# Approximation of the Yolk by the LP Yolk

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June, 1991
revised May, 1993
revived November, 2005

**Abstract**

If $n$ points are sampled independently from an absolutely continuous distribution with support a convex subset of $\Re^2$, then the center and radius of the ball determined by the bounding median lines (the LP yolk) converge with probability one to the center and radius of the yolk. The linear program of [McKelvey (1986] is therefore an effective heuristic for computing the yolk in large samples. This result partially explains the results of numerical experiments in [Koehler (1992], where the bounding median lines always produced a radius within 2% of the yolk radius.

# 1 Introduction

If $\{x_1, x_2, ..., x_n\} \subseteq \Re^m$ are $n$ points in an $m$ dimensional Euclidian space, the "yolk" is defined to be the ball with smallest radius which intersects all median hyperplanes in the sample. The "LP yolk" is defined to be the ball with the smallest radius which intersects all "bounding" median hyperplanes, where a bounding median hyperplane is defined to be one that passes through at least $m$ of the sample points. A linear programming formulation is given in [McKelvey (1986] to compute the LP yolk (hence the name), and in the same article it is claimed that this computation will also yield the yolk. However [Stone and Tovey (1992] give counterexamples in which the yolk and the LP yolk are different.

In this paper, we show that in a two dimensional space, the result in [McKelvey (1986] is approximately correct, in the following sense: We assume that the points $x_i$ are drawn independently from a measure $\mu$, where $\mu$ is assumed to be absolutely continuous with respect to Lebesgue measure, and its support, $S$, is assumed to be convex. Then we show that with probability one, as the sample size tends to infinity, the LP yolk converges to the yolk.

# 2 Notation

Let $X = \{x \in \Re^2 : ||x|| = 1\}$ be the set of unit length vectors in $\Re^2$. For any $x \in X$, and $c \in \Re$, we define $H(x,c) = \{z \in \Re^2 : x \cdot z = c\}$ to be the hyperplane (line) normal to $x$ which passes through the point $cx$. Define $H^+(x,c) = \{z \in \Re^2 : x \cdot z \geq c\}$ and $H^-(x,c) = \{z \in \Re^2 : x \cdot z \leq c\}$ to be the positive and negative closed half spaces defined by $H(x,c)$.

For any $x \in X$, a distributional median line for $\mu$ is defined by the unique $c \in \Re$ for which $\mu[H^+(x,c)] \geq 1/2$ and $\mu[H^-(x,c)] \geq 1/2$. The existence follows from absolute continuity of $\mu$, and the uniqueness follows from the support, $S$, of $\mu$ being convex. For any vector $x \in X$, let $h_x$ denote the unique distributional median hyperplane normal to $x$.

Similarly, for any integer $n$ and any sample $\{x_1, x_2, ..., x_n\} \subseteq \Re^2$ of size $n$ a sample median line is defined by any $c \in \Re$ for which $|\{i : x_i \in H^+(x,c)\}| \geq n/2$ and $|\{i : x_i \in H^-(x,c)\}| \geq n/2$. There always exists a median line in any direction $x$. If $n$ is odd, then it is unique. We assume $n$ is odd. For any vector $x \in X$, let $h_x^n$ denote the unique sample median hyperplane normal to $x$.

For any sets $A \subseteq \Re^2$, and $B \subseteq \Re^2$, let $d(A, B)$ denote the distance between the sets. For any $x \in X$, and $\delta > 0$, we define $B(x, \delta) = \{y \in \Re^2 : d(\{y\}, h_x) \leq \delta\}$ to be the $\delta$ band normal to $x$ around (and parallel to) $h_x$. For any $x \in X$, $y \in \Re^2$, and $\theta > 0$, define

$$W(y, x, \theta) = \bigcup_{x' \in X, x' \cdot x \leq \cos(\theta)} H(x', x' \cdot y)$$

to be the union of all lines passing through $y$, normal to a vector at an angle less than or equal to $\theta$ from $x$. This is a double wedge of angle $2\theta$ parallel to $h_x$ through $y$. For any $\delta, \theta > 0$, and compact subset $U \subset \Re^2$, define

$$G(\theta, \delta, U) = \inf_{x \in X, y \in B(x,\delta) \cap U} \mu(W(y, x, \theta))$$

to be the greatest lower bound of the measure of a $\theta$ wedge normal to $x$ originating in the interesection of $U$ and some band $B(x, \delta)$.

# 3 Results

First we start with three technical lemmata. The first says that for a large enough sample the sample median lines in any direction approach (uniformly) the corresponding distributional median lines. The second shows that the minimum measure, $\mu$, over all wedges of angle $\theta$ normal to $x$ originating at a point within $\delta$ of $h_x$ is greater than 0. The third lemma says that for $\delta$ small enough, and for any $\theta$, with probability one, in a large enough sample, every such wedge will contain at least one sample point.

**Lemma 3.1** *For any $\delta > 0$, with probability one, as $n \to \infty$, $h_x^n \subseteq B(x, \delta)$ for all $x \in X$.*

*Proof:* For all $x \in X$, let $c_x$ define the distributional median line, $h_x = H(x, c_x)$. Then the class of sets $\{H^-(x, c_x + \delta)\}_{x \in X} = \{H^+(x, c_x - \delta)\}_{x \in X}$ is of polynomial discrimination, with the $\mu$ measure of all elements in the class bounded strictly above 1/2. It follows from results of [Pollard (1984, p. 18], that with probability one, for a large enough sample, for all $x \in X$, $|\{i : x_i \in H^+(x, c_x - \delta)\}| \geq n/2$ and $|\{i : x_i \in H^-(x, c_x + \delta)\}| \geq n/2$. Thus, $d(h_x^n, h_x) < \delta$ for all $x \in X$. I. e., $h_x^n \subseteq B(x, \delta)$. [Alternatively, we could apply the Glivenko-Cantelli Lemma for fixed $x$, and then use compactness of $X$ to get the result.] ∎

**Lemma 3.2** *For any $\theta > 0$, and nonempty compact $U$, there exists $\delta > 0$ such that $G(\theta, \delta, U) > 0$.*

*Proof:* First note that $\mu(W(y, x, \theta))$ is a continuous function of all its arguments. This follows directly from absolute continuity of $\mu$.

Second, let $C(p, \epsilon) \equiv \{z | d(z, p) < \epsilon\}$ denote the circle of radius $\epsilon$ around $p$. We claim that there exists $\epsilon > 0$ such that for all $x \in X$, there exists a point $p_x \in h_x$ such that $C(p_x, \epsilon)$ is contained in the region of support of $\mu$. To see this, for any $x \in \Re^2$ define $\phi(x) = \min(d(x, S^c), 1)$, where $S^c$ is the complement of $S$. Now $\phi$ is a continuous function on $\Re^2$, and $\Gamma(x) = h_x$ is lower hemicontinuous (in fact continuous) correspondence from $X$ to $\Re^2$. For any $x \in X$ define $M(x) = \sup\{\phi(y) : y \in \Gamma(x)\}$. It follows from convexity of $S$, and from absolute continuity of $\mu$ that $M(x) > 0$ for all $x \in X$. It follows from the maximum theorem in [Berge (1963, p. 115, Theorem 1] that $M(x)$ is lower semi continuous. Since $X$ is compact, it follows that $M(x)$ achieves its minimum, which must be strictly greater than 0. Setting $\epsilon$ less than this minimum, this establishes the existence of a circle of guaranteed minimum radius for each median hyperplane $h_x$, within which the density of $\mu$ is positive.

Third, let $\delta = \epsilon/2$. Let $x$ be arbitrary. Let $C$ denote the circle $C(p_x, \epsilon)$. Let $y$ be any point $y \in B(x, \delta)$. The point $y$ has wedge $W(y, x, \theta)$, which intersects $C$ (see figure 1) in a region of variable size: for instance, when $y$ is sufficiently far from $C$ its wedge is large and contains all of $C$. Even when $y$ is close to or in $C$, however, this region $W(y, x, \theta) \cap C$ must have area at least $\delta^2 \sin\theta/2$. This is because its area is least when $y \in C$, but even in these cases (see figure 1) $W \cap C$ always contains an isosceles triangle with sides $\delta$ and apex angle $\theta$. (The altitude of the triangle from the apex is parallel to $h_x$.)

Since $C$ is contained in the support of $\mu$, and $\mu$ is absolutely continuous with respect to Lebesgue measure, and $W \cap C$ has positive Lebesgue measure, we have $\mu(W(y, x, \theta)) \geq \mu(W(y, x, \theta) \cap C) > 0$.

For any $x \in X$, compact $U$, and $\delta, \theta > 0$, define

$$G(x, \theta, \delta, U) = \inf_{y \in B(x,\delta) \cap U} \mu(W(y, x, \theta))$$

Now fix $U$ and fix $x \in X$. Recall $\theta > 0$ is already fixed, and $\delta > 0$ is as constructed above. Since $\mu(W(y, x, \theta))$ is continuous on the compact set $B(x, \delta) \cap U$, it attains its minimum. Since (as we have just shown) it is strictly positive, this implies that

Figure 1: $W(y,x,\theta) \cap C(p_x,\epsilon)$ (The shaded isosceles triangle has acute angle $\theta$ and two sides of length $\delta$)

$$\inf_{y \in B(x,\delta) \cap U} \mu(W(y,x,\theta) > 0.$$

*A fortiori*, the result follows for all $y \in B(x,\delta)$.

Summarizing, for any $\theta > 0$ and compact $U$ we have constructed $\delta > 0$ such that $G(x,\theta,\delta,U) > 0$ for any $x \in X$. But now by the theorem of the maximum it follows that $G(x,\theta,\delta,U)$ is a continuous function of $x$ which is everywhere positive. Since $X$ is compact, it follows that $G(\theta,\delta,U) = \min_{x \in X} G(x,\theta,\delta,U) > 0$. ■

**Lemma 3.3** *For any $\theta > 0$, and any compact $U$, there is a $\delta > 0$ such that with probability one, as $n \to \infty$ every wedge $W(y,x,\theta)$, for every $y \in B(x,\delta) \cap U$, for all $x \in X$, contains at least one point.*

*Proof:* This follows from results in [Pollard (1984] using the fact that the class of sets $W(y,x,\theta)$ is of polynomial discrimination, with measure bounded uniformly away from 0 (by Lemma 2). ■

Now let $n$ points ($n$ odd) be drawn from $\mu$. For any $z \in \Re^2$, let $r_n(z)$ denote the radius of a $z$ centered yolk for the sample, and $lr_n(z)$ denote the radius of a $z$ centered LP yolk for the sample. Let the sample yolk have center $c_n$ and radius $r_n$, and let the sample LP yolk have center $lc_n$ and radius $lr_n$. Note that since $n$ odd implies there is a unique median in every direction, the sample yolk and LP yolk centers are unique. Let $r(z)$ denote the $z$ centered radius of the distributional yolk (i. e., the yolk for $\mu$). Finally, let $r$ and $c$ denote the distributional yolk radius and center, respectively.

Next, we show that the centers of the sample yolk and sample LP yolk are both almost surely within a bounded distance of $c$. This will enable us to work within the compact set $U$ of Lemma 2.

**Lemma 3.4** *Let $\xi > 0$. Then as $n \to \infty$, w.p.1, eventually*

1. $lr_n \leq r_n \leq r + \xi$;
2. $c_n \in C(c, 2r + 2\xi)$;
3. $lc_n \in C(c, 4r + 4\xi)$.

*Proof:* Let $U^1$ denote the distributional yolk $C(c, r)$ which intersects all lines $h_x$. Let $U^2$ be a circle of slightly more than twice the radius, $C(c, 2r + 2\xi)$. Applying Lemma 1 with $\delta = \xi$ we have $h_x^n \subseteq B(x, \xi)\ \forall x \in X$ a.s. Therefore $d(c, h_x^n) \leq r + \xi\ \forall x \in X$ a.s., since by construction $d(c, h_x) \leq r\ \forall x$. Thus $r_n \leq r + \xi$ a.s., though of course $c_n$ is not necessarily equal to $c$. Every bounding median line is a median line, whence $lr_n \leq r_n$. This proves part 1 of the lemma.

For any point $p$ outside $U^2$, we have $d(p, c) > 2r + 2\xi$. Consider $x_p \equiv \frac{p-c}{||p-c||}$, the normalized vector between $p$ and $c$. Its sample normal median hyperplane $h_{x_p}^n$ satisfies $d(c, h_{x_p}^n) \leq r + \xi$, and is perpendicular to line segment $pc$. Hence $d(p, h_{x_p}^n) > 2r + 2\xi - (r + \xi) = r + \xi \geq r_n$. Hence there is a sample median hyperplane farther than $r_n$ from $p$, whence $p$ can not be the sample yolk center. This proves part 2 of the lemma.

Now let $U^4 \equiv C(c, 4r + 4\xi)$. For the last part of the lemma we must prove that as $n \to \infty, lc_n \in U^4$ a.s. As in the preceding paragraph, for any $p \notin U^4$ consider $x_p = \frac{p-c}{||p-c||}$. Applying Lemma 1 with $\delta = \xi$, we find $d(p, h_{x_p}^n) > 3r + 3\xi$. If $h_{x_p}^n$ were a bounding median line our proof would be complete, for we would know $lr_n(p) \geq d(p, h_{x_p}^n) > 3r + 3\xi \geq r + \xi \geq lr_n$, and $p$ could not be $lc_n$.

Unfortunately, $h_{x_p}^n$ might pass through only one sample point, denoted $z$. We have two cases.

Figure 2: distance from $p$ to median line $\tilde{h}^n_{x_p}$ exceeds $lr_n$

Case 1: $z \in U^2$. Apply Lemma 3 to $z$ with $\theta = \pi/6$, with compact set $U^2$. Lemma 3 then says we can "wiggle" $h^n_{x_p}$, holding it tacked at $z$, and we will bump into another sample point before the line has rotated more than $\theta$. Observe that the wiggled median line remains median when it bumps into a second point. So we get a bounding median line $\tilde{h}^n_{x_p}$ passing through $z$ and at angle less than $\pi/6$ from $h^n_{x_p}$.

A direct geometric calculation shows $d(p, \tilde{h}^n_{x_p}) > lr_n$, whence $p$ can not be the LP yolk center $lc_n$. (Moreover, by uniform convergence this applies to all $p \notin U^4$).

The calculation is: Let $t$ be the intersection of segment $pc$ and $\tilde{h}^n_{x_p}$, and let $s$ be the closest point on $\tilde{h}^n_{x_p}$ to $p$ (see figure 2). Let $\alpha$ denote the angle between $h^n_{x_p}$ and $\tilde{h}^n_{x_p}$. Since $\alpha \leq \pi/6$, we have $d(p,s) \geq \sqrt{3}d(p,t)/2$. Also, let $q$ be the intersection of segment $pc$ with $h^n_{x_p}$. By Lemma 1, $d(q,c) \leq r+\xi$. By assumption $z \in U^2$ so $d(z,q) \leq 2(r+\xi)$. Then $\alpha \leq \pi/6$ implies $d(t,q) \leq d(z,q)/\sqrt{3} \leq 2(r+\xi)/\sqrt{3}$. So $d(t,c) \leq (r+\xi)(1+\frac{2}{\sqrt{3}})$.

Since $p \notin U^4, d(p,c) > 4(r+\xi)$. So $d(p,t) > (4-1-\frac{2}{\sqrt{3}})(r+\xi) = (3-2/\sqrt{3})(r+\xi)$. Therefore, $d(p, \tilde{h}^n_{x_p}) \equiv d(p,s) \geq (\sqrt{3}/2)(3-2/\sqrt{3})(r+\xi) = (3\sqrt{3}/2 \ - 1)(r+\xi) > 1.5(r+\xi) > lr_n$ *a.s.* and we are done with case 1.

For case 2, where $z \notin U^2$, observe that by convexity of support there exists, independent of choice of $p$ and $x$, a circle $C(c,\delta)$ contained in the support of $\mu$, because $c$ is the distributional yolk center. With probability 1, as $n \to \infty$, eventually there exists at least one sample point within $\delta$ of $c$. So we can "wiggle" $h^n_{x_p}$ towards $c$, holding it tacked at $z$, and bump into another sample point before going more than $\delta$ past $c$. A simple calculation shows that $z \notin U^2$ renders $z$ sufficiently far from $c$ that the resulting bounding median line $\tilde{h}^n_{x_p}$ is far enough away from $p$. In particular, for the subcase where $h^n_{x_p}$ separates $c$ and $p$, we have $d(z,c) > 2(r+\xi)$, $d(p, \tilde{h}^n_{x_p}) > d(z, pc) > r+\xi$ as desired. The other case, where $h^n_{x_p}$ is on the other side of $c$ from $p$, is similar. This ends the proof of the lemma. ■

The next result establishes that $lr_n(z)$ and $r_n(z)$ approach each other pointwise with probability one as the sample size tends to infinity.

**Theorem 3.5** *For all* $\xi > 0$, *and for all* $z \in U^4 \equiv C(c, 4(r+\xi))$, $\lim_{n\to\infty}(lr_n(z) - r_n(z)) = 0$ *a.s.*

*Proof:* For any $0 < \eta < 1$, pick $\theta > 0$ to satisfy $\cos(\theta) \geq \eta$. Apply Lemma 2 with this $\theta$ and compact $U = U^4$, to get a $\delta$ small enough to satisfy the conclusion of the lemma. For any given sample of size $n$, and for any $z \in \Re^2$, if $lr_n(z)$ is not equal to $r_n(z)$ then from [Tovey (1992], there must be one (or more) points $y$ at distance $r_n(z)$ from $z$, with the property that the line through $y$, normal (perpendicular) to the line segment between $z$ and $y$, is a median line. Pick one such point $y$. Let $x = (y-z)/||y-z||$ be the unit length vector in direction $y - z$. Let $h = h^n_x$ be the sample median line normal to $x$, passing through $y$. Now $h$ must be parallel to $h_x$. As $n \to \infty$, we also know from Lemma 1, with probability 1, that eventually $h$ is within $\delta$ of $h_x$. This means $y$ is in the slice $B(x, \delta)$.

We can also bound the distance of $y$ to $c$. Since $z \in U^4$ by the hypothesis of the theorem, trivially $d(z,c) \leq 4(r+\xi)$. Now, the distance from $z$ to $y$ is $r_n(z)$. By Lemma 1, $r_n(z) \leq 5(r+\xi)\ \forall z \in U^4$, eventually. Therefore $d(c,y) \leq d(z,y) + d(z,c) \leq 5(r+\xi) + 4(r+\xi) = 9(r+\xi)$. We conclude that $y \in U^9 = C(c, 9(r+\xi))$.

Putting the last two paragraph's conclusions together, this means that $y$ and $h$ meet the conditions of Lemma 3, with respect to the compact set $U^9$.

By Lemma 3, for increasing $n$ we are guaranteed with probability 1 that every wedge $W(y, x, \theta)$, where $y \in B(x, \delta) \cap U^9$ has at least one point in it. Thus, we can wiggle $h$ by no more than $\theta$ and bump into another point. Observe that when we wiggle $h$, keeping it tacked at $y$, it (the hyperplane) remains a median hyperplane (with respect to the sampled points) until and including the instant we "bump into" another point. Call the wiggled median line $\tilde{h}$. Clearly, $\tilde{h}$ is a bounding median line. Let $\tilde{h}$ be normal to $x' \in X$. It follows that the radius of the $z$ centered LP yolk must be at least as large as the distance from $z$ to $\tilde{h}$. This is the length of the projection of $y - z$ onto $x'$. Hence, using $y - z = x \cdot ||y - z||$, we get

$$lr_n(z) \geq x' \cdot (y - z) = (x' \cdot x)||y - z||$$

$$\geq \cos(\theta)||y - z|| = \cos(\theta)r_n(z) \geq \eta r_n(z)$$

It follows that for any desired $0 < \eta < 1$, as the sample size grows, eventually $lr_n(z) \geq \eta r_n(z)$ w.p.1. Since it follows by definition that $lr_n(z) \leq r_n(z)$, it follows that with probability one, $\lim_{n\to\infty}(lr_n(z) - r_n(z)) = 0$. Finally, it should be noted that for a given $\eta$, the same sample size will work for all $z$.
■

Our main theorem is that as the sample size goes to infinity, the LP yolk approaches the yolk with probability one. More specifically, we show that the LP yolk center approaches the yolk center, and the LP yolk radius approaches the yolk radius.

**Theorem 3.6** *Under the conditions for Lemmas 1 and 2,*

$$Pr[\lim_{n\to\infty}(lc_n - c_n) = 0] = 1,$$

*and*

$$Pr[\lim_{n\to\infty}(lr_n - r_n) = 0] = 1.$$

*Proof:*
By argument similar to that for Lemma 3.1 $r_n(z)$ approaches $r(z)$ uniformly in $z$ with probability 1 as $n$ goes to infinity. By Lemma 4, $lc_n \in U^4$ and $c_n \in U^4$, which permits us to focus on $z \in U^4$. That is,

$$r_n = \min_{z\in U^4} r_n(z)$$

and

$$lc_n = \min_{z\in U^4} lr_n(z).$$

Further, from Theorem 4, $r_n(z) \to lr_n(z)\ \forall z \in U^4$ uniformly with probability 1 as $n$ goes to infinity. So for any $\delta > 0$, and $0 < \eta < 1$, with probability 1 as $n$ goes to infinity, we have simultaneously for all $z \in U^4$,

$$r(z) - \delta \leq r_n(z) \leq r(z) + \delta$$

and

$$\eta r_n(z) \leq lr_n(z) \leq r_n(z)$$

Define $D(\delta) = \{z \in U^4 : r(z) - \delta \leq r + \delta\}$ and $E(\delta, \eta) = \{z \in U^4 : \eta \cdot (r(z) - \delta) \leq r + \delta\}$. Since $r(x)$ is a convex function and $U^4$ is compact, these sets are convex, compact sets. Then by Lemma 4, for all $\delta$, $\eta$, $c \in D(\delta) \subseteq E(\delta, \eta)$, $c_n \in D(\delta)$, and $lc_n \in E(\delta, \eta)$. Further, for any $\epsilon > 0$ we can find $\delta > 0$, $0 < \eta < 1$ for which the radius of $E(\delta, \eta)$, (and hence $D(\delta)$) is less than $\epsilon$. It follows that with probability one, $lim_{n \to \infty} lc_n = lim_{n \to \infty} c_n = c$. This shows the first result.

To get the second result, just note that, since by Lemma 4 $lc_n \in U^4$, the last displayed equation implies that $\eta r_n(lc_n) \leq lr_n(lc_n) = lr_n \leq r_n(lc_n)$ which implies $(lr_n - r_n(lc_n)) \to 0$ as $n \to \infty$. But by definition, $lr_n \leq r_n \leq r_n(lc_n)$. Thus, $lr_n - r_n(lc_n) \leq r_n - r_n(lc_n) \leq 0$. So with probability one, $(lr_n \to r_n)$ as $n \to \infty$. ■

# 4 Conclusions and computational issues

We remark that the support of $\mu$ could have "holes" and the result could still hold true. We only use the convexity of support in proving the existence of uniform size balls of positive density on the median lines (in Lemma 2), and to get a positive density ball at the distributional yolk center (in Lemma 4).

The linear programming formulation of [McKelvey (1986] has the remarkably attractive computational feature of having only three variables. Hence its dual will have only three constraints and is extremely easy to solve with off-the-shelf software, since the number of constraints in a linear program is the most crucial parameter affecting computation time.

How many variables will the dual have? Let $n$ be the number of points. Then there are at most $\binom{n}{2}$ limiting lines (passing through two points). Of these, only the medians will add to the number of variables in the dual. It has been proved there aren't more than $O(n^{1.5})$; it is known to be possible to have as many as $O(n \log n)$ ([Erdös et al. (1973]); it has been conjectured that there aren't more than $o(n^{1+\epsilon})$ ([IBID]); and computational experience suggests between $2n$ and $3n$ as the typical number. If we take a liberal upper bound of $5n$, based on computational experience with actual data, then a standard commercial linear programming package such as CPLEX, running on an ordinary PC, would solve an LP of this type in much less than a second, for problems with a few thousand points. Data from the U.S. House or Senate, indeed from any assembly of representatives would therefore be very easy to process. Indeed, problems with $\approx 10^8$ points would be tractable on a PC. Cases of this magnitude may not arise at present, but larger data sets, representing, for example, preferences of internet users, may well become available in the future. There would of course have to be a model-building step which generates the median bounding lines. This preprocessing step may require more computational effort than the LP yolk computation itself.

The LP yolk provides a lower bound approximation of the yolk, in the sense that $lr_n \leq r_n$. Given the LP yolk one can also easily find an upper bound approximation, namely $C(lc_n, r_n(lc_n))$. ( As shown in [Tovey (1992] the value $r_n(lc_n)$ can be computed in time $O(n^2)$. Indeed all that is needed once the LP has been solved is to check all lines normal to the line segments $(lc_n, x_i)$, and take the maximum length of a segment whose normal line is a median line.) Theorem 6 assures that the yolk radius is tightly bracketed, $lr_n \leq r_n \leq r_n(lc_n)$, the second inequality being close because $lc_n$ is close to $c_n$.

Finally, we remark that it is an open question as to how often the LP yolk exactly coincides with the yolk. Computational experiments of [Koehler (1992] indicate this event may occur frequently.

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