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Title of my talk: On Market Risk Premia

Material for talk is contained in Pages 13 - 37 of this paper

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Private Information and Diverse Beliefs: How Different?

by

Mordecai Kurz, Stanford University
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<u>Abstract:</u> The paper contrasts implications of theories which explain diverse belief by asymmetric private information (in short PI) with theories which postulate agents use heterogenous probability models (in short HB). We focus on the use of such theories in solving problems where agents forecast economic aggregates such as the S&P500, growth rate of GDP, inflation rates, etc. We study implications of the two asset pricing theories with a model where each agents trades for two periods. Such a model has been used extensively in the literature on asset pricing (e.g. Brown and Jennings (1989), Grundy and McNichols (1989), Allen, Morris and Shin (2005)).

On conceptual grounds we find that there is no basis to assuming PI about economic aggregates. In contrast, HB theories are naturally suited. Since PI is not observed, models with PI offer little in the way of testable hypotheses, making it possible to prove anything with PI. In contrast, agents who hold HB are happy to reveal their forecasts hence there is vast data on market belief which can be used to test hypotheses about the effect of diversity.

The two asset pricing models differ on several central issues. (i) Models with PI introduce artificial "noise" such as *unobserved* shocks to the supply of securities to explain diversity of forecasts but diversity does not persist. It vanishes as the number of trading rounds increases. In contrast, HB theories have a natural renewal mechanism to explain the perpetual persistence of diversity; (ii) under both theories the market belief fails iterated expectations but, in economic substance, the Beauty Contest implications of the two are different. Under PI today's price depends upon today's market belief about tomorrow's *unobserved supply shock* while under HB it depends upon today's market belief about tomorrow's *market beliefs. Tomorrow's market beliefs are mostly mistaken and are a central cause of market uncertainty, called "endogenous uncertainty."* (iii) PI implies an equilibrium map with infinite memory while HB implies a map which is time invariant; (iv) models with HB offer a rich theory of market risk premium which depend upon the market state of belief. We also offer some empirical support for the theoretical propositions developed; (v) contrary to PI which offers no restrictions on possible private information, an HB theory offers restrictions on belief. We thus explain the rationality conditions of the theory of *Rational Beliefs* imposed on the model at hand and show these establish tight restrictions on the model's parameters.

JEL classification: D82, D83, D84, G12, G14, E27.

Keywords: private information; Bayesian learning; updating beliefs; heterogenous beliefs; asset pricing; Rational Beliefs.

For Correspondence:
Mordecai Kurz, Joan Kenney Professor of Economics
Department of Economics, Landau Building
Stanford University
Stanford, CA. 94305 - 6072
Email: mordecai@stanford.edu

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Mordecai Kurz[2]

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People disagree on most issues and have diverse views about future events. A large and growing body of work has used this diversity to explain various market phenomena and there are two broad theories inspired by it. One follows the Harsanyii doctrine viewing people as Bayesian decision makers who hold the same probability belief but who have asymmetric private information used for forecasting. A sample of papers which are applicable here includes Phelps (1970), Lucas (1972), Diamond and Verrecchia (1981), Singleton (1987), Brown and Jennings (1989), Grundy and McNichols (1989), Wang (1994), He and Wang (1995), Hellwig (2002), Woodford (2003), Allen, Morris and Shin (2005) and others. An alternative view holds that there is nothing to justify a common prior and diversity of probability models is natural. This is particularly true of forecasting aggregates such as the S&P 500, GNP growth rate, inflation and interest rates yet there is a vast diversity of such forecasts. A sample of papers includes Harrison and Kreps (1978), Varian (1985), (1989), Harris and Raviv (1993), Detemple and Murthy (1994), Kurz (1994), (1997a), Kurz and Motolese (2001), Kurz Jin and Motolese (2005a) (2005b), Motolese (2001), Nielsen (1996), Wu and Guo (2003), (2004). In particular, Kurz's (1994), (1997) theory of belief diversity stresses the impossibility of perfect learning. It suggests our environment is non-stationary with technological and institutional changes occurring faster than we can learn them. How different are these two theories? What are the differences in their theoretical and empirical implications?

This paper explores the circumstances when a private information approach is less appropriate than a diverse beliefs approach and highlights the theoretical and empirical implications of the two theories. We explore in much more details the diverse beliefs approach which is not as well known as the private information approach. To that end we keep the exposition simple, concentrating on ideas

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[2] Department of Economics, Serra Street at Galvez, Stanford University, Stanford, CA. 94305-6072, USA

and concepts. We are concerned with theories where optimizing agents forecast aggregates such as the S&P500 returns, inflation rates, GDP growth etc. Our main conclusions are that models with private information are not appropriate for these tasks and offer contrived solutions to the problems at hand. On the other hand, theories where agents have heterogeneous beliefs and use diverse forecasting models constitute a natural setting for problems of this type. We argue private information models have very limited empirical implications and hence with private information one can prove almost anything. This is contrasted with the fact that models with heterogenous beliefs have strong empirical implications and testable hypotheses since market beliefs are observable, .

To explore the key ideas we first outline a simple model used to study asset pricing with private information. In Section 3 we adapt the model to the case of heterogenous beliefs without private information. After fully developing the equilibrium asset pricing theory under diverse beliefs we compare the results to those obtained under private information. We also explore the restrictions on beliefs proposed by the theory of Rational Beliefs (see Kurz (1994), (1997a)). For the rest of the paper we use the notation of PI for "private information" and HB for "heterogenous beliefs."

1. Asymmetric Information and Asset Pricing

The model we discuss is an adaptation of the short lived trader model used by Brown and Jennings (1989), Grundy and McNichols (1989), Allen, Morris and Shin (2005) and others. There is a unit mass of traders, indexed by the unit interval [0, 1] and only one homogenous asset with unknown intrinsic value Q. The economy is static with one period divided into three dates (no discounting): in date 1 traders receive a public and a private signal about the asset value and after the disclosure of information they trade. In date 2 they trade again. In date 3 uncertainty is resolved, the true liquidation value Q of the asset is revealed and traders receive this value for their holdings. The initial information of traders is simple: Q is distributed normally with $E(Q) = y$ and variance $\frac{1}{\alpha}$. At date 1 each trader observes a private signal about Q, $x^i = Q + \varepsilon^i$ where ε^i are, independently normally distributed across all i and j with mean 0 and variance $\frac{1}{\beta}$. Since these facts are common knowledge, agents know that the true unknown value Q is "in the market" at all time since by the law of large numbers the mean of all private signals is the future value Q. All have the same CARA utility over wealth W, with constant absolute coefficient of risk aversion. They maximize expected utility of $u(W^i) = -e^{-(1/\tau)W^i}$

where $W^i = S^i p_1 + D_1^i(p_2 - p_1) + D_2^i(Q_t - p_2)$. Trader i starts with a supply S^i of the stock, (D_1^i, D_2^i) are i's demands in the first and second rounds and (p_1, p_2) are market prices in the two rounds. Aggregate supplies (S_1, S_2) of shares traded in each of the rounds *are random, unobserved and normally distributed*. This noise is crucial since it ensures traders cannot deduce from prices the true value of Q. In a noisy Rational Expectations Equilibrium (in short, REE) traders maximize expected utility and markets clear after traders deduce from prices all available information. Indeed, Brown and Jennings (1989) show the equilibrium price at date 1 is

(1a) $$p_1 = \kappa_1(\lambda_1 y + \mu_1 Q - S_1)$$

and since S_1 is normally distributed p_1 is also normally distributed. (1a) shows that since Q and S_1 are both unknown, prices are not fully revealing. Since Q is fixed, more rounds of trading generate more price data from which traders deduce added information about Q. But with additional noise shocks the inference problem becomes more complicated. That is, at date 2 the price p_2 contains more information about Q but p_2 depends upon *two* unobserved noise shocks (S_1, S_2). As in Brown and Jennings (1989), the price function takes the form

(1b) $$p_2 = \hat{\kappa}_2(\hat{\lambda}_2 y + \hat{\mu}_2 Q - S_2 + \psi S_1).$$

Since the realized noise S_1 is not known at date 2, we condition instead on the known price p_1 to infer information about S_1. Hence, we actually use date 2 price function which takes an equivalent form

$$p_2 = \kappa_2(\lambda_2 y + \mu_2 Q - S_2 + \xi_{21} p_1).$$

By (1a) equivalence is seen from $\kappa_2 = \hat{\kappa}_2$, $\lambda_2 = (\hat{\lambda}_2 + \lambda_1\psi)$, $\mu_2 = (\hat{\mu}_2 + \mu_1\psi)$ and $\xi_{21} = -\frac{\psi}{\kappa_1}$. Denote by (H_1^i, H_2^i) the information of i in the two rounds. Brown and Jennings (1989, Appendix A) show that due to the form of the equilibrium price map the payoff is normally distributed. Moreover, there exist constants (G_1, G_2) determined by the covariance matrix of the model's random variables such that the demand functions of trader i are

(2a') $$D_2^i(p_2) = \frac{\tau}{\mathrm{Var}^i(Q|H_2^i)}[E^i(Q|H_2^i) - p_2].$$

(2b') $$D_1^i(p_1) = \frac{\tau}{G_1}[E^i(p_2|H_1^i) - p_1] + \frac{(G_2 - G_1)}{G_1}[E^i(D_2^i|H_1^i)].$$

The second term in (2b') is the "hedging demand" arising from risk perception of traders at date 1 about price change at date 2. The hedging demand in a noisy REE complicates the inference problem and raises problems regarding existence of equilibrium. As a result, most writers ignore this demand

and study what is known as *the myopic-investor economy for which the demand functions are*

(2a) $$D_2^i(p_2) = \frac{\tau}{\sigma_Q^2}[E^i(Q|H_2^i) - p_2].$$

(2b) $$D_1^i(p_1) = \frac{\tau}{\sigma_{p_2}^2}[E^i(p_2|H_1^i) - p_1]$$

where $\sigma_Q^2 = Var^i(Q|H_2^i)$ and $\sigma_{p_2}^2 = Var^i(p_2|H_1^i)$. It is typically assumed σ_Q^2 and $\sigma_{p_2}^2$ are independent of i. In multiperiod models these demand functions take the form $D_t^i(p_t) = (\tau/\sigma_{p_{t+1}}^2)[E^i(p_{t+1}|H_t^i) - p_t]$.

Now, average (2a)-(2b) on i, equate aggregate demand to supply and deduce

(1c) $$p_2 = \bar{E}_2(Q) - \frac{\sigma_Q^2}{\tau}(S_1 + S_2), \quad p_1 = \bar{E}_1(p_2) - \frac{\sigma_{p_2}^2}{\tau}S_1.$$

$\bar{E}_2(Q)$ is date 2 average market forecast of Q and $\bar{E}_1(p_2)$ is average market forecast of p_2.

(2a)-(2b) and (1c) depend only upon the condition that prices are normally distributed but not upon any PI assumption. Hence, the difference between the PI and HB theories result from differences between their implications to the conditional expectations in (2a)-(2b). For example, (2a) shows that demand depends upon date 2 expectations which are updated based on the information deduced from p_2 and p_1. This is different from date 1 information which consists of public signal, private signals and inference from p_1 only. Allen, Morris and Shin (2005, Appendix A) present computations of a closed form solution. To get an idea of the inference involved we review the steps they take.

What does a trader learn in round 1? Given prior belief $Q \sim N(y, \frac{1}{\alpha})$ trader i observes $p_1 = \kappa_1(\lambda_1 y + \mu_1 Q - S_1)$. Since $S_1 \sim N(0, \frac{1}{\gamma_1})$ all he infers from date 1 price is that

$$\frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y) = Q - \frac{S_1}{\mu_1} \sim N(Q, \frac{1}{\mu_1^2\gamma_1}).$$

But now, his second piece of information is the private signal $x^i = \theta + \varepsilon^i$. Using a standard Bayesian inference from these three sources, his posterior belief becomes

(3a) $$E^i(Q|H_1^i) = \frac{\alpha y + \beta x^i + \mu_1^2\gamma_1 \frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y)}{\alpha + \beta + \mu_1^2\gamma_1} = \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x^i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1}$$

(3b) with precision $\alpha + \beta + \mu_1^2\gamma_1$.

Averaging (3a) over the population we can see that the average market forecast at date 1 is then

$$\bar{E}_1(Q|H_1) = \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta Q + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1} \equiv \frac{\alpha y + (\beta + \mu_1^2\gamma_1)Q}{\alpha + \beta + \mu_1^2\gamma_1} - \frac{\mu_1\gamma_1 S_1}{\alpha + \beta + \mu_1^2\gamma_1}.$$

In round 2 a trader observes p_2 which is a function of the same three variables and of p_2. Given p_1 and the fact that $S_2 \sim N(0, \frac{1}{\gamma_2})$, he infers from $p_2 = \kappa_2(\lambda_2 y + \mu_2 Q - S_2 + \xi_{21}p_1)$ that

$$\frac{1}{\kappa_2\mu_2}(p_2 - \kappa_2\lambda_1 y - \kappa_2\xi_{21}p_1) = Q - \frac{S_2}{\mu_2} \sim N(Q, \frac{1}{\mu_2^2\gamma_2}).$$

He now updates (3a)-(3b). Since supply shocks are i.i.d. the updated posterior is standard

$$E^i(Q|H_2^i) = \frac{[\frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x^i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1}](\alpha + \beta + \mu_1^2\gamma_1) + \frac{1}{\kappa_2\mu_2}(p_2 - \kappa_2\lambda_2 y - \kappa_2\xi_{21}p_1)(\mu_2^2\gamma_2)}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}.$$

Simplification leads to

(4b) $$E^i(Q|H_2^i) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 - \mu_2\gamma_2\lambda_2]y + \beta x^i + [\frac{\mu_1\gamma_1}{\kappa_1}p_1 + \frac{\mu_2\gamma_2}{\kappa_2}p_2 - \mu_2\gamma_2\xi_{21}p_1]}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}$$

(4c) $$\mathrm{Var}(Q|H_2^i) = \frac{1}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}.$$

To compute (1c) we average (4b) to conclude that

(5a) $$\bar{E}_2(Q) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 - \mu_2\gamma_2\lambda_2]y + \beta Q + [\frac{\mu_1\gamma_1}{\kappa_1}p_1 + \frac{\mu_2\gamma_2}{\kappa_2}p_2 - \mu_2\gamma_2\xi_{21}p_1]}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}$$

(5b) $$\bar{E}_1(p_2) = \kappa_2(\lambda_2 y + \mu_2\bar{E}_1(Q) + \xi_{21}p_1).$$

When (5a)-(5b) are inserted into (1c) we end up with two equations in the two unknown prices which can now be computed. The final step is to match coefficients of the price functions (1a)-(1b) in order to identify $(\kappa_1, \lambda_1, \mu_1, \kappa_2, \lambda_2, \mu_2, \xi_{21})$. For details of these computations see Allen, Morris and Shin (2003) , Appendix A.

How much information is revealed by prices? The model is static but multiple trading rounds provide opportunities to deduce more information from prices about Q, if revealed after N rounds. As trading proceeds, the memory of all past prices is preserved since prices depend upon all unobserved supply shocks. In such a case the price system can never be a finite memory Markovian process.

The model has, indeed, been extended to multi period trading where Q is revealed N periods later (see Brown and Jennings (1989), Grundy and McNichols (1989), He and Wang (1995) and Allen, Morris and Shin (2003)). In these models the inference depends upon the model used. A key difference is the hedging demand. The issue is framed as *an interpretation of the traders* as long or short lived. A "short lived" trader lives one period only. He first trades in date 1, gains utility from p_2 and leaves the economy. He is replaced by a new short lived trader who knows the information of the first trader but trades in date 2. This trader gains utility from Q revealed in date 3. Neither trader has a hedging demand. A "long lived" trader lives through *both* periods, trades in dates 1 and 2 hence has a hedging demand and this extra hedging demand complicates the inference problem[3].

Leaving aside the distinction between long and short lived traders, note that in either case *the number of trading rounds is an arbitrary modeling construct*. It would thus be instructive to examine the limit behavior of the model. In a third round of trading the price map becomes

$$p_3 = \kappa_3(\lambda_3 y + \mu_3 Q - S_3 + \xi_{31} p_1 + \xi_{32} p_2).$$

Hence, the independent supply shock leads to an updating rule which is again standard

$$E^i(Q|H_3^i) = \frac{E^i(Q|H_2^i)(\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2) + \frac{1}{\kappa_3\mu_3}(p_3 - \kappa_3\lambda_3 y - \kappa_3\xi_{31}p_1 - \kappa_3\xi_{32}p_2)(\mu_3^2\gamma_3)}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2 + \mu_3^2\gamma_3}.$$

Individual and market forecasts can be expressed in terms of the random supply shocks. They can easily be extended to N rounds of trade and take the general form

$$E^i(Q|H_N^i) = \frac{\alpha y + \beta x^i + \sum_{j=1}^{N} \mu_j^2\gamma_j Q}{\alpha + \beta + \sum_{j=1}^{N}\mu_j^2\gamma_j} - \frac{\sum_{j=1}^{N}\mu_j\gamma_j S_j}{\alpha + \beta + \sum_{j=1}^{N}\mu_j^2\gamma_j} \tag{6}$$

[3] For discussion of the "long lived" traders see He and Wang (1995) and Appendix A of Allen, Morris and Shin (2003). For a simple exposition of the hedging demand in a two period economy see Brown and Jennings (1989).

A standard argument is now used to conclude the μ_j converge. Also, for simplicity assume the precisions γ_j of S_j are the same over time. The independence property of the noise is then used to reason that by (6), and by the law of large numbers the first term converges to Q and the second term converge with probability 1 to 0. Hence, with probability 1 individual and market forecasts converge to the true Q and the effect of the public signal y disappears in the limit. R*epeated trade leads in the limit to a full revelation of the true value Q*. Moreover, in the limit we have p = Q and traders do not forecast prices at all. This result contradicts an impression given by Allen, Morris and Shin (2005) who argue that, with PI, the effect of the public signal y on the price lingers on forever.

To conclude, the theory of markets with asymmetric PI has advanced our knowledge significantly but here we are concerned with its limits. It is clear that with information asymmetry agents make different forecasts. But PI is a very sharp sword. As a result, when diverse forecasting is an important component of any theory, the temptation is to assume PI as a quick vehicle to model the problem. A large literature in Economics and Finance has done just that to study many problems. It has become so common that for some, thinking of agents with different opinions is *synonymous* to thinking of them as having PI. We argue below that in a broad set of applications this assumption has no merit. We identified three areas of forecasting where the alternative model of diverse beliefs is better suited:

(i) Market prices such as interest rates, indices of stock prices, foreign exchange rates ;

(ii) Macroeconomic variables such as rates of GNP growth, inflation, unemployment, monetary policy actions;

(iii) Exogenous shocks like productivity shocks, aggregate factor supplies etc.

These categories highlight the fact that models with asymmetric PI, to which we object, include early work such as Phelps (1970) and Lucas (1972). Some recent papers includes Romer and Romer (2000), Hellwig (2002), Woodford (2003), Amato and Shin (2003), Bacchetta and van Wincoop (2005a) and many others. We doubts the validity of such applications[4]. For the purpose of comparison with theories under diverse beliefs, we interpret the single asset in the Brown and Jennings (1989) or

[4] To illustrate, Kurz (1997b) explains the volatility of foreign exchange rates and the forward discount bias in foreign exchange markets by demonstrating that these are consequences of diverse beliefs of traders about future exchange rates. In rejecting the REE framework he assumes agents hold diverse Rational Beliefs which are restricted as explained in Section 4.5 below. In such a market the center of uncertainty is the uncertainty of traders about future *beliefs* of other traders. Bacchetta and van Wincoop (2005a) adopt the same idea by using a noisy REE but assume that at each date traders have random private information about future aggregate money supply. Our argument here is that in the context of exchange rates determination such an assumption does not have empirical validity and leads to an implausible explanation of the forward discount bias.

Allen, Morris and Shin (2005) to be an index such as the S&P500 or an exchange rate. We thus turn to an evaluation of the use of a PI assumption in models where agents make forecasts in the three categories of variables listed above.

2. When Should the Assumption of Asymmetric Information Be Questioned?

We take up, one at a time, arguments which cast doubt on the validity of the PI assumption. Typical problems studied with PI include market volatility, volume of trade, market risk premia, foreign exchange dynamics, business cycles and the effects of monetary policy. Apart from lack of plausibility, we also argue that the explanations offered for these phenomena, driven by the assumption of PI, are unconvincing. Thus, PI offers a distorted "solution" for such problems.

(i) What is the data that constitutes "private" information? If forecasters of GNP growth or future interest rates use PI, one must be able to specify the data to which such forecasters have an exclusive access. Forecasters of macroeconomic variables, including the Federal Reserve system itself, always state the sources of data they use and universally claim they use only published data. More important, a reasonable person should be able to state what is the PI to which a forecaster of future GNP growth or interest rates have an exclusive use. That is, if we are to say that a forecaster used PI to forecast future value of the S&P500 we need to be able to say what that PI is. Without such identification a model with PI cannot make sense. Indeed, all empirical implication the model has are deduced from restrictions imposed by that information. As illustrated in the Section I, a model with PI specifies an unknown parameter θ about which agents receive private signals x_t^i with $i = 1, 2, \dots$ For this to have meaning one must know what the x_t^i are or what they could conceivably be. When agents forecast variables in the three categories above, no such imaginary data exist.

(ii) *Asymmetric information imply a Secretive Economy*. Forecasters take pride in their models and are eager to make their forecasts public. As a result, there are vast data files on market forecasts of most variables mentioned above. These include data of the Blue Chip Economic Indicators (BLU), Blue Chip Financial Forecasts (BLUF), the Survey of Professional Forecasters (SPF), forecasts by individual firms engaged in forecasting and even detailed forecast data of the staff of the Federal

Reserve System itself. Such data are being used more and more in economic research as (e.g. Romer and Romer (2000), Swanson (2006), Kurz (2005), Kurz and Motolese (2005))

In addition to making public their forecast data, forecasters stress their opinions are different from others. In discussing public information they explain their own interpretation of such information, the weight they place on it and their disagreement with others' use of that same information. Trade journals are used to debate forecasting techniques and in public competitions prizes are awarded to the best forecaster in specified categories. With PI giving advantage to some, other forecasters would not compete since there is nothing to compete about. In short, in an HB environment, forecasters view their work as model formulation and interpretation of information, not as a reflection of secret information to which they are privy. Such behavior is not compatible with an equilibrium with PI.

In contrast, an equilibrium with PI is secretive. Individuals are careful not to divulge their PI about state variables since it would deprive them of the advantage they have in possessing information. In such equilibrium all private forecast data of any state variable (e.g. productivity, GNP growth etc.) are treated as source of *new information.* Agents use forecast data of other forecasters to update their posterior beliefs about that state variable. Had such PI been deduced from forecasts, the mean market forecast would change. Since in reality all forecasters happily reveal their forecasts, the economy must converge to an equilibrium with uniform information. The eagerness of agents to reveal their forecasts is thus not compatible with PI being the cause of the persistent divergence of opinions and forecasts.

(iii) *For the problems considered, asymmetric information is not sufficient.* Implicit in (ii) above is the fact that in REE with PI, there is tension between information asymmetry and revelation\learning. If prices reveal PI one needs to have noise to prevent such revelation. Noise must be unobserved and the cause for the noise is often unspecified. When specified, it can take strange forms such as an unobserved random supply of the asset in the market. But then, the implications of the theory do not depend only upon the assumed PI but, more important, on the investigator's noise created to prevent the revelation of PI. However, the problem does not end there. In (6) we have seen that repeated trading overcomes the effect of noise and leads to full revelation. Since in the model the number of rounds of trade is actually contrived, the empirical implications of the model are affected by one more component which is artificial. Finally, there are other channels that affect the revelation of PI and have an impact on the implications of the theory. For example, as argued in (ii), private forecast data exists and is extensively used (otherwise the data would not be collected). Given the assumption of PI some information can then be deduced from private forecasts. Hence, any implications of theories based on PI cannot depend only upon prices and must depend upon other channels for inference. Without credible and observable ways to measure these channels of revelation, the theory lacks empirical implications. Also, there are other formulations of the private information model in real time (e.g. Judd and Bernardo (1996), (2000), Bacchetta and van Wincoop (2005a)) but we cannot review them here.

(iv) If private signals are unobserved, how could common knowledge of the structure be attained? To permit a deduction of PI from public data the structure of the private signals must be common knowledge. For example, they may take the form $x^i = \theta + \varepsilon^i$ where ε^i are pure noise, independent across traders. But then one asks the simpler question: if these signals are not publically observed, how does the common knowledge come about? How does agent i knows that his own signal takes the form $x^i = \theta + \varepsilon^i$ and that x^i is an unbiased estimate of θ? How does i knows that the signal of k takes the form $x^k = \theta + \varepsilon^k$? Are these not merely devices used by the investigator to enable a closed form solution of the Bayesian inference problem, rather than an empirically verifiable hypothesis?

(v) Why are private signals more informative than audited public signals? One peculiar assumption which drives the results of Morris and Shin (2002), Allen, Morris and Shin (2005), Bacchetta and van Wincoop (2005a) and many others, is explained in the simple model in Section I. It says that traders get a public signal y which is the mean value of the unknown Q. Knowing the prior mean of Q is clearly inferior to knowing the true Q. It is then assumed that there is a continuum of agents on [0 , 1] with $x^i = \theta + \varepsilon^i$ and with ε^i i.i.d. Hence, if you knew all private signals you would use the law of large numbers to aggregate them and learn the true Q. In an REE it is assumed there is some agent who aggregates the information and hence equilibrium price becomes a function of the true Q, which nobody knows. But this procedure raises two questions.

(a) First, why do private signals contain more precise information than the information in the carefully and professionally audited statements? Does it make sense to postulate that audited statements are less reliable than the sum of the collection of all the fragmentary signals that individuals obtain?

(b) Second, who is doing the aggregation? How did he acquire the knowledge of the i.i.d. structure to arrive at an aggregation? If there is somebody who aggregates, the theory must recognize his incentives to use this information. If he is a neutral agent with a sacred duty not to exploit the public he should simply announce Q. Or else, he has a motive to use his monopoly information.

(vi) *With asymmetric information you can prove anything*. A typical model with PI about financial markets or about the macro-economy is based on the fact that crucial components of the theory can never be observable. We shall never observe the private signals agents receive about GNP growth or about future value of the S&P500. This lack of observability is contrasted with insurance or automobile markets where driving records or health records can confirm the assumption that agents have PI not available ex ante to firms in the insurance market. But if there is no way to ever obtain data on the crucial component of the theory, the theory cannot be falsified: for any hypothesis about market behavior one can find a pattern of PI that would induce that behavior as an equilibrium behavior. The theory offers no real restrictions.

But the problem is deeper. With the weapon of "private" but unobserved information you can reject any other theory of agent heterogeneity with which you disagree. Consider any heterogeneity which is truly the result of some cause we denote by Z. It is always *possible* that agents in that market had PI. You can then construct imaginary private information they are alleged to have, with which you can reject the theory Z despite the fact that Z is true and explains the data well. Thus, equipped with a tool of PI a scholar is actually free of the scientific method. This is perhaps the important motivation for this paper. It insists that the ultimate weapon of PI should be granted only if a scholar can provide direct compelling empirical evidence in support of such informational assumption. The burden of proof is on the one who uses such assumption.

3. Modeling Asset Pricing Under HB without PI

We now turn to a detailed discussion of the alternative paradigm of heterogenous belief rather than PI. What are the differences between these two paradigms and do these differences matter?

3.1 Adaptation of the Earlier Model

To adapt the earlier model with PI to a market with public information only and HB we clearly reject the common knowledge assumptions made before. But then what is the common knowledge basis of traders with diverse beliefs? Our *unequivocal* answer is *past data on observable variables*. Traders know they all observe the same data. They have diverse beliefs about the future because they have diverse interpretations of past data. Hence, a mechanical adaptation of the two- period PI economy is meanigless for an economy with HB. A meaningful model with HB must be anchored in real time with past data available at each date. But in a real time context the announced liquidation is artificial. To permit a comparison of the two theories we thus adapt the earlier model by preserving its key assumptions. Leaving aside private information, the key trading assumptions underlying the model in Section 1 are simple: (i) at date 1 agents cannot trade futures contracts for delivery of the stock at date 2; (ii) at date 1 traders must form beliefs about date 2 price and some random value Q.

Our adaptation is then based on two principles. First, we maintain the trading assumptions above. Second, and more essential, we require that *our model generates exactly the same demand functions as (2a) -(2b)* so the comparison is reduced to differences between the implied probabilities used. Our solution is to let trading take place in real time but the surprising fact is that this can be accomplished by simply going to a traditional infinite horizon economy with a continuum of traders. In this formulation we abandon the liquidation value altogether and $\{Q_t, t = 1, 2, \ldots\}$ are the usual *dividends paid by the portfolio*. We also fix the artificial timing of the liquidation so that dividends are paid at the start of the period, not the end of it. We thus change the notation from Q_t to D_t. Before proceeding it is important to explain the crucial role of past data.

The infinite repetition introduces the driving force of the diverse beliefs theory, which is the lack of knowledge of the true probability of the process $\{D_t, t = 1, 2, \ldots\}$. The model is given an interpretation via a collection of real assets represented by a stock portfolio such as the S&P500. The assets in the background experience changes in productivity, innovation and organization all of which cause the process $\{D_t, t = 1, 2, \ldots\}$ to be non-stationary with unknown probability. Since D_t is the date t dividend, we assume that the long history $\{D_k, k = 1, 2, ..., t\}$ is known at date t. Given the data, agents compute the finite dimensional distributions of the observations and all compute the same empirical moments. Using standard extension of measures they all deduce from the data a unique probability measure on infinite sequences which is denoted by m. It can be shown that m is stationary (see Kurz (1994)) and we call it “the stationary measure.” This is the *empirical knowledge shared by all agents*. To conform to the earlier model we assume that the long run data reveals the D_t are conditionally normally distributed with mean μ and precision $\frac{1}{\alpha}$[5]. To simplify define $d_t = D_t - \mu$. Being a non-stationary process, this long term average probability is not the true probability of the process, merely an average over an infinite sequence of changing regimes. A theory of belief diversity crucially flows from the fact that agents do not know the true probability distribution of the d_t's. That is, the stochastic process $\{d_t, t = 1, 2, \ldots\}$ *has an unknown probability Π* which is different from m. All traders know is the stationary probability m deduced from data. The distinction between m and Π is central to our development and is explored later. Here we note that at date t all traders have the information, H_t, on which they condition their beliefs consisting of past values of d_k and prices.

Turning now to our infinite horizon model, trader i buys stocks θ_t^i at date t and receives the dividend $d_t + \mu$ at t. We continue to assume the riskless rate is constant over time so that there is a technology by which an agent can invest the amount B_t at date t and receive with certainty the reward $B_t R$ at date t+1. The definition of consumption is then standard

(7a) $$c_t^i = \theta_{t-1}^i [p_t + d_t + \mu] + B_{t-1}^i R - \theta_t^i p_t - B_t^i.$$

Given some initial values (θ_0^i, B_0^i) the utility of an agent is then

(7b) $$U = E^i_{(\theta^i, B^i)} \sum_{t=1}^{\infty} -\beta^{t-1} e^{-(\frac{c_t^i}{\tau})}$$

An optimal strategy is a sequence of portfolios (θ_t^i, B_t^i) which maximizes the expected utility of the agent. Now, to proceed we shall conjecture that, conditionally on the state variables of the economy, the equilibrium price p_t is normally distributed but in Theorem 2 below we shall confirm and prove this conjecture. In that context we shall also clarify the moments of this distribution. With these modification the demand functions for stocks become

(8) $$\theta_t^i(p_t) = \frac{\tau}{R\sigma_\varepsilon^2}[E_t^i(p_{t+1} + d_{t+1} + \mu - Rp_t].$$

where $\sigma_\varepsilon^2 = Var^i[p_{t+1} + d_{t+1} + \mu - Rp_t | H_t]$ which is constant and assumed to be the same for all agents. We later clarify the exact value of σ_ε^2. Note that (8) is exactly the same as (2a)-(2b). The differences consist of our new interpretation of d_t, the information available to the market and the traders' beliefs,

[5] It would probably be more realistic to assume that the values Q_t grow and the growth rate of the values has a mean μ rather than the values themselves. This added realism is useful when we motivate the model later but is not essential for the analytic development.

which is our central topic. We turn to it now.

We have stressed that disagreements arises from diverse interpretation of the same empirical record. Thus, for simplicity we make the realistic assumption that the empirical frequencies of past values recorded imply a first order Markov, inducing a stationary dividend process with transition

(9) $$d_t = \lambda_d d_{t-1} + \rho_t^d \qquad , \qquad \rho_t^d \sim N(0, \frac{1}{\alpha}).$$

Since the implied stationary probability is denoted by m, we write $E^m[d_{t+1}|d_t] = \lambda_d d_t$.

Is the stationary model (9) with probability m the true data generating mechanism? Since the economy undergoes rapid changes in technology, products and institutions $\{d_t, t = 1, 2, \ldots\}$ is then a non-stationary process under a true and unknown probability Π. Hence, a stationary Markov empirical record such as (9) is simply an average over different regimes. We can offer the simple analogy by suggesting that computing the empirical frequencies of the past is like taking a long data set over many different regimes and running a single regression which estimates the average over different structures. However, for long data sets, this is all that agents would ever agree on with certainty. In short, traders do not believe the empirical distribution of the past is adequate to forecast the future. All surveys of forecasters show that subjective judgment about the data contributes more than 50% to the final forecast (e.g. Batchelor and Dua (1991)). Given this environment, each trader forms his own beliefs about future d_t and other state variables to be explored in the next section. With such complexity how do we describe an equilibrium? For such a description do we really need to give a full, detailed, development of all the diverse theories of the traders?

3.2 Heterogeneity of belief: we do not ask Why, we Ask How!

Diverse beliefs is the result of the fact that agents have only incomplete knowledge of the structure of a complex economy. Hence, the concept of rationality must be modified. The theory of Rational Beliefs (in short, RB see, Kurz (1994), (1997a)) defines a trader to be *rational* if his model cannot be falsified by the data and if simulated, will reproduce the empirical distribution. More generally, when agents do not know the true structure it is not feasible for then to hold Rational Expectations. Hence, an agent cannot be declared irrational if he does not hold such expectations. In that case a meaningful concept of "rationality of belief" must embrace a wide collection of models. Such a conclusion raises an important methodological question. In formulating an asset pricing theory should we also provide a detailed description and motivate the subjective models of each trader? With diversity of traders such a task is formidable. But if the objective is an understanding of the dynamics of asset prices, is such a detailed description necessary? An examination of the subject reveals that, although a very intriguing question, such a detailed task is not needed. Instead, to describe an equilibrium all that we need is to specify how the beliefs of the traders affect their subjectively perceived transition functions of the state variables. Once these are specified, the Euler equations are fully specified and market clearing leads to a pricing and equilibrium dynamics. To carry out such a program we follow the structure developed in Kurz, Jin and Motolese (2005a),(2005b). We now outline this development for agents in our simple economy.

3.3 Market Belief as a State Variable: Diverse Opinions vs. Asymmetric Information

To motivate our development we note that in markets without PI agents willingly reveal their forecasts. Hence, in formulating our theory we now assume that market forecast data are public. The crucial difference between markets with and without PI is that when individual forecasts of a state variable are revealed in a market *without* PI, others *do not see such forecasts as a source of new data and do not update their own beliefs about that state variable.* In such a market, a forecaster uses knowledge about the forecasts of others to alter his forecasts of endogenous variables, such as prices, since these depend upon the market belief. In short, the difference between an equilibrium with PI and an equilibrium with symmetric information but HB is that in the latter *agents do not learn from others and do not update their beliefs about state variables based on the opinions of others.* But then, how do we describe the individual and market beliefs?

The key analytical step we have taken (see Kurz (1994), Kurz (1997a), Kurz and Motolese (2001), Kurz, Jin and Motolese (2005a),(2005b)) is to treat individual and market beliefs as state variables, generated within the economy. Endogenous variables (e.g. prices) depend upon state variables. For example, the primitive is an agent's belief about future productivity growth. If an agent knows that the market expects high productivity growth, the agent's forecast of productivity growth is not changed but he recognizes that the market belief has an effect on present day prices and other endogenous variables. Future endogenous variables likewise depend upon future market belief. Since market belief exhibit persistence, the agent also knows that today's market belief is useful for forecasting future endogenous variables. How is this equilibrated? Kurz and Motolese (2001), and Kurz, Jin and Motolese (2005a),(2005b) describe individual beliefs about a state variable with a state of belief variable which uniquely pins down the conditional probability or transition function of next period's state variable. The market state of belief is then defined naturally as the average of individual beliefs. Future endogenous variables are then forecasted by forecasting the market beliefs using the known equilibrium map. A great simplification is attained by assuming a large economy where individual agents are *"anonymous"* so the beliefs of any one of them has no effect on prices. Hence, equilibrium endogenous variables depend only upon the *distribution* of market beliefs.

We thus introduce the concept of *a state of belief of i* denoted by g_t^i. It describes the perception of trader i hence it pins down his transition functions of observables. Anonymity implies each trader knows his own g_t^i and all distributions of such variables. Hence, he knows g_τ^j but cannot link it to specific past traders $\tau < t$. We specify the dynamics of g_t^i by

(10) $$g_t^i = \lambda_Z g_{t-1}^i + \rho_t^{ig} \quad , \quad \rho_t^{ig} \sim N(0, \sigma_g^2).$$

where ρ_t^{ig} are correlated across i reflecting correlation of beliefs across individuals. The state of belief is a tool used to express the trader's subjective evaluation of the difference between a key parameters of date t distribution and the corresponding parameter in the empirical distribution m. Specifically, the date t perception of trader i about d_{t+1} which we denote by d_{t+1}^i is expressed by

$$d_{t+1}^i = \lambda_d d_t + \lambda_d^g g_t^i + \rho_t^{id} \quad , \quad \rho_t^{id} \sim N(0, \frac{1}{\hat{\alpha}}).$$

The assumption that $\hat{\alpha}$ is the same for all traders is made for simplicity. The belief of the trader is then seen to express his deviation from the empirical forecast

(11) $$E^i[d_{t+1}^i | H_t, g_t^i] - E^m[d_{t+1} | H_t] = \lambda_d^g g_t^i.$$

(11) reveals how g_t^i are measured in reality. For any state variable X_t we have data on a trader's subjective forecasts of X_{t+1} (in (11) it is d_{t+1}) since he informs us of $E^i[X_{t+1}^i | H_t, g_t^i]$. We can then use standard econometric techniques to construct the stationary forecast $E^m[d_{t+1} | H_t]$ hence we can construct the difference in (11). The data which this construction makes available are the basis of the work of Fan (2005) and Kurz and Motolese (2005). A trader who believes the empirical distribution is the truth, is described by $g_t^i = 0$, hence he believes that $d_{t+1} \sim N(\lambda_d d_t, \frac{1}{\alpha})$.

The average market state of belief Z_t is the average of the g_t^i over the population. However, due to the correlation across traders the average of ρ_t^{ig} over i does not vanish as the law of large numbers is not operative. We write it in the form

(12) $$Z_{t+1} = \lambda_Z Z_t + \rho_{t+1}^Z.$$

The distribution of ϱ_{t+1}^Z is unknown. Correlation across agents exhibits time dependency and non stationarity and this property is inherited by the { Z_t , t = 1, 2, ...} process. Since Z_t are observable we *expand the empirical distribution* and consider the joint process $\{(d_{t+1}, Z_{t+1}), t = 1, 2, \ldots\}$. Traders are now assumed to know the *joint empirical distribution* of these variables. For simplicity we assume that this distribution is described by the system of equations

$$\text{(13a)} \quad d_{t+1} = \lambda_d d_t + \rho^d_{t+1} \qquad \text{(13b)} \quad Z_{t+1} = \lambda_Z Z_t + \rho^Z_{t+1} \qquad \begin{pmatrix} \rho^d_{t+1} \\ \rho^Z_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma^2_d, & 0, \\ 0, & \sigma^2_Z \end{bmatrix} = \Sigma \right), \quad \text{i.i.d.}$$

Such measures typically reveal large variances (σ^2_d, σ^2_Z). If a trader does not believe (13a)-(13b) is the truth, he formulate his own model\belief. Since he knows d_t and Z_t, how does his own state of belief g_t^i pin down his forecasts of the economy's state variables (d^i_{t+1}, Z^i_{t+1})? To show this we define the *perception model of the trader.* In this case it takes the symmetric form

$$\text{(14a)} \quad d^i_{t+1} = \lambda_d d_t + \lambda^g_d g_t^i + \rho^{id}_{t+1} \qquad \text{(14b)} \quad Z^i_{t+1} = \lambda_Z Z_t + \lambda^g_Z g_t^i + \rho^{iZ}_{t+1} \qquad \begin{pmatrix} \rho^{id}_{t+1} \\ \rho^{iZ}_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \hat\sigma^2_d, & \hat\sigma_{Zd}, \\ \hat\sigma_{Zd}, & \hat\sigma^2_Z \end{bmatrix} = \Sigma^i \right).$$

The assumptions $\lambda^g_Q \geq 0$ and $\lambda^g_Z \geq 0$ give the "state of individual belief" a natural interpretation. When $g_t > 0$ agent i believes date t+1 dividend will be above normal, reflected by the empirical distribution, and when $g_t < 0$ the opposite is true. Although model (14a)-(14b) is later given Bayesian foundations, this is not essential. Most traders follow rules of behavior which may not be Bayesian. For the moment we stress that the state of belief g_t^i expresses the agent's own theory of how different the current state of the economy is from the long term average expressed by (13a)-(13b):

$$E_t^i \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} - E_t^m \begin{pmatrix} d_{t+1} \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda^g_d g_t^i \\ \lambda^g_Z g_t^i \end{pmatrix}.$$

Why is this subjective modeling important? Because we have data on forecasting with which we can measure Z_t and test if the empirical implications of the data support the theory.

The perception models (14a)-(14b) explain why the market belief operator which is defined by $\bar{E}_t(Q_{t+1}) = \int_0^1 E_t^i(Q_{t+1}) di$ is not a proper probability. To explain let $X = D \times Z$ be a product space where (d_t, Z_t) take values and by G^i the space of g_t^i. Since i conditions on his own g_t^i, his unconditional probability is a measure on the space $((D \times Z \times G^i)^\infty, \mathscr{F})$ where $\mathscr{F}$ is a sigma field. Hence, the market conditional belief operator is an average over conditional probabilities, each conditioning on a different state variable. Hence, one cannot write down a probability space for the market belief and we have the following result:

Theorem 1: The market belief operator violates iterated expectations: $\bar{E}_t(d_{t+2}) \neq \bar{E}_t\bar{E}_{t+1}(d_{t+2})$.

Proof: Since

$$E_t^i(d_{t+2}) = \lambda_d E_t^i(d_{t+1}) + \lambda_d^g E_t^i(g_{t+1}^i) = \lambda_d[\lambda_d d_t + \lambda_d^g g_t^i] + \lambda_d^g \lambda_Z g_t^i$$

It follows that

(15a)
$$\bar{E}_t(d_{t+2}) = \lambda_d^2 d_t + \lambda_d^g(\lambda_d + \lambda_Z)Z_t.$$

On the other hand we have from (14a) that

$$\bar{E}_{t+1}(d_{t+2}) = \lambda_d d_{t+1} + \lambda_d^g Z_{t+1}$$

hence

$$E_t^i \bar{E}_{t+1}(d_{t+2}) = \lambda_d[\lambda_d d_t + \lambda_d^g g_t^i] + \lambda_d^g[\lambda_Z Z_t + \lambda_Z^g g_t^i].$$

Aggregating now we conclude that

(15b)
$$\bar{E}_t\bar{E}_{t+1}(d_{t+2}) = \lambda_d^2 d_t + \lambda_d^g(\lambda_d + \lambda_Z + \lambda_Z^g)Z_t.$$

Comparison of (15a) and (15b) shows that $\bar{E}_t(d_{t+2}) \neq \bar{E}_t\bar{E}_{t+1}(d_{t+2})$. ■

A final comment on how one must think of the market belief Z_t. From the point of view of a trader Z_t is a state variable like any other state variable. News about Z_t are used to forecast prices and assess market risks in the same way aggregate information like non-farm payroll is used to assess risk of recessions. Market belief may be wrong as it may forecast recessions that never occur. Traders do not treat Z_t as a "signal" about unobserved PI since they know that others do not know any more than they do. They do not "learn" from data on Z_t in the sense that they do not use it to update forecasts of future *exogenous* variables. Z_t is used to forecast future *endogenous* variables.

3.4 Combining the Elements: the Implied Asset Pricing Theory Under Diverse Beliefs

We now derive equilibrium prices under HB. First we assume as before that the conditional variance of d_{t+1} is the same for all traders and we denote it by σ_d^2. By (8a)-(8b) we can write the demand functions of the two type of traders at date t as

(16)
$$\theta_t^i(p_t) = \frac{\tau}{R\sigma_\varepsilon^2}[E_t^i(p_{t+1} + d_{t+1} + \mu - Rp_t].$$

For an equilibrium to exist we need some stability conditions. First we require the interest rate r to be positive, $R = 1 + r > 1$ so that $0 < \frac{1}{R} < 1$. Now we add:

Stability Conditions: We require that $0 < \lambda_d < 1 \quad , \quad 0 < \lambda_Z + \lambda_Z^g < 1$.

The first requires $\{d_t , t = 1, 2, ...\}$ to be stable and have an empirical distribution. The second is *a* stability of *belief* condition. It requires i to believe (d_t , Z_t) is a stable. To see why take expectations of (14b), average over the population and recall Z_t are market averages of the g_t^i. This implies that

$$\bar{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t.$$

Theorem 2: For the model with HB and under the stability conditions specified, there is a unique equilibrium price function which takes the form $p_t = a d_t + bZ_t - cS$.

Proof: With aggregate supply S being 1, aggregating (16) over all traders at date t leads to

(17) $$\frac{R\sigma_\varepsilon^2}{\tau} = [\bar{E}_t(p_{t+1} + d_{t+1} + \mu) - Rp_t].$$

Now use the perception models (14a)-(14b) about the state variables, average them over the population and use the definition of Z_t to deduce the following relationships which are the *key implications of treating individual and market beliefs as state variables*

(18a) $$\bar{E}_t(d_{t+1} + \mu) = \lambda_d d_t + \mu + \lambda_d^g Z_t$$

(18b) $$\bar{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t$$

Now solve for date t price to deduce

(19) $$p_t = \frac{1}{R}[\bar{E}_t(p_{t+1})] + \frac{1}{R}[\lambda_d d_t + \mu + \lambda_d^g Z_t] - \frac{\sigma_\varepsilon^2}{\tau}$$

(19) shows that equilibrium price is the solution of a linear difference equation in the two state variables (d_t , Z_t). Hence, a standard argument (see Blanchard and Kahn(1980), *Proposition*, page 1308) shows that the solution is

(20a) $$p_t = a d_t + bZ_t - c$$

To match coefficients use (20a) to insert (18a) - (18b) into (19) and conclude that

(20b) $$a = \frac{\lambda_d}{R - \lambda_d} > 0$$ because of the stability conditions.

(20c) $$b = \frac{\lambda_g^d}{R - (\lambda_Z + \lambda_Z^g)}[1 + \frac{\lambda_d}{R - \lambda_d}] > 0$$ because of the stability conditions.

(20d) $$c = \frac{\sigma_\varepsilon^2 R}{\tau r} > 0 .$$

The stability conditions also ensure that (20a) - (20d) is the unique solution as asserted. ■

3.5 Bayesian Foundations for the State Space Approach to Individual Belief

The treatment of individual belief as state variables is the main analytical tool used in this paper. We now provide a justification for this procedure but before doing so remind the readers that *we are not in a standard environment in which Bayesian inference is natural*. The postulated environment is dynamically complex, in which few parameters are fixed and those which are fixed are known to agents from the empirical distribution. This fact raises two issues that we discuss first.

(i) *Tractability in describing market equilibrium.*

In a changing environment there is no standard Bayesian procedures to learn an unknown sequence of parameters. Hence, it is less important to explain why agents disagree and more important to be able to describe their disagreement so that an equilibrium analysis is *tractable*. A description (1) of the individual state of belief in the form $g^i_{t+1} = \lambda_Z g^i_t + \rho^{ig}_{t+1}$ where $\rho^{ig}_{t+1} \sim N(0, \sigma^2_g)$ leads to a tractable description of an equilibrium and Sections 3.1 - 3.4 illustrate it. Hence, our first motive for the state variable approach is that it is tractable and permits useful analysis.

(ii) *Subjective Interpretation of Quantitative and Non Quantitative Market Data*.

It is common to think that diverse beliefs arise, to some degree, from diverse "interpretations" of data. But what does that mean in a world with public information? It is natural to say that lack of subjective interpretation of the data means a trader believes (13a)-(13b) are the truth.

One procedure used to express diverse interpretation of data uses structural models which are estimated by putting subjective weights on past data. The idea is that in a non- stationary world agents always need to estimate the regime in place and recent data is one source of information about the current regime (for an example see Cho, Williams and Sargent (2002)). The objection is that this procedure leads to a time invariant rule of, say, re-estimating a parameter each date by using only the last J observations. But then, why J? Also, the procedure is risky since with probability 1 one must encounters a run of "bad" data which would lead to a sharp change in the estimates with drastic consequences. Such procedure would be rejected by a rational learner.

A different approach which reflects a real practice and which we favor, is based on the fact that public data such as the d_t are accompanied with *a great deal of non-quantitative information* about abnormal market conditions. For example, a report about inflation may include an evaluation of abnormal productivity features, legal conditions of unions, new sources of energy, etc. Since d_t are dividends of a firm, a financial report includes a long list of unusual conditions the company faces such as pending patents, future technologies, legal suits, etc. Qualitative information cannot be compared over time and does not constitute conventional"data." Professional forecasters and financial markets pay a great deal of attention to such information. It often explains the diverse judgments of forecasters.

One way to formalize this practice is to denote the list of non quantitative variables at date t by $(C_{t1}, C_{t2}, \ldots, C_{tK})$. The list itself changes in time. These often offer *contradictory* perspectives in the sense that if, for example, C_{t1} materializes it would imply bright prospects for d_{t+1} while C_{t2} may lead to a negative assessment of d_{t+1}. Consider now a subjective map which projects the implications of each factor, by itself, into *abnormal* future values of d_{t+1}. We list the *subjective* projections of agent i as $(\Psi^i_{t1}, \Psi^i_{t2}, \ldots, \Psi^i_{tK})$. That is, if C_{t1} was the only information in addition to d_t, the agent would predict that $(d_{t+1} - \lambda_d d_t)$ equals Ψ^i_{t1}. Some of the $(C_{t1}, C_{t2}, \ldots, C_{tK})$ may be *random variables*, hence some of the $(\Psi^i_{(t)1}, \Psi^i_{(t)2}, \ldots, \Psi^i_{(t)K})$ are, themselves, subjective expected values of trader i. Now a trader needs to assess the joint impact of $(C_{t1}, C_{t2}, \ldots, C_{tK})$. A subjective forecast is then implied by a set of weights, or subjective probabilities, $(\pi^i_{t1}, \pi^i_{t2}, \ldots, \pi^i_{tK})$ used by i. The thought process of trader i then results in a translation of the public *qualitative* information into a subjective *quantitative* measure

$$\Psi^i_t = \sum_{k=1}^{K} \pi^i_{tk} \Psi^i_{(t)k} .$$

Since by (13a) the long term average of $(d_{t+1} - \lambda_d d_t)$ is zero, we argue later that rationality requires the Ψ_{t+1} are zero mean random variables. Although *public data* consist only of d_t, the procedure outlines shows that in a world with diverse beliefs traders endogenously create subjective quantitative measures which reflect their beliefs. We incorporate such a measure in the Bayesian procedure below.

(iii) *A Bayesian Model: Beliefs are Markov State Variable*

To express the non stationary environment we assume that under the true probability Π, Q_t has a transition of the form

$$d_{t+1} - \lambda_d d_t = b_t + \rho^d_{t+1} \qquad , \qquad \rho^d_{t+1} \sim N(0, \frac{1}{\beta}) \tag{21}$$

where b_t is a time varying mean value function. Traders do not observe b_t and hence, at the starting date t (which can be t = 1) they have a *prior belief* about b_t (before observing d_{t+1}) which is

$$b_t \sim N(\varphi, \frac{1}{\alpha}) .$$

The changing parameter b_t leads to two problems. First, when $d_{t+1} - \lambda_d d_t$ is observed trader i updates his belief to the posterior $E^i_{t+1}(b_t | d_{t+1})$. But such an update gives him merely a sharper estimate of b_t which is not sufficient since at t+1 he has a second problem of *estimating* b_{t+1}. In other words, how does he go from $E^i_{t+1}(b_t | d_{t+1})$ to a new prior $E^i_{t+1}(b_{t+1} | d_{t+1})$ of b_{t+1}? Absent any new information and given the constant φ, this prior belief of b_{t+1} has to be the earlier estimated posterior $E^i_{t+1}(b_t | d_{t+1})$. That is, if the only information available is $d_{t+1} - \lambda_d d_t$ the t+1 posterior $E^i_{t+1}(b_t | d_{t+1})$ is also the prior for b_{t+1}. This highlights the fact that standard Bayesian procedure does not allow for the fact that the value of b_t changes. This dual problem is clearly unique to our dynamic conditions since in a standard Bayesian framework a parameter does not change. Since Bayesian learning draws its inference from the past, it cannot offer a method of updating one's belief about a future value of a parameter with respect to which no data has been observed. An new source of private assessment must then emerge.

To supplement data on Q_t we assume that before trading resumes at t+1, all traders receive a vector of non quantitative information $(C_{(t+1)1}, C_{(t+1)2}, \ldots, C_{(t+1)K})$ which lead to subjective measures Ψ^i_{t+1} which imply some estimates of b_{t+1}. Now our trader has two independent sources for a prior on b_{t+1}: $E^i_{t+1}(b_t | d_{t+1})$ and Ψ^i_{t+1} which must be reconciled. Under a Bayesian approach we thus assume:

Assumption (*): Trader i uses a subjective probability μ to form his date t+1 prior belief which is then

(22) $$E^i_{t+1}(b_{t+1} | d_{t+1}, \Psi^i_{t+1}) = \mu E^i_{t+1}(b_t | d_{t+1}) + (1-\mu)\Psi^i_{t+1} \qquad 0 < \mu < 1 .$$

This assumption is the new added element that permits $E^i_{t+1}(b_t | d_{t+1}, \Psi^i_t)$ to be upgraded into a prior belief at date t+1, $E^i_t(b_{t+1} | d_{t+1}, \Psi^i_{t+1})$, before d_{t+2} is observed. We continue to assume that the empirical distribution (13a)-(13b) is known and can now show the following:

Theorem 3: Suppose $\Psi^i_t \sim N(0, \frac{1}{\gamma})$ and Assumption (*) holds. Then for large values of t the posterior $E^i_{t+1}(b_{t+1} | d_{t+1}, \Psi^i_{t+1})$ is a Markov state variable such that if we define $g^i_t = E^i_t(b_t | d_t, \Psi^i_t)$ and $\mu = \lambda_Q$ then the dynamics (10) holds: (22) implies (10).

Proof: We pick a starting date t when data d_t is known and the trader generates a subjective measure of qualitative data Ψ^i_t. He then forms a prior on b_t, *assumed* to be $b_t \sim N(\mu\varphi + (1-\mu)\Psi^i_t, \frac{1}{\alpha})$. Now we move on to t+1 and d_{t+1} is observed. The trader updates the prior in a standard Bayesian manner:

$$E^{i}_{t+1}(b_t|d_{t+1},\Psi^i_t) = \frac{\alpha(\mu\varphi+(1-\mu)\Psi^i_t) + \beta[d_{t+1}-\lambda_d d_t]}{\alpha+\beta} \quad , \quad 0\le\mu\le 1.$$

But before date t+1 trading he generates the subjective measure Ψ^i_{t+1} of qualitative data. By the Assumption (*) the expected parameter b_{t+1} under the new prior at t+1 is

$$E^{i}_{t}(b_{t+1}|d_{t+1},\Psi^i_{t+1}) = \mu E^{i}_{t}(b_t|d_{t+1},\Psi^i_t) + (1-\mu)\Psi^i_{t+1} \quad , \quad 0\le\mu\le 1.$$

Denote by $\zeta = \frac{1}{\mu^2}$ and $\xi = \frac{1}{(1-\mu)^2}$. Then the prior is

$$b_{t+1} \sim N\left(E^{i}_{t}(b_{t+1}|d_{t+1}\Psi^i_{t+1}), \frac{1}{\zeta(\alpha+\beta)+\xi\gamma}\right)$$

and forecasts of $d_{t+2}-\lambda_d d_{t+1}$ are made. We now move to t+2, the trader observes $d_{t+2}-\lambda_d d_{t+1}$ and based on this observation he updates his belief to

$$E^{i}_{t+1}(b_{t+1}|d_{t+2},\Psi^i_{t+1}) = \frac{(\zeta(\alpha+\beta)+\xi\gamma)[\mu E^{i}_{t}(b_t|d_{t+1},\Psi^i_t)+(1-\mu)\Psi^i_{t+1}] + \beta[d_{t+2}-\lambda_d d_{t+1}]}{\zeta(\alpha+\beta)+(\xi\gamma+\beta)}.$$

Before the start of date t+2 trading the trader generates a new value Ψ^i_{t+2} leading to t+2 belief that

$$E^{i}_{t+2}(b_{t+2}|d_{t+2},\Psi^i_{t+2}) = \mu E^{i}_{t+1}(b_{t+1}|d_{t+2},\Psi^i_{t+1}) + (1-\mu)\Psi^i_{t+2} \quad , \quad 0\le\mu< 1.$$

When $d_{t+3}-\lambda_d d_{t+2}$ is observed the updated belief is then

$$E^{i}_{t+2}(b_{t+2}|d_{t+3},\Psi^i_{t+2}) = \frac{[\zeta^2(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{1}\zeta^n-\beta][\mu E^{i}_{t+1}(b_{t+1}|d_{t+2},\Psi^i_{t+1})+(1-\mu)\Psi^i_{t+2}]+\beta[d_{t+3}-\lambda_d d_{t+2}]}{\zeta^2(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{1}\zeta^n}.$$

Next the agent generates a new value Ψ^i_{t+3} leading to t+3 belief $E^{i}_{t+3}(b_{t+3}|d_{t+3},\Psi^i_{t+3})$. By induction we iterate forward to conclude that

$$E^{i}_{t+N}(b_{t+N}|d_{t+N+1},\Psi^i_{t+N}) = \frac{[\zeta^{N-1}(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n-\beta]}{\zeta^N(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n}[\mu E^{i}_{t}(b_{t+N-1}|d_{t+N},\Psi^i_{t+N-1})+(1-\mu)\Psi^i_{t+N}] + $$

$$+ \frac{\beta[d_{t+N+1}-\lambda_d d_{t+N}]}{\zeta^N(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n}.$$

Now take the limit. Since $\zeta>1$, as N increases $\zeta^N\to\infty$ hence we find that for large t

$$E_{t+1}^{i}(b_{t+1}|d_{t+2},\Psi_{t+1}^{i}) = \mu E_t^{i}(b_t|d_{t+1},\Psi_t^{i}) + (1-\mu)\Psi_{t+1}^{i}$$

But by definition we have

(23) $$E_{t+1}^{i}(b_{t+1}|d_{t+1},\Psi_{t+1}^{i}) = \mu E_t^{i}(b_t|d_{t+1},\Psi_t^{i}) + (1-\mu)\Psi_{t+1}^{i}$$

We thus conclude that for large t, the contribution of each new observation of dividends is nill hence

$$E_t^{i}(b_t|d_{t+1},\Psi_t^{i}) = E_t^{i}(b_t|d_t,\Psi_t^{i}).$$

Inserting this last equation in (23) we finally have the desired conclusion for large t

(24) $$E_{t+1}^{i}(b_{t+1}|d_{t+1},\Psi_{t+1}^{i}) = \mu E_t^{i}(b_t|d_t,\Psi_t^{i}) + (1-\mu)\Psi_{t+1}^{i}.$$

Now identify $g_t^{i} = E_t^{i}(b_t|d_t,\Psi_t^{i})$, $(1-\mu)\Psi_{t+1}^{i} = \rho_{t+1}^{ig}$ and $\mu = \lambda_Q$ to see that (24) is actually (10). ■

Theorem 3 shows that as the d_t data set increases, there is nothing new to learn. The posterior does not converge but *the law of motion* of the posterior converges to a time invariant stochastic law of motion defined by (10). The posterior fluctuates forever, providing the foundations for the dynamics of market belief but the fluctuations follow a simple Markov transition. New data d_t and Ψ_t^{i} alter the conditional probability of the trader, but these do not change the dynamic law of motion of g_t^{i}.

3.6 Equilibrium Risk Premium and the Forecastability of Excess Returns

3.6a Equilibrium Results

We now explore the often misunderstood problem of measuring market risk premium under heterogenous beliefs. We shall see in a moment that there are many different concepts involved here and the main issue is one of choosing the concept which is most appropriate to any application. The definition of actual, realized, equity risk premium is not under dispute: it is defined by

(25a) $$\pi_{t+1} = \frac{p_{t+1} + d_{t+1} + \mu - Rp_t}{p_t}.$$

We find it convenient to define the premium per share held which is also the excess return per share

(25b) $$\pi_{t+1} = p_{t+1} + d_{t+1} + \mu - Rp_t.$$

(25a) -(25b) are random variable measuring the actual excess return of stocks over the riskless bond. The need is to measure the premium as a known quantity which is recognized by market participants at each date. One such measure is the subjective expected excess returns by agent i, defined naturally by

(26a) $$\frac{1}{p_t} E_t^{i}(p_{t+1} + d_{t+1} + \mu - Rp_t).$$

We know from the demand functions (8) that trader i's premium *completely determines the size of trader i's equilibrium stock holdings*. Moreover, under HB the individual premia vary across traders and this diversity is the central reason for volatility and trade.

Aggregating over i defines the market premium as the average market expected excess returns. This perceived market premium reflects what the market expects, not necessarily what the market gets. From (8) we deduce that this market premium to be equal, in equilibrium, to

$$\frac{1}{p_t}\bar{E}_t(p_{t+1} + d_{t+1} + \mu - Rp_t) = (\frac{R}{p_t})\frac{\sigma_\varepsilon^2}{\tau} \tag{26b}$$

Note that neither the individual perceived premium nor the market perceived premium are necessarily correct. Hence the last question is whether we have a measures of the risk premium which is objective, common to all traders and would be measured by an objective econometrician? The answer is yes and this fact is central to our theory. An econometrician who studies the long term time variability of the premium would measure it by the empirical distribution of the random variable in (25a) or (25b), hence using the conditional expected value under the stationary measure this premium is defined by

$$E_t^m[\pi_{t+1}] = \frac{1}{p_t}E_t^m[p_{t+1} + d_{t+1} + \mu - Rp_t]. \tag{26c}$$

(26c) is the way researchers have measured the risk premium and therefore we shall refer to it as "the risk premium." To measure the time variability of the risk premium we then estimate the time dependent variables which cause $E_t^m[\pi_{t+1}]$ to vary. This provides an objective assessment of the equilibrium dynamics of the risk premium.

We now compare (26a), (26b) and (26c) by using (14a) - (14b) for the expectations of agent i, (13a) - (13b) for m, and (18a) - (18b) for the market. Focusing on the premium per share, note first that (26a), the equilibrium price map and (13a) - (13b) imply that

$$E_t^i(p_{t+1}+d_{t+1}+\mu-Rp_t) = = (a+1)(\lambda_d d_t+\lambda_d^g g_t^i)+b(\lambda_Z Z_t+\lambda_Z^g g_t^i)+\mu-c-Rp_t \tag{27a}$$

As to the market perceived premium the equilibrium price map and (18a) - (18b) imply that

$$\bar{E}_t(p_{t+1}+d_{t+1}+\mu-Rp_t) = (a+1)(\lambda_d d_t+\lambda_d^g Z_t)+b(\lambda_Z+\lambda_Z^g)Z_t+\mu-c-Rp_t \tag{27b}$$

Finally we see that (26c), the equilibrium price map and (13a) - (13b) imply that

$$E_t^m[p_{t+1}+d_{t+1}+\mu-Rp_t] = (a+1)(\lambda_d d_t)+b(\lambda_Z)Z_t)+\mu-c-Rp_t. \tag{27c}$$

We can then sum up as follows:

$$\bar{E}_t(p_{t+1}+d_{t+1}+\mu-Rp_t) = R\frac{\sigma_\varepsilon^2}{\tau} \tag{28a}$$

(28b) $$E_t^i[p_{t+1} + d_{t+1} + \mu - Rp_t] = R\frac{\sigma_\varepsilon^2}{\tau} + [(a+1)\lambda_d^g + b\lambda_Z^g](g_t^i - Z_t).$$

(28a)-(28b) are not surprising: since $(a+1)\lambda_d^g + b\lambda_Z^g > 0$ the individual premium depends upon the difference between the state of belief of individual i and the belief of the market. A trader's perceived premium increases with $(g_t^i - Z_t)$. But the much more surprising result is the risk premium itself

(29a) $$E_t^m[p_{t+1} + d_{t+1} + \mu - Rp_t] = R\frac{\sigma_\varepsilon^2}{\tau} - (\lambda_d^g + \lambda_Z^g)Z_t$$

The conclusion in (29) is, perhaps, the most important result of this paper. Since $\lambda_d^g + \lambda_Z^g > 0$ it says that the risk premium declines with the market belief. That is, if we run a regression of excess returns on the observable variables in the economy, the effect of the market belief on excess return *is negative*. This key result remains true if we compute the risk premium per dollar invested rather than the risk premium per share. That is so since if we divide by the price we have

(29b) $$\frac{1}{p_t}E_t^m[p_{t+1} + d_{t+1} + \mu - Rp_t] = \frac{1}{p_t}[R\frac{\sigma_\varepsilon^2}{\tau} - (\lambda_d^g + \lambda_Z^g)Z_t].$$

The price map (20a) shows that the price increases with Z_t hence the negative effect on excess return continues to hold. The significance of the results (29a)-(29b) arises from two factors. First, these results show that in a market with heterogenous beliefs there is an inherent risk that arises in the market just due to the diversity of beliefs in the market.

(29a)-(29b) are central implications of the theory. They exhibit what we call the *Endogenous Uncertainty* component of the risk premium (see Kurz (1997a)) which inherently depends upon market belief and which consists of two parts. One part is the effect of market beliefs on the mean premium $R\frac{\sigma_\varepsilon^2}{\tau}$. The second part is the effect of market belief on the time variability of the risk premium, reflected in $-(\lambda_d^g + \lambda_Z^g)Z_t$ with a sign which is very revealing. To examine σ_ε^2 we recall from (7a)-(8) that $\sigma_\varepsilon^2 = Var_t^i(p_{t+1} + d_{t+1})$ and we now compute this conditional variance as follows:

$$\begin{aligned}\sigma_\varepsilon^2 &= Var_t^i((a+1)(\lambda_d d_t + \lambda_d^g g_t^i + \rho_{t+1}^{id}) + b(\lambda_Z Z_t + \lambda_Z^g g_t^i + \rho_{t+1}^{iZ}))\\ &= Var^i[(a+1)\rho_{t+1}^{id} + b\rho_{t+1}^{iZ}] = ((a+1),\ b)^T\Sigma^i((a+1),\ b).\end{aligned}$$

The direct effect of market belief on the mean risk premium is then seen through the effect of ρ_{t+1}^{iZ} in the above expression. The increased volatility of the return on the risky stock raises the permanent mean premium due to the variance of the market belief Z_t. But the second effect on the time variability of the risk premium arises due to the fact that the premium fluctuates over time due to changes in Z_t, as seen in (29b). The sign of this effect is of great interest: it is positive when $Z_t > 0$ and negative when $Z_t < 0$. This is surprising since when $Z_t > 0$ the market expects *above normal* future dividend and in that case the risk premium on the stock *declines*. When the market holds bearish belief about future dividend ($Z_t < 0$) the risk premium *rises*. The importance of this result is not only related to its theoretical implications but also because it is empirically testable since we have data on Z_t. Such a test was indeed performed by Kurz and Motolese (2005). We briefly review this test.

3.6b Empirical Test of the Endogenous Time Variability of the Risk Premium

Kurz and Motolese (2005) estimated the risk premia in the futures markets, the bond markets and the stock market. We provide a short sample of their results. The variable explained in each case is the actual, realized, holding returns of a specified asset: Federal Funds futures, 3 month Treasury Bills, 6 month Treasury Bills and a portfolio consisting of the S&P500. Data availability made it possible to estimate the model for holding period from 1 month to 6 month for Fed Funds futures, from 1 month to 12 months for Treasury Bills and for one year for the S&P500. Due to differences in data availability the models for the bond and stock market are different. Let us start with the bond market.

The market belief about future interest rates was computed from forecast data compiled by the Blue Chip Financial Forecast for the period 1980:1 to 2003:10. For each interest rate and for each horizon the authors constructed the long term stationary forecast by using a real time stationary econometric model which utilized a vast set of past economic variables. The market belief about future interest rate associated with asset i and horizon h which is denoted by $Z_t^{(i,h)}$ is then computed as the the constructed econometric stationary forecast minus the given Blue Chip mean market forecast. To understand what this means consider $Z_t^{(i,h)} > 0$. It says that at date t the market believes that interest rate on bonds with maturity i will be lower than normal at t + h. Since a lower interest rate is beneficial to a long position in the bond, the orientation of $Z_t^{(i,h)}$ is chosen to be beneficial to a long position as is the convention in the theory. That is, we could define $Z_t^{(i,h)}$ as the data on the Blue Chip mean market forecast *minus* the constructed econometric stationary forecast. In that case $Z_t^{(i,h)}$ would be oriented in a negative direction to a long position in the bond.

Data on the cross sectional distribution of the forecasts allowed the estimation of the cross sectional variances of the individual states of belief, denoted $\sigma_t^{(i,h)}$. To permit agents to hold beliefs about the entire term structure Kurz and Motolese (2005) introduced $SZ_t^{(6-F,h)} = Z_t^{(6,h)} - Z_t^{(F,h)}$ which measures the belief in a change in the *slope* of the yield curve. To control for other possible variables which affect the time variability of the premium, the following variables were used

NFP_{t-1} –year over year growth rate of Non Farm Payroll at t-1, forecasting on-coming recession

CPI_{t-1} – rate of inflation at t-1

F_t^{Cum} – measured cumulative intensity of monetary policy up to last month

$R_t^{Fj}, j = 1, 2, 3$ – three principal components of past interest rates on 18 maturities estimated at t. These variales summarizes information in past yield curves.

Tables 1A-1C summarize the results. To preserve space we present estimates only for two horizons for each asset but the results are the same for all horizons (for details see Kurz and Motolese (2005)). In all equations the sign of the parameter of the market belief variable and the belief in the slope of the yield curve are always negative, in accord with the prediction of the theory. Also, a higher cross sectional variance also reduces the risk premium. Hence the empirical results can be summed up as follows:

(1) Increased market belief in lower future interest rate reduces the risk premium on such investment;

(2) Belief in future steeper yield curve reduces the risk premium on each maturity

(3) Markets with greater diversity of opinions are less risky with reduced risk premium.

Table 1A: Predictability of Excess Returns, Fed Funds

	Constant	NFP_{t-1}	CPI_{t-1}	F_t^{Cum}	R_t^{F1}	R_t^{F2}	R_t^{F3}	$\sigma_t^{(F,h)}$	$Z_t^{(F,h)}$	$SZ_t^{(6-F,h)}$	R^2
h=4	**0.503* (0.231)**	**-0.177* (0.040)**	**0.087* (0.036)**	**-0.007 (0.033)**	**0.234* (0.107)**	**-0.013 (0.046)**	**-0.098 (0.057)**	**-0.945* (0.493)**	**-0.573* (0.126)**	**-0.871* (0.273)**	**0.289**
h=6	**0.633* (0.312)**	**-0.232* (0.052)**	**0.169* (0.047)**	**-0.005 (0.042)**	**0.373* (0.141)**	**0.052 (0.068)**	**0.039 (0.108)**	**-0.988* (0.488)**	**- 0.930* (0.188)**	**- 1.661* (0.482)**	**0.436**

Table 1B: Predictability of Excess Returns, 3 Months Treasury Bills

	Constant	NFP_{t-1}	CPI_{t-1}	F_t^{Cum}	R_t^{F1}	R_t^{F2}	R_t^{F3}	$\sigma_t^{(3,h)}$	$Z_t^{(3,h)}$	$SZ_t^{(6-F,h)}$	R^2
h=6	**0.820* (0.174)**	**-0.185* (0.026)**	**0.078* (0.025)**	**0.006 (0.021)**	**0.360* (0.072)**	**0.032 (0.036)**	**-0.044 (0.042)**	**-0.820* (0.189)**	**- 0.516* (0.095)**	**-0.636* (0.153)**	**0.447**
h=10	**1.272* (0.133)**	**-0.168* (0.025)**	**0.067* (0.020)**	**0.027 (0.015)**	**0.561* (0.063)**	**-0.016 (0.030)**	**-0.013 (0.025)**	**-0.887* (0.120)**	**-0.437* (0.063)**	**- 0.413* (0.149)**	**0.663**

Table 1C: Predictability of Excess Returns, 6 Months Treasury Bills

	Constant	NFP_{t-1}	CPI_{t-1}	F_t^{Cum}	R_t^{F1}	R_t^{F2}	R_t^{F3}	$\sigma_t^{(6,h)}$	$Z_t^{(6,h)}$	$SZ_t^{(6-F,h)}$	R^2
h=6	**1.964* (0.370)**	**-0.388* (0.063)**	**0.175* (0.050)**	**0.012 (0.047)**	**0.828* (0.158)**	**0.045 (0.081)**	**-0.052 (0.087)**	**-2.508* (0.441)**	**-1.434* (0.195)**	**-1.384* (0.348)**	**0.519**
h=10	**2.717* (0.269)**	**-0.387* (0.056)**	**0.092* (0.038)**	**0.072* (0.032)**	**1.141* (0.133)**	**-0.100 (0.063)**	**-0.014 (0.052)**	**-1.867* (0.258)**	**- 0.906* (0.129)**	**- 0.733* (0.279)**	**0.677**

To understand the effect of market beliefs on the time variability of risk premia Kurz and Motolese (2005) present three graphs of the difference between the model prediction and the actual data. One can see that the model is able to assess with great accuracy the *turning points* in the data

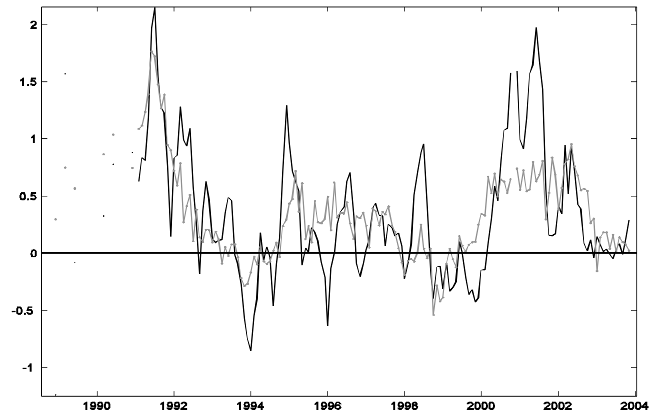

Figure 1 Excess Returns on Fed Funds Futures contract 6 months ahead. The gray line (green in color) represents the fitted values from regression.

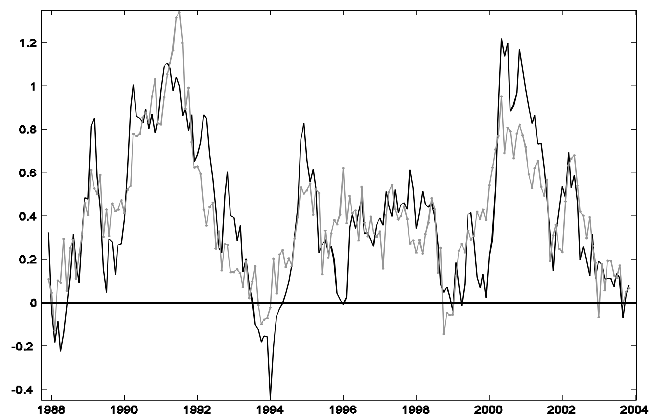

Figure 2 Excess Returns on 3 Months T-Bills 10 months ahead. The gray line (green in color) represents the fitted values from regression.

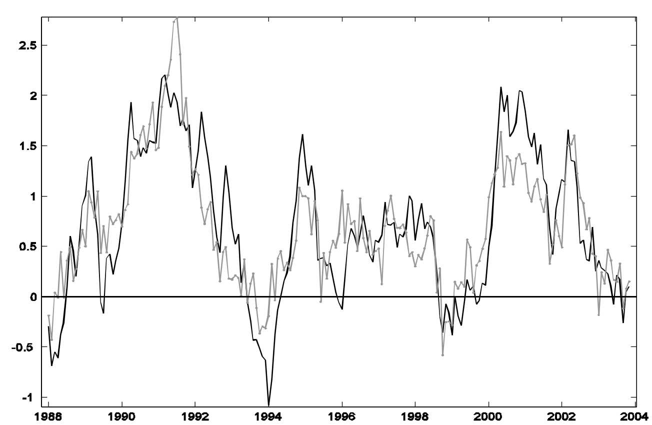

Figure 3 Excess Returns on 6 Months T-Bills 10 months ahead. The gray line (green in color) represents the fitted values from regression).

4. Contrasting the Theories: Heterogeneous Beliefs vs. Private Information

In Section 2 we discussed our objections to the excessive use of the assumption of PI. We now complete the comparison between the two theories and their empirical implications *based on the specific models that we have presented*.

4.1 Causes and Persistence of Heterogeneity

Market data reveal persistent heterogeneity of beliefs. Hence, to be relevant, a theory must exhibit non transitory diversity. A theory must then contain a renewal mechanism so that whatever updating agents make, the renewal is sufficient to maintain belief diversity. Under PI, heterogeneity arises from asymmetric information. Such diversity has no effect without the exogenous noise: without it an REE is fully revealing. With revelation prevented by the noise, does heterogeneity persists? The answer is NO since (6) shows that with time and repeated observations agents learn all they want to know. *Hence, heterogeneity under PI is transitory*.

In a market with HB agents never learn the true structure of the economy, they learn only the empirical distribution of past variables. We thus have a simple theorem: in an equilibrium with HB there is one true stochastic law of motion of state variables while traders hold diverse beliefs about that dynamics. *Hence, most traders are wrong most of the time*. The implication is that under HB prices and allocations are determined by the distribution of mistakes agents make in forecasting. Prices are functions of observed exogenous fundamentals and of the market's belief about the future. But the market belief is the aggregation of individual assessments, including all mistaken assessments. Hence price volatility is greater than the volatility of the exogenous shocks. Kurz (1974), (1997a) calls this component of market risk "Endogenous Uncertainty." Since under HB agents do all the learning they can from past data, both heterogeneity and price volatility are persistent.

4.2 Empirical Implications

Examination of the model in Section 1 supports our observations in Section 2. Given the fact that neither the private signals nor the supply noise are observable, the proposed theory under PI lacks testable implications. Hence, restricted to two or three rounds of trade, any dynamic pattern which is compatible with the pricing formula is possible. The model's key empirical implication is the pricing formula, discussed below. But then our critical observations come into play: why is it that private signals are so good that the price is actually a function of something that nobody knows? And how did the common knowledge of these facts come to be in such a secretive world? We conclude that the implications of the pricing maps are contrived by the implausible assumptions.

In contrast to the model under PI the model under HB is entirely testable since all central components of the theory are observable, including the market belief. Data on the variables Z_t are constructed exactly as required by averaging (11). In a standard asset pricing equilibrium one can then write down the Euler equations of the agents, aggregate them and use the market data on returns, asset prices and market beliefs for a full identification. For examples of work where this has been done see Fan (2005) and Kurz and Motolese (2005). These papers show that with data on asset returns and market belief an asset pricing theory leads to specific testable restrictions. Moreover, by studying the excess returns on different categories of assets (i.e. stocks, bonds, etc.) one derives sharp estimates of the market risk premia of different assets and the effect of market beliefs on such premia. A sample of these results was demonstrated above where we offered empirical evidence in support of the hypothesis that any risk premium contains a crucial Endogenous Uncertainty component which affect both the long term mean premium as well as the time variability of the premium. This component is directly determined by the distribution of beliefs in the market. This work also reveals a more general principle which applies to all empirical work on intertemporal phenomena such as portfolio analysis, investment behavior, etc.. In such phenomena aggregate market demand functions depend upon market belief. Hence if one does not use data on market belief in such estimation then an important state variable is missing and an error in specification is implied.

4.3 Sharp Differences in Asset Pricing Implications

The difference between the pricing map (1a)-(1c) under PI and (20a) under HB reflects central differences between the two theories. The map (1a)-(1c) under PI has infinite memory, a generic property of any learning since if some components of a finite memory process are unobserved, the process without full observability must be of infinite memory[6]. Since Bayesian learning is driven by unobserved shocks, it starts with a prior and progresses with inference requiring all past prices, which are proxies for the shocks. The infinite memory of prices arises despite the fact that exogenous

[6] Some (e.g. Bacchetta and van Wincoop (2005b)) bypass this theorem by not carrying out the full inference and instead assuming that the information structure in a noisy REE is of finite memory. Such analysis is not meaningful.

shocks have no memory at all. In addition, considering the PI model in real time, price volatility over time is driven only by the "fundamentals" in the market: changes in Q_t and the supply shocks.

In contrast, the pricing theory under HB leads to a an invariant price map, defined over the economy's state variables, including the market state of belief. This is a generic result since under HB all learning was done in the past and the two observed state variables summarize the common knowledge in the market. Hence, the pricing process is non-stationary only to the extent that state variables are non stationary. Under HB the price does not reflect the unknown intrinsic value of the object, and p_t forever fluctuate around their fundamental values. Moreover, price volatility is greater than the volatility of the fundamental values Q_t, a property named "Endogenous Uncertainty."

4.4 Beauty Contest Implications

A great deal has been written about the Keynesian Beauty Contest metaphor. In the context of the PI model with N rounds of trading we can rewrite (1c) in the form

(30) $$p_1 = \bar{E}_1\bar{E}_2\ldots\bar{E}_N(Q) - \frac{Var_1(p_2)}{\tau}S_1$$

Allen, Morris and Shin (2005) associate this equation with the Beauty Contest since the price reflects the average forecast *of the true fundamental value Q*. This is too narrow interpretation of the Beauty Contest. An examination of this idea as explained by Keynes (see Keynes (1936), page 156) shows that the crux of Keynes's conception is that there is little merit in the idea of *fundamental values*. Hence what matters in pricing an asset is what the market believes the future price of that asset will be rather than what the intrinsic value of the asset is. Moreover, Keynes insists future price depends upon future market beliefs which may be right or wrong but have no necessary relation to fundamental values. This idea is not reflected in the Allen, Morris and Shin's (2005) formulation. Indeed, under PI the price is a function of the unknown intrinsic value Q as seen in (1a)-(1b). If the Beauty Contest is a statement that the price today is determined by today's investors' forecasts of the forecasts of other investors tomorrow, when such forecasts are "right" or "wrong", then the Allen, Morris and Shin (2005) does not rise to this level of subtlety. Finally, we consider the number of trading rounds to be an artificial modeling construct and as the number of rounds increases the PI market leads to full revelation hence, with time, p = Q. When this is the case, traders do not engage in any "Beauty Contest."

Another line of thinking in the literature about the Beauty Contest often stresses the role of *higher order* expectations. This is misleading since higher order expectations are direct properties of conditional expectations. (1a)-(1b) imply iterated higher order expectations and this is true with and without PI or HB (see also Townsend (1978),(1983)).

Let us now examine the view of the Beauty Contest under HB and keep in mind the fact that *in this model the unknown value Q_t is announced at the end of each date*. To make our discussion comparable with the PI model we write equation (1c) in the form which is common to both theories

(31a) $$p_t = \bar{E}_1(p_{t+1}) - \frac{G_1}{\tau} S .$$

Now insert into (31a) the equilibrium map under HB, $p_t = a d_t + b Z_t - c S$, where price always depend upon (d_t, Z_t, S), all of which agents observe and know, to conclude that

(31b) $$p_t = a\bar{E}_t(d_{t+1}) + b\bar{E}_t(Z_{t+1}) - (c + \frac{G_1}{\tau}) S .$$

Compare (31b) with the price map under PI which depends upon the unknown and unobserved values of Q and of the supply shocks

$$p_1 = \kappa_1(\lambda_1 y + \mu_1 Q - S_1)$$
$$p_2 = \hat{\kappa}_2(\hat{\lambda}_2 y + \hat{\mu}_2 Q - S_2 + \psi S_1) .$$

In (31b) the price at t is an invariant function of the market expected values of d_{t+1} and of Z_{t+1}. This last key element of the theory shows that Endogenous Uncertainty is a risk inherent in the returns on an asset. The source of this market uncertainty are *the possible mistakes markets make in pricing assets too high or too low*, leading to excess volatility. The term $\bar{E}_t[Z_{t+1}]$ in the price map (31b) says the price today depends upon today's fear of market participants about the mistaken assessment the market may collectively make tomorrow. This risk is central to any theory with HB. This, in our view, is the essence of the Keynesian Beauty Contest. Also, with N rounds of trading, iterated expectations with higher order beliefs arise naturally in all conditional expectations and should not be exaggerated.

In short, the difference between a PI perspective of the Beauty Contest and an HB perspective is in their economic content which is found in the price maps. Under PI the price is a function of the true fundamental value which is unknown to anyone! Under HB, date t price depends upon the information available at date t which consists of d_t and Z_t.

4.5 Difference in Rationality Restrictions: Rational Beliefs

An equilibrium of an economy with agents who have the same information but hold heterogenous belief is not REE. Equilibria with PI are typically Noisy REE. This entails differences between the two theories which are inherent in the HB paradigm. In addition, up to now we compared the PI model with the HB model without asking for restrictions on information or added restrictions on beliefs. We criticized the PI theory for permitting arbitrary, unobservable, private signals and introducing contrived random supply shocks. These make it possible to prove anything with a PI model. One might make an analogous argument against HB, claiming that with HB one can prove anything. This last argument is false for two reasons. First, since market belief is observable, any hypothesis about market belief is entirely testable. Secondly, we have anchored individual beliefs to be about *deviations from the empirical distribution* and this, by itself, places restrictions on beliefs. We agree that we need to go further and seek additional a priori restrictions on beliefs of in order to further narrow the empirical implications of the theory.

The theory of Rational Belief (in short, RB) due to Kurz (1994), (1997a) proposes natural restrictions on beliefs. The RB theory explains the emergence of diverse beliefs and excess volatility. In a sequence of papers since 1994 the theory has been applied to various markets (e.g. Kurz (1996), (1997a), (1997b), Kurz and Schneider (1996), Kurz and Wu (1996), Kurz, Jin and Motolese (2005b), Motolese (2001), (2003) Nielsen (1996), (2003), Wu and Guo (2003), (2004)). In relation to the equity risk premium, Kurz and Beltratti (1997), Kurz and Motolese (2001), and Kurz, Jin and Motolese (2005a) explain the equity premium by asymmetry in the distribution of beliefs.

A belief is said to be an RB *if it is a probability model which, if simulated, reproduces the empirical distribution known from the data*. In essence, an RB is a model which cannot be rejected by the empirical evidence at hand. In our model beliefs are specified by perception models (14a) -(14b). For these to be RB they need to induce the same empirical distribution as (13a)-(13b). But this amounts to the requirement that

(25) The empirical distribution of $\begin{pmatrix} \lambda_Q^g g_t^{\,i} + \rho_t^{iQ} \\ \lambda_Z^g g_t^{\,i} + \rho_{t+1}^{iZ} \end{pmatrix}$ = the distribution of $\begin{pmatrix} \rho_t^{Q} \\ \rho_{t+1}^{Z} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_Q^2, & 0, \\ 0, & \sigma_Z^2 \end{bmatrix} \right)$, i.i.d.

To compute the implied data generated by the model, one treats the $g_t^{\,i}$ symmetrically with other random variables. From (10), the unconditional variance of $g_t^{\,i}$ is

$$\mathrm{Var}(g^{\,i}) = \frac{\sigma_g^2}{1 - \lambda_Z^2} .$$

Hence, we have the following rationality conditions which follow from (25):

$$\text{(i)}\ \frac{(\lambda_Q^g)^2\sigma_g^2}{1-\lambda_Z^2} + \hat{\sigma}_Q^2 = \sigma_Q^2 \qquad \text{(ii)}\ \frac{(\lambda_Z^g)^2\sigma_g^2}{1-\lambda_Z^2} + \hat{\sigma}_Z^2 = \sigma_Z^2 \qquad \text{(iii)}\ \frac{\lambda_Q^g\lambda_Z^g\sigma_g^2}{1-\lambda_Z^2} + \hat{\sigma}_{ZQ} = 0$$

$$\text{(iv)}\ \frac{(\lambda_Q^g)^2\lambda_Z\sigma_g^2}{1-\lambda_Z^2} + \mathrm{Cov}(\hat{\rho}_t^{iQ}, \hat{\rho}_{t-1}^{iQ}) = 0 \qquad \text{(v)}\ \frac{(\lambda_Z^g)^2\lambda_Z\sigma_g^2}{1-\lambda_Z^2} + \mathrm{Cov}(\hat{\rho}_t^{iZ}, \hat{\rho}_{t-1}^{iZ}) = 0.$$

The first three conditions pin down the covariance matrix in (14a)-(14b). The last two pin down the serial correlation of the two terms $(\hat{\rho}_t^{iQ}, \hat{\rho}_t^{iZ})$. An inspection of (14a)-(14b) reveals the only choice left for a trader are the two free parameters $(\lambda_Q^g, \lambda_Z^g)$. But under the RB theory these are not completely free either. The requirements $\hat{\sigma}_Q^2 > 0$, $\hat{\sigma}_Z^2 > 0$ place two strict conditions on $(\lambda_Q^g, \lambda_Z^g)$:

$$|\lambda_Q^g| < \frac{\sigma_Q}{\sigma_g}\sqrt{1-\lambda_Z^2} \qquad |\lambda_Z^g| < \frac{\sigma_Z}{\sigma_g}\sqrt{1-\lambda_Z^2}.$$

Finally, to ensure the covariance matrix in (14a)-(14b) is positive definite one must impose an additional condition. One such condition

$$\frac{1-\lambda_Z^2}{\sigma_g^2} > \frac{(\lambda_Z^g)^2}{\sigma_Z^2} + \frac{(\lambda_Q^g)^2}{\sigma_Q^2}$$

is sufficient. We then see that the "free" parameters $(\lambda_Q^g, \lambda_Z^g)$ are restricted to a rather narrow range. In short, the restrictions imposed by the RB theory on the beliefs of agents are incomparably stronger than the a-priori restrictions which unobserved asymmetric information imposes on their behavior.

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