these proposals to satisfy some nice property such as remaining within a reasonable distance of the ideal points. This is not the case; the democratic process permitting counterproposals allows discussions to resemble "chaotic dynamics." While counterintuitive, this assertion may have been anticipated by veterans of departmental politics.

To explain, McKelvey [7] used some differential topology to prove the remarkable fact that if the core is empty, it is possible to go, via majority votes, from *any proposal* to *any other proposal*; there are no restrictions on the beginning and final choices.

Theorem 1. *(McKelvey [7]) Suppose the ideal points of the voters have an empty core for the majority vote. For any two proposals \mathbf{p}_b and \mathbf{p}_f , there exists a sequence of counterproposals $\{\mathbf{p}_j\}_{j=1}^N$ which start at the beginning proposal $\mathbf{p}_b = \mathbf{p}_1$ and progress to*

the final proposal $\mathbf{p}_f = \mathbf{p}_N$ by majority votes; e.g., for each $j = 1, \dots, N - 1$, proposal \mathbf{p}_{j+1} beats \mathbf{p}_j by a majority vote.

An immediate consequence of this so-called “chaos theorem” is that there exist proposals beginning at \mathbf{p}_b which pass through any number of specified proposals in any specified order, only to return to \mathbf{p}_b . To illustrate with the Fig. 2 ideal points, a sequence of proposals can be found starting at the diamond, moving high in the second quadrant (assuming that negative values for coordinates have interpretations), next over to the dagger, then progressing all the way to an point in the extreme right side in the fourth quadrant before ending at the original diamond proposal. No wonder some departmental meetings seem interminable and cyclic in substance. (Richards [11] relates the “voting chaos theorem” with “dynamical chaos” by mimicking mathematical approaches used to demonstrate dynamical chaos to recover the voting result.)

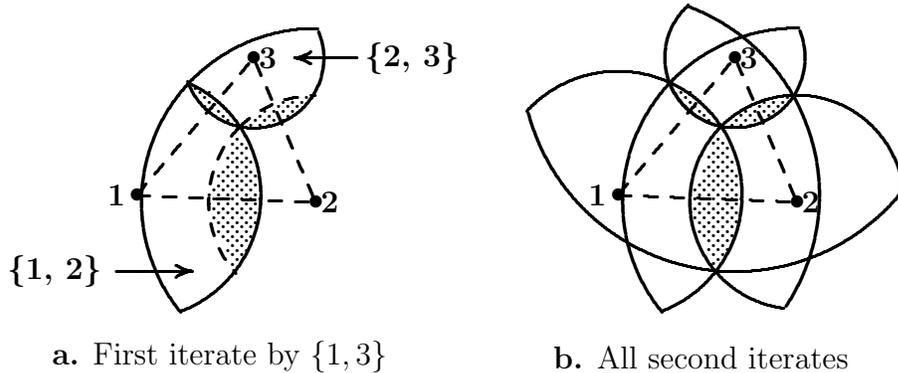


Fig. 4. Second iterate

A way to develop insight into the mathematical structure is to compute some of these agendas. To provide intuition, the shaded region in Fig. 4a contains all points that can beat the original proposal from Fig. 2. To find the possible second counter proposals, start with the points in the upper left shaded leaf. The points that can be reached in two steps passing through this leaf are inside the solid curved arcs of Fig. 4a; this region resembles a pinched pea pod and includes voter three’s ideal point. Notice the dynamic; the first step, a successful amendment made by coalition $\{1, 3\}$, defines a point in this leaf. At the second step either voter one or three creates a coalition with voter two; the possible proposals are determined by a circle using voter two’s ideal point as the center that passes through the choice determined at the first step. Figure 4a describes the extreme situation by using the far tip of the upper left leaf.

To find all points that can be reached in two steps, compute similar regions for the other two shaded leaves of the original trefoil; these three superimposed pinched pea pods are in Fig. 4b. As an illustration of what we learn from this geometry, these ideal points make it possible to go from the diamond proposal to any specified voter’s ideal point with just two “amendments.” To compute what can happen in three steps, use the same approach. Notice how the regions of successful proposals expand quite rapidly.

This chaos theorem supports my earlier “for a price . . . ” claim. Namely, with an empty core and any initial proposal, there always are ways to make counterproposals, each accepted by a majority vote, to eventually reach any specified point. Another interpretation is that, even with the best intensions, voters could keep “improving” a proposal, each by a majority vote, only to reach a final version where *everyone* strongly prefers the original.

3.3. Rate of convergence. But what is the minimum value of N ? For instance, although some meetings may seem incessant, they never allow $N = 10^{100}$ proposals if only because this would require more time than allowed by the age of the universe. Consequently, should the minimal value of N required to go from \mathbf{p}_b to \mathbf{p}_f be sufficiently large, the chaos theorem loses practical significance.

One of my former Ph.D. students, Maria Tataru, investigated this growth rate question in her dissertation [20]. But first she extended the McKelvey “chaos theorem” from the majority vote to any q -rule.

Theorem 2. (*Tataru [20, 21]*) *Suppose the ideal points in R^n for a finite number of voters fail to admit a core for a q -rule. For any two proposals \mathbf{p}_b and \mathbf{p}_f , there exist N proposals $\{\mathbf{p}_j\}_{j=1}^N$ where $\mathbf{p}_b = \mathbf{p}_1$, $\mathbf{p}_f = \mathbf{p}_N$, and where \mathbf{p}_{j+1} beats \mathbf{p}_j with the q -rule; $j = 1, \dots, N - 1$.*

Tataru’s proof differs from McKelvey’s in that, rather than using differential topology, she emphasizes the set-orbit structure introduced by the symmetry of the circles; the symmetries form a group, and she determined the associated orbits. Her orbit structure approach had the advantage of giving her a handle on the number of terms needed to go from one point to another; in this manner she showed that upper and lower bounds on the minimum value of N depend linearly on the distance between \mathbf{p}_b and \mathbf{p}_f . As suggested by Fig. 4, the location of the ideal points determine the size of regions of counterproposals; e.g., size matters, larger regions allow smaller values of N . In turn, a smaller bound on the minimum value of N means that the setting is more chaotic.

Theorem 3. (*Tataru [20, 21]*) *With a finite number of voters whose ideal points fail to admit a core for a q -rule, upper and lower bounds for the value of N can be found which are linear in the distance $\|\mathbf{p}_b - \mathbf{p}_f\|_2$. These bounds are determined by the locations of the ideal points.*

In her thesis [20], Tataru describes how the configuration defined by the ideal points determines the coefficients of the linear expression. To describe her result in terms of three voters, in Fig. 5 let $h = \min(h_1, h_2, h_3)$ be the minimum of the three altitudes defined by the triangle. While she has sharper estimates, Tataru proved that

$$C + \frac{\|\mathbf{p}_b - \mathbf{p}_f\|_2}{h} \leq \text{Minimum } N \leq C + 3 \frac{\|\mathbf{p}_b - \mathbf{p}_f\|_2}{h} \quad (3.1)$$

where C is a constant needed to handle situations where \mathbf{p}_b and \mathbf{p}_f are close to each other; e.g., in Fig. 2, if \mathbf{p}_b is the diamond proposal and \mathbf{p}_f is a point close to \mathbf{p}_b but outside the trefoil, it takes $N = 2$, rather than one iterate, to achieve the goal. (Tataru’s result is a limit theorem so it is more applicable for large $\|\mathbf{p}_b - \mathbf{p}_f\|_2$ values.)

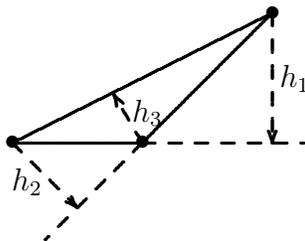


Fig. 5. The minimum altitude and growth rates

According to Eq. 3.1, the smallest bounds on N occur when the ideal points define an equilateral triangle; this setting permits more chaotic behavior. At the other extreme, $h \rightarrow 0$ requires the location of the ideal points to approximate a straight line setting where a core exists; here $N \rightarrow \infty$. Consequently, the closer the ideal points approximate settings where a core exists, the more subdued the successful proposals and counterproposals. While it remains theoretically possible to reach any desired point, it requires an unrealistic number of steps. Tataru’s growth estimates, then, establish a nice link between admissible chaotic behavior and the stability of a core point. Incidentally, while it is easy to form an agenda moving from any beginning \mathbf{p}_b to a final \mathbf{p}_f , I do not believe that an algorithm specifying the “optimal” path (i.e., with a minimum N) has been found or even investigated. But by following the lead of Fig. 4, finding such an algorithm seems to be a doable and reasonable project.

4. GENERIC PROPERTIES OF THE CORE

The core plays a central role in understanding voting behavior for q -rules. As asserted earlier, the combination of Plott’s construction (see Fig. 3) and Prop. 1 ensures that a core exists for any q -rule with any number of issues and voters. But, what about structural stability? Will the core persist with slight changes to the ideal points? Namely, can a trivial shift in just one individual’s preferences push a group’s discussion from the stability setting of a core to McKelvey’s chaotic framework?

To demonstrate that a core need not persist, I use the easily proved fact that with preferences defined by the Euclidean distance, the points preferred by q voters are the points in the convex hull defined by their ideal points. Thus, for $n = 5$, $q = 3$, the core is the intersection of the convex hulls defined by the $\binom{5}{3} = 10$ triplets. We use this fact to show that the Fig. 6 configuration, where voter two’s ideal point is only slightly changed from that in Fig. 3b, has an empty core.

By moving voter two’s ideal point, the Fig. 6 convex hull defined by voters $\{1, 2, 4\}$ (denoted by the dashed lines) meets the convex hull defined by voters $\{3, 4, 5\}$ only in voter four’s ideal point. Similarly, this hull meets the hull defined by voters $\{2, 3, 5\}$ only in voter two’s ideal point. Thus, the common intersection of all hulls — the core — is empty. To see, for instance, that voter three’s ideal point no longer is a core point, notice that the majority coalition of $\{1, 2, 4\}$ prefers any point on the line segment starting from and perpendicular to the line connecting voters’ two and four ideal points and ending in voter three’s ideal point.

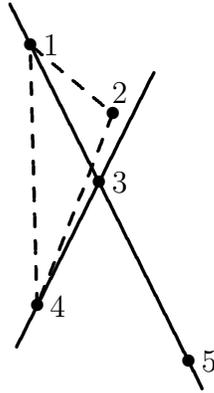


Fig. 6 Dropping the core

4.1. Geometry and the existence of the core. The geometry of convex hulls helps us appreciate when a core can, or cannot, persist with small changes in voter preferences. Intuitively, the more hulls there are, the more difficult it is for them to have a non-empty intersection. But since a $q \geq \lceil \frac{n+1}{2} \rceil$ rule defines $\binom{n}{q}$ possible coalitions and convex hulls, and since $\binom{n}{q}$ increases as q decreases to $\frac{n}{2}$ (the majority rule), we must anticipate that the closer q is to a majority rule, the more difficult it is for a core to exist or persist.

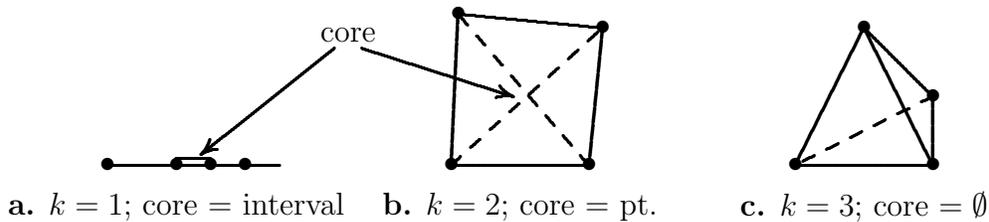


Fig. 7. Changes in the core with dimensions

The dimensionality of issue space also plays a crucial role. With $k = 1$ (a single issue), the core is nonempty because any convex hull involving $q \geq \lceil \frac{n+1}{2} \rceil$ points must include the median voter's ideal point. This need not be true for $k \geq 2$. Indeed, as the majority rule for $n = 3$ requires $q = 2$, the convex hulls are the edges of the triangle defined by the ideal points; as these edges never have a common intersection, a core never exists.

To trace what happens to the core with changes in dimensions, consider the special case of $n = 4$ and $q = 3$. When $k = 1$, the convex hull of any three points (Fig. 7a) must include the two interior ideal points, so the core is the closed interval defined by these two points; this core clearly persists with changes in these points. If $k = 2$ (Fig. 7b), the convex hulls now are triangles consisting of any two edges of the quadrilateral with a common vertex and a diagonal; these hulls meet at the intersection of the diagonals to create a core in a two-dimensional issue space; again, changes in the ideal points only

move the diagonals, so the core persists. (Similarly, an ideal point in the interior of the triangle defined by the other three ideal points must be a bliss point.) But for $k = 3$ dimensions, the four ideal points define a tetrahedron where the hulls are the faces; as the faces never have a common intersection, a core does not exist. So, with changes in k , the core for this example progresses from an interval, to a point, to the empty set.

In general, larger q values define fewer hulls; as these hulls require more points their larger size and dimension make it is easier to have a stable core. Indeed, as the hulls for a q_1 rule are subsets of hulls for a q rule where $q > q_1$, results about the geometric structure of the core, and others such as Prop. 1, follow immediately. Incidentally, these notions probably can be combined to determine which issue space dimensions allow a structurally stable core, but I doubt whether anyone has tried this approach.

4.2. Stability of a core. The natural question is to understand when a core will persist with small changes in the ideal points. Restating the dimensionality comments in common terms, more issues provide more reasons for voters to disagree, so it is more likely for chaos to ensue. Using these terms, this concern was brought to my attention by R. Kieckhefer, a Prof. of the History of Religions at Northwestern University. He pointed out that several times when a simple majority ($q = \lfloor \frac{n+1}{2} \rfloor$) was used to select a pope for the Catholic Church, the precarious instability of the voting system, generated by raising new issues (so k increases) to induce voters to change opinion, caused the church to erupt into dissension and conflict with a pope and an anti-pope vying for power. To achieve stability, in 1179 the Third Lateran Council adopted the current $q = \lfloor \frac{2n+1}{3} \rfloor$ rule; stability was achieved – for a while. This history suggests that the persistence of the core involves finding an upper bound on k , the number of issues, in terms of the values of q and n . As shown next, this is the case.

The importance of this problem caused it to attract considerable research attention. Schofield [18] and then McKelvey and Schofield [8, 9] obtained some bounds on k values. While their conclusion described only a subset of the relevant k values, it was correctly greeted as a major advance. Banks [1] found an error in these papers which, unfortunately, invalidated the conclusions and reopened the problem. The problem was finally completed in Saari [12]; the result is described next.

But first, why should a voter’s preferences be defined in terms of Euclidean distances? After all, a voter placing more importance on one issue than another might measure “closeness” with ellipses. More generally, rather than the Euclidean distance where a person with ideal point \mathbf{q} strictly prefers \mathbf{x} to \mathbf{y} iff $\|\mathbf{x} - \mathbf{q}\|_2 < \|\mathbf{y} - \mathbf{q}\|_2$, that is, iff $-\|\mathbf{x} - \mathbf{q}\|_2 > -\|\mathbf{y} - \mathbf{q}\|_2$, we determine what happens if the j th person’s preferences are defined in terms of an *utility function* $U_j : R^k \rightarrow R$, where \mathbf{x} is strictly preferred to \mathbf{y} iff $U_j(\mathbf{x}) > U_j(\mathbf{y})$, and \mathbf{x} is *indifferent* to \mathbf{y} iff $U_j(\mathbf{x}) = U_j(\mathbf{y})$. Assume that these utility functions are smooth and strictly convex. The Euclidean preferences become the special case $U_j(\mathbf{x}) = -\|\mathbf{x} - \mathbf{q}\|_2$. Other choices might define ellipsoids for level sets to capture individual scaling effects for certain issues. But notice, by generalizing to utility functions, the geometry defining the core may change.

Rather than changing ideal points, we now must determine when a core exists for an open set of utility functions. The topology imposed on the utility functions is the Whitney Topology. (There are several references for this topology and the singularity

theory used next; e.g., see Golubitsky and Guillemin [3] or Saari and Simon [17].) In the following theorem, “generic” means that an assertion holds for a residual set of utility functions in this topology; that is, for a countable intersection of open, dense sets. As one can show (e.g., see Saari and Simon [17]), if the proposals are restricted to a compact subset of R^k , the residual sets can be replaced with open dense sets. To interpret these comments in simpler terms, if k satisfies the specified bounds, then, in general, a core persists even with small changes in preferences. But, if k does not satisfy the bounds, then slight changes in preferences cause the core to disappear. To state the theorem, recall that a “bliss point” is a core point that is a voter’s ideal point. After the formal statement, an easily used, intuitive description is given.

Theorem 4. (Saari [12]) *a. For a q -rule, bliss-core points exist generically iff*

$$k \leq 2q - n. \quad (4.1)$$

b. Nonbliss core points exist, generically, for $k \leq 2$ where $q = 3, n = 4$. For $n \geq 5$, if $4q < 3n + 1$, then nonbliss core points exist generically iff

$$k \leq 2q - n - 1. \quad (4.2)$$

For super-majorities where $4q \geq 3n + 1$, let α be the largest odd integer where $\frac{q}{n} > \frac{\alpha}{\alpha+1}$. Nonbliss core points exist generically iff

$$k \leq 2q - n - 1 + \frac{\alpha - 1}{2}. \quad (4.3)$$

c. For any k and n , there exists a q -rule where core points exist generically. In particular, the unanimity rule $q = n$ exists in all dimensions.

Unless $4q \geq 3n + 1$, which is a super-majority where a successful vote requires more than three-fourths support, expect stable cores of some sort to exist for $k \leq 2q - n$. To interpret this inequality, notice that the maximal value of $k = 2(q - \frac{n}{2})$ corresponds to the number of voters who would have to change their minds to reverse the outcome. In other words, the bound on k identifies the number of issues, one per voter, that need to be raised to change the election outcome. To illustrate with the two-thirds vote adopted by the Catholic Church, with $n = 100$ cardinals, $q = 67$. For the $n - q = 33$ voters on the losing side to reverse the outcome, they must convince $67 - 33 = 34$ voters to change their minds; this number of required defections agrees with $k = 2q - n = 2(67) - 100 = 34$.

To further illustrate with the majority vote and an odd number of voters, as it takes only one voter to change the conclusion, a stable core exists only with a single issue. (Here $q = \frac{n+1}{2}$, so $k = 2\frac{n+1}{2} - n = 1$.) The situation only slightly improves with an even number of voters; it takes two voters to change an outcome, so the core persists for up to two issues. (Here $q = \frac{n}{2} + 1$ so $k \leq 2(\frac{n}{2} + 1) - n = 2$ issues.) As it is difficult to imagine elections with only two issues, these assertions underscore the precarious nature of this standard voting method. Yet, the conclusion seems to conflict with reality; we can enjoy stability in two-candidate elections for, say, mayor. To explain, notice that Thm. 4 describes what happens if there is a freedom to advance different proposals; in contrast, a mayoral or gubernatorial election involves specific candidates so it imposes stability. So, instability requires permitting any proposal, or candidate, to join; an illustration is the 2003 California recall election for governor where 135

competing proposals (candidates) including a self-described “porn-queen” were thrown in the mix. Unintentionally reflecting the mathematics, the press commonly described the situation as “chaotic!”

It remains to describe the Eq. 4.3 “super-majorities.” This equation shows that a super-majority election provides a slight stability bonus by adding extra dimensions to the “number of voters who need to reverse opinions” computation. With a three-fourths rule, the bonus allows the number of issues allowing stability for a bliss and non-bliss core point to agree. With a five-sixth rule, the non-bliss core points exist for a dimension $k = 2q - n - 1 + \frac{5-1}{2} = 2q - n + 1$.

The number of extra dimensions, however, is “slight” when using reasonable super-majorities. After all, the extreme 90% rule, where $\frac{q}{n} > \frac{9}{10}$, adds only four extra dimensions to permit the core to persist with $k = 2q - n - 1 + 4 = 2q - n + 3$ independent issues. The mathematical significance of these bonus dimensions is that as the q rule approaches unanimity, the issue space dimension $k \rightarrow \infty$, as it must. But for the more widely used super-majorities and for Euclidean preferences, use the $k = 2q - n$ value corresponding to the “number of voters who need to reverse opinions” computation. (For Euclidean preferences, an exception occurs for $q = n - 1$ to partially reflect the transition to $k = \infty$ for $q = n$.) Notice a peculiarity; these bonus dimensions never involve bliss points. The reason is given in the next section.

5. OUTLINE OF PROOF

Although the proof of Thm. 4 is technical and long (it requires about 30 published pages), the basic ideas are natural; this is where singularity theory and configurations from the Newtonian N -body problem play a role. To start, with n voters and a specified proposal \mathbf{x} , alternatives preferred by the j th voter are determined by the gradient $\nabla U_j(\mathbf{x})$. Actually, it is not the gradient which matters, but the direction $\nabla U_j(\mathbf{x}) / \|\nabla U_j(\mathbf{x})\|$. (The strict convexity assumption forces the preferred choices to be strictly in the half plane which includes the normal vector $\nabla U_j(\mathbf{x})$.) Thus, the gradient conditions can be described as an arrangement of points on the sphere S^{k-1} .

We need to determine when the core condition is, or is not, robust. To motivate the approach, consider the well known fact that, generically, the critical points of a smooth mapping $F : R^2 \rightarrow R^1$ are isolated. As with the core, this statement combines domain points and the gradient. So, use the five dimensional Jet Space $J^1 = (x, y, z, A, B)$ where $x, y, z, A, B \in R^1$ and J^1 is endowed with the appropriate topology. This J^1 space is intended to capture the domain, range, and first derivative terms, so $j^1(F)(x, y) = (x, y, F(x, y), \nabla F(x, y)) \in J^1$ is a mapping from R^2 to the five-dimensional J^1 .

To describe the generic properties of critical points, we need to find a J^1 subspace which characterizes these points. This is easy; critical points are determined by the three-dimensional subspace $\Sigma = (x, y, z, 0, 0)$ where the gradient is zero. Thus, all critical points of F are given by $[j^1(F)]^{-1}(\Sigma)$. Since Σ has dimension three, or codimension two, the inverse function theorem tells us that when $j^1(F)$ satisfies the appropriate determinant conditions, $[j^1(F)]^{-1}(\Sigma)$ is a codimension two, or a $2 - 2 = 0$ dimensional set in R^2 ; that is, the critical points of F are isolated.

To describe the appropriate determinant conditions, first consider a smooth mapping $G : R^k \rightarrow R^m$ and let Σ be a smooth s -dimensional submanifold of R^m . Locally, Σ is given by $g^{-1}(\mathbf{0})$ where g is an appropriately chosen smooth mapping $g : R^m \rightarrow R^{m-s}$. By composition of maps, when $G(\mathbf{x}) \in \Sigma$, we have that $g(G(\mathbf{x})) = \mathbf{0} \in R^{m-s}$. Consequently, $G^{-1}(\Sigma)$ is given locally by $[g(G)]^{-1}(\mathbf{0})$; this formulation of the problem tells us that $Dg_{G(\mathbf{x})}[D_{\mathbf{x}}G]$ must have full rank $m - s$.

We now need to understand what conditions need to be imposed on G so that $Dg_{G(\mathbf{x})}[D_{\mathbf{x}}G]$ has maximal rank. As $g^{-1}(\mathbf{0})$ defines a local portion of Σ , the kernel of $Dg_{G(\mathbf{x})}$ is the s -dimensional tangent space $T_{G(\mathbf{x})}\Sigma$. Consequently, if $\mathbf{v} \in R^k$ is such that $D_{\mathbf{x}}G(\mathbf{v}) \in T_{G(\mathbf{x})}\Sigma$, then \mathbf{v} is in the kernel of $Dg_{G(\mathbf{x})}[D_{\mathbf{x}}G]$. Thus, to satisfy the rank condition, the image of $D_{\mathbf{x}}G$ must include a $(m - s)$ dimensional linear subspace not in the tangent space $T_{G(\mathbf{x})}\Sigma$. Putting these statements together, an argument now shows that for $Dg_{G(\mathbf{x})}[D_{\mathbf{x}}G]$ to have maximal rank,

$$\text{Span}[D_{\mathbf{x}}G(R^k) \cup T_{G(\mathbf{x})}\Sigma] = R^m. \quad (5.1)$$

This is called a *transverse intersection* of Σ .

So, for the motivating example of finding properties of critical points of F , we need to determine whether $j^1(F)$ has a transverse intersection with the described Σ . Namely, does the combination of Σ with the image of

$$D_{(x,y)}j^1(F) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right); \nabla F; D^2F$$

determining whether D^2F has rank two, but in general the computation can be messy and difficult. This reality underscores the importance of the following result by René Thom [22] (also see [3]). To have a compact statement, the notion of a transverse intersection is extended.

Definition 3. $G : R^k \rightarrow R^m$ has transverse intersection with submanifold $\Sigma \subset R^m$

1. if $\text{Image}(G) \cap \Sigma = \emptyset$, or if
2. Eq. 5.1 is satisfied.

We now can state Thom's theorem.

Theorem 5. (Thom) Let $\Sigma \subset R^m$ be given. Generically (i.e., for a countable intersection of open dense sets), a mapping $F : R^k \rightarrow R^m$ has a transverse intersection with Σ .

Because of the extended definition of a transverse intersection, to use this theorem we need to establish that the first case, where the image of all mappings miss Σ , does not hold. Here we just need to prove that a mapping G exists which meets Σ .

To illustrate the "empty set" condition, suppose we wish to find the generic property of functions $F : R^2 \rightarrow R^1$ where the critical point occurs along the line $x = 1$. The appropriate J^1 submanifold is $\Sigma_1 = \{1, y; z; 0.0\}$; it has codimension three. According to Thom's theorem, generically, mappings have a transverse intersection with Σ_1 . If Eq. 5.1 is satisfied, the set of critical points, $(j^1(F))^{-1}(\Sigma_1)$, would have codimension three in R^2 ; they would be sets of dimension $2 - 3 = -1$ which cannot exist. Consequently, generically, a function does not have a critical point along the line $x = 1$. While

examples with this property are easy to create, slightly perturbing them destroys the property. As shown next, Plott's plots share this mathematical structure.

5.1. Core and the N -body problem. To describe the core conditions, the mapping consists of the utility functions for all n agents; it is

$$\mathcal{U}(\mathbf{x}) = (U_1(\mathbf{x}), \dots, U_n(\mathbf{x})) : R^k \rightarrow R^n. \tag{5.2}$$

where the appropriate jet space is

$$J^1 = (\mathbf{x}; \mathbf{y}; \mathbf{A}_1, \dots, \mathbf{A}_n), \quad \mathbf{x} \in R^k; \mathbf{y} \in R^n, \mathbf{A}_j \in R^k, j = 1, \dots, n$$

and the jet map is

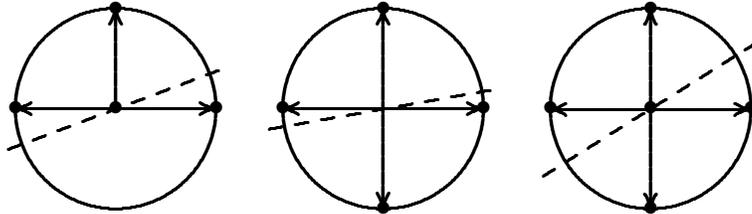
$$j^1(\mathcal{U})(\mathbf{x}) = (\mathbf{x}; \mathcal{U}(\mathbf{x}); \nabla U_1(\mathbf{x}), \dots, \nabla U_n(\mathbf{x})).$$

We must first define subspaces Σ which characterize the core conditions and then determine the codimension of Σ .

The construction is based on the positioning of the gradients, which are treated as points on S^{k-1} . To see what is involved, suppose $n = 4, q = 3$ (the majority rule), and $k = 2$. For a point to be a core point, any line passing through it cannot have more than $q - 1 = 2$ points on one side; if q or more points were on one side, they would define a winning coalition with a preferred alternative. So, position the points on the circle so that for any line passing through the origin (the dashed line in Fig. 8), at most two points are on either side. With a bliss point, as depicted in Fig. 8a, this condition is satisfied with the open condition that any two of the remaining three vectors are separated by more than $\frac{\pi}{2}$. This defines

$$\Sigma_2 = \{(\mathbf{x}; \mathbf{y}; \mathbf{A}_1, \dots, \mathbf{A}_n) \mid \mathbf{A}_1 = \mathbf{0}, \mathbf{A}_j \cdot \mathbf{A}_k < 0, j, k = 2, 3, 4\}.$$

Since the restrictions on the $\mathbf{A}_j, j = 2, 3, 4$, vectors define an open condition, Σ_2 has codimension two (determined by $\mathbf{A}_1 = \mathbf{0}$); thus, generically, the core consists of isolated points; e.g., the location of the bliss point cannot define a curve.



a. $n = 4$, bliss point **b.** $n = 4$, symmetry **c.** $n = 5$, Plott

Fig. 8. Finding Σ

I leave it to the reader to show for the non-bliss point setting of Fig. 8b, this condition, where no three points are on the same side of any dashed line forces the points to be positioned precisely 90° apart. As this defines conditions such as $\mathbf{A}_1 = -\mathbf{A}_3; \mathbf{A}_2 = -\mathbf{A}_4$, the corresponding Σ has codimension two; thus the core consists of isolated points and persists with small changes in preferences.

The Plott configuration for $n = 5, q = 3$ depicted in Fig. 8c differs. Since no more than $q - 1 = 2$ points can be on the same side of any line drawn through the center,

even placing points equally spaced $\frac{2\pi}{5}$ radians apart (a codimension four construction) fails the requirement; the condition can be satisfied only with a bliss point. Namely, the corresponding Σ must have $\mathbf{A}_1 = \mathbf{0}$, a codimension two condition, with a symmetry condition on the remaining points where $\mathbf{A}_2 = -\mathbf{A}_4$; $\mathbf{A}_3 = -\mathbf{A}_5$; this defines a codimension four setting. Generically, then, the core is empty; e.g., similar to requiring a critical point at $x = 1$, it can happen but it will not persist.

This construction answers a question about the $\alpha/2$ bonus dimensions of Thm. 4; can they occur with bliss points? Yes, just place a voter's ideal point at the core point. But, while such core points exist, they do not generically. To explain, because a bliss point already contributes codimension k to the Σ structure, all constraints on the remaining vectors must define open conditions; the constraints defining the "bonus" dimensions, however, are lower dimensional.

The challenge, then, is to determine whether n points can be positioned on S^{k-1} so that no more than $q - 1$ of them are on the same side of any hyperplane passing through the origin *and* the codimension of the associated Σ is no more than k . For $k = 2$, the construction uses a circle, so the analysis is easy. For $k = 3$ the analysis is slightly harder; e.g., the symmetric positioning of four points on S^2 is a tetrahedron, but how should five, or six, points be positioned and can this be done using "open" conditions?

For $k > 3$, the challenge is more interesting. This is where insights from my research in the Newtonian N -body problem helped. To suggest the connection, central to the N -body problem are configurations known as "central configurations." An interesting fact is that there exist solutions which maintain these equilibrium configurations for all time. Now think of n equal masses placed on a sphere S^{k-1} where a plane is passed through the origin. Should there be two or more points on one side of the plane than the other, the configuration most surely is not in "equilibrium." Considerations of this type motivated the final constructions needed to prove the theorem.

6. CONCLUSION

Voting is something we all do often, yet, as shown here and in Saari [14], the process is fraught with dangers. But voting is only one of many mathematical concerns from the social and behavioral sciences. While the consequences and modeling of these difficulties belong to the social sciences, many of these issues are highly mathematical. Indeed, I expect that the only way many of these crucial issues will ever be resolved is through the muscle power of mathematics. In other words, more mathematicians need to get involved. There are delightful rewards; the mathematics can be fascinating, and the results often prove to be of interest and importance to many people.

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UNIVERSITY OF CALIFORNIA, IRVINE, IRVINE, CA. 92697
E-mail address: dsaari@uci.edu