Discussion Point

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Don Saari asked me to post this answer on the conference web site.

A question was raised by the discussant about the physical meaning of negative eigenvalues produced by the Hamiltonian used in the Trueblood & Busemeyer (2011, Cognitive Science) quantum model of probabilistic inference. The answer to the discussant was that one can always add a constant to all of the eigenvalues of the Hamiltonian without changing any model predictions. Thus one can simply add a constant the makes all the eigenvalues positive if one wishes.

Take for example the Pauli matrix used in physics for spin

$$\sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues are -1,+1. We can make the eigenvalues both positive by

$$\sigma_{zz} = \begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$$

which now has eigenvalues +9,+11. However the predicted probabilities produced by σ_z will be the same as σ_{zz} and so this has no observable effect. The proof is described below.

First of all, we note that if the matrix H is Hermitian ($H^\dagger = H$) then we can diagonlize it as $H = V \cdot L \cdot V^\dagger$ with$V$representing the orthonormal matrix of eigenvectors ($V^\dagger V = I = VV^\dagger$) and L is a diagonal matrix of real eigenvalues. It is also true that if H is Hermitian, then the matrix exponential $U = exp\left(-i \cdot H\right)$ is unitary such that $U^\dagger U = UU^\dagger = I$. The matrix $U = exp\left(-i \cdot H\right)$ can be diagonalized as follows: $U = V \cdot e^{-i \cdot L} \cdot V^\dagger$ where $V$is the matrix of eigenvectors of $H$and L is the diagonal matrix of eigenvalues of H. Then it follows that

$$\begin{aligned} U^\dagger U &= V \cdot e^{+i \cdot L \cdot} V^\dagger \cdot V \cdot e^{-i \cdot L} \cdot V^\dagger \\ &= V \cdot e^{+i \cdot L} e^{-i \cdot L} V^\dagger \\ &= V \cdot V^\dagger = I. \end{aligned}$$

A similar argument is used to establish $UU^\dagger = I$.

Now suppose H has a diagonal matrix of eigenvalues L. Then we can set new eigenvalues equal to $K = L + k \cdot I$, where k is a real number, which produces a new

Hermitian matrix $J = V \cdot K \cdot V^{\dagger} = V \cdot (L + k \cdot I))V^{\dagger} = V \cdot L \cdot V^{\dagger} + k \cdot I = H + k \cdot I$. The unitary matrix produced by this addition equals

$$U_K = exp\left(-i \cdot K\right) = exp\left(-i \cdot (H + k \cdot I)\right).$$

If two Hermitian matrices $H_1,\ H_2$ do not commute then $exp\left(H_1 + H_2\right) \neq exp\left(H_1\right) \cdot exp\left(H_2\right)$, but if they do commute then $exp\left(H_1 + H_2\right) = exp\left(H_1\right) \cdot exp\left(H_2\right)$. Now it is the case that H commutes with I so that we can write

$$U_K = exp\left(-i \cdot H\right) \cdot exp\left(-i \cdot k \cdot I\right) = e^{-i \cdot k} \cdot exp\left(-i \cdot H\right) = e^{-i \cdot k} \cdot U_H,$$

where $e^{-i \cdot k}$ is a scalar, that is a common phase factor. This common phase factor has no effect once we take the squared modulus to compute probabilities because $\left|e^{-ik}\right| = 1$.

To see this, suppose the initial state equals ψ and it evolves according to the unitary operator U_K and define the projector for some event as the projection matrix M. Then the probability of this event after unitary evolution equals

$$p\left(M\right) = \left\|M \cdot U_K \cdot \psi\right\|^2 = \left\|e^{-ik} \cdot M \cdot U_H \cdot \psi\right\|^2 = \left\|M \cdot U_H \cdot \psi\right\|^2.$$

In sum, adding a constant to all the eigenvalues simply produces a common phase shift which disappears when we take the squared modulus.