IN SEARCH OF THE UNCOVERED SET

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Abstract

This paper pursues a number of theoretical explorations and conjectures pertaining to the uncovered set in spatial voting games. It was stimulated by the article on "The Uncovered Set and the Limits of Legislative Action" by Bianco, Jeliazkov, and Sened (*Political Analysis*, 2004) that employed a grid-search computational algorithm for estimating the size, shape, and location of the uncovered set. At the same time, the work has been greatly facilitated by access to the *CyberSenate* spatial voting software being developed by Joseph Godfrey. I bring to light theoretical considerations (in particular, the size and location of the yolk, the role of "phantom voters" and the distinction between "local" and "distant" covering in a spatial context) that account for important features of the Bianco et al. results (for example, the straight line boundaries of uncovered sets displayed some of their figures, the "unexpectedly large" uncovered sets displayed in other figures, and the apparent sensitivity of the location of uncovered sets to small shifts in the relative sizes of party caucuses) and present theoretical insights of more general relevance to spatial voting theory.

An earlier version of this paper was presented at the Annual Meeting of the Public Choice Society, New Orleans, March 10-13, 2005. I thank Joseph Godfrey, Bernard Grofman, and other participants at the panel on "Using *CyberSenate* Software to Analyze Spatial Voting Games" for useful comments, particularly pertaining to the connection between "phantom voters" and the Shapley-Owen value. I also thank Bill Bianco, Ivan Jeliazkov, and Itai Sened for provoking me into pursuing this line of inquiry.

IN SEARCH OF THE UNCOVERED SET

In early versions of their paper on "The Uncovered Set and the Limits of Legislative Action," Bianco, Jeliazkov, and Sened (2004 and henceforth BJS) made the emphatic claim that "*no one knows what the uncovered set looks like for a given* [two-dimensional] *spatial voting game or real-world situation*." This claim was not precisely true (for example, we had known for some time what the uncovered set looks like in the three-voter case) and it was dropped from the final version of their article. Nevertheless BJS's claim was substantially true until they put their grid-search computational algorithm to use. The evidence presented in their figures for a variety of spatial voting scenarios is therefore of great interest for the theory of spatial voting and social choice.

Here I pursue a number of theoretical explorations and conjectures stimulated by the BJS paper — and, for good measure, I have also appropriated their original title. I had a number of interactions with the authors as their paper developed and, when they submitted it to *Political Analysis*, its editor invited me to prepare a comment that might appear with it. Though what follows is much more substantial (and took much longer to complete) than either I or the editor initially had in mind, it remains in part a commentary on BJS and is best read with their paper at hand.

In pursuing these explorations, I have been enormously helped by access to early versions of *CyberSenate*, another computer program for analyzing spatial voting games being developed by Joseph Godfrey. This software allows researchers to create configurations of ideal points by point and click methods, or generate them by Monte Carlo methods, or derive them from empirical data (e.g., interest group ratings, NOMINATE scores, etc.). Indifference curves, median lines, Pareto sets, win sets, yolks, cardioid bounds on win sets, uncovered set approximations (based on an algorithm similar to that of BJS), and other constructions can be generated on screen. *CyberSenate* produced most of the figures that follow, plus many dozens of additional figures posted on the web that illustrate many points made in this article.[1] More important, *CyberSenate* has been indispensable in developing and testing theoretical ideas. While the BJS paper is informative regarding the macro-relationship between ideal point configurations and uncovered sets, my aim here is to explore the micro-structure that underlies the BJS findings.

By way of introduction, Section 1 sketches out the line of thinking that led to the concepts of covering and the uncovered set proposed in Miller (1980). Section 2 provides a brief summary of earlier finding concerning the uncovered set in a spatial context. Section 3 runs through some theoretical preliminaries. Section 4 lays out new findings (based on *CyberSenate* simulations and experiments) concerning the size and location of the yolk and makes particular use of BJS Figure 1. Section 5 introduces the concept of "phantom voters" and makes extensive use of BJS Figure 2. Section 6 sets out general characteristics of the covering relation in a spatial context and, in particular, notes that covering may operate either "locally" or "at a distance." Section 7 combines theoretical deduction and induction from much *CyberSenate* work to set out findings and conjectures

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[1] These figures may be found at *http://research.umbc.edu/~nmiller/RESEARCH/WEBLIST.htm* and, as web postings, can make advantageous use of color. Almost all of these figures were initially produced by *CyberSenate*, with labeling and other embellishments subsequently added. Citations to these figures are provided at appropriate points in the text of this article in this manner: [FIGURE X]. I am very much indebted to Joseph Godfrey of the WinSet Group LLC for making early versions of *CyberSenate* available to me. It will become more generally available to researchers in the near future.

concerning uncovered points and the general character of uncovered sets. Section 8 sets out findings and well-founded conjectures concerning the size and location of the uncovered set and specifically addresses BJS's graphical results presented in their Figures 1, 2, and 5. Section 9 concludes by offering more speculative conjectures concerning the structure of the uncovered set in a spatial context. Consistent with BJS and the capabilities of *CyberSenate*, I focus entirely on two-dimensional spatial voting games with Euclidean preferences, though most points probably generalize to higher dimensions. I do not in these remarks address the issue of whether and how the uncovered set may be relevant for the empirical study of legislative politics and "the limits of legislative action."

In general, I try to bring to light theoretical considerations that account for important features of the BJS results — for example, the straight line boundaries of the uncovered sets displayed in their Figure 2, the "unexpectedly large" uncovered sets displayed in their Figures 1 and 5, and the sensitivity of the location of uncovered sets to smalls shifts in the relative sizes of the party caucuses — as well as to present theoretical insights of more general relevance. These focus primarily on the location and size of the yolk and the uncovered set. As BJS note, two bounds on the uncovered set have been known for many years: (i) it lies within the Pareto set and (ii) it lies within a circle centered on the yolk with a radius four times that of the yolk. I show that the first bound operates through "local covering" and is overgenerous in principle (the actual bound is the Pareto set after "phantom voters" have been excluded have been excluded from the configuration of ideal points) though not in practice (phantom voters arise only in the presence of empirically unlikely collinearities in ideal points). The second bound operates through "distant covering" and is distinctly overgenerous, as the uncovered set typically is contained within a circle centered on the yolk with a radius of about twice that of the yolk. BJS's observation (p. 270) that the uncovered set "can be much larger than our expectations based on conventional wisdom and previous work" is really an observation that the yolk can be much larger than would be expected if ideal points did not form into the kind of distinct clusters displayed in their figures.

1. Origins of the Uncovered Set

I had the first inkling of the uncovered set idea during the winter of 1970 as graduate student at Berkeley. I was revising and extending a seminar paper (and prospective dissertation chapter) on "Voting in Committees" that I had written the previous year in a course on Formal Models in Politics taught by Michael Leiserson and Robert Axelrod (probably one of the first such political science courses taught anywhere outside of Rochester). We had read a book by Otomar Bartos on *Simple Models of Group Behavior* (1967) that included a chapter on "dominance structures" in societies and employed adjacency matrices and directed graphs — and tournaments in particular — to represent and analyze such structures. It occurred to me that, given an odd number of voters with strong preferences, the same analytical device could be used to represent the majority preference relation and that some of the deductions Bartos presented concerning social dominance structures (many of which were standard graph-theoretic theorems) had analogs for majority voting. In my seminar paper, I set out to use the tournament technology to systematize and extend Duncan Black's work (1948, 1958) on committee voting. Tournament diagrams drove home a point that I had not adequately appreciated: if you arrange four or more points around a circle and draw arrows around the perimeter so as to create a cycle encompassing all the points, this cycle also has an "internal structure" — that is, arrows must also extend across the interior of the circle. Moreover, given five points or more, cycles of given length may have different internal structures.

By the time I had finished the paper, I had discovered with some disappointment that Michael Taylor (1968) had already had much the same idea by applying graph theory to social choice problems. Nevertheless, the tournament technology allowed me to make some modest advances in the "Black program." For example, Black (1958: 40-41) had provided an example with three voters and four alternatives in a cycle of majority preference showing that, for each alternative, there was some voting order under "ordinary committee procedure" (what we now call *amendment procedure*) that would lead to its winning under sincere voting. Using the tournament technology, I was able to extend this result to any odd number of voters (indeed, the tournament device made it unnecessary to specify a particular number of voters) and to any number of alternatives in a cycle. (This became Proposition 4 in Miller, 1977.)

In a last-minute addendum to the seminar paper, I made a first stab at deriving a similar kind of result under strategic voting. Using a three-dimensional strategic form for a voting game, I was able to show that the same result held under strategic voting with three voters and three alternatives. I could also show that the voting orders that led to victory by a given alternative were different under sincere and strategic voting and, in particular, that under sincere voting the last alternative in a three-alternative cycle to enter the voting wins, but under strategic voting the first such alternative wins. But I had no good idea as to how to proceed beyond three voters and three alternatives in the strategic voting case.

By the winter of 1970, Robin Farquharson's *Theory of Voting* (1969) had been published and I studied it eagerly. His "Table of Results" in Appendix I confirmed my results for both sincere and strategic (or "sophisticated") voting in the three-voter three-alternative case. But I was completely mystified by his statement in the Preface that these "results . . . can be readily extended to cover any desired number" of voters and/or alternatives, since I had been using essentially the same kind of strategic form analysis that Farquharson was manifestly using in the body of his text, and I had found it to be impossibly burdensome to extend to a larger number of voters and/or alternatives. But Farquharson also introduced the "voting tree" device (a highly compressed extensive form) to concisely represent binary voting games. At some point that winter, while staring at such a voting tree, I had an epiphany and discovered the kind of backwards induction that readily solves binary voting games and that was later definitively characterized by McKelvey and Niemi (1978). With this tool, I could easily determine the strategic voting outcome for any number of voters and relatively many alternatives, and I enthusiastically began using it to derive Black-style propositions for strategic voting. But an apparent anomaly quickly turned up: when I extended the number of alternatives in a cycle from three to four, I discovered that there was always one alternative in the cycle that could not win under any voting order. This was surprising and somewhat disconcerting, since I expected sincere and strategic voting results to run in parallel. Something was preventing one alternative from winning, regardless of the voting order. Evidently it had to do with the fact that, when internal structure is considered, the alternatives in a four-element cycle (unlike those in a three-element cycle) occupy distinct positions in the tournament structure. But this structural asymmetry did not preclude a degree of symmetry with respect to sincere voting outcomes, so the discrepancy remained puzzling. I further discovered that, given a cycle of five (or more) alternatives, different internal structures are possible, some of which made it impossible for certain alternatives to win while others did not. In my subsequent dissertation chapter, I simply presented the following proposition: "Under ordinary [amendment] procedure, there may be a motion [alternative] in the Condorcet [top cycle] set that is not the sophisticated voting decision under any voting order."

In 1976, I had a revise and resubmit decision on a paper largely derived from this dissertation chapter. My main task was to revise the presentation of the material (on the basis of wise editorial guidance provided by *AJPS* editor Phillips Shively) rather than its substance. But in revisiting the analytical issues, I noticed two additional points concerning the strategic voting anomaly. First, each alternative *y* that could not win under any voting order (in the relatively small cyclic tournaments I was examining) was "dominated" in a particularly strong way by some other alternative *x* — not only did *x* beat *y* but *x* also beat everything that *y* beat. Given an arrangement of *x*, *y*, and the alternatives beaten by *y* into a three-level structure with *x* at the top and with downward-pointing arrows representing majority preference, in my own mind it seemed natural to say that *x* "covers" *y* — and, for better or worse, the terminology stuck. The second point I noticed was that if *x* is unanimously preferred to *y*, then *x* is majority preferred to all alternatives to which *y* is majority preferred — that is, *x* covers *y*. I incorporated the latter point into the revised paper (Miller, 1977, Proposition 10), and I then turned quickly to explore the covering relation more thoroughly and discovered that it had many interesting properties. That exploration led to a paper that I presented at the 1978 APSA meeting, which drew almost no attention (perhaps because it was one of six papers squeezed into a two-hour panel, one of which was Kenneth Shepsle's early statement of the "structure-induced equilibrium" idea) but which did attract considerable attention once it was published (Miller, 1980).[2]

In Miller (1980), I focused primarily on discrete alternatives and unrestricted preferences, but I also ventured some observations and conjectures concerning covering and the uncovered set in the spatial context, especially in light of McKelvey's (1976 and 1979) "global cycling theorem" and his famous concluding comments (in the earlier article) about its implications for agenda control. Since at the time I did not have the analytical tools at hand to pursue this aspect of covering effectively, I sent my uncovered set paper to Richard McKelvey and invited him to apply his expertise to the problem. I like to think that this helped lead to McKelvey (1986).

2. Earlier Findings

In Miller (1980), I made two points relevant to determining the size and location of the uncovered set in a spatial context. First, I noted (p. 80) that, in any setting, unanimity implies covering, from which it follows that the uncovered set is always a subset of the Pareto set. In a two-

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[2] I have noticed that authors sometimes give Miller (1977) as the original citation for "Miller's uncovered set." While these recollections indicate that such a citation is not entirely off-the-mark, I did not use the term, identify the concept, or explore its general ramifications in print until Miller (1980). These recollections also tend to contradict BJS's statement (p. 259) that the uncovered set originated as a cooperative game-theoretic concept. In any event, my original motivation was to understand strategic voting in an explicitly non-cooperative setting.

dimensional spatial voting game with Euclidean preferences, the Pareto set is the convex hull of voter ideal points, so the uncovered set in any event lies within this set. Secondly, I conjectured (footnote on p. 84) that in a spatial context the uncovered set "would be a relatively small subset of [the Pareto set], centrally located in the distribution of ideal points, and that it would shrink in size as the number and diversity of ideal points increase."

McKelvey (1986) introduced the concept of the *yolk*, i.e., the set of points bounded by the smallest circle that intersects every median line in a two-dimensional spatial voting game, and he showed that, if voters have Euclidean preferences, the uncovered set must lie within the circle centered on c with a radius of $4r$, where c is the center of the yolk and r is its radius. [FIGURE 1]

Shortly thereafter Hartley and Kilgour (1987) established the precise boundaries of the uncovered set given any configuration of three voters with Euclidean preferences in a two-dimensional space. They showed that, in the event ideal points form the vertices of an equilateral triangle, the uncovered set coincides with the Pareto set and that, otherwise, the uncovered set excludes portions of the Pareto triangle in the vicinity of the one (if the Pareto triangle is acute) or two (if it is obtuse) relatively extreme ideal points. Moreover, as an obtuse Pareto triangle becomes "flatter," the uncovered set becomes smaller, not only absolutely but relative to the (shrinking) area Pareto triangle. In the limit, the ideal points become collinear and the uncovered set and the ideal point of the median voter coincide. [FIGURES 2A-2D]

When the concept was first propounded, there was a widespread intuition that the yolk would be centrally located and would tend to shrink in size as the number of voters increases. However, it was difficult to confirm these intuitions or even to state them in a theoretically precise fashion. Feld et al. (1988) took a few very modest first steps. Tovey (1990) took a considerably larger step by showing that, if ideal point configurations are random samples drawn from any "centered" continuous distribution, the expected size of the yolk approaches zero as the number of ideal points increases without limit. More recently, Banks et al. (2004) showed that the same kind of result applies to the uncovered set itself.

Not much more was learned about either the yolk or the uncovered set in a spatial context until BJS provided the first pictures of the uncovered set in more general two-dimensional configuration in their Figure 2 (for contrived five-voter configurations) and Figures 1, 4, and 5 (for empirical U.S. House data) in their recent article. [FIGURES 3A-3E] Based on the computational results displayed in their figures, BJS (pp. 270-271) made three claims concerning the location and size of the uncovered set.

(1) "The uncovered set can be much larger than our expectations based on conventional wisdom and previous work," as all their figures seem to illustrate.

(2) The uncovered set is not necessarily "centrally located." If ideal points are polarized (as in the contemporary House), "the uncovered set does not lie in the center of the distribution of legislators' ideal points but is skewed toward the majority caucus," as illustrated by BJS Figure 1, the first panel of Figure 4, and all panels of Figure 5.

(3) "The size, shape, and location of the uncovered set are very sensitive to the distribution of ideal points." With respect to size, this sensitivity is quite dramatically illustrated by their

Figure 2 and is less dramatically illustrated by comparing panels in Figure 5. With respect to location, such sensitivity is illustrated by the first panel of their Figure 4 and the last two panels of their Figure 5.

We may also observe that BJS Figure 2 is distinctive in that the uncovered set has straight line boundaries that coincide with certain median lines. Furthermore, in each panel the uncovered set appears to coincide with the Hartley-Kilgour construction for the three-voter case — in some way, the two additional ideal points (to the left and right) have no apparent effect on the size and location of the uncovered set.

3. Theoretical Preliminaries

I follow BJS in focusing on a two-dimensional majority-rule spatial voting game. Some degree of familiarity with standard terminology and notation — such as that found in BJS — is assumed. In this spatial context, I refer to alternatives as *points*. The set X of all alternatives is the set of all points in the space. There is a finite odd number $n \geq 3$ of voters with Euclidean preferences — that is, each voter i has an ideal point x_i in the space and prefers any point closer to his ideal point to one that is more distant, so that the set of points $P_i(x)$ that i prefers to x is the set of points bounded by a circle that is centered on x_i and passing through x (so all indifference curves are concentric circles around ideal points).

If some majority of $m = (n+1)/2$ voters prefers x to y, I say "x beats y." The *win set* $W(x)$ is the set of all points in X that beat x. The set of points that a particular majority of voters prefers to x is the intersection of all sets $P_i(x)$ such that i belongs to that majority. $W(x)$ is the union all such majority preference sets. Thus the boundary of a win set is everywhere demarcated by segments of individual voter indifference curves (segments of circles in the Euclidean context). In a spatial context, x beats essentially all points not in $W(x)$.[3] [FIGURES 4A-4C]

I call a configuration of ideal points *diverse* if all points are distinct — that is, no two ideal points precisely coincide. A key feature of spatial voting games is whether the configuration of ideal points exhibits any *collinearity* — that is, whether three or more ideal points lie precisely on the same straight line. Collinearity always exists when ideal points coincide but may also in be found in diverse configurations.[4] Collinearity produces a variety of peculiarities — in particular, the "phantom voter" phenomena discussed in Section 5.

Let us first consider a one-dimensional policy space defined by a line L (= X), in which the n ideal points necessarily are collinear and where voter m has the median ideal point x_m. It is well known that, in this setup, the win set of any point x is the interval between x and its *reflection* x''

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[3] There are some majority preference ties but, in order to simplify exposition, I overlook technical issues pertaining to points that lie on the boundaries of sets.

[4] Collinearity may be deemed "exceptional." For example, if hypothetical ideal points are "randomly thrown" into a policy space, collinearity would almost never occur. Of course, we can deliberately contrive collinear configurations (as BJS do in their Figure 2). In empirical work, where ideal point locations estimated from interest group rating scales or similar data are typically expressed in whole numbers, it is likely that several legislators have identical scores on a given dimension, producing non-diversity and other collinearities.

through x_m, i.e., the point on L such that x and x'' are equidistant from x_m — which is to say that $W(x) = P_m(x)$. From this it follows that $W(x_m)$ is empty and that x_m is the *Condorcet winner* that beats every other point on L.

Next suppose that we embed this one-dimensional configuration of n ideal points on L in a two-dimensional space X (so all ideal points remain collinear). With respect to points on L only, the same relationships hold — that is, x is beaten by all points on L that lie on the interval between x and its reflection x'' through x_m, and x_m beats every other point on L (in fact, it beats every other point in X). [FIGURE 5]

Finally suppose that ideal points are scattered in any fashion across the two-dimensional space X. Each voter i has an *induced ideal point* x^L_i on L — that is, a most preferred point on L. Given Euclidean preferences, x^L_i is the point on L closest to i's ideal point, so x^L_i lies at the intersection of L and P_i^L where P_i^L is the line through i's ideal point perpendicular to L. The n induced ideal points appear on L in some (possibly weak) order and (since n is odd) we can identify the median induced ideal point(s) x^L_m, where P_m^L intersects L.[5] With respect to points on L only, it again follows that x is beaten by all points on L that lie on the interval between x and its reflection x'' through x^L_m, and that x^L_m beats every other point on L (but *not* every other point in X). We call a line L though x such that x is the median induced ideal points, and therefore beats every other point on L, a *dividing line* through x. Note that the induced ideal points on any other line L' parallel to L are determined by the same perpendicular lines P_i^L i, etc., that determine the induced ideal points on L. [FIGURES 6A-6D]

Any line L through the space partitions the set of voter ideal points into three subsets: those that lie on one side of L, those that on the other side of L, and those that lie on L itself. But P_m^L partitions the ideal points in a special way — namely, no more than half of the ideal points lie on either side of it. P_m^L is therefore a *median line* (which we henceforth label M). If n is odd, at least one — and typically only one — ideal point x_j lies on each median line M and fewer than half lie on either side of M. Put otherwise, a majority of ideal points — and typically a bare majority of $m = (n+1)/2$ — lies on and to either side of M. If it is important to specify what ideal point(s) a median line passes through, we label it M_j, M_{ik}, etc. [FIGURE 7]

We can repeat the operations described above starting with any other line L through the space, from which it follows that there is a median line (and, if n is odd, no more than one) perpendicular to any line drawn through the space and that at least one median line passes through every point in the space.

We saw above that, if point x lies off any median line (e.g., P_m^L), x is beaten by points on that median line (in particular, by e.g., x^L_m). It follows that a point x in the space is unbeaten (and is a Condorcet winner) only if it lies on every median line, which is possible if and only if all median lines intersect at the single point x (which itself must be an ideal point). This in turn can hold only in the presence of a sufficient (and unlikely) degree of "Plott symmetry"in the distribution of ideal

[5] Note that, even if ideal points are diverse, it is possible that two (or more) distinct ideal points may lie on the same line perpendicular to L, so the corresponding induced ideal points coincide.

points (see Plott, 1967, and the elaboration that allows for non-diversity by Enelow and Hinich, 1983). [FIGURES 8A-8B]

Consider an ideal point x_j and a median line M_j that passes through x_j but through no other ideal point. Thus $m-1$ ideal points lie on either side of M_j. Let us rotate the median line M "clockwise" about x_j as the pivot point until it hits a second ideal point x_i, giving median line M_{ij}. While $m-1$ ideal points still lie on one side of M_{ij}, only $m-2$ lie on the other side (since x_i lies on M_{ij}). It is clear that, if this "clockwise" rotation of M were to proceed beyond M_{ij}, M would no longer be a median line since then a bare majority of m ideal points (including x_j) would lie strictly on one side of M. Thus M_{ij} is what Miller et. al. (1989) called a *limiting median line*. Such a line always passes through (at least) two ideal points. [FIGURES 9A-9B]

Now, once again starting with the original M_j, let us rotate the median line M "counter-clockwise" until it hits a second ideal point x_k, forming a second limiting line M_{jk}. All median lines "between" M_{ij} and M_{ik} are *non-limiting* median lines that pass through the single ideal point x_i. [FIGURE 9C]

Let α_{ijk} be the angle between pairs of limiting median lines through x_j, i.e., M_{ij} and M_{ik}. Typically, α_{ijk} is a quite small angle. In the limiting case $\alpha_{ijk} = 0°$, in which case $M_{ij} = M_{ik} = M_{ijk}$ and (at least) three (collinear) ideal points (e.g., x_i, x_j, and x_k) all lie on this "stand-alone" limiting median line, with no "intermediate" non-limiting median lines. (In this event, voters *i* and *j* may be "phantoms," as discussed in Section 5.) [FIGURE 9D] Additional (pairs of) limiting median lines through x_j that hit pairs of ideal points other than x_i and x_k may also pass through x_j, giving a total angle $A_j = \sum \alpha_{ijk}$, where the summation is taken over all pairs of limiting median lines through x_j. [FIGURE 9E]

As we have seen, if n is odd and in the absence of collinearity, $m-2$ ideal points lie on one side of a limiting median line and $m-1$ on the other. I say that a point *faces* a limiting median line M_{ij} if it lies on the more populated side and that x is *behind* M_{ij} if it lies on the less populated side. Thus, in the absence of collinearity, a bare majority of m ideal points lies on or behind any limiting median line and a one-greater than bare majority of $m+1$ ideal points lies on or faces it.

The *yolk* is the set of points bounded by the smallest circle that intersects every median line. The *location* of the yolk is given by its center c, which indicates the generalized center (in the sense the median) of the configuration of ideal points. The *size* of the yolk is given by its radius r, which indicates the extent to which the configuration of ideal points departs from one exhibiting a degree of Plott symmetry sufficient for the existence of a Condorcet winner. This circle is inscribed within the *yolk triangle* formed by three median lines to which the circle is tangent.[6] [FIGURES 10A-10C]

To get a preliminary sense of the size, shape, and location of a win set $W(x)$ in the spatial context, consider the special case in which there is only one voter i. In this event, the center of the yolk is i's ideal point, the yolk radius is zero, and $W(x)$ coincides with $P_i(x)$, which (given Euclidean preferences) is the circle with a center at c and a radius of d, where d is the distance from x to c. [FIGURE 11]

[6] These are typically, but not always, limiting median lines; see Tovey (1992) and Koehler (1992). An atypical case produces a "Tovey anomaly." [FIGURE 10D]

In the general case of multiple voters with diverse ideal points and for a point x outside the yolk, the boundary of $W(x)$ is (more or less well) approximated by the similar same circle centered on the center of the yolk c and a radius of d. The accuracy of this approximation depends on the size of the yolk, as given by its radius r, according to the following *2r rule*: point x beats all points more than $d + 2r$ from the center of the yolk, and x is beaten by all points closer than $d - 2r$ to the center of the yolk; put otherwise, the boundary of $W(x)$ everywhere falls between two circles centered on the yolk radii of $d + 2r$ and $d - 2r$ respectively (the inner constraint disappears if $d < 2r$).[7] [FIGURES 12A-12C]

To more precisely characterize a win set $W(x)$ as a two-dimensional whole, consider all possible lines through x and all median lines perpendicular to them. $W(x)$ emanates along each such line L from x to its reflection point through the median induced ideal point on L. The points follow from this: first, $W(x)$ is *starlike* relative to x, i.e., every point on the straight line from x to a point y in $W(x)$ belongs to $W(x)$; and second, $W(x)$ is (typically) *polarized* — more precisely, that if x is beaten by any point y on L, x beats every point on L on the opposite side of x from y and, unless L is a dividing line through x, *vice versa*.[8] Thus majority rule in a spatial voting game operates "locally," in that every point x (i) beats a neighboring point and, if beaten at all, (ii) is beaten by a neighboring point. [FIGURE 13A]

A win set $W(x)$ can be described more intuitively with respect to the finite number of limiting median lines. $W(x)$ is composed of a number of *petals* that extend from x to its reflection point through each limiting median line that x faces. Each petal represents the set of points preferred to x by the bare majority of m voters whose ideal points lie on or behind each limiting median line x faces. Each petal is formed by the intersection of the preferences sets $P_i(x)$ for all voters in this bare majority. (Typically this coincides with the intersection of the preference sets of the two voters whose ideal points lie on the limiting median line.) Thus the boundary of $W(x)$ is composed of segments of individual indifference curves through x. There is a kink in the boundary of the petal where individual indifference curves intersect and the identity of the voter whose indifference curve demarcates the win set boundary changes. Since all voter preference sets are convex (indeed circular), all petals are convex. [FIGURE 14A]

Petals of $W(x)$ extending through different limiting median lines that x faces may intersect and thereby form *subpetals*. Points in such subpetals beat x by a margin greater than a bare majority, the exact size of which depends on the number of petals included in the intersection.[9] [FIGURE

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7 Tighter bounds on $W(x)$, especially in the vicinity of x itself, are provided by the outer and inner cardioids described in McKelvey (1986) and Miller et. al. (1989). [FIGURES 12D-12E]

8 A dividing through x can be characterized as a line through x that nowhere intersects $W(x)$. Typically only a small finite number of dividing lines pass through x (see below). However, if x is itself a (non-phantom) ideal point, an infinite number of dividing lines may pass through x, creating a "non-polarized" win set $W(x)$. [FIGURE 13B] Of course, if x is a Condorcet winner, every line through x is a dividing line.

9 Like a (bare majority rule) petal, such a subpetal extends from x to its reflection through a *limiting quota line* Q_{ij}, i.e., a line that passes through two ideal points and has $q > m$ ideals points on and (relative to x) behind it. Note that a limiting median line M_{ij}, that x lies behind (rather than faces) is a limiting quota line for $q = m + 1$. Limiting quota lines for $q > m + 1$ are typically distinct from median lines, but they may coincide in the presence of collinearity.

14B] The union of two or more intersecting petals constitutes a *leaf* of $W(x)$. Leaves of $W(x)$ are separated by dividing lines through x. Typically an odd number of dividing lines pass through x and the resulting "pie slices" of the space around x alternately enclose leaves of $W(x)$ and opposing regions that (by polarization) include only points that x beats.[10] [FIGURES 14C and 14D]

A win set $W(x)$ is *orderly* if it lies entirely to one side of some (dividing) line through x. (Typically, this is the only dividing line, and $W(x)$ has a single leaf.) [FIGURES 15A and 15B] A win set $W(x)$ is *disorderly* if it is not confined to any half space about x. [FIGURE 15C] We may informally characterize a win set $W(x)$ as "highly disorderly" if it has multiple small leaves (typically each composed of a single petal) that "point in all directions" from x. [FIGURE 15D]

The orderliness of a win set $W(x)$ largely — but not entirely — depends on the distance d from x to the center of the yolk in relation to size of the yolk, i.e., on the ratio d/r. If $r = 0$, all win sets are "perfectly" orderly, i.e circles. Fixing r at some positive constant, let us consider what happens to $W(x)$ as d decreases — that is, as x moves from the outer reaches of the space toward the center of the configuration of ideal points. So long as x remains outside the Pareto set, $W(x)$ is orderly with a single leaf pointing toward the yolk and probably encompassing it,[11] because there are no (limiting) median lines "behind" x — that is (to state the matter informally), there remains an escape path from x to outer reaches of the space that does not cross any (limiting) median line.[12] [FIGURE 15A] Unless there are only three voters (so the Pareto boundary is formed by median lines), $W(x)$ typically remains orderly even after x penetrates the Pareto set. [FIGURE 15B] $W(x)$ becomes disorderly only when x is "surrounded" by (limiting) median lines (and thus has no "escape path"), for now some petals of $W(x)$ point away from the yolk. [FIGURE 15C] Thus it is quite possible for $W(x)$ to be orderly and $W(y)$ to be disorderly even if x is closer to c. While every win set intersects the yolk, if x lies outside the yolk and has a disorderly win set, some of its leaves may fail to intersect the yolk. $W(x)$ becomes highly disorderly once x is "surrounded" by limiting median lines — in particular, once x is inside in the yolk.[13] [FIGURE 15D]

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[10] However, petals may be separated by a single dividing line, in which event they are adjacent but non-intersecting and thus constitute separate leaves. (This occurs in the exceptional event that the median line M that is perpendicular to L and passes through x is itself a "stand alone" limiting median that passes through ideal points in addition to x, producing a "pie slice" of zero width.) In this event, an even number of dividing lines may pass through x and $W(x)$ may have an even number of leaves. [FIGURE 14E]

[11] $W(x)$ always intersects the yolk; $W(x)$ includes c if $d > 2r$; and $W(x)$ contains the yolk if $d > 3r$.

[12] The parenthetical qualifier "limiting" is not logically necessary but facilitates visualization of the stated condition.

[13] It should be noted that the orderliness of its win set is in no sense indicative of the "stability" of a point in any game theoretical sense. To the contrary, a (status quo) point x with an orderly win set is typically beaten by a great many points, many of which are distant from x, and nearby points may beat x by large majorities. However, these points all lie "on the same side" of x — namely, the side facing the center of the yolk. Conversely, a point x with a highly disorderly win set $W(x)$ is not exceptionally "unstable." To the contrary, such a point is centrally located, is beaten by relatively few points, all which are close to x and that typically beat it only by small majorities. (Indeed, even small "imperfections" in voting may render such points "stable"; see Tovey, 1991.) However, the few points that beat x lie "on all sides" of it.

4. The Location and Size of the Yolk

It is clear that the location and (especially) size of the yolk play critical roles in analyzing spatial voting games. As we have just seen, the orderliness of win sets depends on the size of the yolk. As noted earlier, McKelvey (1986) showed that the uncovered set must lie within a $4r$ bound from the center of the yolk. As we shall see, the uncovered set generally tracks the yolk as ideal point configurations vary. Moreover, the characteristics of the yolk are important parameters in their own right. Indeed, it is worth remarking that, while BJS (and others) claim that the work of Shepsle and Weingast (1984) highlights the role of the uncovered set in demarcating the boundaries of "enactability" in a sophisticated voting body, what Shepsle and Weingast actually focus on is not the uncovered set per se but the set of points not covered by some status quo or "starting point" q. Subsequently Feld et al. (1989) showed that the size of this set depends on the size of the yolk; indeed, almost all of the "agenda propositions" in Feld et al. refer to the size of the yolk and the set of points not covered by the status quo — and none refers to the uncovered set per se. Thus an important side benefit the BJS computational procedure is that it can provide pictures of yolks in large ideal point configurations. As BJS claim in their footnote 14 (I believe accurately), their Figure 1 showing the yolk for a large configuration of ideal points is the first of its kind.[14]

In general, the size of the yolk declines (relative to the size of the Pareto set or some other measure of the "span" of an ideal point configuration) as the number of ideal points n increases. In particular, if ideal point configurations are random samples (with n odd) drawn from any "centered" continuous distribution (e.g., normal, uniform, etc.), the expected size of the yolk approaches zero as the number of ideal points increases without limit (Tovey, 1990). More intuitively, if the underlying distribution has a well-defined center, finite random samples drawn from it have imperfectly defined centers that become more perfectly defined as sample size increases.

But Tovey's theoretical result leaves two important questions open. The first concerns the *rate* at which the yolk shrinks as the number of voters increases. For example, does a yolk with $n = 101$ or $n = 435$ (to pick two n's of political relevance in the U.S.) look more like a yolk in a committee-sized configuration of $n = 9$ to $n = 25$ ideal points or in an electorate-sized configuration of $n = 100{,}000$ to $n = 100{,}000{,}000$ ideal points? The second concerns the impact of "non-random" *clustering* within configurations of ideal points, such as we might expect to see in empirical ideal point data (and certainly do see in BJS Figures 1, 4, and 5), on the size and location of the yolk. *CyberSenate* allows us to address both of these questions.

CyberSenate can generate configurations of ideal points drawn randomly from both bivariate normal and bivariate uniform distributions, compute and display all limiting median lines, display the yolk, and compute its size and location. Figures 1 and 2 show computed yolk sizes for 192 ideal

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[14] Though BJS seem to say in the text (p. 261) that their Figure 1also displays median lines, it does not in fact do so; however, I have seen other BJS figures that do display median lines in large-n configurations. It is a bit unfortunate that yolks are not similarly displayed in the panels of BJS Figures 4 and 5, but one can fairly readily estimate approximate yolks by sketching in approximate "outermost" median lines that are candidates for forming the yolk triangle. *CyberSenate* graphics can display all limiting median lines — and also check for Tovey anomalies before calculating and displaying the yolk. [FIGURES 16A-16E]

point configurations, half drawn from each type of distribution with a standard deviation of 15 in each dimension, and with various *n*'s (all odd) ranging from 3 to 435 on a log scale.[15] [Also FIGURES 17A-17D]

We see that, for every *n*, expected yolk size is a bit larger when voters are drawn from a uniform rather than normal distribution with the same standard deviation. It is evident (and unsurprising) that yolk sizes are quite stable from sample to sample in large configurations but are highly variable in small configurations. Nevertheless, it is clear that, once a low threshold of about $n = 9$ is crossed, the expected yolk radius shrinks as the number of voters increases and, given configurations of several hundred voters, the expected yolk radius is about one quarter (and yolk area about 6%) of that for small configurations.

More specifically, for $n = 101$ (e.g., the U.S. Senate) and with configurations randomly drawn from a bivariate normal distribution with an SD of 15 in each dimension, the average yolk radius is about 1.8. If we take the Pareto set of the configuration to be approximately the area enclosed by a circle centered on *c* with a radius of 45 (i.e., three SDs), the yolk can be expected to occupy about 0.15% of the Pareto set. For $n = 435$ (e.g., the U.S. House), the expected yolk radius is about 0.9, so the yolk can be expected to occupy about 0.04% of the Pareto set. Generally, the expected yolk *radius* appears to follow an inverse square root law with respect to sample size (in the manner of sampling error more generally), so expected yolk *area* follows an simple inverse law.[16]

With respect to the second question, it is evident that "non-random" clustering of ideal points may greatly increase the expected size of the yolk. For example, the yolk displayed in BJS's Figure 1 displaying 106th U. S. House ideal points is considerably larger than would be expected if ideal points were normally distributed. [FIGURE 3B vs. FIGURE 18D] We can put a ruler to the diagram as it appears on the printed page and find that the yolk has a diameter of about one third of an inch and the (approximately circular) Pareto set has a diameter of about 3 inches, so the yolk occupies about 1% of the Pareto set — or about 25 times the area it would be expected to occupy in the event ideal points were normally distributed. But the configuration of ideal points in is far

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15 More samples were drawn for smaller configurations than for larger ones, both because computations take more time for the large ones (upwards of 10 minutes for each configuration with $n = 435$) and also because (as is evident from the data displayed in Figures 1 and 2) there is much more variability in yolk sizes in the small configurations. See Miller and Godfrey (2005) for more details. Bräuninger (2004), using generally similar simulations, provides a very similar plot of expected yolk radii for $n = 3$ through $n = 101$ (with samples of all intermediate odd-numbered sizes). [FIGURES 17E-17F]

16 It follows from the $2r$ rule on win sets that, given large "random" ideal point configurations, win sets of points at some distance from the yolk come very close to being perfect circles. [FIGURES 18A-18D] Moreover, Tovey's (1990) theoretical result noted above implies that vanishingly small yolks should occur frequently with very large electorate-sized *n*'s. While it is beyond the practical computing power of *CyberSenate* to work with electorate-sized *n*'s, such yolk sizes can be estimated on the basis of the inferred inverse law. For *n* greater than about 100 drawn from a bivariate normal distribution, this law appears to be approximately RATIO = $1.3/n$, where RATIO is the expected yolk area divided by the "Pareto area," where the latter is defined as above in the text. Note that in fact, given large *n*'s, the true Pareto area will (almost certainly) be considerably larger than this, because large samples of ideal points will (almost certainly) include extreme outliers (well beyond three SDs from the mean), so the true ratios will be even smaller.

from a (typical) random draw out of an underlying bivariate normal (or uniform) distribution — rather it displays two distinct "clusters" of ideal points (evidently party groups).

With respect to the location of the yolk, it is worth first noting that there is no reason to expect that the center of the yolk will coincide with either the "center" of the policy space (however that might be defined) or with the "center of gravity" (i.e., mean) of the ideal points. Rather the center of the yolk indicates the generalized center in the sense of the *median* of the configuration of voter ideal points. Furthermore, while we generally think of the median as a "stable" measure of central tendency, we should also remember (as BJS do not in their footnote 28) that there is one circumstance in which the median is highly "unstable" and (unlike the mean in the same circumstance) shifts radically in response to very small changes in the overall distribution. In the one dimensional case, this occurs when data is polarized into two quite widely separated clusters of nearly equal size. Consider a simple one-dimensional spatial model of party polarization. If the most liberal Republican is considerably more conservative than the most conservative Democrat and if the two party groups have little internal dispersion relative to the polarization between them and are closely balanced in size, the median ideal point lies within whichever group has bare majority status, and accordingly will shift back and forth across the wide (and empty) ideological center whenever a few seat change party hands and majority control shifts back and forth between the two parties (see Grofman et al., 2001).

We see essentially this kind of polarization along one dimension in BJS Figure 1 (as well as their Figures 4 and 5). While the two party clusters largely overlap with respect to the vertical dimension, they are highly polarized with respect to the horizontal dimension.[17] Thus we should expect to see the same kind of instability in the location of the center of the yolk that we find with respect to the median in the one-dimensional case. More specifically, almost every median line lies so that just about P % of the points in the left cluster together with $100 - P$ % of the points in the right cluster lie above it and $100 - P$ % of the points in the left cluster and P % in the right cluster lie below it. These median lines therefore all pass through a small area about halfway between the two clusters, forming the "bow tie" pattern shown in Figure 3. [Also FIGURE 19A] (The "bow tie" is narrower or wider depending on the magnitude horizontal polarization between the clusters relative to their vertical dispersion.) But in addition, there must be at least one median line that lies more or less vertically along the centrist face of the majority cluster (with about 99% of the majority cluster to one side and about 1% of the majority cluster plus 100% of the minority cluster on the other side). Since it intersects all median lines, the yolk must lie within the "majority half" of the "bow tie" (which includes the yolk triangle) and be nestled against the centrist face of the majority cluster. Its size depends on the polarization between the clusters relative to their heights. In Figure 3, the majority cluster on the left has just one more ideal point than the minority cluster on the right, so if

[17] The overlap and polarization are almost perfect if the axis system is rotated about 20° counterclockwise. Note that all concepts and analyses presented here are independent of the axis system, which can be rotated in any fashion without affecting any conclusions. I will refer to the dimensions in the BJS figures as "horizontal" and "vertical" as if this rotation had taken place.

just one point is shifted from the left cluster to the right cluster (e.g., as a result of the “turnover” of a single seat), the yolk likewise switches sides. [FIGURE 19B]

5. Phantom Voters

Given median lines, we can demarcate any win set in the manner described in Section 4. Conversely, given any win set $W(x)$, we know that for every line L through x, there is a median line perpendicular to L that intersects it at the midpoint between x and the boundary of $W(x)$. It follows that, given any win set $W(x)$, we can deduce the location of all median lines and in turn deduce the size and shape of all other win sets (Feld and Grofman, 1991). Thus, (i) the set of all median lines and (ii) any single win set are equally informative concerning the configuration of ideal points. However, neither is fully informative, because collinearities may produce “phantom voters” whose presence is undetectable on the basis of either median lines or win sets.

Suppose we know the ideal point configuration, so that when we demarcate win set boundaries from median lines, we can identify the induced ideal point(s) on each median line. This allows us to identify the voter that “controls” part of a win set boundary — that is, the voter whose indifference curve through x demarcates part of the boundary of $W(x)$.

Given any point x, let us consider all possible lines L passing through x — or, perhaps easier to visualize, let us consider an arbitrary line L through x, which we then rotate 180° about x. We initially suppose that the configuration of ideal points exhibits no collinarities.

Starting with an arbitrary line L through x, we identify the median line M perpendicular to L. Almost certainly M is a non-limiting median line passing through a single ideal point x_i, in which case x'_i is the unique median induced ideal point on L. As we have seen, x is beaten by all points on L that lie between x and its reflection point through x'_i so the boundary of $W(x)$ on L is the reflection point. In turn, this reflection point is the intersection point (other than x itself) of voter i’s indifference curve through x with L. [FIGURE 20A] If we continuously rotate L about x while allowing the perpendicular median line to adjust accordingly, we can trace out the locus of reflection points of x through the median induced point on L and thereby trace out the boundary of $W(x)$.

As we begin to rotate L about x, the perpendicular median lines initially are other non-limiting median lines through x_i, so x'_i initially remains the median induced ideal point and the locus of reflection points on L is simply a portion of $P_i(x)$. However, as the rotation of L continues, the perpendicular median line through x'_i in due course hits a second ideal point x_j, giving the limiting median line M_{ij} perpendicular to a unique line L_{ij} through x. Voters i and j share the same induced ideal point on L_{ij} and jointly occupy the median position. The boundary of $W(x)$ on L_{ij} is the reflection of x through this jointly held median induced ideal point and thus the boundary of $W(x)$ on L_{ij} lies on the common intersection of $P_i(x)$, $P_j(x)$, and L_{ij}. [FIGURE 20B] But as the rotation of L continues, i’s and j’s induced ideal points again diverge, and j now assumes the median position. Thus “control” of the boundary of $W(x)$ shifts from i to j, producing a kink in the boundary of $W(x)$ at the reflection point on L_{ij} where i’s and j’s indifference curves intersect, and this continues until the rotation of L hits the next limiting median line M_{jk} perpendicular to another distinctive line L_{jk} through x. These points are summarized in Figure 4. [Also FIGURE 20C]

Recall that the angle between M_{ij} and M_{jk} is α_{ijk}. It is evident that the angle between L_{ij} and L_{jk} is likewise α_{ijk}. [FIGURE 21] Recall also that $\mathrm{A}_j = \Sigma\,\alpha_{ijk}$, where the summation is taken over all pairs of limiting median lines through x_j. Since the sum of the angles between L_{ij} and L_{ij} taken over all adjacent pairs of limiting median lines is 180°, it follows that $\Sigma\,\mathrm{A}_j = 180°$. Thus $\mathrm{A}_j\,/180° = \rho_j$ is the fraction of lines L through x on which voter j occupies the median induced ideal point, where x can be any point in the space. Thus ρ_j is in some sense the share of (preference-based) "voting power" held by voter j given the configuration of ideal points in a spatial voting game.[18]

To this point, we have assumed that there are no collinearities in the configuration of ideal points. But let us now suppose that (i) ideal points x_i, x_j, and x_k are distinct but collinear (with x_j in the intermediate position) and also that (ii) the line on which these three ideal points lie is itself a median line — necessarily the "stand alone" limiting line M_{ijk}. Consider the unique line L_{ijk} through x perpendicular to M_{ijk}. The induced ideal points x'_i, x'_j, and x'_k coincide, jointly occupying the median position, and the reflection of x through this median point lies on the common intersection of $P_i(x)$, $P_j(x)$, $P_k(x)$, and L_{ijk}. However, if L_{ijk} is rotated infinitesimally in either direction, the induced ideal points diverge and x'_j is the unique median. [FIGURE 22] Thus, at least in this vicinity of $W(x)$, voters i and k (i) occupy the induced median position only instantaneously on the distinctive line L_{ijk} and, even then, (ii) share the honor with each other and with j. If there are no other lines L through x on which i (or k) uniquely occupies the median induced ideal point, i (and k) is what we call a "phantom voter."[19]

A *phantom voter* may be defined in any of the following equivalent ways: a voter i is a phantom if and only if (i) his indifference curves do not uniquely demarcate any segment of the boundary of any win set[20]; (ii) only "stand alone" limiting median lines pass through his ideal point x_i; (iii) x_i does not lie on any non-limiting line; and (iv) $\rho_i = 0$. Phantom voters appear only in the presence of collinearities, but collinearity is not sufficient to create phantom voters. Phantom voters appear in pairs and can be jointly removed from (or added to) a configuration of ideal points, or their ideal points shifted in certain ways, without affecting the location of median lines and the shape of win sets.[21]

Consider a *one-dimensional* spatial voting game with any odd number of voters with diverse Euclidean preferences on a line L. As we noted earlier, every win set $W(x)$ coincides with the

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[18] Indeed, ρ_j is the *Shapley-Owen spatial voting power index* for voter j; see Grofman et al. (1987), Owen and Shapley (1989), Godfrey (2005), and additional citations therein. *CyberSenate* can compute ρ_j for all voters (as an approximation based on a sample of lines L).

[19] These "phantom voters" are to be distinguished from the "phantom voters" discussed by Border (1983). Border's phantom voters (whose "phantom ideal points" lie at the intersection of median lines that are orthogonal to fixed issue axes) don't actually exist but seem to appear, while our phantom voters do actually exist but seem to disappear.

[20] However, as we have seen, phantom indifference curves touch win set boundaries at isolated points.

[21] In a non-diverse configuration in which two (or more) ideal points coincide, one (or more) of these voters may be a phantom (in conjunction with another voter) but it is arbitrary which is deemed to be the phantom. This special case necessitates the qualification "*uniquely* demarcates" in (i).

preference set $P_m(x)$ of the median voter. Thus, *all* voters other than the median voter are phantoms. Given any win set $W(x)$, we can determine the location of x_m at the midpoint of $W(x)$ and, given the location of x_m, we can demarcate any other win set $W(y)$. What we cannot deduce from $W(x)$ is the location (or even the existence) of any other ideal points — except that we know that, if they do exist, an equal number lie on either side of the median point. Win sets are unaffected if such voters are added to or deleted from the configuration or their locations are shifted in any way that does not change the location of x_m. (Precisely for this reason, a standard shortcut in one-dimensional Euclidean spatial modelling is to let the median voter stand in analytically for a full committee, legislature, or electorate.)

Now suppose we embed L and the collinear configuration of ideal points in a two-dimensional space. All median lines pass through x_m, and every line through x_m is a median line. There is a single "stand-alone" limiting median line (i.e., L) that passes through all other ideal points, so all other voters remain phantoms.

Next suppose that ideal points are distributed over two dimensions but the Plott symmetry condition holds — that is, there is a distinctive voter with ideal point x_c such that all other ideal points form pairs x_i and x_j such that x_i, x_c , and x_j are collinear with x_c lying between x_i and x_j. Every such line passing through three ideal points a limiting median line, and only such median lines pass through ideal point other than x_c. $W(x)$ again coincides with $P_c(x)$, and all voters except the one with ideal point x_c remain phantoms.

Figure 2 in BJS provides excellent illustrations of these points. [FIGURE 3A] For easy reference, let's label the six panels (a) through (f) in the natural progression. Let's also label the leftmost ideal point 1 and, proceeding clockwise in panel (a), label the others 2 through 4 and label the central point 5. [FIGURE 23A] Finally, let's also create two additional panels: in (g) point 2 is rotated beyond 3 into the southeast quadrant, and in (h) point 2 is rotated until it coincides with 4. [FIGURES 23B-23C] Note that collinearities exist in every panel. In (a), ideal points 1, 5, and 3 are collinear, as are 2, 5, and 4; in (b) through (h) 1, 5, and 3 remain collinear; in (f) 1, 5, 3, and 2 are collinear (with 2 and 3 coinciding), and in (h) 2, 4, 5 are collinear (with 2 and 4 coinciding).

The configuration in panel (a) meets the Plott symmetry condition, with point 5 playing the role of x_c. Voters 1, 5, and 3 lie on one "stand alone" limiting median line, as do voters 2, 5, and 4, but no non-limiting median line pass through 1, 2, 3, or 4. For all x, $W(x) = P_5(x)$, and all voters but 5 are phantoms.[22] It is precisely for this reason that the configuration in panel (a) produces perfectly circular win sets. [FIGURE 24A]

In panels (b) through (e), voters 1, 5, and 3 remain on the same limiting median line and no other median lines pass through 1 and 3, so voters 1 and 3 remain phantoms and continue to play no role in demarcating win sets. [FIGURE 24B] In panel (f), points 2 and 3 coincide, so either 2 or 3 may be deemed a phantom along with voter 1. [FIGURE 24C]

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[22] Note, however, that the presence or absence of phantom voters determines what *subpetals* exist within a (majority-rule) win set. Put substantively, voters who are phantoms with respect to majority rule may be non-phantoms with respect to more demanding decision rules. (The same point holds in the one dimensional case.)

In each panel of BJS Figure 2, if ideal points 1 and 3 are jointly deleted, or if any number of additional ideal points are placed in pairs toward opposite ends of the horizontal median line, median lines and win sets are unaffected. Also, if ideal point 1 is moved inward along the horizontal median line until it coincides with 5 or is moved outward any distance, or if 3 is moved inward until it coincides with the median line through 2 and 4 or is moved outwards any distance, median lines and win sets again are unaffected.[23]

Now consider the additional panel (g) in which point 2 rotates beyond 3 into the southeast quadrant. While points 1, 5, and 3 remain collinear, and the line through them remains a "stand alone" limiting median line, new limiting median lines appear through 1 and 2 and through 3 and 4 (as panel (g) is drawn; through 3 and 2, if 2 were in the interior of the Pareto set) with the result that non-limiting median lines also pass through both 1 and 3. As a result, voters 1 and 3 are no longer phantoms and their indifference curves now demarcate parts of win set boundaries. [FIGURE 24E] Finally, consider additional panel (h) in which points 2 and 4 coincide. This turns the line through 4 (and 2) and 5 into a second "stand alone" limiting median line while no non-limiting median lines pass through 5, so that the initially "all-powerful" voter 5 is converted into a phantom (along with either of 2 and 4).[24] [FIGURE 24F]

Let us call the convex hull of non-phantom voters the *effective Pareto set* — that is, it is the Pareto set after phantom voters have been removed from the configuration. In panel (a), the effective Pareto set is $\{x_5\}$; in panels (b) through (f), the effective Pareto set is the triangle with vertices at 1, 2, and 4; in panels (g) and (h), the effective Pareto set is the same as the Pareto set — in (g) because there are no phantom voters and in (h) because neither phantom voter uniquely lies at a vertex of the Pareto set. [FIGURES 23A-24F]

6. Covering in a Spatial Context

To say that x covers y is to say that x beats y and x also beats every point that y beats. Put otherwise, y is beaten by y and by every point that beats x; thus $W(x)$ is a proper subset of $W(y)$. In the spatial context, this covering relationship manifests itself geometrically, with the boundary of $W(y)$ literally enclosing $W(x)$.

It follows from the characteristics of win sets and the phenomenon of phantom voters outlined in the preceding sections that there are two distinct modes of covering in the spatial context, one of which entails "local covering" while the other allows only "covering at a distance."[25]

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[23] In like manner in panel (f), if point 2, instead of coinciding with point 3, lies inside of 3 on the median line through 1, 5, and 3, voter 2 is a a non-phantom and 3 phantom; if 2 lies outside of 3, 2 is a phantom and 3 a non-phantom. [FIGURES 24D]

[24] The observations made in the previous footnote with respect to 2 and 3 also apply to 2 and 4 in panel (h).

[25] An early statement of this distinction was presented in the unpublished manuscript (Miller, 2002) cited by BJS. In this early statement, I identified two distinct types of local covering — the covering of points outside the Pareto set and the covering of all points other than the Condorcet winner in the presence of Plott symmetry — between which I could see no theoretical connection. [FIGURES 25A-25B] Moreover, it appeared to me that local covering could operate *only* in these two circumstances. Once I applied the *CyberSenate* tool to the configurations displayed in panels

In the *local* mode, if point *x* covers point *y*, *x* also covers (and *y* is also covered by) every point on the straight line between *x* and *y*, so *x* covers (and *y* is covered by) neighboring points. In this sense, the covering relation operates "locally," in the manner of majority preference. It follows that, if *x* locally covers *y*, *both win sets have essentially the same shape and W*(*x*) *is simply a slightly shrunken replica of W*(*y*) *with the latter enclosing the former*. It is also evident that *x* covers *y* locally only if *x* is (at least slightly) closer to the center of the yolk than *y* is.

It is clear that if *W*(*x*) and *W*(*y*) are to any degree disorderly, so that *x* and *y* each beat and are beaten by neighboring points lying "on all sides," neither win set can be a subset of the other if *x* and *y* are near each other. Thus, the reach of local covering is limited in that it can operate only with respect to orderly win sets. But, while orderliness of win sets is necessary for local covering, it is far from sufficient. Rather, a kind of unanimity among voters is required — specifically, *a point y is locally covered by neighboring point x if and only if x is closer to the ideal points of every voter whose indifference curve through y uniquely demarcates part of the boundary of W*(*y*).

Local covering most obviously occurs if point *y* lies outside the Pareto set, so *y* is covered by a neighboring Pareto-superior point. In this event, *W*(*y*) is orderly and all petals in its single leaf have a common intersection comprising points unanimously preferred to *y*.[26] If and only if *x* lies in this common intersection, *x* is closer to *every* ideal point than *y* is, so *W*(*x*) is an everywhere shrunken replica of *W*(*y*) and lies entirely inside it. Moreover, *x* can be arbitrarily close to *y*. An illustration of such local covering of a point outside the Pareto set is provided in Figure 5. [Also FIGURE 26A] As BJS point out (p. 260), it has long been known that *x* covers *y* if *x* is unanimously preferred to *y*. We now see that, in the spatial context, such covering operates locally.

Comparison of the italicized language above with definition (a) of phantom voters tells us that, if some voters are phantoms, local covering may occur within parts of the Pareto set. More particularly, a Pareto-optimal point is locally covered if it lies outside the effective Pareto set defined with respect to non-phantom voters only.

In panel (a) of BJS's Figure 2, where the Plott symmetry condition holds, voter 5 is the only non-phantom, so the effective Pareto set coincides with 5's ideal point and all other points are locally covered. All win sets are perfectly orderly, and covering is identical to the preferences of voter 5 (and to majority preference). [FIGURE 26B]

(b) through (f) of BJS Figure 2, I discovered that local covering operated within parts of the Pareto set, even in the absence of the Plott symmetry exhibited by panel (a). It was evident that voters 1 and 3 in some sense didn't matter. Pursuing this puzzle led to the concept of phantom voters introduced in the previous section, which in turn provided a theoretical unification of what had appeared to be two distinct types of local covering and also identified other circumstances in which local covering may occur. The distinction between local and distant covering is in a sense implicit in the early analysis of Hartly and Kilgour (1987) in that they initially (pp. 177-178) focus on the (relatively easy) matter of analyzing (local) covering resulting from the Pareto relation before turning their attention to the (relatively difficult) matter of analyzing (distant) covering within the Pareto set.

[26] This smallest (unanimity-rule) subpetal of *W*(*y*) extends through the Pareto frontier (the limiting quota line for $q = n$). Since *y* and *c* must lie on opposite sides of the Pareto frontier, *x* must be closer to *c* than y is.

In panels (b) through (f) of BJS's Figure 2, voters 1 and 3 remain phantoms (in panel (f), either 2 or 3 may be deemed a phantom), so the effective Pareto set is the convex hull of ideal points 2, 4, and 5 (or, equivalently, of 3, 4, and 5). A point *y* outside of this effective Pareto set is locally covered by an effectively Pareto-superior point *x* (i.e., any point in the subpetal created by the intersection of preference sets $P_i(y)$ for all non-phantom voters), as shown in Figure 6. [Also FIGURE 26C] The fact that phantom voter 1 prefers *y* to *x* does not preclude local covering, since 1's indifference curve through *y* nowhere demarcates the boundary of *W*(*y*). In contrast, win sets of points inside the effective Pareto set become disorderly (though it remains true that the indifference curves of voters 1 and 3 nowhere demarcate their boundaries), so local covering cannot operate. [FIGURE 26D]

While the same collinearity remains in the hypothetical panel (g) of BJS Figure 2, voters 1 and 3 are no longer phantoms, so the effective Pareto set coincides with the actual Pareto set, inside of which all no local covering occurs (even in regions where win sets are orderly). [FIGURE 26E]

Finally, in hypothetical panel (h), voter 5 together with either 2 or 4 are phantoms, but since neither phantom (uniquely) defines a vertex of the Pareto set, the effective Pareto set coincides with the actual Pareto set, inside of which all no local covering can occur. Indeed, the boundary lines of the (effective) Pareto set are all median lines, so all win sets inside the (effective) Pareto set are disorderly). [FIGURE 25F]

The second mode of covering entails only "action at a distance" in that, if point *x* covers point *y*, *x* and *y* cannot be neighboring — rather, they must be some distance apart (*x* being substantially closer to the center of the yolk than *y* is). Usually the most relevant covering relationships operate at a distance and, in particular, only covering at a distance can operate within the effective Pareto set.

If point *x* covers point *y at a distance*, *x* covers *y* but does *not* cover every point on the straight line between *x* and *y*, and *W*(*x*) is not a slightly shrunken replica of *W*(*y*). Typically *W*(*x*) is somewhat (or totally) differently *shaped* from *W*(*y*) but, at the same time, *W*(*x*) is sufficiently *smaller* than *W*(*y*) (because *x* is sufficiently closer to the center of the yolk than *y* is) that *its differently shaped boundary (perhaps with multiple leaves) can nevertheless be enclosed within (a single leaf of) W*(*y*).

Clearly for covering to operate a distance, *x* must be at some distance from *y*. Typical covering at a distance for the case in which *y*, *x*, and *c* are aligned is displayed in Figure 7.[27] [FIGURES 27A-27C] A natural question to ask about covering at a distance is how great a distance is required.

The first point to note is that what really matters is not the distance between *x* and *y* per se but rather that *x* must be considerably closer to the center of the yolk than *y* is. The 2*r* rule discussed in Section 3 allows us to specify the minimum distance that is *sufficient* for covering at a distance.

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[27] It might at first blush be expected that, if points x'', x', and *y* are collinear and if x'' and x' both cover *y*, all points between x'' and x' also cover *y*. But this is not true, for reasons that are evident once one takes account of the generally odd shape of win sets. [FIGURES 28A-28D]

Like so much else, this distance is a function of the size of the yolk. Point *x* beats every point *y* that is *2r* further from the center of the yolk than *x* is, and *y* beats every point *z* that is 2*r* further from the center of the yolk than *z* is. So point *x* covers at a distance any points *y* more than 4*r* further from the center of the yolk than *y* is (but may not cover any closer point). [FIGURE 29A-29B]

However, *x* may cover *y* at a distance even if *y* is less that 4*r* more distant from *c* than *y* is (as Figure 7 illustrates). In fact, we cannot specify a minimum distance that is *necessary* for covering at a distance, since this distance may be arbitrarily small, as illustrated by Figure 8, in which three ideal points form an elongated Pareto set. [FIGURE 29C] A point *y* near the "extreme" ideal point 3 cannot be locally covered (since *y* is in the yolk triangle and *W*(*y*) is disorderly) but *y* is covered at a distance by a point *x* that is only slightly closer to the yolk than *y* is. Indeed, it is evident that, as *y* moves toward ideal point 3, the distance from *x* to *y* sufficient for covering further diminishes and converges on zero as *x* and *y* converge on the vertex.[28]

The typical case of covering at a distance lies between these two extremes. Given configurations with more than a few ideal points distributed in a more or less random manner, it appears to be the case — based on a variety of considerations and *CyberSenate* analysis of many examples — that point *x* typically covers *all* points *y* that are about 3.5 *r* further from the center of the yolk than *x* is, and that *x* typically covers *most* points *z* if that are about 2.5 *r* further from the center of the yolk than *z* is.[29]

7. Uncovered Points and the Uncovered Set

The following points have long been known to be uncovered in spatial voting games. First, a *Condorcet winner* (if it exists) is the unique uncovered point. Second, the *center of the yolk* is uncovered (because no point can be closer to the center of the yolk). Third, the *strong point* (or *Copeland winner*) is uncovered (since by definition it has a smaller win set than any other point).

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[28] Figure 7 illustrates the kind of covering that Hartley and Kilgour (1987) analyze. Clearly *y* is also covered by other more distant points in the vicinity of the center of the yolk.

[29] It may be useful to conclude this discussion of local and distant covering by considering their analogs in the context of finite alternatives and arbitrary preference profiles (in the manner of Miller, 1980). The analog of local covering results if *x* is ranked just above *y* in every individual orderings (so that, except that *x* belongs to W(*y*), *W*(*x*) and *W*(*y*) coincide). It follows that, if *y* is ranked above *z* in a majority of orderings, *x* is also ranked above *z* in the same majority of orderings, so *x* covers *y*. Local covering also results if *x* is ranked well above *y* is some orderings (though in this case *W*(*x*) may be considerably smaller than *W*(*x*). Furthermore, we can augment the preference profile by adding pairs of *directly opposed* (but otherwise unrestricted) orderings without changing any majority preference relationship (cf. Feld and Grofman, 1986), so *x* still covers *y*. (Such pairs of orderings are approximate analogs to pairs of phantom ideal points in a spatial context. However, such pairs of orderings are "phantoms" regardless of the nature of the other orderings in the profile, while in the spatial context, no voters can have wholly opposed preferences, so whether or not pairs of voters are phantoms depends on the configuration of other ideal points.) The analog of covering at a distance results if *x* is ranked well above *y* in some particular majority *S* of orderings, with result that *W*(*x*) is much smaller than *W*(*y*). It is then very likely that an alternative *z* that is ranked below *y* in a majority of orderings is also ranked below *x* in the particular majority *S* of orderings, so that *x* covers *y* in a manner analogous to spatial covering at a distance.

Fourth, any point *x* that lies at the intersection of two orthogonal median lines is uncovered.[30] Fifth, if there are just three voter ideal points (so the Pareto set is a triangle), every point in the Pareto set is uncovered if and only if it forms an equilateral triangle; otherwise points in the Pareto region close to the one or two "extreme" ideal points that form vertices with angles of less than 60° are covered (Hartley and Kilgour, 1987). [FIGURES 2A-2C] Finally, work with many *CyberSenate* examples makes it evident that no point in the yolk can be covered. On the one hand, there can be no local covering within the yolk (as all win sets are disorderly) and, on the other hand, there is not enough "room" within the yolk to allow covering at a distance.

Let *UC*(*x*) be the set of all points *not* covered by point *x*. We want to consider the location, size, and shape of such sets. *W*(*x*) is the set of all points *y* such that *y* beats *x*, i.e., that beat *x* in one step. *UC*(*x*) is the set of all points *y* such that either (i) *y* beats *x* or (ii) *y* beats some *z* that beats *x,* i.e., that beat *x* in one or two steps. Clearly *UC*(*x*) is a superset of *W*(*x*). In principle, *UC*(*x*) may be demarcated by (i) forming *W*(*x*), (ii) forming *W*(*z*) for all points *z* in *W*(*x*), and (iii) forming the union of *W*(*x*) and all such *W*(*z*).[31]

UC(*x*) shares some properties with *W*(*x*): (i) *UC*(*x*) intersects the yolk; (ii) *UC*(*x*) is approximately centered on *c*; and (iii) the size of *UC*(*x*) increases with the distance of *x* from *c*. In addition, (iv) *UC*(*x*) appears to be *starlike* relative to *x* (which is to say, if *x* fails to covers *y*, *x* also fails to covers all points on the straight line between *x* and *y*).[32]

However, *UC*(*x*) also differs from *W*(*x*) in important respects. By definition, *UC*(*x*) encompasses *W*(*x*) and (if $r > 0$) is considerably larger than *W*(*x*). Second, *UC*(*x*) encompasses (rather than merely intersects) the yolk. Moreover, *UC*(*x*) is (to speak loosely) more "orderly" and "compact" (in the sense legislative districting, not the mathematical sense) than *W*(*x*), in that it is never divided

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[30] On these points, see Feld et al. 1987. The fourth point follows because *x* beats all points on both median lines, and every point off either median line is beaten by a point on it, so *x* beats every other point in at most two steps. However, the parallel argument in three (or more) dimensions requires three (or more) steps, and thus does not established that *x* is uncovered. In fact, if the number of dimensions equals or exceeds the number of ideal points, *x* may lie outside the Pareto set (Tullock, 1967, Chapter 2; Erikson, 2002) and thus be covered.

[31] While step (2) in principle requires an infinite number of operations, in small and/or symmetric ideal point configurations, it is sufficient to find *W*(*z*) only for all points *z* that lie at the (unique) tip of each petal. (For example, see Hartley and Kilgour, 1987, p. 178, and the figures in Shepsle and Weingast, 1984, pp. 66-67. [Also FIGURE 30A] In such cases, *UC*(*x*) is formed entirely out of win sets and is thus everywhere demarcated by individual indifference curves. In somewhat more complex configurations, it is sufficient to find *W*(*z*) only for all points z that lie at kinks in the leaves of *W*(*x*). Kelly (1987) conjectures that in general only a finite number of points *z* need be considered. (*W*(*x*) is the first element and *UC*(*x*) the second element in the sequence of "voting sets" that Kelly discusses.) However, work with *CyberSenate* examples indicate that this conjecture is wrong (though it appears that only points on the boundary of *W*(*x*) need be considered) and that portions of the boundary of some sets *UC*(*x*) are not be formed out of win set boundaries (and indifference curves) but are produced by continuous mappings from parts of the boundary of *W*(*x*). [FIGURE 30B]

[32] See Kelly, 1987, and also Taturu, 1996. However, *CyberSenate* work suggests that this holds only because, as noted in the previous footnote, Kelly's conjecture regarding the existence of a "finite basis" for implementing step (2) in the formation of *UC*(*x*) is not true.

into separate leaves. Point *x* typically lies in the interior of *UC*(*x*), as *x* can lie on the boundary of *UC*(*x*) if and only if *x* covers another point locally, i.e., if and only if *x* lies on or outside the boundary of the effective Pareto set. If *x* lies on the boundary of the effective Pareto set, *UC*(*x*) has a cusp with an angle of essentially 0° at *x*. As *x* moves further outside the effective Pareto frontier, the angle at the cusp widens. [FIGURES 31A-31B]

Given $r > 0$, the sets *W*(*c*) and *UC*(*c*) are of particular interest. Since by construction the center of the yolk lies at distance *r* from the three median lines that form the yolk triangle, three petals of *W*(*c*) extend outwards to a distance of 2*r* from *c* and none extends further, and *UC*(*c*) in turns can extend outwards to a distance of no more 4*r* from *c*. [FIGURES 32A-32B]

A similar 4*r* bound applies to all such sets, so the boundary of *UC*(*x*) always lies between the two circles centered on *c* with radii of *d* and $d + 4r$. [FIGURE 32C] But given the generalization made at the end of Section 6, the boundary of *UC*(*x*) typically lies between the circles centered on *c* with a radii of about $d + 2r$ and $d + 3.5r$ (unless *x* lies on or outside the boundary of the effective Pareto set and covers neighboring points).

Recall that *X* is the set of all points *x* in the space. The *uncovered set UC*(*X*) is the set of all points not covered by *any* other point — put otherwise, *UC*(*X*) is the intersection of *UC*(*x*) for all points *x* in *X*. Equivalently *UC*(*X*) is the set of all points that beat every other point in one or two steps. Since *UC*(*X*) is a subset of *UC*(*c*), *UC*(*X*) also lies within the circle centered on *c* with a radius of 4*r* — indeed, the well-known McKelvey (1986) 4*r* bound on *UC*(*X*) is actually a bound on *UC*(*c*). We also know that *UC*(*X*) is contained in the effective Pareto set.

8. The Size and Location of the Uncovered Set

Clearly it is difficult in a spatial context to demarcate the boundary of *UC*(*X*) precisely, since this requires forming the intersection of an infinite number of sets, each of which is the union of many, and perhaps an infinite number, of other sets. However, we know there is a 4*r* circular bound on *UC*(*c*) and thus also on *UC*(*X*). Two questions then arise. First, how generous is the 4*r* bound on *UC*(*c*)? Second, how well does *UC*(*c*) approximates *UC*(*X*) — put otherwise, to what extent is *UC*(*c*) is "pared down" by forming intersections with *UC*(*x*) for points *x* other than *c*

The first point to make is that in small-*n* configurations *UC*(*X*) may be dramatically pared down relative to *UC*(*c*) as a result of local covering. Consider the three-voter case fully analyzed by Hartley and Kilgour (1987). In the event the ideal points form the vertices of an equilateral triangle, *UC*(*c*) everywhere extends beyond the Pareto triangle to a maximum of about 3.6*r* from *c*, so the 4*r* bound on *UC*(*c*) is not greatly overgenerous. But all covered points are also locally covered, so the boundary of *UC*(*X*) nowhere results from covering at a distance by *c* or any other point, with the result that the 4*r* bound on *UC*(*x*) is greatly overgenerous, as *UC*(*X*) coincides with the Pareto triangle. [FIGURE 33A]

If the Pareto triangle departs from equilateral perfection in a sufficiently extreme way, the 4*r* bound itself can exclude portions of the Pareto triangle, but in this event *UC*(*c*) itself puts much tighter bounds on *UC*(*X*) within the Pareto set. Consider the case displayed in Figure 9. [Also FIGURE 33B] While *UC*(*c*) now comes very close to touching the 4*r* at two points *c*, *UC*(*c*) within the Pareto triangle extends only about 2.2*r* from *c*. On the other hand, covering by other points near *c* pares *UC*(*c*) down only slightly, so *UC*(*X*) is only slightly smaller than the intersection of *UC*(*c*) and the Pareto set. The same pattern holds with respect to both extreme vertices in the event that the three-voter Pareto triangle is obtuse. [FIGURE 33C]

These theoretical considerations fully account for the particular features of the uncovered sets displayed in panels (b) through (f) of BJS Figure 2 and, in particular, their predominant straight-line boundaries. It can be checked that in each panel (b) through (f) the uncovered set fits within the effective Pareto triangle formed by ideal points 2, 4, and 5. Within this triangle, the boundary of the uncovered sets follows from the formal analysis of Hartley and Kilgour (1987) summarized above. Thus BJS Figure 2 provides good evidence that their algorithm works correctly.[33]

Now consider the configuration of five ideal points forming the vertices of a regular pentagon, as displayed in Figure 10. *UC*(*c*) falls short of the five ideal points but elsewhere protrudes a bit beyond the edges of the pentagon. Of course, local covering by points on the Pareto frontier pares *UC*(*X*) back to the edges of the pentagon. But in fact this local covering is redundant as covering at a distance by other points in the vicinity of *c* pares *UC*(*X*) further, so that its boundary everywhere falls just short of the edges of the pentagon, as also displayed in Figure 10.[34] [Also FIGURE 35A]

Given more or less randomly generated configurations of a modestly large number of ideal points, the boundaries of *UC*(*X*) are determined entirely by covering at a distance, and *UC*(c) is substantially pared down by intersection with *UC*(*x*) for points *x* other than *c*. An example is provided in Figure 11, which mimics the ideal point configuration displayed in BJS Figure 1 but with the number of voters is scaled down to $n = 25$. [Also FIGURE 38A] Starting with the center of the yolk *c* and working outward, Figure 11 displays the yolk, *W*(*c*), *UC*(*X*) [approximation], *UC*(*c*) [approximation], and the McKelvey 4*r* circle. Perhaps the most striking thing about Figure 11 —

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[33] However, the BJS algorithm entails approximation, which shows up in micro-features of BJS Figure 2. The uncovered set boundaries that actually coincide with the edges of the triangle formed by ideal points 2, 4, and 5 appear to bulge a bit beyond them. This results because points slightly outside the triangle are (in the underlying continuous policy space) locally covered by points on the edge of the triangle, but the latter points may not appear in the finite grid even when some of the former ones do. (BJS Theorem 2 tells us that this problem will diminish as the grid is refined.) For the same reason, the uncovered set boundary appears everywhere to be slightly irregular, and the true shape of the very small uncovered set in panel (b) is pretty much hidden from view. (In the hypothetical extra panels (g) and (h) of the figure in which point 2 has rotated into the southeast corner of the space, voters 1 and 3 are no longer phantoms and the effective Pareto set coincides with the actual Pareto set, so *UC*(*X*) expands outside of the triangle with vertices at 2, 4, and 5.) [FIGURES 34A-34B]

[34] At my request, Bianco several years ago used the BJS algorithm to generate the uncovered set for such a regular pentagon. [FIGURE 35B] Their picture closely matched the boundaries that I had previously estimated using tedious (pre-*CyberSenate*) "ruler and compass (plus plausible interpolation)" techniques as well as those subsequently produced by *CyberSenate*. [FIGURE 35A] Increasing the number of ideal points forming the vertices of a regular ploygon reduces the size of both *UC*(*c*) and *UC*(*X*) relative to the area of the polygon. [FIGURE 36]. This finding as it pertained to *UC*(*c*), confirmed by hand drawings more than 20 years ago, was the basis for the remarks by Shepsle and Weingast (1984) in their footnote 15 (p. 63). [FIGURES 37A-37F]

and also (as they note) BJS Figure 1 — is that the uncovered set is considerably smaller than required by the McKelvey $4r$ circular bound. Figure 11, by showing $W(c)$ and the approximate boundary of $UC(c)$, indicates how this reduction occurs.[35] It is evident that the $4r$ circular bound on $UC(c)$ is not overly generous — $UC(c)$ comes close to the $4r$ circle at several points. But it is also true that $UC(c)$ is somewhat "star-shaped" with points emanating from the central "core" of $UC(c)$ that approach the $4r$ circle, while the central core itself has a radius of about $2r$. For points x near c, $UC(x)$ likewise has points emanating from a central core, but the points of such sets $UC(x)$ tend to offset one another rather than overlap. Thus when the intersection of all such sets is formed to get $UC(X)$, the points are pared away, leaving an uncovered set $UC(X)$ that is essentially the common core of all the sets $UC(x)$ and is therefore more compact than the individual $UC(x)$ sets and that nowhere extends much beyond $2r$ from c.

Additional *CyberSenate* work indicates that, event when ideal point configurations are not clustered in the manner of Figure 10 (and BJS Figure 1), this pattern is typical — that is, even when the boundaries of the uncovered set are determined entirely by covering at a distance, $UC(X)$ is much smaller than the well-know $4r$ bound suggests. [FIGURES 38B-38D]

For configurations with a fair number (e.g., $n > 15$) of ideal points, such work supports the following observations:

(a) the parts of the boundary of $UC(c)$ closest to c are about $2r$ from c;
(b) the parts of the boundary of $UC(c)$ furthest from c are about $3.5r$ from c;
(c) all points that are covered at a distance are covered by points close to c;
(d) for points x close to c, (a) and (b) above also hold to good approximation;
(e) however, for different points x close to c, the parts of the boundary of $UC(x)$ furthest from c lie in different directions from c; and therefore
(f) the uncovered set $UC(X)$ is approximately equal to the "central core" of $UC(c)$.

Thus, for such two-dimensional spatial voting games, we can make the following claims with respect to the size and location of the uncovered set.

(1) $UC(X)$ contains, and is approximately centered on, the yolk.

(2) The boundary of $UC(X)$ is irregular but approximately circular.

(3) The "radius" of $UC(X)$ is about $2r$ or somewhat greater.

We can combine these conclusions with the estimates of the size of the yolk for politically relevant magnitudes of n presented in Section 4. For $n = 101$, with configurations randomly drawn from a bivariate normal distribution, the expected "radius" of the uncovered set is about 3.6, where the standard deviation (in each dimension) of the distribution is 15, so the uncovered set can be

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35 Here and elsewhere, I first formed $W(c)$, then formed the union of $W(z)$ for all z at the tips of $W(c)$ and at selected other points z on the boundary of $W(c)$ such that $W(z)$ pushed out beyond the previously established union of such win sets, and finally smoothed out by interpolation portions of the $UC(c)$ boundary not formed by indifference curves (as described in footnote 31). [FIGURE 30B] The match between the location, size, and the polygon-like shape of the uncovered sets displayed in Figure 11 and BJS Figure 1is striking.

expected to occupy about 0.7% of the Pareto set. For $n = 435$, the expected $UC(X)$ "radius" is about 1.8, so $UC(X)$ can be expected to occupy about 0.2% of the Pareto set.[36]

Several qualifications need to be made to these generalizations. First, if most ideal points lie on or near the Pareto frontier, the radius $UC(X)$ expands to about $3r$. This is most apparent in configurations in which ideal points form the vertices of a (more or less) regular polygon. [FIGURES 35A and 36]

Second and more relevant empirically, "non-random" clustering of ideal points produces larger uncovered sets (relative to the Pareto set), but this occurs, not because the uncovered sets are larger relative to the yolk (the $2r$ generalization holds well in both Figure 11 and BJS's Figure 1), but because such clustering increases the size of the yolk itself (relative to the Pareto set), in the manner discussed in Section 4.

With these findings at hand, we can review the three BJS (pp. 270-271) claims summarized in Section 2. While these claims all appear to be accurate, they follow from the theoretical first principle I have tried to elucidate here, and the BJS findings therefore provide additional evidence that their computational procedure works properly. Let us consider each claim in turn.

What leads BJS to their first claim is actually that the clustering of ideal points found in their data produces a *yolk* "much larger than our expectations based on conventional wisdom and previous work" (though it needs to be said that there was little relevant previous work). Indeed (and consistent with the findings summarized above), their work if anything indicates that the *size of the uncovered set relative to the yolk* is much *smaller* than expectations based on conventional wisdom and previous work suggested. In particular (and as BJS note on p. 261 with respect to their Figure 1), the familiar McKelvey $4r$ circular bound on the uncovered set is over-generous (even when the effective Pareto set and local covering play no role in demarcating its boundary).

With respect to the second BJS claim, the uncovered set is (more or less) centered on the yolk, but there is no reason to expect that the center of the yolk will coincide with the "center" of either the policy space or the ideal point configuration. As we saw in Section 4, given ideal points polarized into two clusters in the manner of all BJS figures based on empirical data, most median lines form into a "bow-tie" pattern intersecting approximately midway between the point; however, but at least one median line lies along the centrist face of the majority cluster, and yolk must intersect it. Thus the yolk lies within the majority side of the "bow tie" pattern and is nestled against the centrist face of the majority cluster. Now we also know that the uncovered set is approximately centered on the yolk and has a "radius" of about twice that of the yolk. Thus one side of the uncovered set must penetrate well into the majority cluster itself, while the other side extends approximately to the midpoint between the two clusters. This is pretty much what we observe in all of the BJS figures.

Note that this conclusion holds independent of the relative size of the majority and minority clusters — in particular, it holds even if the majority cluster has only a bare majority of ideal points.

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[36] The observation made in the last sentence of footnote 16 applies here as well.

This consideration makes the third BJS claim unsurprising. If the majority and minority clusters contain almost the same number of ideal points, and if a few ideal points are deleted from the former and/or added to the latter so that the clusters exchange majority and minority status, the yolk flips from one side to the other. Since it tracks the yolk, the uncovered set does likewise.

We may note one final point concerning the panels of BJS Figure 5: the uncovered sets appear to be noticeably larger in the earlier 1949-1970 period than in the more recent 1979-2000 period. We can provide a ready explanation that combines theoretical principles and political context. In the earlier (pre-Southern realignment period), the Democratic (majority) cluster is considerably more spread out in the vertical dimension than the Republican minority cluster. This produces an imbalance in the bow-tie pattern of median lines such that it is especially wide on the Democratic majority side, with the result that the size of the yolk (and uncovered set) increases, as is illustrated by Figure 12. [Also FIGURE 39A] Alternatively, we may think of the 1949-1970 House as a three-party/cluster system of Northern Democrats, Southern Democrats, and Republicans, none of which was of majority size. Thus many median lines pass through two clusters but entirely miss the third. The yolk therefore is very large, rather resembling the yolk in a typical small scale three-voter configuration, as is illustrated in Figure 13.[37] [Also FIGURE 39B]

9. The Structure of the Uncovered Set

I conclude with some conjectures concerning the structure of the uncovered set that are supported by considerable work with CyberSenate but remain relatively speculative.

The Shape of the Uncovered Set. The uncovered set is not convex — in particular, its boundary may have distinct indentations and other irregularities. However, it is starlike relative to c and other points in the vicinity of c. Given a "random" configuration of a modestly large number of ideal points, the uncovered set is approximately circular but with flattish segments on its boundary. Given "non-random" clustering of ideal points, the uncovered set assumes a more distinctively polygon shape. [FIGURES 3B and 38A] Of course, wherever local covering is controlling, the boundary of the uncovered set is formed out of straight line segments.

The Boundary of the Uncovered Set. Boundaries of win sets are everywhere formed out of segments of individual indifference curves. In a small ideal point configuration, the boundary of a set $UC(x)$ of points not covered by x is formed out of win sets (and individual indifference curves); it therefore has kinks but is otherwise smooth. In the general case, however, portions of the boundary of $UC(x)$ do not correspond to win set boundaries; rather as a point z travels along the boundary of some leaf of $W(x)$, the tip of $W(z)$ traces out a portion of the boundary of $UC(x)$. [FIGURES 30B] The boundary of the uncovered set itself is nowhere formed by win set boundaries. As $UC(X)$ is the intersection of an infinite number of sets $UC(x)$, its boundary (within the effective Pareto set) appears to be created by a continuous mapping from points on a somewhat complex loop in the vicinity of the center of the yolk to points on the $UC(X)$ boundary, such that each former point

[37] Further CyberSenate graphics illustrating the effects of other clustering patterns on the location and size of the yolk may found at *http://research.umbc.edu/~nmiller/research/clusters.graphics.pdf*

covers the corresponding latter point at a distance and is least distant point to do so.[38] [FIGURES 40A-40B] Such a mapping produces a smooth boundary on the uncovered set. Of course, where the effective Pareto frontier forms the boundary of *UC*(*X*), the boundary is composed of straight line segments, producing kinks in the boundary.

The Internal Structure of the Uncovered Set. The uncovered set is composed of a "central nucleus" and an "outer shell." If the boundary of *UC*(*X*) is determined entirely by covering at a distance, the "central nucleus" includes all points each of which uniquely covers some other point at a distance and, in particular, uniquely covers a point on the boundary of *UC*(*X*). This "central nucleus" appears to lie within about 0.5r of the center of the yolk and contains the loop described in the preceding paragraph.. The "outer shell" is composed all other points in *UC*(*X*), which cover only points beyond the boundary of *UC*(*X*) that are also covered by points in the "central nucleus." Thus, the points in the outer shell belong to the uncovered set, not because they do any "essential" covering, but only because they are not themselves covered by any points in the "central nucleus."[39] However, if the effective Pareto frontier forms part of the boundary of *UC*(*X*) through local covering, points outside but neighboring that part of the boundary of *UC*(*X*) are *not* covered at a distance by points in the "central nucleus." Rather they are locally covered by neighboring points lying on the boundary of *UC*(*X*) (and the effective Pareto frontier). So, in this event, part of the boundary of *UC*(X) forms third — though partial and "thin" — distinct ring of *UC*(*X*).

[38] Hartley and Kilgour (1987) have pinned this down in the case of $n = 3$. They show is a continuous mapping between a sequence of points in the vicinity of the center of the yolk and the points they cover at a distance that form the boundary of *UC*(*X*) within the Pareto triangle. Each such portion of the boundary takes the form of an ellipse with foci at two ideal points. (See Hartley and Kilgour, 1987.)

[39] See BJS's second approach for enhancing the efficiency of their algorithm (p. 275). This consideration suggests that *UC*(*X*) is starlike with respect to all points in the "nucleus" and that it therefore also "connected" in the sense conjectured by BJS in their footnote 17.

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Figures

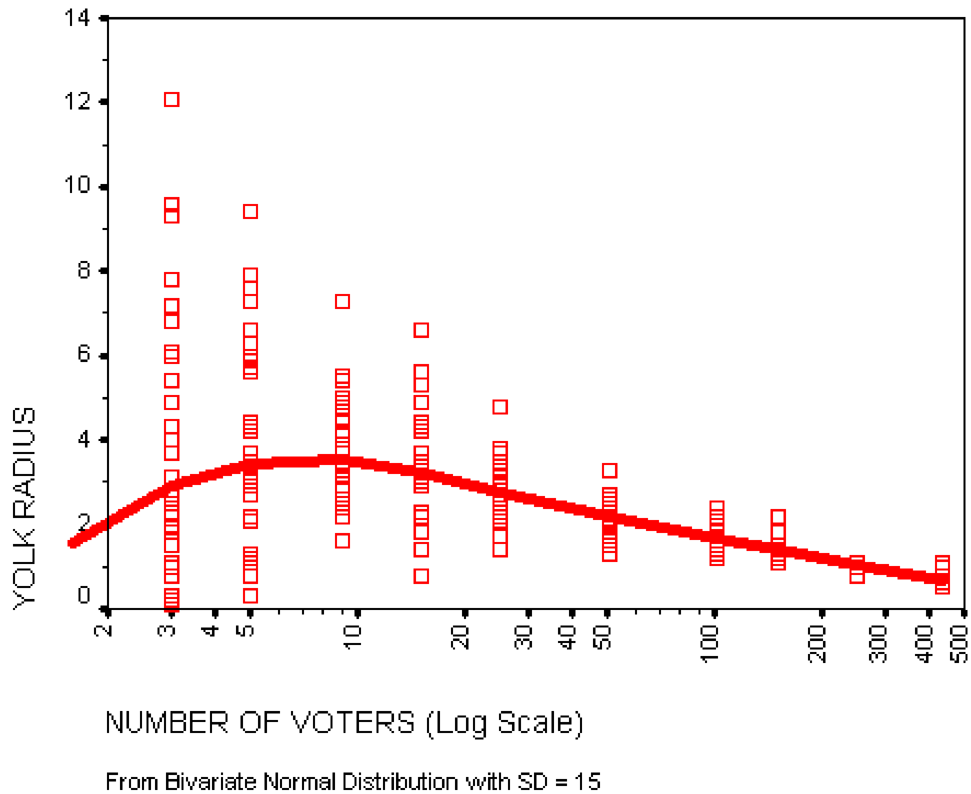

Figure 1

Expected Yolk Radius by the Number of Ideal Points, with *n* = 3 through *n* = 435 Drawn Randomly from a Bivariate Normal Distribution with SD = 15

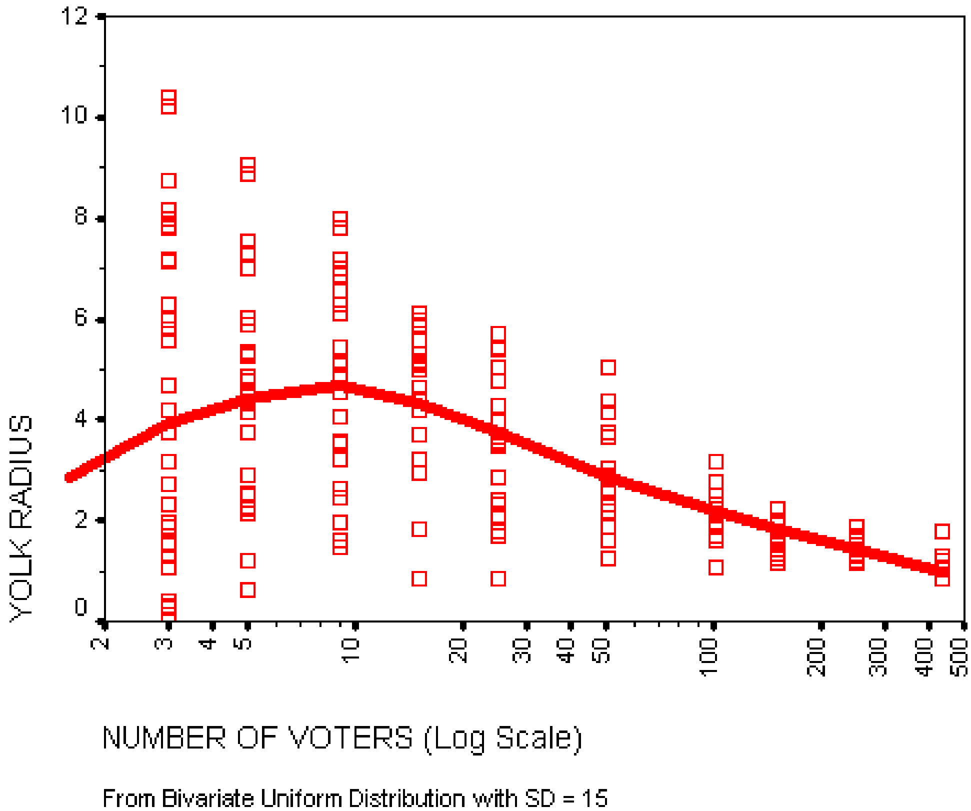

Figure 2

Expected Yolk Radius by the Number of Ideal Points, with *n* = 3 through *n* = 435 Drawn Randomly from a Bivariate Uniform Distribution with SD = 15

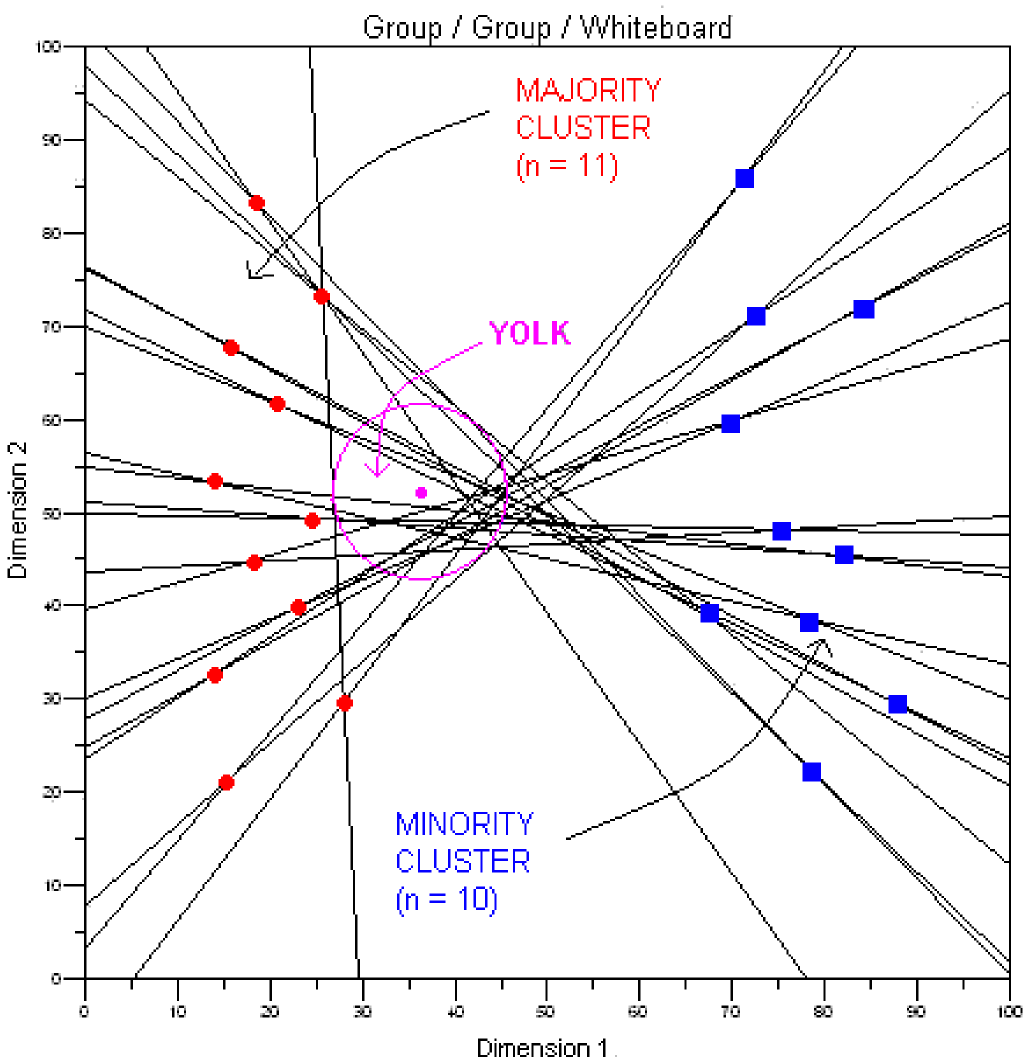

Figure 3

The Location and Size of the Yolk with Two Closely Balanced Clusters of Ideal Points

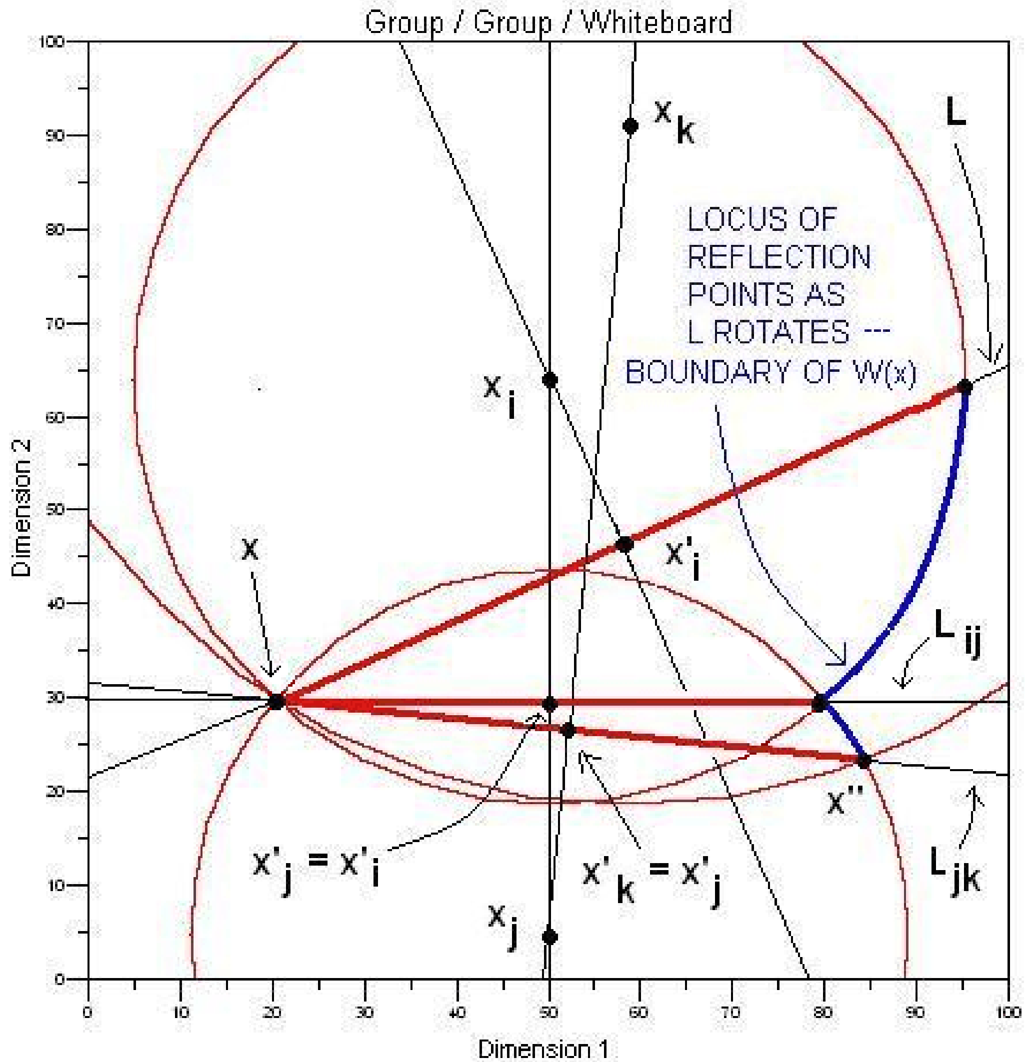

Figure 4

Rotating Line L through Point x

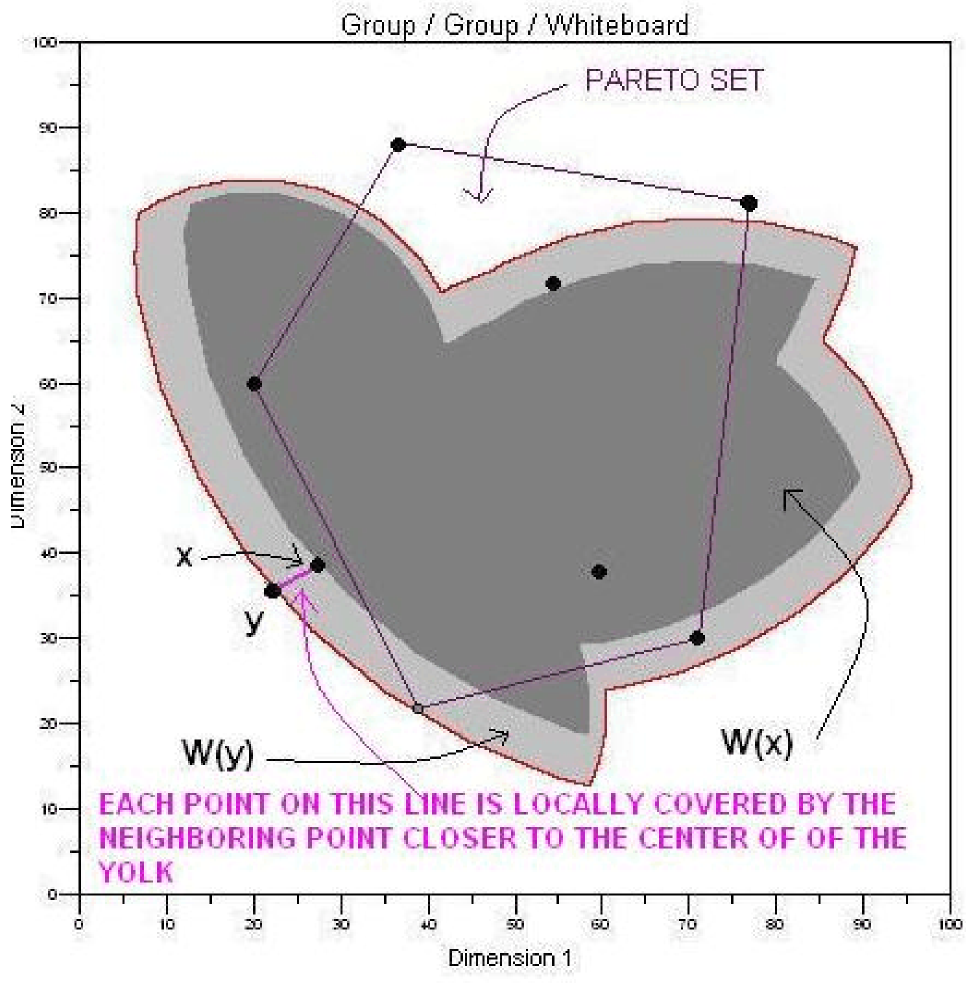

Figure 5

Local Covering of a Point Outside the Pareto Set

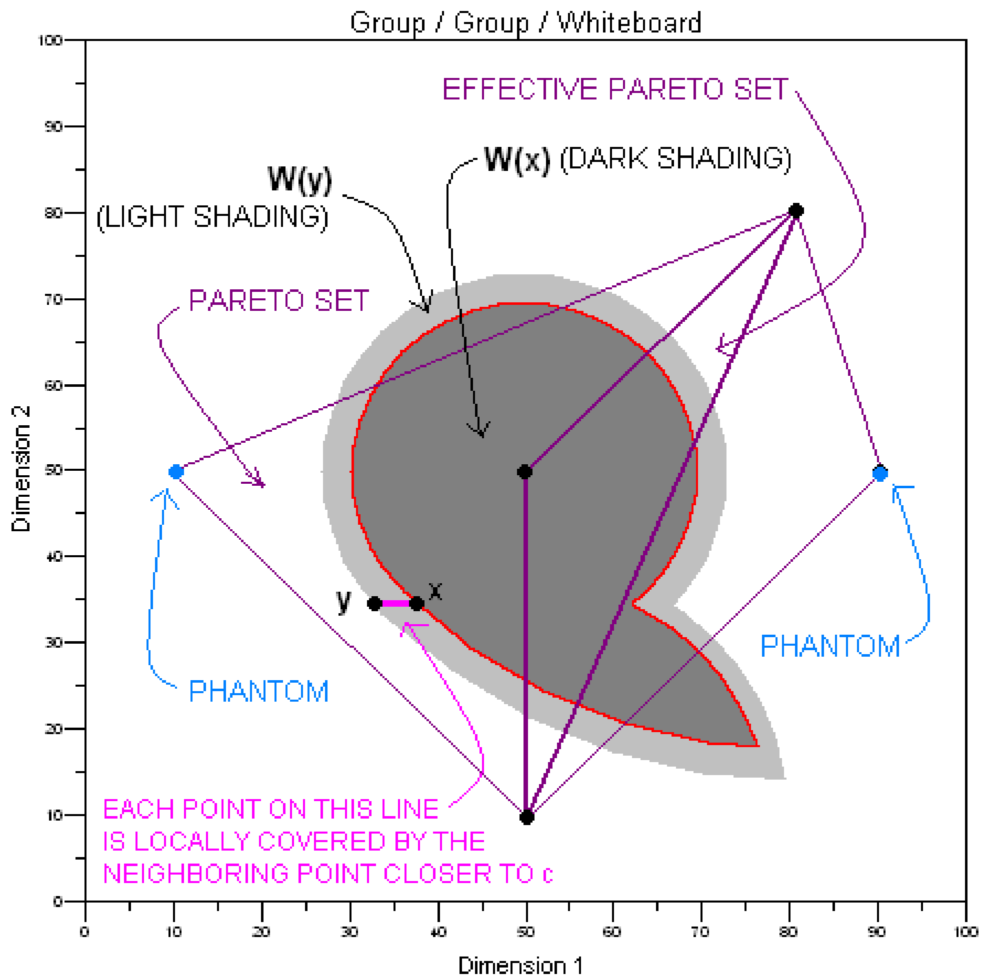

Figure 6

Local Covering of a Point Inside the Pareto Set but Outside the Effective Pareto Set

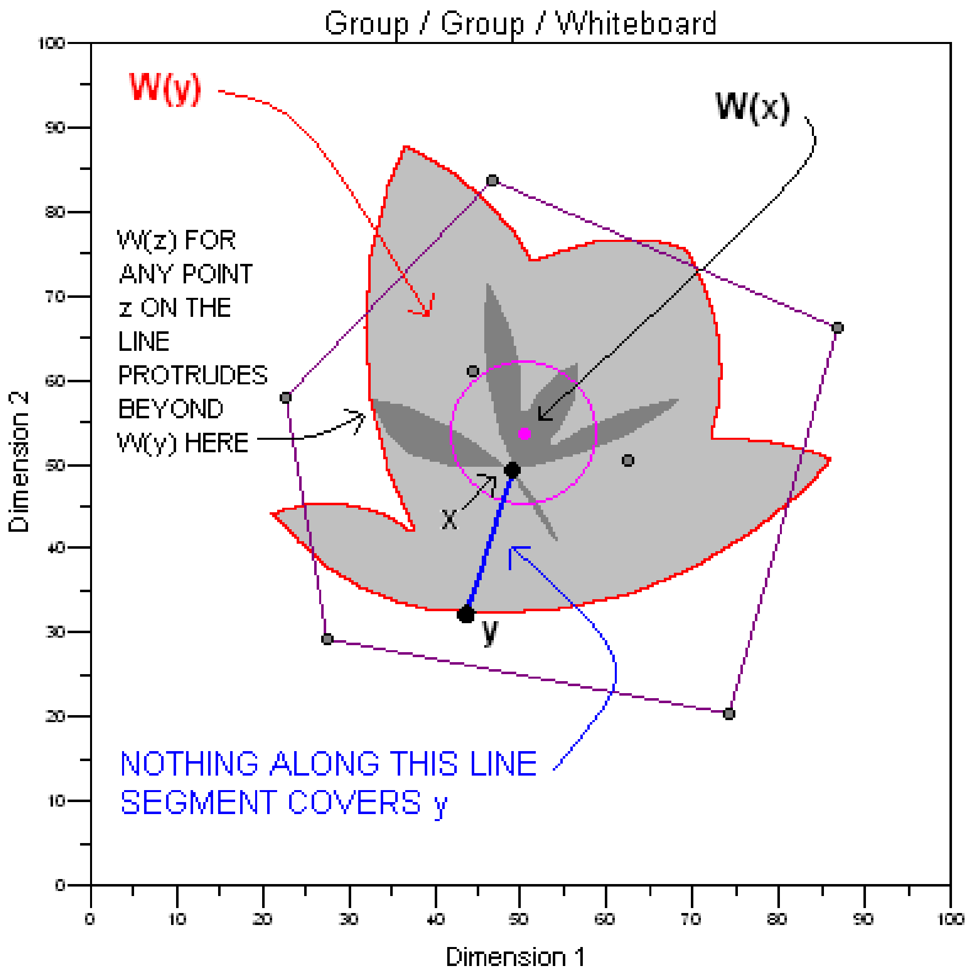

Figure 7

Typical Covering at a Distance

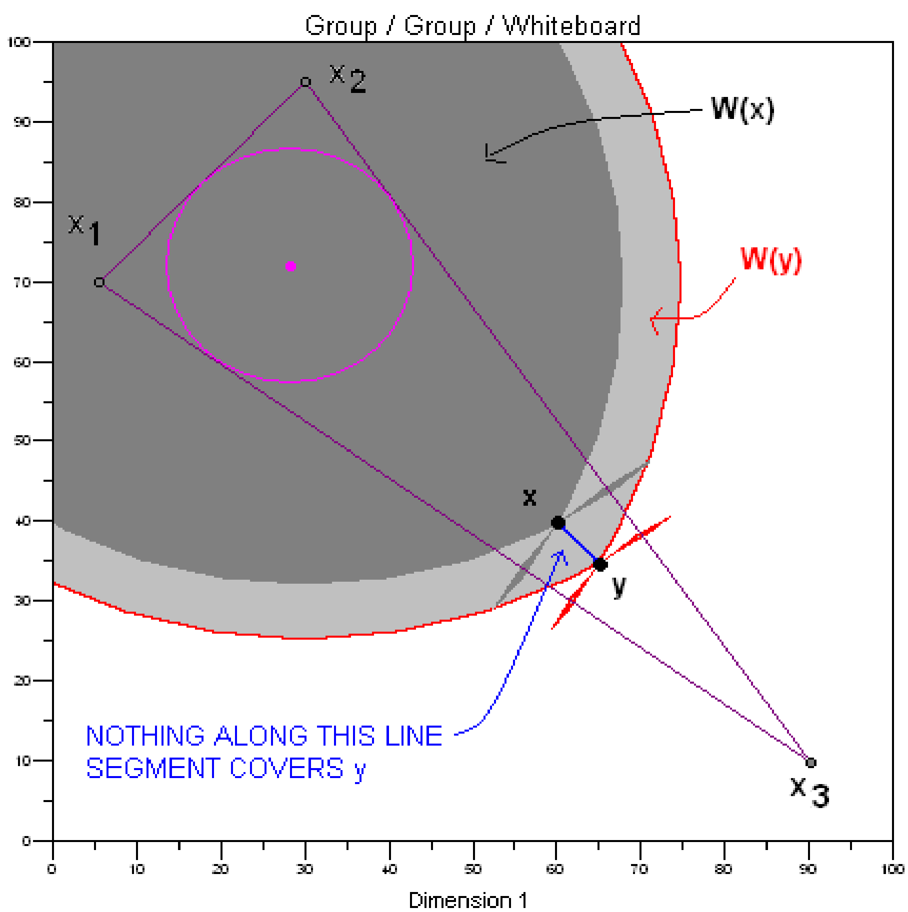

Figure 8

Atypical Covering at a (Small) Distance

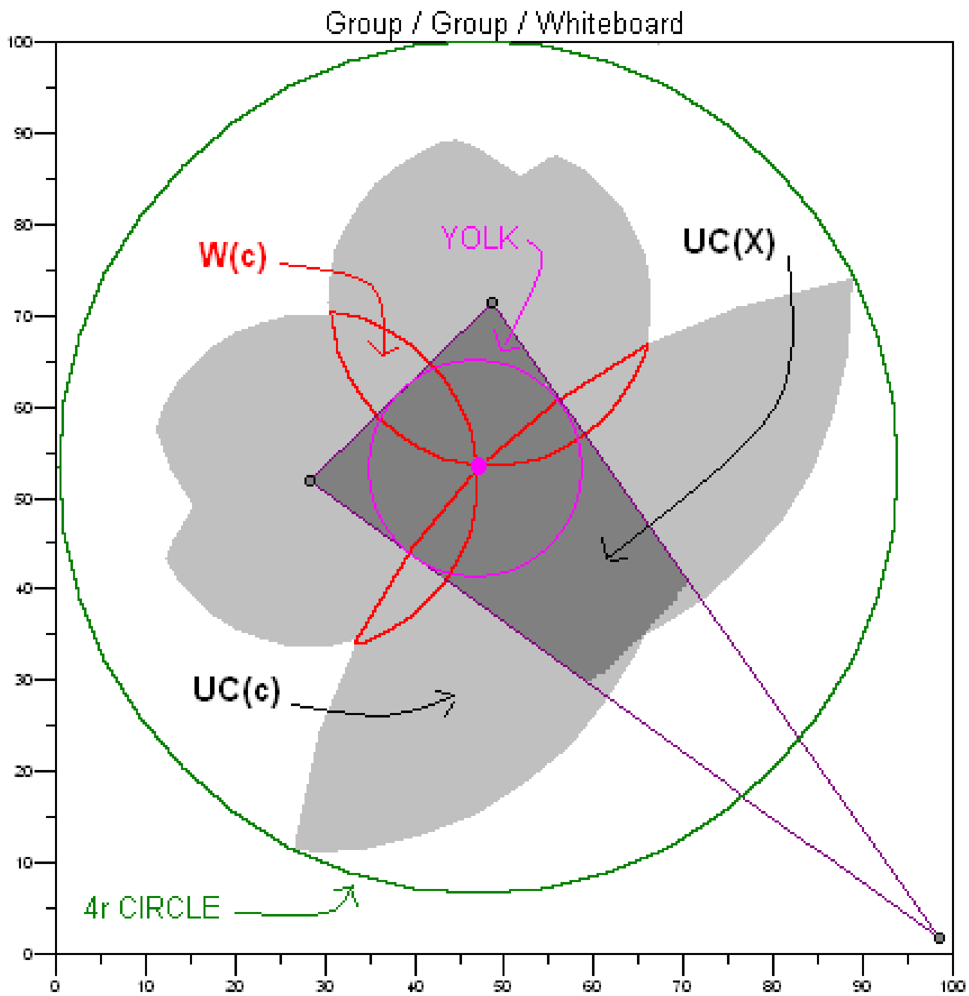

Figure 9

***UC*(*c*) and *UC*(*X*) versus the 4*r* Circle [Highly Acute Pareto Triangle]**

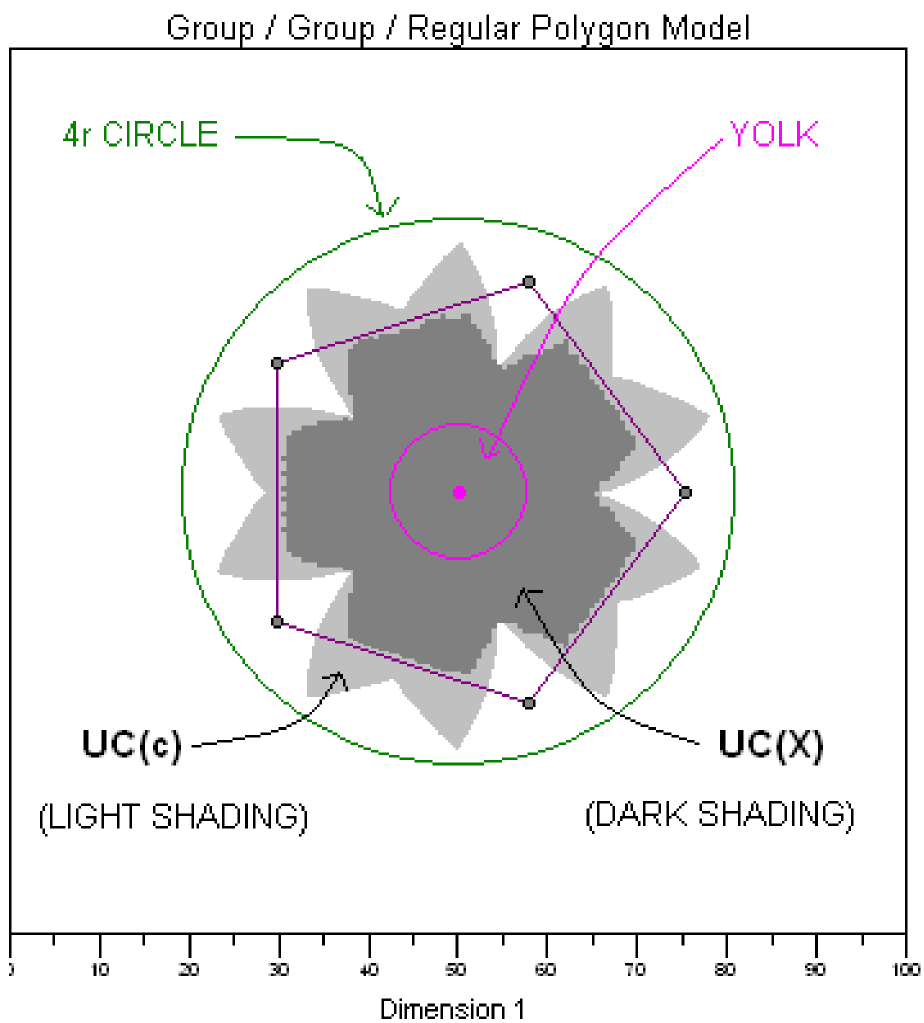

Figure 10

***UC*(*c*), *UC*(X), and 4*r* Circle [Regular Pentagon Configuration]**

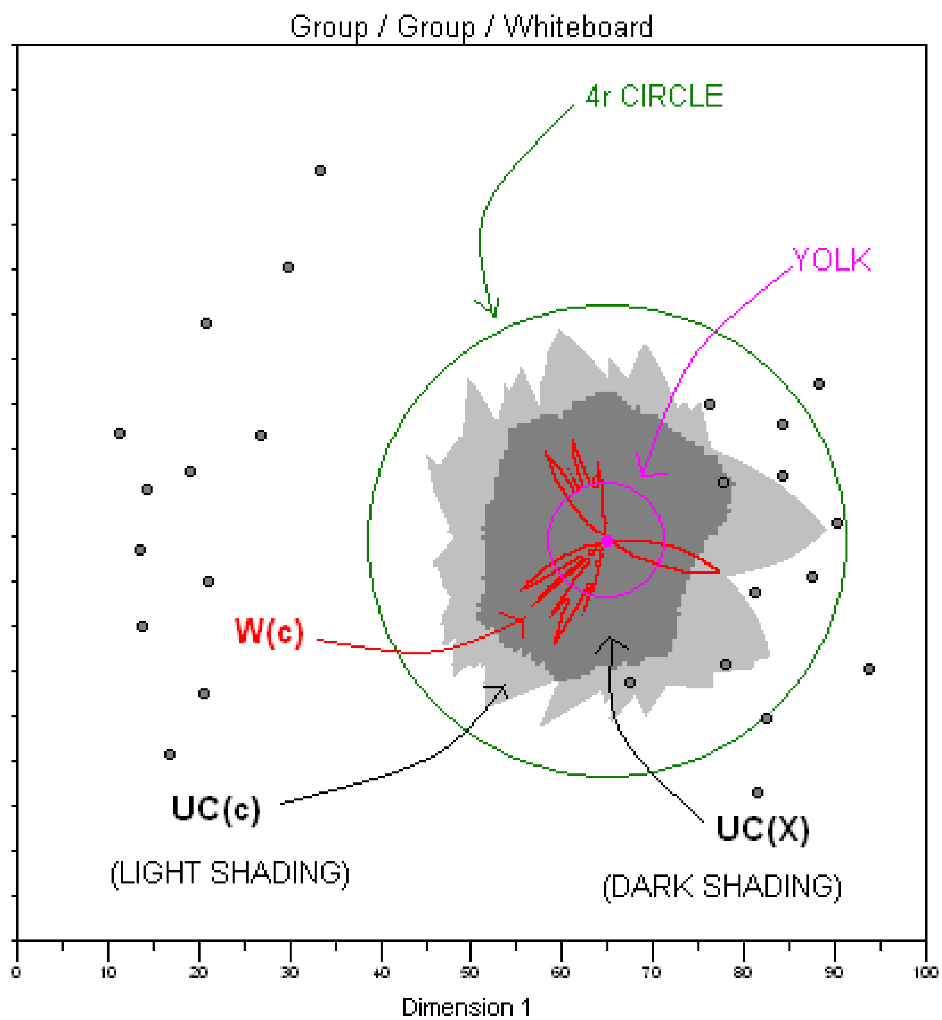

Figure 11

The Yolk, *W*(*c*), *UC*(*c*), *UC*(*X*), and 4*r* Circle for a Scaled-Down Version of BJS Figure 1 (*n* = 25)

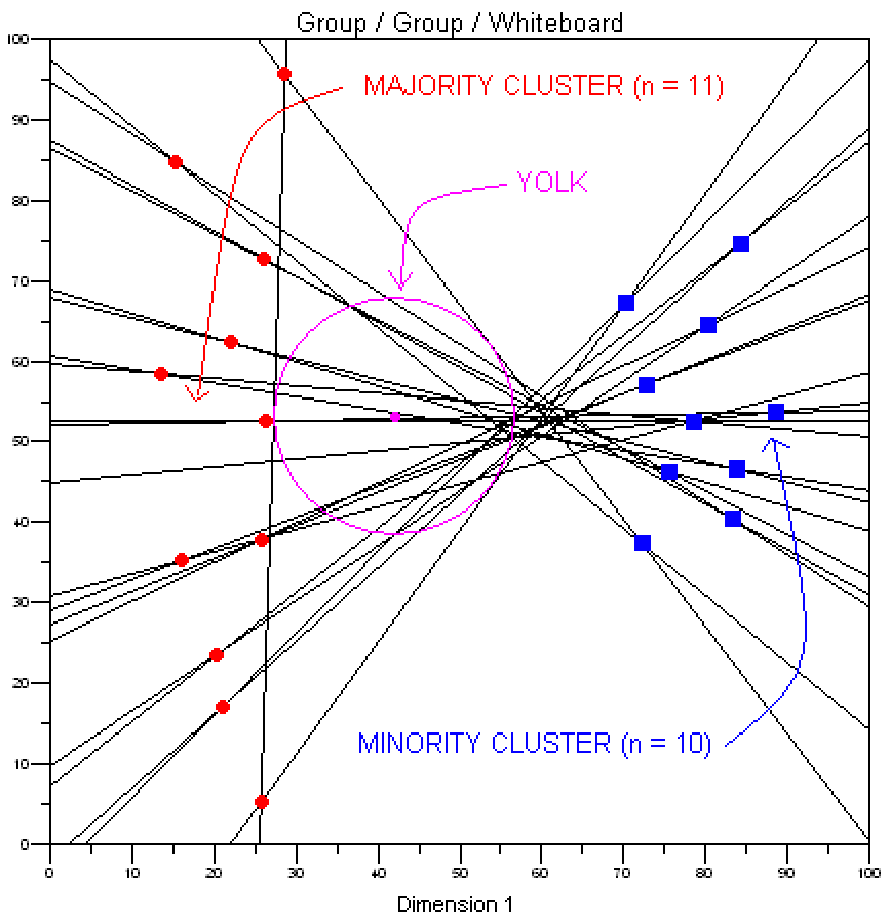

Figure 12

The Location and Size of the Yolk with Two Clusters of Ideal Points Where the Majority Cluster Has Greater Vertical Dispersion

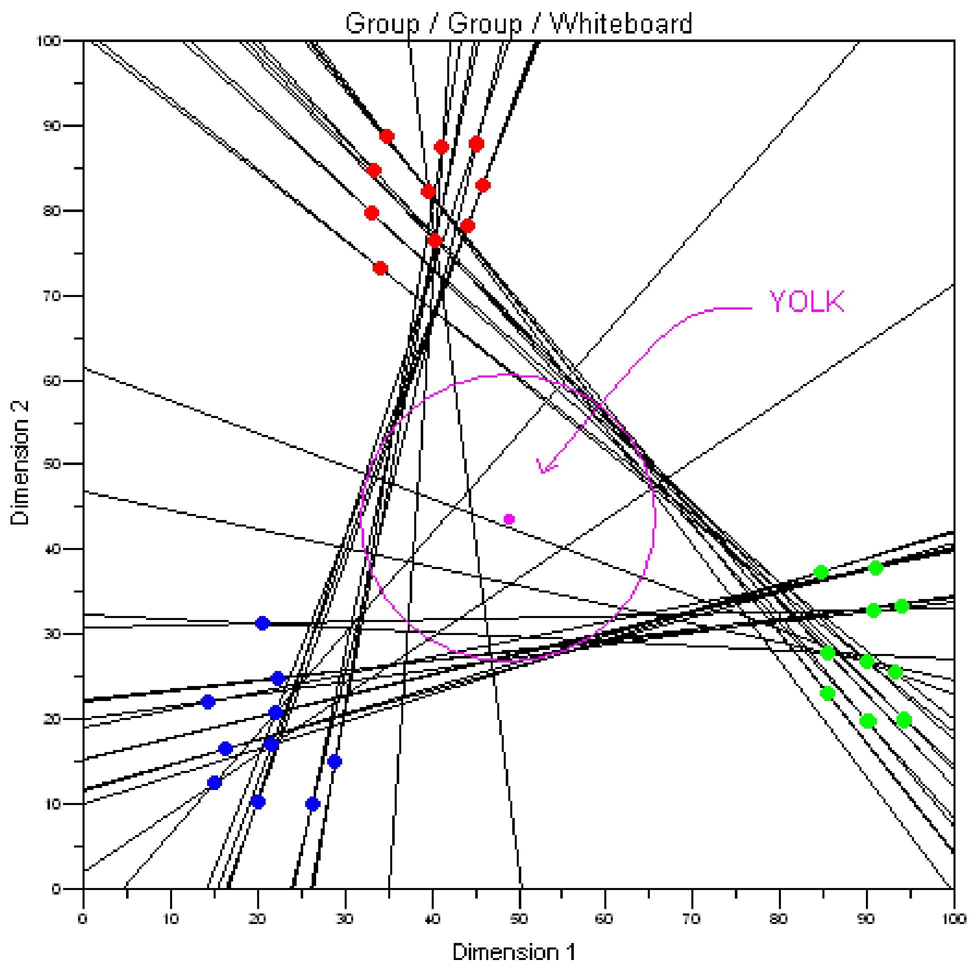

Figure 13

The Location and Size of the Yolk with Three Minority Clusters