

# **The Economics of Controlling a Biological Invasion**

by

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## **Abstract**

The paper develops a simple economic model of a biological invasion. The natural growth of the invasion is non-convex and the immediate cost of controlling the invasion depends on the level of current control as well as the current size of the invasion. Greater control raises control costs today while reducing damages - now and in the future. In addition, by decreasing the size of the invasion, increased control today raises the marginal cost of control in the future. As a consequence, the optimal path of an invasion is not necessarily monotonic. When the marginal control cost declines sharply with the size of invasion, it may be optimal to allow an invasion to grow naturally before it is controlled. We characterize conditions under which it is optimal to eradicate an invasive species (immediately and eventually) and conditions under which it is optimal to manage an invasion without complete eradication.

**Keywords:** Biological invasion, invasive species, eradication, renewable resource economics

**JEL Classifications:** Q20, D99

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## The Economics of Controlling a Biological Invasion

### 1. Introduction

Throughout history the spread of plants, animals and other organisms has been governed by natural ecological processes and has accompanied the movement of humans. As human beings have become more mobile there has been an increase in the introduction of species to areas outside their native habitat. Sometimes such introductions are harmless, or even beneficial. In other instances, introduced species become invaders with few natural enemies to limit their growth.

The invasion of ecological systems by non-indigenous species is now recognized as a growing global problem that imposes significant economic and ecological damages. In the United States alone, the total costs of non-indigenous species have been estimated to be at least \$137 billion per year [Pimentel, et. al., 2000]. Approximately one-fourth of the value of the country's agricultural output is lost to non-indigenous plant pests or the costs of controlling them [Simberloff, 1996]. Invasive species also cause significant ecological harm. They can alter ecosystem processes, act as vectors of disease, and reduce biodiversity [Vitousek, et. al., 1996]. Worldwide, out of 256 vertebrate extinctions with an identifiable cause, 109 are known to be due to biological invaders. In comparison, 70 such extinctions are known to be caused by human exploitation [Cox, 1993]. It is estimated that forty percent of the threatened or endangered species in the United States are at risk due to pressures from invading species [The Nature Conservancy, 1996; Wilcove et. al., 1998].

Although the number of harmful invasive species in the United States is in the thousands [Pimentel, et. al, 2000] a single invasive species can cause tremendous economic and/or ecological damage. For example, the costs of controlling the zebra mussel (*Dreissena polymorpha*) in the Great Lakes are expected to reach \$5 billion in 2001 [USGS, 2000] while the Russian wheat aphid (*Diuraphis noxia*) caused an estimated \$600 million (1991\$) in crop damages between 1987 and 1989 [OTA, 1993]. The invasion of Guam by the brown tree snake (*Boiga irregularis*) demonstrates the ecological harm an

invader can cause. Since its introduction in the mid-twentieth century it has caused the extinction of 12 of the island's bird species [Savidge, 1987; Rodda, Fritts and Chiszar, 1997].

Biological invasions occur almost everywhere. They comprise a significant component of global environmental change [Vitousek et. al., 1996] and are viewed by some as one of the most important issues in natural resource management today [Williams and Meffe, 2000]. As a consequence, the past decade has seen a sizeable growth in public policy directed toward invasive species.<sup>1</sup>

In spite of the growing concern with invasive species, scientific understanding of the relation between economics and ecology in the control of biological invasions is not well-developed. The purpose of this paper is to examine the economics of controlling a biological invasion. We analyze the optimal control policy in a dynamic model of invasion by a single species. The aim is to understand the intertemporal economic and biological tradeoffs that determine the extent to which an invasion should be controlled at any point of time. A clear understanding of how economic and ecological factors interact to determine the optimal policy is important because a wide range of outcomes is possible, even in a simple one-dimensional model. The paper focuses on the conditions under which it is optimal to eradicate an invasive species and conditions under which eradication is not optimal. The results characterize both of these possible outcomes in terms of joint properties of the biological growth of the invasion, control costs, damages and the social discount rate.

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<sup>1</sup>In 1990 the U.S. Congress passed the Nonindigenous Aquatic Nuisance Prevention and Control Act to deal with the increasing problems of invasive species in waterways. In 1996 the act was amended to become the National Invasive Species Act. In 1992 the Alien Species Prevention and Enforcement Act was passed to protect Hawaii from the introduction of prohibited plants, plant pests and injurious animals that may be contained in the mail. On Feb 3, 1999, President Clinton signed Executive Order 13112 which created the National Invasive Species Council. Subsequently, the Council drafted the National Invasive Species Management Plan to develop a national strategy for combating problems of invasive species. There has also been increasing concern at the international level. In 1997, the Global Invasive Species Programme was established by the Scientific Committee for Problems of the Environment (SCOPE), in collaboration with the United Nations Environment Programme, The World Conservation Union, DIVERSITAS (an international programme on biodiversity science) and the Commonwealth Agricultural Bureau International.

Most introductions of non-indigenous species occur as a result of commerce, travel, agriculture or other human activity. The majority (between 80 and 95 percent) of non-native species never become established in their new environment [Williamson, 1996]. Once established, however, the spread of an invasive species is typically characterized by three distinct phases [Shigesada and Kawasaki, 1997, pp 26-27]. The first is an initial establishment phase during which little or no expansion occurs. This is followed by an expansion phase where the population and range of the invasive species increases. Finally, there is a saturation phase as the invasion approaches geographical, climatic, or ecological limits to its range.<sup>2</sup>

The classic ecological model of the spread of an invasion is the reaction-diffusion model of Kolmogorov, Petrovsky, and Piscounov [1937], Fisher [1937] and Skellam [1951]. The model of a biological invasion developed in this paper is an aggregate model that abstracts from spatial considerations inherent in reaction-diffusion models. In our model, the state of the invasion at each point in time is defined by its size. Depending on the context, the size of an invasion may be either the area contained within the frontal boundary of the invasion, or the population or biomass of the invasive species. This aggregation is analogous to the way that standard bioeconomic models of natural resource harvesting aggregate over spatial and other life history characteristics that influence resource growth. At the same time, the model of this paper accommodates invasions that exhibit almost any pattern of growth

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<sup>2</sup>In the reaction-diffusion model growth and spread jointly determine the density,  $n(y,z,t)$ , of a species at location  $(y,z)$  at time  $t$  according to the partial differential equation  $\frac{\partial n}{\partial t} = G(n) + D(\frac{\partial^2 n}{\partial y^2} + \frac{\partial^2 n}{\partial z^2})$ . Local growth is governed by the growth function  $G(n)$  while the coefficient  $D$  that determines how fast the species disperses in space. Fisher [1937] analyzed the case of logistic growth,  $G(n) = rn(1-n/K)$ , while Skellam [1951] considered Malthusian growth,  $G(n) = rn$ . Kolmogorov, Petrovsky, and Piscounov [1937] considered the general class of growth functions that satisfy  $G(0) = G(1) = 0$ ,  $G(n) > 0$  for  $0 < n < 1$ ,  $G'(0) > 0$ ,  $G'(n) < G'(0)$  for  $0 < n < 1$ . The frontal boundary of an invasion is defined to be the radial distance at which the species density equals a critical threshold,  $n^*$ . For example,  $n^*$  may define a detection threshold below which the species density is low enough to avoid detection. For our purposes it useful to think of  $n^*$  as an economic threshold below which the species density is not sufficient to cause damage. In what has become a classic result in the ecological theory of biological invasions Kolmogorov, Petrovsky, and Picounouv [1937] showed that the frontal boundary of an invasion governed by (3.1) expands asymptotically (as  $t \rightarrow \infty$ ) at a constant rate  $2\sqrt{G'(0)D}$ . For the case of Malthusian growth this was proved by Kendall [1948]. Mollison [1977] provides a useful review.

and spread, including those where the invasion is governed by a non-convex biological growth function. In particular, our model is consistent with invasions that follow a pattern of establishment, expansion, and saturation as suggested by historical evidence.

Control of an invasive species takes the form of reducing the size of the invasion by chemical, biological, manual, or other means. In this paper, control is an aggregate measure of the means used to reduce the size of an invasion. Control costs are an increasing and convex function of the amount of control. Control costs may also depend on the invasion size. The evidence from historical attempts to eradicate invasive species indicates that it may cost as much to remove the last one to ten percent of an invasion as it does to control the initial ninety to ninety-nine percent [Myers, et. al., 1998]. This means that unit control costs can escalate as the size of an invasion is reduced and control costs may not be jointly convex due to complementarities between the invasion size and control.

Together, control costs that depend on the invasion size and non-convex invasion growth have important implications for the optimal control of an invasion. Under non-convex growth the optimal amount of control may increase or decrease as the size of invasion grows, there may be multiplicities or discontinuities in the optimal policy, and there are more likely to be corner solutions where there is no control or complete eradication. When control costs depend on the invasion size an increase in the current control has two opposing effects on future net benefits. It lowers future damages and it increases future control costs. The latter creates an economic incentive to postpone control and can lead to outcomes where the invasion follows a nonmonotonic time path under an optimal policy. The optimal management of an invasion may exhibit cyclical or complex dynamics, as is the case when an invasion is allowed to grow unchecked for a number of periods and only after it becomes large enough is it controlled. When costs depend only on the amount of control, as with some forms of chemical control, we show that the optimal invasion size is monotonic over time and convergent. If eradication is optimal from one invasion size, it must necessarily be optimal for any invasion of a smaller size.

The analysis in this paper is related to the literature on optimal management of renewable resources (see, Clark, 1990) where the objective is to maximize the discounted sum of social welfare obtained from harvesting a useful biological resource. Particularly relevant to our problem is the analysis of conservation and extinction of renewable resources on optimal paths. The literature includes models where the growth function is non-convex (non-convex feasible set for the optimization problem) as well as models where the utility depends not only on the size of the harvest but also on the total stock size (see, for example, Olson and Roy, 1996). More generally, the literature on optimal economic growth has also analyzed related dynamic optimization problems including ones where the feasible set is non-convex (among others, Majumdar and Mitra, 1982, Dechert and Nishimura, 1983) as well as models where the utility function depends on both consumption and capital stock (see, for example, Nyarko and Olson, 1991, and the collection of papers in Majumdar, Mitra, and Nishimura, 2000). There is a key distinction between our analysis and the literatures on economic growth and renewable resources. In the latter, the physical or biological capital stock contributes to the production of a good that yields positive social welfare. Hence, conservation of the resource is positively correlated with its growth rate and extinction is not optimal if the productivity of the resource is higher than the discount rate. With a biological invasion, the capital stock or size of the invasion contributes in a negative way to social welfare. This difference turns out to reverse the relationship between biological productivity and incentives to preserve the invasive species. Under the assumptions of our model, the faster the growth rate of an invasion, the greater is the incentive to eradicate it.

There has been relatively little work on the dynamic economics of invasive species control. An early precursor to this paper is Jaquette's [1972] analysis of a discrete time biological population control model. In a finite horizon setting, Jaquette examines the existence and monotonicity properties of an optimal policy. Other early applications of dynamic programming to the problem of pest management are reviewed by Shoemaker [1981]. These tend to focus on issues such as pesticide resistance and intra-seasonal pest management. Wilman [1996] and Knowler and Barbier [2000] examine models with an

invasive predator whose prey is harvested for its economic value. Spatial dimensions of pest control strategies are examined in Brown, Lynch and Zilberman. Sharov and Leibhold [1998] examine the economics of using barrier zones to control the spread of invasive species. Costello and McAusland [2002] consider the links between trade, protectionism and damage arising from exotic species introductions. Jensen [2002] examines optimal protection and damage mitigation in a model where the probability of invasion is exponentially distributed, but where there is no growth and spread of an established invasion. In a separate paper (Olson and Roy [2002]), we examine the economics of controlling a stochastic biological invasion when the costs of control are independent of the invasion size.

The paper is organized as follows. Section two develops the model. The basic properties of a controlled invasion are discussed in section three. Section four studies the economic and ecological conditions for eradication of an invasive species. Circumstances under which eradication does not make sense are examined in section five. Section six develops an example to illustrate the main results. Concluding remarks are given in Section seven and all proofs are in the appendix.

## 2. The model

Let  $y_t$  represent the size of the biological invasion at the beginning of time  $t$  and let  $a_t$  represent the amount of control at time  $t$ . The invasion that remains at the end of period  $t$  is given by  $x_t = y_t - a_t$ . The invasion is assumed to grow and spread according to an invasion growth function  $y_{t+1} = f(x_t)$ . The invasion growth function is assumed to satisfy the following properties:

- A1.  $f(x)$  has a continuous derivative,  $f'_x(x)$ .
- A2.  $f(0) = 0$ .
- A3.  $f'_x(x) \geq 0$ .
- A4.  $f'_x(0) > 1$ .
- A5. There exists an  $K \in (k, 4)$  such that  $f(x) < x$  for all  $x > K$  and  $f'_x(K) > 0$ .

Assumption A2 implies that once an invasion is eradicated it cannot recur. This paper does not address situations where re-invasion is a serious concern. Assumption A3 says that the invasion growth function is increasing in the size of the invasion. To be successful an invasive species must necessarily be able to sustain an invasion. Assumption A4 implies that an invasion can be sustained from an isolated occurrence of the species. Assumption A5 simply reflects the fact that the spread of any invasion is bounded by climatic, geological or ecological factors.

The costs of control and damages caused by the invasion are denoted by  $C(a,y)$  and  $D(x)$ , respectively. Control costs include both the direct costs of control and any indirect costs that may be associated with control, such as adverse effects arising from the use of chemicals. Derivatives are indicated by relevant subscripts, e.g.  $C_a$  represents the partial derivative of  $C$  with respect to  $a$ . Let  $\mathbf{S} \subseteq \mathbb{U}_+^2$  be the set defined by  $\{(a,y) \mid 0 \leq a \leq y \leq K\}$ . Costs and damages are assumed to satisfy the following:

- B1.  $C$  and  $D$  are twice continuously differentiable.
- B2.  $C(0,y) = 0$  for all  $y$  and  $D(0) = 0$ .
- B3.  $C_a(a,y) \leq 0$ ,  $C_y(a,y) \neq 0$ , and  $C_a(a,y) + C_y(a,y) \leq 0$  on  $\mathbf{S}$ .  $D_x(x) \leq 0$ .
- B4.  $C$  is convex in  $a$ .  $D$  is convex.

Assumption B2 rules out fixed costs and it also implies that  $C_y(0,y) = 0$ . Assumption B3 implies that damages are increasing in the size of the invasion, the costs of control increase as control increases, and that a given amount of control is cheaper to achieve from larger invasions. The assumption that  $C_a(a,y) + C_y(a,y) \leq 0$  means that if  $y \neq y'$  it is less costly to reduce the size of the invasion from  $y$  to  $x$  than it is to reduce the size of the invasion from  $y'$  to  $x$ . Assumption B4 gives standard convexity conditions. We do not assume that  $C$  is jointly convex in  $a$  and  $y$ . Hence, our model allows for nonconvexities in both the biological growth function and in the control cost function. It is assumed that A1-A5 and B1-B4 hold throughout the paper.



A *policy*,  $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2, \dots)$ , is a sequence of decision rules,  $\mathbf{B}_t$ , that specify a plan for controlling the biological invasion as a function of the previous history,  $h_t = (y_0, a_0, x_0, \dots, a_{t-1}, x_{t-1}, y_t)$ . That is,  $a_t = \mathbf{B}_t(h_t)$  and  $x_t = y_t - \mathbf{B}_t(h_t)$ . A *stationary Markov policy* is associated with a pair of decision rules that specify the control and the size of the invasion that remains at the end of each period as a function of the size of the invasion at the beginning of the period. Associated with each initial state,  $y_0$ , and each policy  $\mathbf{B}$  is a discounted sum of social costs  $V_{\mathbf{B}}(y_0) = \sum_{t=0}^{\infty} \delta^t [C(a_t, y_t) + D(x_t)]$ , where the sequence  $\{a_t, x_t\}$  is generated by the invasion growth function,  $f$ , and the policy,  $\mathbf{B}$ , in an obvious manner. The objective of the dynamic optimization problem is to minimize the discounted sum of costs and damages over time subject to the transition equation that governs the growth and spread of the invasion. The optimal value satisfies:

$$V(y_0) = \text{Min} \sum_{t=0}^{\infty} \delta^t [C(a_t, y_t) + D(x_t)] \quad \text{subject to } y_t = a_t + x_t \text{ and } y_{t+1} = f(x_t). \quad (2.1)$$

Under A1-A5 and B1-B4, standard dynamic programming arguments imply that there exists a stationary optimal value that satisfies the recursion  $V(y) = \text{Min} [C(a_t, y_t) + D(x_t) + \delta V(f(x_t))]$  subject to  $0 \leq x_t \leq y_t$ ,  $y_t = a_t + x_t$  and  $y_{t+1} = f(x_t)$ , and that there exists a *stationary Markov optimal policy* whose decision rules are  $X(y) = \text{Arg Min}\{[C(y-x, y) + D(x) + \delta V(f(x))]: 0 \leq x \leq y\}$  and  $A(y) = y - X(y)$ . A sequence  $(y_t, x_t, a_t)_0^{\infty}$  that solves (2.1) is an optimal program from  $y_0$ . Given an initial invasion of size  $y_0 = y$  and a selection  $x(y)$  from the stationary optimal policy  $X(y)$ , an optimal program is defined recursively by  $y_{t+1} = f(x(y_t))$ ,  $x_t = x(y_t)$ ,  $a_t = a(y_t)$ ,  $t = 0, 1, 2, \dots$

### 3. Controlled Invasions and their Basic Properties

This section characterizes the basic properties of an optimal policy and the optimal value. The initial results characterize the sensitivity of the optimal value  $V(y)$  and optimal policy  $X(y)$  to the size of the invasion.

**Lemma 1.**  $V(y)$  is continuous and non-decreasing.

Lemma 1 formalizes the intuitive notions that incremental changes in the size of an invasion are associated with small changes in social cost and that larger invasions involve higher social costs.

Sensitivity of the optimal policy depends on how the costs of control vary with control and the invasion size. In a nonconvex model the optimal policy may be multivalued. That is, there may be more than one optimal control from a given invasion size. Consequently, our characterization is based on the properties of a correspondence. Let  $x \in X(y)$  and  $x' \in X(y')$  where  $y \neq y'$ . A correspondence  $X(y)$  is an *ascending* correspondence if  $\min[x, x'] \in X(y)$  and  $\max[x, x'] \in X(y')$ . Similarly,  $X(y)$  is *descending* if  $\max[x, x'] \in X(y)$  and  $\min[x, x'] \in X(y')$ .

**Lemma 2.** (a) If  $C_{aa}(a, y) + C_{ay}(a, y) \leq 0$  on  $\mathbf{S}$ , then  $X(y)$  is an ascending correspondence and the maximal and minimal selections from  $X$  are non-decreasing functions. If the inequality is strict then every selection from  $X$  is non-decreasing. (b) Assume  $C_{aa}(a, y) + C_{ay}(a, y) \neq 0$  on  $\text{int } \mathbf{S}$ . If there exists some  $y < K$  such that  $0 < X(y) < y$  then there is a neighborhood  $N(y)$  of  $y$  such that  $X(\cdot)$  is descending on  $N(y)$  and the maximal and minimal selections from  $X$  are non-increasing functions on  $N(y)$ .

The economic requirement of the first part of Lemma 2 is that a change in control has a larger effect on marginal costs than a change in the size of the invasion.<sup>3</sup> This provides an economic criterion for the optimal size of the invasion to evolve monotonically over time. Since the optimal invasion size is bounded, every invasion with a monotonic time path must necessarily converge to a positive steady state or zero (eradication). If two invasions differ only in their initial size, then the invasion that is larger today will be (weakly) larger at all points in the future.

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<sup>3</sup>Jaquette [1972] proves a result similar to Lemma 2(a), but imposes additional, and inessential assumptions that  $C(a, y)$  and  $f(x)$  are convex functions.

When the size of the invasion has a large effect on the marginal cost of control, as in part (b), this may result in a non-monotonic optimal policy for the size of the invasion.<sup>4</sup> An example of this occurs when the marginal costs of control for a small invasion are sufficiently high that the optimal policy involves no control while the invasion is small. As the invasion grows larger, marginal costs decrease and at some point it may become optimal to reduce the invasion back to very small levels, from which no control is once again optimal.

Some invasions cause minimal damage and control is not cost effective. It is therefore useful to first identify the circumstances under which control makes sense. There are different ways to view the control of an invasion. One may be interested in control from an invasion of a particular size, control of an invasion of any size, or one may be concerned about control of the invasion immediately or at some future date. This motivates the following definitions.

**Definition.** (a) An invasion is a *controlled invasion from*  $y$  if there exists some  $t$  such that  $A(y_t) > 0$ , where  $y_t$  is optimal from  $y$ . (b) An invasion is *currently controlled from*  $y$  if  $A(y) > 0$ . (c) An invasion is *controlled globally* if  $A(y) > 0$  for all  $y$ . (d) An invasion is *interior* if it is controlled globally and  $X(y) > 0$  for all  $y$ .

Each successive definition of control is more restrictive in the sense that (d)  $\supset$  (c)  $\supset$  (b)  $\supset$  (a).

The next result characterizes the economic conditions that are sufficient for different types of control. Define the  $t^{\text{th}}$  iterate of  $f(\cdot)$  and its derivative by  $f^0(y) = y$ ,  $f^t(y) = f^{t-1}(f(y))$ ,  $t = 1, \dots$  and  $f_x^t(y) = df^t(x)/dx$ .

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<sup>4</sup>This is true even in a convex model. The non-monotonicity of the optimal policy arises solely from the fact that  $C_{aa} + C_{ay} \neq 0$ .

**Lemma 3. a.** If there is an  $n \in \mathbb{N}$  such that  $C_a(0, f^n(y)) < \sum_{i=n}^{\infty} \delta^i D_x(f^i(y)) f_x^i(y)$  then the invasion is a controlled invasion from  $y$ .

**b.** If  $C_a(0, y) < D_x(y)$  then the invasion is currently controlled from  $y$ .

**c.** If  $C_a(0, y) < D_x(y) + \delta [\inf_a \{C_a(a, f(y)) + C_y(a, f(y))\}] f_x(y)$  for all  $y > 0$  then the invasion is controlled globally.

The first part of Lemma 3 provides a weak criteria for control to be optimal at some point. It says that if the invasion is allowed to grow unrestricted and, if at some future date the marginal cost of starting to control the invasion is less than the discounted stream of future marginal damages from that time onward into the indefinite future, then it is optimal to control the invasion at some point. The second part of Lemma 3 says that if the current marginal damages from an invasion of size  $y$  exceed the current marginal costs of starting to control the invasion at size  $y$  then control is optimal when an invasion is size  $y$ . This is because even a myopic social planner would undertake positive control in such a situation. If the inequality in part (b) is true for all  $y > 0$ , then the invasion is globally controlled. The latter holds for example, if the costs of the initial increment in control are negligible or  $C_a(0, y) = 0$ , and if the marginal damages from an invasion are always strictly positive, i.e.  $D_x(y) > 0$  for all  $y > 0$ . Part (c) of Lemma 3 provides a somewhat weaker condition for an invasion to be globally controlled by comparing the marginal cost of an arbitrarily small control to the savings in current marginal damage and future marginal damages and control costs. When a change in control has a larger effect on marginal costs than a change in the invasion size, then the requirement of Lemma 3c simplifies to  $C_a(0, y) < D_x(y) + \delta C_a(0, f(y)) f_x(y)$ .<sup>5</sup> A final observation is that in many instances the marginal costs of control will be decreasing in  $y$ . In such cases all of the conditions for controlling an invasion will be more likely to hold, the larger the invasion.

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<sup>5</sup>When  $C_{aa} + C_{ay} \leq 0$  then  $\inf_a C_a(a, y) + C_y(a, y) = C_a(0, y) + C_y(0, y) = C_a(0, y)$ , where the last equality follows from B2.

In the following let  $(y_t, x_t, a_t)_0^4$  be an optimal program from  $y_0$ . The next result characterizes the intertemporal tradeoffs between marginal costs and damages along an optimal program.

**Lemma 4. a.** If  $a_t > 0$  then  $C_a(a_t, y_t) \# D_x(x_t) + \star [C_a(a_{t+1}, y_{t+1}) + C_y(a_{t+1}, y_{t+1})] f_x(x_t)$ .

**b.** If  $x_t > 0$  and  $a_{t+1} > 0$  then  $C_a(a_t, y_t) \$ D_x(x_t) + \star [C_a(a_{t+1}, y_{t+1}) + C_y(a_{t+1}, y_{t+1})] f_x(x_t)$ .

**c.** If  $0 < x_t < y_t$  and  $a_{t+1} > 0$  then

$$C_a(a_t, y_t) = D_x(x_t) + \star [C_a(a_{t+1}, y_{t+1}) + C_y(a_{t+1}, y_{t+1})] f_x(x_t). \quad (3.1)$$

Since the value function in nonconvex models may not be differentiable, Lemma 4 cannot be obtained by applying standard envelope theorem arguments such as those of Benveniste and Scheinkman [1979].

Majumdar and Mitra [1982] use variational methods to obtain the Euler equation in a nonconvex growth model. Our version of Lemma 4 generalizes the envelope theorem of Benveniste and Scheinkman by using an alternative approach based on the principle of optimality and the fact that Dini derivatives of  $V$  exist everywhere.

**Corollary to Lemma 4c.** If  $0 < x_t < y_t$  and  $0 < x_{t+1} < f(x_t)$  for all  $t$  then

$$C_a(a_t, y_t) = D_x(x_t) + \sum_{i=1}^{\infty} \delta^i \left[ D_x(x_{t+i}) + C_y(a_{t+i}, y_{t+i}) \right] \prod_{j=0}^{i-1} f_x(x_{t+j}).$$

This has a simple interpretation when the costs of control are independent of the size of the invasion. For an interior policy the optimal control equates the marginal costs of control with the discounted sum of marginal damages over time multiplied by the compounded marginal growth of the invasion. This is a simple cost-benefit criterion which balances the cost of removing a unit of the invasion against the discounted sum of current and future damages associated with that increment of the invasion. When the costs of control depend on the size of the invasion, the stream of future damages

must be adjusted to account for the influence of the invasion size on future control costs. In addition, given that it is less costly to reduce the size of the invasion to  $x$  from  $y$  than it is from  $y' > y$  (assumption A3), the dynamic optimum always involves at least as much control as the static optimum.

#### 4. The Economics of Eradication

In this section, we consider the conditions under which it makes sense to eradicate an invasive species. The term eradication can have two meanings. In general it applies when the species is eradicated in the long run and the invasion is controlled in a manner that reduces its size to zero in the limit. It can also have a narrower meaning in cases where the species is fully exterminated in the current period. Eradication in the general sense includes both immediate eradication and the possibility that the species is fully eliminated within a finite number of periods.

It is intuitively clear that a first step towards eradication is a reduction in the size of invasion. We begin by giving a result about when it is economic for the current control to do so.

**Proposition 1.** Suppose that for  $y_0 = y > 0$ , the invasion is currently controlled from all  $y' \in [y, f(y)]$  and for all  $x \in (f^{-1}(y), y)$ ,  $C_a(y - x, y) < D_x(x) + \delta \left[ \inf_{0 \leq a \leq f(x) - x} C_a(a, f(x)) + C_y(a, f(x)) \right] f_x(x)$ .

Then every optimal program  $(y_t, x_t, a_t)_0^\infty$  satisfies  $y_1 = f(x_0) < y_0$ .

To interpret this result, first examine the static case where  $\delta = 0$ . Consider the set of controls that lead to an increase in the size of the invasion next period. If marginal damages exceed marginal costs for every control in this set, then can never be optimal to allow the invasion to increase in size.<sup>6</sup> In the dynamic case, marginal control costs are compared to the current and future marginal damages, adjusted

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<sup>6</sup>The non-convex structure of the dynamic optimization problem is the reason the inequality has to hold over the entire range of control that would lead to an increase in the size of the invasion.

for the effect of the invasion size on future control costs. The second term on the right hand side of the inequality in Proposition 1 is a lower bound on the effect that a reduction in the invasion today has on future damages and control costs.

We now characterize the economic and biological conditions under which eradication is optimal in the general sense where the size of invasion is reduced to zero in the long run. Whether eradication is optimal or not depends on the initial size of the invasion. If Proposition 1 holds for every invasion whose size is between zero and some positive level,  $\bar{y}$ , then if the initial invasion size is below  $\bar{y}$ , it is optimal to reduce the size of invasion in every period and over time the invasion is eradicated.

**Corollary to Proposition 1.** If there exists a  $\bar{y} > 0$  such that the invasion is currently controlled from every  $y \in (0, \bar{y})$ , and if

$$C_a(y - x, y) < D_x(x) + \delta \left[ \inf_{0 \leq a \leq f(x) - x} C_a(a, f(x)) + C_y(a, f(x)) \right] f_x(x)$$

for every  $y \in (0, \bar{y}]$  and for all  $x \in (f^{-1}(y), y)$ . Then, (eventual) eradication is optimal from every invasion of size  $y \in (0, \bar{y}]$ . If this condition holds for  $\bar{y} = K$ , then global eradication is optimal.

The condition in the corollary implies that for any invasion of size below  $\bar{y}$ , it is less costly to reduce the size of the invasion than to incur the current and future damages should the invasion be allowed to grow. When discounted growth exceeds one from every invasion size below  $\bar{y}$ , then eradication is optimal even if marginal damages are very low. The rationale is that if a fast growing invasion increases future damages and control costs more rapidly than the rate of discount then it makes sense not to allow the invasion to grow and the invasion should be eradicated from its current level. The condition for eradication becomes stronger with higher values of  $\delta$ . Thus, it is generally more likely for eradication to be optimal from small invasions than from large invasions.

If  $C$  is submodular and the marginal costs of control vary more with the amount of control than with the size of the invasion, then the optimal policy for controlling the invasion is monotone (Lemma 2.a). In that case, the efficiency of eradication depends on the economic and biological conditions at steady states and a tighter condition for eradication is possible.

**Proposition 2.** Assume  $C_{aa}(a,y) + C_{ay}(a,y) \leq 0$  on  $\mathbf{S}$ . If there exists a  $\bar{a} > 0$  such that the invasion is currently controlled from every  $y \in (0, f(\bar{a})]$ , and if

$$C_a(f(x)-x, f(x)) < D_x(x) + \bar{a} C_a(0, f(x)) f_x(x) \quad (4.1)$$

for all  $x \in (0, \bar{a}]$ , then (eventual) eradication is optimal from every invasion of size  $y_0 \in (0, \bar{a}]$ .

A steady state policy is one where  $a = f(x)-x$  and the invasion remains constant over time. The condition in Proposition 2 balances the marginal costs of steady state control against the current marginal damages plus the lower bound on future marginal damages and control costs. If the current and future marginal damages from steady state control are higher for all invasions smaller than  $\bar{a}$  then eradication is optimal from all such invasions.

Note that if  $\bar{a} = K$ , then (4.1) is a condition for global eradication. It is possible, however, to provide a more direct condition for global eradication by ensuring that there is no positive steady state.

**Proposition 3.** If  $C_{aa}(a,y) + C_{ay}(a,y) \leq 0$  on  $\mathbf{S}$  and if

$$C_a(f(x)-x, f(x)) < D_x(x) + [C_a(f(x)-x, f(x)) + C_y(f(x)-x, f(x))] f_x(x)$$

for all  $x \in (0, K]$ , then every optimal program converges to zero and eradication is globally optimal.

There are three differences between Propositions 2 and 3. First, Proposition 2 relies on a lower bound on future marginal damages while this is not necessary in Proposition 3. Second, Proposition 3 is a result



about global eradication so the condition in the Proposition is required to hold for every possible invasion size. On the other hand, Proposition 2 can be used to evaluate the efficiency of eradication from small invasions; the conditions need not apply when the invasion is large.

Next, we characterize the circumstances under which immediate eradication is optimal, i.e., where the invasion is fully eradicated in the current period.

**Proposition 4.** Suppose that for some  $y \in (0, K]$ , the invasion is currently controlled from every  $y_0 \in (0, f(y))$  and that

$$C_a(y, y) < D_x(0) + \delta \inf_{0 \leq a \leq f(x) - x, 0 \leq x \leq y} \left[ \left( C_a(a, f(x)) + C_y(a, f(x)) \right) f_x(x) \right]$$

Then, immediate eradication is optimal from  $y$ .

The criterion for immediate eradication balances the costs of removing the last unit of the invasion against the current and future damages that would be caused should the invasion be allowed to remain. The second term on the right hand side of the inequalities is a lower bound on the future damages associated with an arbitrarily small invasion. It is possible that the condition for immediate eradication may be satisfied for large  $y$  and not for small  $y$ . That is, immediate eradication may be optimal for a large invasion even if it is not optimal for a smaller invasion. Since Propositions 1 and 4 are not mutually exclusive there can exist circumstances where: eradication of small invasions is optimal, eradication is not optimal for medium size invasions, and eradication is optimal for large invasions.

These results on the economics of eradication have the following implications. First, eradication is more likely to be an optimal policy for invasions that have a higher discounted growth rate than it is for invasions that grow slowly. This might seem counter-intuitive, but it is because the benefits from control today are higher when an invasion expands rapidly. In addition, the benefits from control today are magnified further into the future when the discount rate is lower. Second, for some invasions economic

considerations may favor eradication when the invasion is small, but not when the invasion is large. In such cases a rapid response may be necessary for eradication to be justified. Finally, in the special case where the marginal costs of control at  $a=0$  are insignificant, the criteria for eradication in Propositions 1, 2 and 4 essentially involve static benefit cost considerations that balance current marginal costs and damages. This is a consequence of the fact that  $C_a(0,y) = 0$  and  $C_a(a,y) + C_y(a,y) \geq 0$  imply  $\inf_{0 \leq a \leq f(x)-x} C_a(a,f(x)) + C_y(a,f(x)) = 0$ . Hence, the lower bound on future marginal social costs is relatively weak when marginal control costs are insignificant.

## 5. The Economics of Noneradication.

In this section we characterize the economic and biological environments under eradication that are not optimal. Under these circumstances the optimal policy either involves no intervention, or suppression in order to reduce damages. Our first result rules out immediate eradication as an optimal strategy.

**Proposition 5.a.** If  $D_x(0) + \delta C_a(0,0) f_x(0) < C_a(y,y)$  then  $X(y) > 0$ .

**b.** If  $C_a(0,y) > D_x(y) + \delta \sup_{0 \leq a \leq f(x), 0 \leq x \leq y} [C_a(a, f(x)) + C_y(a, f(x))] f_x(x)$  then  $X(y) = y$ .

If the damages from an arbitrarily small invasion are less than the marginal costs of removing the entire invasion, then it is always optimal to allow some of the invasion to remain. There should be no control in the current period if, given the current invasion size, the marginal costs of initiating control exceed the maximum current and future marginal damages that can occur. This proposition focuses on the optimal policy from a given invasion size. For example Proposition 5b can be used to provide a condition for no control for small invasions. In general, however, the proposition does not rule out the possibility that eradication is optimal in the long run. That question is the focus of the next result.

**Proposition 6.** If  $C_{aa}(a,y) + C_{ay}(a,y) \leq 0$  on  $\mathbf{S}$  and  $D_x(0) + \delta C_a(0,0)f'_x(0) < C_a(0,0)$  then  $X(y) > 0$  for all  $y$  and, in addition, for all  $y$  sufficiently close to zero  $A(y) = 0$  and  $X(y) = y$ .

For an arbitrarily small invasion, if the damages compounded indefinitely at the discounted expected intrinsic growth rate are less than the marginal costs of eradicating the invasion then the optimal policy is not to control the invasion at all when it is sufficiently small. This implies that eventual eradication is not an optimal strategy from an invasion of any size.

Proposition 6 comes very close to providing necessary and sufficient conditions for sufficiently small invasions to be uncontrolled. This can be seen by a comparison of Proposition 6 with Proposition 1 evaluated as the invasion size approaches zero.

A final possibility is that eradication is optimal from small invasions but that it is not optimal if the invasion grows to be large. The last result can be used to help identify such outcomes.

**Proposition 7.a.** Suppose there exists an  $\hat{x}$  such that every invasion of size  $y \leq f(\hat{x})$  is currently controlled and that

$$C_a(y - \hat{x}, y) > D_x(\hat{x}) + \delta \sup_{0 \leq x < \hat{x}, 0 \leq a \leq f(x)} \left[ (C_a(a, f(x)) + C_y(a, f(x))) f'_x(x) \right]$$

Then from every initial invasion size  $y \leq f(\hat{x})$ , the invasion size in every period is bounded below by  $f(\hat{x})$ .

**b.** Assume  $C_{aa}(a,y) + C_{ay}(a,y) \leq 0$  on  $\mathbf{S}$ . Suppose there exists an  $\hat{x}$  such that for every  $x \in (0, \hat{x})$ ,

$$C_a(f(x)-x, f(x)) > D_x(x) + \delta [C_a(f(x), f(x)) + C_y(f(x), f(x))] f'_x(x).$$

Then from every initial invasion size  $y \leq f(\hat{x})$ , the invasion size in every period is bounded below by  $f(\hat{x})$ .

If the marginal costs of reducing the size of the invasion over time exceed the current and future marginal damages for every invasion larger than  $f(\hat{x})$ , then it can never be efficient to reduce the invasion size below  $f(\hat{x})$ .

## 6. Application of the Results

This section uses the case of exponential control costs and damages to illustrate the application of the results. The aim is to demonstrate that the conditions are internally consistent and may be easily applied when costs and damages belong to specific functional classes. Consider costs and damages given by  $C(a,y) = (\exp(\alpha a)-1)\exp(-\beta y)$  and  $D(x) = \exp(-\beta x)$ . Control costs increase exponentially in the amount of control, but decrease exponentially with the invasion size. The parameter  $\alpha$  represents intrinsic marginal costs of control, or the marginal costs of control evaluated at their limiting values of  $a = 0$  and  $y = 0$ . Similarly, damages increase exponentially with the size of the invasion that remains at the end of the period, and  $\beta$  represents intrinsic marginal damages, or marginal damages from an incrementally small invasion. Assumption B.3 requires  $\frac{\alpha}{\beta} > 1 - e^{-\alpha K}$ . Finally,  $C_{aa}(a,y) + C_{ay}(a,y) > (<) 0$  as  $\alpha > (<) \beta$ .

Hence, the optimal policy governing the invasion size is monotone if  $\alpha > \beta$ , while if  $\beta > \alpha$  the optimal invasion size is governed by a decreasing policy on the interior of  $S$ . Table 1 summarizes the conditions for eradication or noneradication, given exponential cost and damage functions and any invasion growth function satisfying A.1-A.5.

The condition for both the Corollary to Proposition 1 and Proposition 2 is independent of  $\beta$  so it is essentially the same condition that applies if control costs are independent of the invasion size. Further, the condition applies regardless of whether  $\beta > \beta^*$  or  $\beta < \beta^*$ . This means that in the case of exponential costs and damages, the efficiency of eradicating small invasions does not depend on the monotonicity of the optimal policy.

It is relatively straightforward to extend these results to other cost and damage functions. For example, what is important in most of the results are the marginal damages from an arbitrarily small invasion (either directly or because  $D_x(0)$  is a lower bound on marginal damages from larger invasions). In Table 1, the conditions for Propositions 1, 2, 4, 5.a and 6 apply much more generally to any convex damage function where  $D_x(0) = c$ , including the case of linear damages,  $D(x) = cx$ .

## 7. Conclusion

The results of this paper can provide the foundation for both a normative and a positive analysis of invasive species control. For example, they may help explain why some observed efforts to control invasive species involve a repeated process where periods of inaction are followed by intervals of significant control. In addition, the results show the importance of the initial invasion size in determining the optimal policy. Propositions 1 or 2 may apply when the invasion is small while Proposition 7 may simultaneously apply when the invasion is large. In such cases the optimal policy is path dependent and there is an economic rationale for eradication if the invasion is small, but not if the invasion is large.

The paper also illustrates the information that is needed to evaluate the economic efficiency of invasive species control. Estimates of the costs of control, damages from an invasion, and the invasion growth rate are required. These may be difficult to assess, particularly in the early stages of an invasion. Yet this is precisely the time when prompt action can reduce future consequences.

There are a number of important issues related to invasive species problems that are not addressed in this paper. There are many circumstances where prevention may be the best control. Similarly, our model does not consider the possibility of re-invasion. Clearly, the value of eradicating an invasive species will depend on the likelihood that a new invasion might occur. Finally, many invasive species problems involve private actions where individuals do not consider the consequences for social welfare. The design of policies that mitigate the conflicts between private incentives and social welfare is another interesting aspect of invasive species problems.

### Appendix.

Proof of Lemma 1. The cost functions  $C$  and  $D$  are bounded continuous functions on their relevant domains. Define the operator  $\mathbf{V}(y) = \inf C(a,y) + D(x) + \mathbf{V}(f(x))$  subject to  $y = x + a$ . By the contraction mapping theorem  $\mathbf{V}$  maps the set of bounded continuous functions into itself. Hence,  $\mathbf{V}$  is continuous. We now show that  $\mathbf{V}$  maps non-decreasing functions into non-decreasing functions.

Suppose  $\mathbf{V}$  is non-decreasing. Let  $x$  and  $x'$  be optimal from  $y$  and  $y'$ , respectively where  $y < y'$ . Suppose  $x' < y$ . Then  $x'$  is feasible from  $y$  and  $\mathbf{V}(y) = C(y-x,y) + D(x) + \mathbf{V}(f(x)) \# C(y-x',y) + D(x') + \mathbf{V}(f(x')) \# C(y'-x',y') + D(x') + \mathbf{V}(f(x')) = \mathbf{V}(y')$ , where the first inequality is due to optimality while the second is due to B3. Now suppose that  $x' \geq y$ . Then  $\mathbf{V}(y) = C(y-x,y) + D(x) + \mathbf{V}(f(x)) \# C(0,y) + D(y) + \mathbf{V}(f(y)) = C(0,y') + D(y) + \mathbf{V}(f(y)) \# C(y'-x',y') + D(x') + \mathbf{V}(f(x'))$ . The first inequality is due to optimality, the equality follows from B2, and the second inequality is implied by B3 and the fact that  $\mathbf{V}$  is nondecreasing.

Proof of Lemma 2. a. Let  $x \in X(y)$  and  $x' \in X(y')$  for  $y \# y'$ . Then  $\max[x,x'] \# y'$  and  $\min[x,x'] \# y$ . Note that  $D(\max[x,x']) + D(\min[x,x']) = D(x) + D(x')$ , and likewise  $V(f(\max[x,x'])) + V(f(\min[x,x'])) = V(f(x)) + V(f(x'))$ . Since  $C_{aa}(a,y) + C_{ay}(a,y) \geq 0$  it follows that  $C(y-x,y) - C(y-\min[x,x'],y) \geq$

$C(y'-\max[x,x'],y') - C(y'-x',y')$ . Hence,  $0 \leq C(y-x,y) + D(x) + *V(f(x)) - [C(y-\min[x,x'],y) + D(\min[x,x']) + *V(f(\min[x,x']))] \leq C(y'-\max[x,x'],y) + D(\max[x,x']) + *V(f(\max[x,x'])) - [C(y'-x',y) + D(x') + *V(f(x'))] \leq 0$ , where the first and last inequalities follow from the principle of optimality. This sequence of inequalities implies that  $\min[x,x'] \in X(y)$  and  $\max[x,x'] \in X(y')$ . If  $C_{aa}(a,y) + C_{ay}(a,y) > 0$  the middle inequality above becomes strict when  $x' < x$ . This yields a contradiction so it must be that every selection from the optimal policy correspondence is monotone.

b. The argument is similar to the proof of part a. Let  $x \in X(y)$  and  $x' \in X(y') < y'$  for  $y \neq y'$ . We want to show that  $\max[x,x'] \in X(y)$  and  $\min[x,x'] \in X(y')$ . This follows immediately if  $x \geq x'$  so suppose that  $x < x'$ . Since  $x' < y'$  it must be the case that  $x' \neq y$  for all  $y$  sufficiently close to  $y'$ . The assumption  $C_{aa}(a,y) + C_{ay}(a,y) \neq 0$  implies  $C(y-x,y) + C(y'-x',y') \leq C(y'-x,y) + C(y-x',y)$ . This in turn yields  $0 \leq C(y-x,y) + D(x) + *V(f(x)) - [C(y-x',y) + D(x') + *V(f(x'))] \leq C(y'-x,y) + D(x) + *V(f(x)) - [C(y'-x',y) + D(x') + *V(f(x'))] \leq 0$ , where the first and last inequalities follow from optimality. This sequence of inequalities implies that  $x' \in X(y)$  and  $x \in X(y')$ . Hence  $\max[x,x'] \in X(y)$  and  $\min[x,x'] \in X(y')$ . The assumption that  $X(y') < y'$  is necessary to insure that  $x'$  is feasible from  $y \in N(y')$ .

Proof of Lemma 3. a. Without loss of generality we can take  $n = 0$ . Let  $\{x_t, a_t\}$ ,  $t = 0, \dots$  be optimal from  $y_0 = y$ . Suppose  $a_t = 0$  for all  $t$  and  $x_t = y_t = f^t(y)$ . It follows that for all  $y \in (0, K)$ ,  $f^t(y) \in (0, K)$ ,  $f^t(y) > y$ , and  $f_x^t(y) \geq f_x(K)^t$  for all  $t \geq 1$ . Consider

an alternative sequence  $\{x_t, a_t\}$  where  $a_t = \varepsilon$ , and  $a_t = 0$  for all  $t \geq 1$ . Then  $x_t = y_t = f^t(y - \varepsilon)$  for all  $t \geq 1$ . As  $\{x_t, a_t\}$  is optimal,

$$0 \geq \left[ \sum_{t=0}^{\infty} \delta^t \{C(0, f^t(y)) + D(f^t(y))\} \right] - \left[ C(\varepsilon, y) + D(y - \varepsilon) + \sum_{t=1}^{\infty} \delta^t \{C(0, f^t(y - \varepsilon)) + D(f^t(y - \varepsilon))\} \right]$$

$$= C(0, y) - C(\varepsilon, y) + \sum_{t=1}^{\infty} \delta^t [C(0, f^t(y)) - C(0, f^t(y - \varepsilon))] + \sum_{t=0}^{\infty} \delta^t [D(f^t(y)) - D(f^t(y - \varepsilon))].$$

Dividing by  $\delta$ , and letting  $\delta \rightarrow 0$  implies:  $C_a(0,y) \leq D_x(y) + \sum_{t=1}^{\infty} \delta^t [C_y(0,f^t(y)) + D_x(f^t(y))] f_x^t(y)$ .

Since  $C_y(0,f(y)) = 0$  this contradicts the condition of the proposition. Thus, there must exist some  $t$  for which  $a_t > 0$ .

b. Let  $a^0$  denote the control that minimizes the sum of one-period control costs and damages. The assumption  $C_a(0,y) < D_x(y)$  implies  $a^0 > 0$ . Let  $a^*$  denote an optimal control for the infinite horizon problem. From the definition of  $a^0$ ,  $C(a^0,y) + D(y-a^0) \leq C(a^*,y) + D(y-a^*)$ , with a strict inequality if  $a^* = 0$ . Further, if  $a^* < a^0$ , then  $V(f(y-a^*)) \leq V(f(y-a^0))$  by Lemma 1. From these inequalities it must be that  $a^* \leq a^0 > 0$ , else  $a^*$  could not be optimal in the infinite horizon problem.

c. Since  $\inf_a C_a(a,f(y)) + C_y(a,f(y)) \leq C_a(0,f(y)) + C_y(0,f(y)) = C_a(0,f(y))$  it follows from the condition in part b that  $C_a(0,y) < D_x(y) + \delta C_a(0,f(y)) f_x(y)$  for all  $y$ . Since the condition in part c holds for all  $y$ ,  $C_a(0,f(y)) < D_x(f(y)) + \delta [\inf_a \{C_a(a,f^2(y)) + C_y(a,f^2(y))\}] f_x(f(y))$ . Substituting this in the previous inequality yields  $C_a(0,y) < D_x(y) + \delta [D_x(f(y)) + \delta [\inf_a \{C_a(a,f^2(y)) + C_y(a,f^2(y))\}] f_x(f(y))] f_x(y)$ . Iterating forward and repeating a similar substitution yields  $C_a(0,y) < D_x(y) + \sum_{t=1}^{\infty} \delta^t [C_y(0,f^t(y)) + D_x(f^t(y))] f_x^t(y)$ , which is the condition in part a. Hence, the condition in part c implies part a and the invasion is a controlled invasion from  $y$ .

Let  $T$  be the first  $t$  such that  $a_t > 0$  and suppose that  $T > 0$ . Then  $a_t = 0$  for  $t = 0, \dots, T-1$ , while  $0 < a_T \leq y_T = f(x_{T-1}) = f^T(y)$  and  $x_T = y_T - a_T = f^T(y) - a_T$ . Since  $a_T > 0$  there exists an  $\epsilon > 0$  such that  $f(y_{T-1}) - f(y_{T-1} - \epsilon) < a_T$ . Consider the alternative feasible sequence  $\hat{a}_{T-1} = \epsilon$ ,  $\hat{a}_T = a_T - [f(y_{T-1}) - f(y_{T-1} - \epsilon)]$ , and  $\hat{a}_t = a_t$  for all other  $t$ . Then  $\hat{x}_{T-1} = y_{T-1} - \epsilon$ , and  $\hat{x}_t = x_t$  for  $t \leq T-1$ . Since the sequence  $\{x_t, a_t\}$  is optimal,

$$\begin{aligned} 0 &\geq \sum_{t=0}^{\infty} \delta^t [C(a_t, y_t) + D(x_t) - C(\hat{a}_t, \hat{y}_t) - D(\hat{x}_t)] \\ &= \delta^{T-1} [C(0, y_{T-1}) + D(y_{T-1}) - C(\epsilon, y_{T-1}) - D(y_{T-1} - \epsilon)] \\ &\quad + \delta^T [C(a_T, f(y_{T-1})) + D(x_T) - C(a_T + f(y_{T-1} - \epsilon) - f(y_{T-1}), f(y_{T-1} - \epsilon)) - D(x_T)] \end{aligned}$$

Dividing by  $\delta$ , and letting  $\delta \rightarrow 0$  implies



$$0 \leq -C_a(0, f^{T-1}(y)) + D_x(f^{T-1}(y)) + [C_a(a_T, f^T(y)) + C_y(a_T, f^T(y))]f_x(f^{T-1}(y))$$

Since this contradicts the condition in the proposition it must be that  $T = 0$  and  $A(y) > 0$  for all  $y$ .

**Proof of Lemma 4.** For purposes of exposition the proof is divided into a sequence of subsidiary lemmas.

Define the lower, right and left Dini derivatives of  $V$  at  $y$  by

$$D_+V(y) = \liminf_{\varepsilon \rightarrow 0} \frac{V(y + \varepsilon) - V(y)}{\varepsilon} \quad \text{and} \quad D_-V(y) = \frac{V(y) - V(y - \varepsilon)}{\varepsilon}.$$

**Lemma 4.1.**  $-C_a(y_0 - x_0, y_0) + D_x(x_0) + D_-V(f(x_0)) \geq 0$ .

Proof. Since  $x_{0-}$  is feasible from  $y_0$ , the principle of optimality implies  $C(y_0 - x_{0+}, y_0) + D(x_{0-}) + V(f(x_{0-})) - C(y_0 - x_0, y_0) - D(x_0) - V(f(x_0)) \geq 0$ . Dividing by  $\delta$  and taking  $\liminf_{\delta \rightarrow 0}$  establishes the result.

**Lemma 4.2.** If  $x_0 < y_0$  then  $-C_a(y_0 - x_0, y_0) + D_x(x_0) + D_+V(f(x_0)) \geq 0$ .

Proof. If  $x_0 < y_0$  then  $x_{0+}$  is feasible from  $y_0$  for sufficiently small  $\delta$ . By the principle of optimality it follows that  $\liminf_{\delta \rightarrow 0} [C(y_0 - x_{0-}, y_0) + D(x_{0+}) + V(f(x_{0+})) - C(y_0 - x_0, y_0) - D(x_0) - V(f(x_0))]/\delta \geq 0$ .

The result follows immediately.

**Lemma 4.3.**  $D_+V(f(x_0)) \leq [C_a(a_1, y_1) + C_y(a_1, y_1)]f_x(x_0)$ .

Proof. Since  $x_1$  is feasible from  $f(x_{0+})$  the principle of optimality implies  $V(f(x_{0+})) - V(f(x_0)) \leq C(f(x_{0+}) - x_1, f(x_{0+})) + D(x_1) + V(f(x_1)) - C(f(x_0) - x_1, f(x_0)) - D(x_1) - V(f(x_1))$ . Dividing by  $\delta$  and taking the  $\liminf_{\delta \rightarrow 0}$  on both sides and simplifying completes the proof.

**Lemma 4.4.** If  $a_1 > 0$  then  $D_-V(f(x_0)) \leq [C_a(a_1, y_1) + C_y(a_1, y_1)]f_x(x_0)$ .

Proof. Since  $a_1 > 0$ ,  $x_1$  is feasible from  $f(x_{0-t})$  for sufficiently small  $t$ . By the principle of optimality it follows that  $V(f(x_0)) - V(f(x_{0-t})) \leq C(f(x_0)-x_1, f(x_0)) + D(x_1) + \lambda[V(f(x_1)) - C(f(x_{0-t})-x_1, f(x_{0-t})) - D(x_1) - \lambda V(f(x_1))]$ . Once again, dividing by  $t$  and taking the limit  $t \rightarrow 0$  on both sides and simplifying completes the proof.

The proof of part a of Lemma 4 follows from Lemmas 4.2 and 4.3 while combining Lemmas 4.1 and 4.4 yields part b. Part c is a joint implication of all four lemmas.

Proof of Proposition 1. Suppose not. Then there exists an optimal program  $(y_t, x_t, a_t)_0^4$  where  $y_1 = f(x_0) \leq y$ , i.e.,  $x_0 \leq f^{-1}(y)$ . Since,  $y_1 \in [y, f(y)]$ , it follows that  $x_1 < y_1$ . Therefore, using Lemma 4, we have

$$C_a(y-x_0, y) \leq D_x(x_0) + \lambda [C_a(f(x_0)-x_1, f(x_0)) + C_y(f(x_0)-x_1, f(x_0))] f_x(x_0)$$

which violates the inequality in the statement of the Proposition.

Proof of Proposition 2. Consider  $y_0 \in (0, \cdot]$  and the optimal policy generated by the maximal selection from  $X(y)$ . (Under  $C_{aa}(a, y) + C_{ay}(a, y) \leq 0$ , this policy function is non-decreasing in  $y$ .) It is sufficient to show that the optimal path generated by this policy is strictly decreasing over time. Suppose not. Then there exists an optimal program  $(y_t, x_t, a_t)_0^4$  where  $y_1 = f(x_0) \leq y_0$ . This, in turn, implies that  $x_1 \leq x_0$ . Since,  $y_1 \in [y_0, f(y_0)] \subset (0, f(\cdot))$ , it follows that  $x_1 < y_1 = f(x_0)$ . Therefore, using Lemma 4, we have

$$C_a(y_0 - x_0, y_0) \leq D_x(x_0) + \lambda [C_a(f(x_0)-x_1, f(x_0)) + C_y(f(x_0)-x_1, f(x_0))] f_x(x_0)$$

and using  $C_{aa}(a, y) + C_{ay}(a, y) \leq 0$  and  $f(x_0) \leq y_0$  we have

$$C_a(f(x_0) - x_0, f(x_0)) \leq D_x(x_0) + \lambda [C_a(f(x_0)-x_1, f(x_0)) + C_y(f(x_0)-x_1, f(x_0))] f_x(x_0)$$

which violates the inequality in the antecedent of the proposition.

Proof of Proposition 3. Consider the set of optimal paths generated by the maximal selection from  $X(y)$ . It is sufficient to show that complete eradication occurs on every path generated by this selection. From

Lemma 2, we know that this is a non-decreasing function and therefore, every optimal path generated by this selection is weakly monotone and hence convergent (they are all bounded). Suppose to the contrary that there is an optimal path generated by the maximal selection from  $X(y)$  which is bounded away from zero. Then, it must converge to a strictly positive optimal steady state  $y^* = f(x^*)$ . Note that  $y_0 \in (0, K)$  implies that every optimal program is bounded above by  $K$  so that  $x^*$  and  $y^*$  lie in  $[0, K]$ . If  $x^* \in (0, K)$  then equation (3.1) implies  $C_a(f(x^*) - x^*, f(x^*)) = D_x(x^*) + [C_a(f(x^*) - x^*, f(x^*)) + C_y(f(x^*) - x^*, f(x^*))]f_x(x^*)$  which contradicts the inequality in the proposition. Also,  $y^* = K = f(K)$  is not an optimal steady state as the inequality in the proposition implies  $C_a(0, K) < D_x(K) + [C_a(0, K) + C_y(0, K)]f_x(K) = D_x(K) + C_a(0, K)f_x(K)$  which implies  $C_a(0, K) < [D_x(K)/(1 - f_x(K))]$ . The latter can be used to show that a program where the control is infinitesimal but greater than zero in period 0 and equal to zero every period thereafter dominates a program where  $a_t = 0$  in every period. Hence,  $K$  is not an optimal steady state. Thus, it must be the case that every optimal path converges to 0.

Proof of Proposition 4. Suppose not. Then there exists an optimal program  $(y_t, x_t, a_t)_0^4$ ,  $y_0 = y$ , where  $x_0 > 0$  so that  $y_1 = f(x_0) > 0$ . Since the invasion is currently controlled from every  $y_0 \in (0, f(y))$ , it follows from Lemma 4 that  $C_a(y - x_0, y) \geq D_x(x_0) + [C_a(f(x_0) - x_1, f(x_0)) + C_y(f(x_0) - x_1, f(x_0))]f_x(x_0)$ . The convexity of  $C$  in its first argument and the convexity of  $D$  then imply

$$C_a(y, y) \geq D_x(0) + [C_a(f(x_0) - x_1, f(x_0)) + C_y(f(x_0) - x_1, f(x_0))]f_x(x_0)$$

This contradicts the inequality in the antecedent of the proposition.

Proof of Proposition 5. a. Suppose  $X(y) = \{0\}$ . Consider the alternative of increasing the remaining invasion to  $\tau$ , and then eradicating it in the following period. By the principle of optimality  $C(y, y) + D(0) + [C(0, 0) + D(0)] + \tau^2 V(0) \# C(y - \tau, y) + D(\tau) + C(f(\tau), f(\tau)) + D(0) + \tau^2 V(0)$ . This implies

$C_a(y,y) \neq D_x(0) + * [C_a(0,0)+C_y(0,0)]f_x(0) = D_x(0) + * C_a(0,0)f_x(0)$ , where the equality follows from B3.

This is a contradiction to the condition in 5a.

b. The condition in 5b implies  $0 > -C_a(0,y) + D_x(y) + * \sup_{0 \leq x < y, 0 \leq a < f(x)} [C_a(a,f(x))+C_y(a,f(x))]f_x(x) \leq -C_a(y-x,y) + D_x(x) + * [C_a(A(f(x)),f(x))+C_y(A(f(x),f(x)))]f_x(x)$  where the last inequality is due to B4. But if  $x < y$ , Lemma 4a implies  $-C_a(y-x,y) + D_x(x) + * [C_a(A(f(x)),f(x))+C_y(A(f(x),f(x)))]f_x(x) \leq 0$ , a contradiction.

Proof of Proposition 6.  $C_{aa} + C_{ay} \leq 0$  implies  $C_a(0,0) \neq C_a(y,y)$  for all  $y$ . Hence the condition in 5a holds for all  $y$  and  $X(y) > 0$  for all  $y$ . To prove the second part we want to show that there exists a  $\delta$  sufficiently close to zero, such that  $X(y) = y$  for all  $y \in (0, \delta)$ . Let  $x \in X(y)$  and suppose that  $x < y$ . By Lemma 4a

$$0 \neq -C_a(y-x,y) + D_x(x) + * [C_a(A(f(x)),f(x))+C_y(A(f(x),f(x)))]f_x(x) \\ \neq -C_a(0,x) + D_x(x) + * [C_a(f(x),f(x))+C_y(f(x),f(x)))]f_x(x). \quad (\text{using } C_{aa} + C_{ay} \leq 0 \text{ twice})$$

Define  $H(x) = -C_a(0,x) + D_x(x) + * [C_a(f(x),f(x))+C_y(f(x),f(x)))]f_x(x) \leq 0$ . The condition in 5b implies  $H(0) < 0$  and by the continuity of  $H$  one can pick an  $\delta$  sufficiently close to 0 such that  $H(x) < 0$ . This yields a contradiction.

Proof of Proposition 7. a. Suppose not. Then, there exists  $y_0 \in f(x)$  and an optimal path  $\{y_t\}$  such that  $y_1 < f(x)$ , i.e.,  $x_0 < x$ . Using Lemma 4, we have  $C_a(y_0-x_0,y_0) \neq D_x(x_0) + * [C_a(a_1,f(x_0)) + C_y(a_1,f(x_0))]f_x(x_0)$ . Then  $x_0 < x$  implies  $C_a(y_0-x_0,y_0) \neq C_a(y_0-x_0,y_0) \neq D_x(x_0) + * [C_a(a_1,f(x_0)) + C_y(a_1,f(x_0))]f_x(x_0) \neq D_x(x) + * [C_a(a_{t+1},f(x_t)) + C_y(a_{t+1},f(x_t))]f_x(x_t)$  which violates the condition in part a.

b. Suppose not. Then, there exists optimal path  $\{y_t\}$  such that  $f(x_0) = y_1 < y_0 = f(x)$ , i.e.,  $x_0 < x$ . Lemma 4 implies  $C_a(y_0-x_0,y_0) = C_a(f(x)-x_0,f(x)) \neq D_x(x_0) + * [C_a(a_1,f(x_0)) + C_y(a_1,f(x_0))]f_x(x_0)$ . Since  $X(y)$  is monotone under  $C_{aa} + C_{ay} \leq 0$  and  $x_0 < x$  then  $x_1 \neq x_0$ . Using  $C_{aa} + C_{ay} \leq 0$  twice, this implies  $C_a(f(x_0)-x_0,f(x_0)) \neq C_a(f(x)-x_0,f(x)) \neq D_x(x_0) + * [C_a(f(x_0)-x_1,f(x_0)) + C_a(f(x_0)-x_1,f(x_0))]f_x(x_0) \neq D_x(x_0) + * [C_a(f(x_0),f(x_0)) + C_a(f(x_0),f(x_0))]f_x(x_0)$ . This violates the condition in part b.

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TABLE 1

Optimal control of an invasion: exponential costs and damages

Result(s)	Sufficient condition	Optimal policy
Proposition 1 and Proposition 2	All sufficiently small invasions are currently controlled and $\beta < (\alpha + \beta) f_x(0)$	There exists a $\bar{y}$ such that eradication is optimal from every invasion of size $y \in (0, \bar{y}]$ .
Proposition 3	$\beta < (\exp(-\beta K) + \beta [\alpha - \beta + \beta \exp(-\beta K)]) f_x(x)$ for all $x \in (0, K]$	Eradication is optimal from every invasion size.
Proposition 4	Every invasion smaller than $f(y)$ is currently controlled, $\beta > \alpha$ , and $\beta \exp(-\beta y) < (\alpha + \beta) \inf_{0 \leq x \leq y} \exp(-\beta f(x)) f_x(x)$	Immediate eradication is optimal for an invasion of size $y$ .
Proposition 5.a	$\beta \exp(-\beta y) > (\alpha + \beta) f_x(0)$	Immediate eradication is not optimal.
Proposition 5.b	$\beta > \alpha$ and $\beta \exp(-\beta y) > (\exp(\beta y) + \beta \sup_{0 \leq x \leq y} [\alpha - \beta \exp(-\beta f(x)) + \beta \exp(-\beta f(x))]) f_x(x)$ or $\alpha > \beta$ and $\beta \exp(-\beta y) > (\exp(\beta y) + \beta \sup_{0 \leq x \leq y} [\exp(-\beta f(x)) f_x(x)])$	It is optimal not to control an invasion of size $y$ .
Proposition 6	$\beta > (\alpha + \beta) f_x(0)$	It is optimal not to control an invasion if it is of sufficiently small size.