Rate of Return Dominance in Walrasian Monetary Equilibrium

Vineet Kumar, Ross M. Starr
University of California, San Diego
La Jolla, CA 92093-0508

January 16, 2004

Abstract

Modification of the Walras-Arrow-Debreu model to include money, market segmentation, and transaction costs displaying scale economies allows the derivation of the use of fiat money as the medium of exchange in monetary trade as an outcome of the general equilibrium. As a consequence of endogenously determined transaction costs, money is held as a stock to transfer purchasing power over short periods of time, despite the yield dominance of bonds; bonds are held over longer periods.

"What has to be explained is the decision to hold assets in the form of barren money, rather than of interest- or profit-yielding securities... This ... is really the central issue in the pure theory of money... we have to look ... frictions in the face ... the most obvious sort of friction ... is the cost of transferring assets from one form to another." — John Hicks, "Suggestion for simplifying the theory of money" (1935)

When paper is substituted in the room of gold and silver money, the quantity of the materials, tools, and maintenance, which the whole circulating capital can supply, may be increased by the whole value of gold and silver which used to be employed in purchasing them. —- Adam Smith, "Wealth of Nations" (1776)

1 Walrasian General Equilibrium of a Monetary Economy

It is well-known that a full Arrow-Debreu general equilibrium model cannot sustain money. The present paper derives money as a store of value and a medium of exchange as an outcome of an Arrow-Debreu equilibrium in a model modified in two respects (minor conceptually, but with powerful consequences): budget constraints apply at each transaction separately not only over the whole course of trade, and transaction costs (transactions are a resource using activity). A monetary equilibrium where both money and interest-bearing bonds are held simultaneously is demonstrated in a sequence economy model with a trading post structure. In an economy with N goods, the $N+1^{st}$ good is government-issued fiat money, sustaining its positive price by acceptability in payment of taxes. The most liquid (lowest transaction cost) asset becomes the common medium of exchange, as a consequence of market equilibrium. Monetary trade using the $N+1^{st}$ good as the unique common medium of exchange, is the outcome of a Walrasian general equilibrium with transaction costs and market segmentation. Scale economies in transaction cost create a natural monopoly, accounting for uniqueness of the $N+1^{st}$ good as the common medium of exchange in equilibrium. Though the rate of return on bonds is higher than on money, bonds are not used as a medium of exchange because of their endogenously determined higher transaction cost in equilibrium.

One of the oldest problems in the theory of money is rate of return dominance. Why is barren money held as a stock when its rate of return is consistently lower than riskless bonds denominated in money? Hicks says this question is "the central issue in the pure theory of money." This paper presents a general equilibrium model where both fiat money (government-issued $N+1^{st}$ good) and money-denominated default-free bonds are traded and held in positive quantity, and where money is a common medium of exchange not by assumption but as the result of market equilibrium.

The specification starts from a Walrasian general equilibrium sequence economy. Transaction costs imply differing bid and ask prices for each good. Households buy and sell at different trading posts, carrying assets between them to transfer purchasing power. The most liquid asset, the instrument that provides liquidity at lowest cost (the narrowest bid/ask spread) will be endogenously selected by optimizing transactors in equilibrium as the medium of exchange. Liquidity is priced: its price is the bid/ask spread. Scale economies in the transaction cost structure imply that a high volume good is likely to carry the lowest transaction cost. In the equilibrium displayed, the $N+1^{st}$ good is the high volume trading instrument. Thus, monetary trade using fiat money is an outcome of market general equilibrium. Money and bonds are used to transfer purchasing power over time. The simultaneous presence of money and of bonds with a higher yield — but higher transaction cost — is part of the equilibrium.

Households live three-period lives. Taxes are payable in money in the same period where the household receives its endowment. A typical household is endowed with one of the N goods in one period of its life and desires a different good, preferring consumption spread out over the three periods. There are three possible timing arrangements in household endowments: some receive endowment in the first period of life only, some in the second, some in the third.

There are N real goods, money, and bonds at each date. For each pair of goods (including money and bonds) there is a distinct trading post where the two goods may be traded for each other. Operation of the trading post is costly, a resource using activity. Each trading post announces bid and ask prices for the goods traded there (denominated as rates of exchange between the goods). The trading post recoups its costs from the bid/ask spread under a zero-profit condition. Each household fulfills a budget constraint at each trading post where it trades (market segmentation).

General equilibrium consists of an array of bid and ask prices at each trading post so that markets clear and the posts cover their costs. Determining which trading posts carry active trade is an outcome of the market equilibrium. Inactive trading posts, those with zero trading volume, post bid and ask prices reflecting their marginal costs. A barter equilibrium would display active trade at most of the $\frac{(N+1)N}{2}$ possible trading posts in each period. A monetary equilibrium with a unique common medium of exchange, money, hosts active trade at only $N+1$ posts, those where money, the $N+1^{st}$ good, is traded against bonds and all other goods.

Using a nonconvex (scale economy) transaction cost model for the trading posts, good $N+1$'s high trading volume leads to low transaction cost, reflected in narrow bid/ask spreads. Households are guided by prices to use good $N+1$ as their medium of exchange. The scale economy leads to a corner solution, a locally unique designation of good $N+1$ as the common medium of exchange, despite the higher yields on bonds. Narrow bid/ask spreads at the monetary trading posts — and wide bid/ask spreads at the (inactive) barter posts and at the (inactive) bonds versus goods posts — reflect transaction costs, and result in monetary trade, concentrating activity only at trading posts where good $N+1$ is traded for all other goods. Money is held as a stock because endogenously determined transaction costs make it the low cost medium of exchange and the low cost store of value over shorter time periods. Bonds with higher yield and higher transaction cost are held as a store of value over longer periods.

Hicks sets us a task in formulating a model of a monetary economy: we must account not only for the use and holding of money, but account as well for its presence in wealth-holder portfolios where it is clearly dominated in yield. At the level of the individual wealth-holder in a monetary economy this notion, following Hicks's emphasis on transaction costs, was formalized by Baumol ( ) and Tobin ( ), and a general equilibrium treatment is provided by Heller and Starr ( ). These treatments assume a monetary economy, with a fiat-money instrument trading at a positive price [1]. Formulating a working model is particularly challenging in an overlapping generations model, Wallace ( ), since those models typically sustain a valued fiat money only when there is no other long lived asset dominating money in yield. Wallace and Zhou ( ) suggest a solution to this quandary: an overlapping generations model with money and bonds where money is held because it is assumed to convey strategic advantage in bargaining (as compared to bargaining with bonds). Then the infinite-horizon overlapping generations formulation sustains an equilibrium with positively valued fiat money and with interest-bearing bonds[2].

The present paper formulates an example of a Walras-Arrow-Debreu general equilibrium model over time where money and bonds are both held[3]. The approach emphasizes transaction costs as Hicks, Baumol, and Tobin suggest. Money is a non-interest bearing durable instrument, entering no utility function or production technology (though it enters transaction technology as a low transaction cost instrument). The uniqueness of money as

[1]This paper will use the term 'fiat money' to describe an instrument that does not enter in any utility function or production technology, but will typically trade at a positive price. Note that some authors, notably Dror Goldberg ( ), tighten the definition further, excluding use for payment of taxes; Prof. Goldberg then argues that historically no pure fiat money with positive value has ever occurred.

[2]Of course there is also a Pareto-inferior non-monetary equilibrium where the value of fiat money is nil.

[3]It is of course well known that an Arrow-Debreu model cannot accomodate money. The model is modified here to allow the budget constraint to apply at each of several transactions (creating a demand for a medium of exchange to carry purchasing power between transactions), and to include transaction costs (creating a demand for a low-transaction cost instrument).

a medium of exchange and its positive equilibrium price are outcomes of the equilibrium, not assumptions. They are derived from more elementary considerations: scale economies in transaction cost and acceptability for tax payments, respectively. That is, fiat money is valued in equilibrium because it is acceptable in payment of taxes [4]. Scale economies in transaction costs mean that high volume trading posts are low transaction cost posts. Low transaction costs are reflected in narrow bid/ask spreads giving trading households the price signal to concentrate their trades on the high volume posts. Thus, monetary trade (with high trading volume at the monetary trading posts) concentrating trade only at monetary trading posts (creating the high volume needed to sustain local uniqueness of money as the medium of exchange) is self enforcing as an equilibrium.

2 Introduction to the Model

The economy in our model is constituted by three sets of entities, namely households, goods and trading posts. The economy is three-period, and households are endowed during each period with one of N *ordinary* goods. In addition to these goods, there is fiat money, represented by money (good $N+1$) and money-denominated bonds represented by good $N+2$. We also have futures goods, where $m(t)$ represents a promise to deliver good m in period t.

Households then trade their endowments with each other through trading posts. Each household has a utility function that it seeks to maximize, subject to its budget constraints.

Each trading post is characterized by the goods it exchanges. Trading Post $\{i, j\}$ will trade good i for good j and vice versa, where i, j are elements of the set of goods. The input to the trading post is good 0, which determines the cost of operating the trading post. The bid/ask spread represents the revenue of the trading post. A trading post and its transaction costs formalize two ideas: quid pro quo payment for purchases at each of several transactions; transaction costs. Transaction costs in a real economy are of course incurred in many places: by firms actually making a market (retailers, wholesalers, brokers) and by firms and households as they expend time and materials in the process of making trades. All of these resource-using activities in the process of trade are here modeled as embodied in the resource using activity of the trading post. Household time, effort, gasoline, shoe-leather, are all modeled as the resources, summarized as inputs of good 0, used by the trading post. Households doing the actual trading don't really see into the workings of the trading post. They see the post's bid (or wholesale) price (for the goods or money the household delivers to the post) and its ask (retail) price. The trading post covers its resource costs in equilibrium by retaining bid/ask spread, the difference between these transaction prices.

The monetary character of trade and the co-existence of money and bonds in equilibrium are intended to be results — not assumptions — of the model. Hence there must be a sufficiently rich array of trading posts that barter, monetary trade, and bonds driving out money, are all logical possibilities. Hence we posit trading posts trading ordinary goods for ordinary goods — barter trading posts, trading goods for money (which is good $N+1$)

[4] This is a notion expounded by Adam Smith (1776) who noted "A prince, who should enact that a certain proportion of his taxes be paid in a paper money of a certain kind, might thereby give a certain value to this paper money" (Wealth of Nations, v. I, book II, ch. 2).

— monetary trading posts, goods for bonds (which is good $N+2$) — trading posts where bonds drive out money, and trading bonds for money — the bond market. That is, the array of trading posts is sufficiently varied that the economy can trade by barter (trading ordinary goods for each other at trading posts dealing in them) or can trade in monetary fashion (trading ordinary goods for money, good N+1). In addition, the model admits the possiblity that households could forego money, trading instead in bonds, if households did not hold 'barren' money, but only bonds. The equilibrium demonstrated is characterized by households holding money, using it as a medium of exchange in each period, as a store of value between adjacent periods, and holding bonds as a store of value between more distant periods. This characterization is a result of the price-guided general equilibrium, not an assumption.

That money acts as a store of value over time, as a consequence, not an assumption of the model is formalized through assuming the availability of futures markets. Following Hahn (1971), the futures markets may endogenously be inactive as an outcome of the equilibrium allocation. We suppose that there are trading posts for trade of spot ordinary goods versus futures contracts (for other ordinary goods), money versus (ordinary) goods futures contracts, and bonds for (ordinary) goods futures contracts. In the equilibria we will investigate, these markets are inactive, since their transaction costs will be high relative to the alternative, monetary trade for spot goods, using money and bonds as the carriers of value over time.

Typically in equilibrium the ordinary goods, 1,2,...,N will not be held over time if there is an acceptable alternative means of moving purchasing power between periods. Without formalizing the issue further, this presumably reflects high storage costs or perishability of the ordinary goods. Goods $N+1$ and $N+2$ are assumed to be durable and costless to store.

2.1 A taxation theory of the value of fiat money

$r+1$ is the typical household endowment of good m. In each period, in each good, let the government tax bill from those endowed with the good in that period equal the proceeds of sale of one unit. Let government spending on that good in that period equal receipts from the good. Household utility functions include an argument for tax payments, including a high marginal utility (>1) for tax payments at or below the level of the tax bill, and a low positive marginal utility for payments above the level of the tax bill.

For a household of type ν, endowed with good i, that is receiving its endowment in period ν , let the tax bill, assessed in money, good $N+1$, be τ. Let government spending on good i equal tax receipts from households endowed with i.

Then markets clear (at least with regard to taxes and government spending) and a price of unity for $N+1$ is market clearing.

2.2 Households

Households live a life of three periods (T_1, T_2, and T_3). Each household receives an endowment of a good in one of the periods. Accordingly, a household is represented as $h_\nu \equiv [m, n, \nu]$, where the household receives an endowment of quantity $r+1$ of good m in period ν, and the household's utility function has a positive value only for good n.

The household sells all of his endowment. In a monetary equilibrium, he sells it for money, good N+1. In addition to valuing n in consumption, the household values paying taxes — presumably to avoid the annoyance of dealing with the tax authorities for non-payment. Taxes payable in money in period ν are levied on household $[m, n, \nu]$ equal to the value of the money the household realizes from the sale of one unit of endowment. Government spends the tax receipts from $[m, n, \nu]$ on good m at date ν. The transactions in m at ν are undertaken at the $\{m, N+1\}$ trading post. This simple story sustains the value of fiat money while keeping the algebra of market clearing simple. The household, in effect, sells 1 unit of the good it is endowed with to the government in return for money (good $N+1$) during the period it receives its endowment. In the same period, the household has to pay tax in exactly the same amount that it received from the government. Thus, in effect, the household is left with an endowment of quantity r to trade with other households. All trade in good m however is impersonal — it goes through trading posts. In a monetary equilibrium, sales of household $[m, n, \nu]$'s good m goes through the $\{m, N+1\}$ trading post.

The consumption of each household is denoted by $X \equiv [X_1, X_2, X_3]$, where each X_t is a vector denoting the consumption of goods $1, 2, ..., N$ in period t.

Each household's utility function is given by

$$u^{[m,n,\nu]}(X) = \sum_{t=1}^{3} \beta^{t-1} \sqrt{X}_{t,n} + \sum_{t=1}^{3} \{\mu_1 \cdot \min[0, Y_t - \tau_t] + \mu_2 \cdot \max[0, Y_t - \tau_t]\}$$

where

$X_{t,n}$ is the consumption of good n during period t by the household

β is the discount factor, corresponding to the rate of interest

$\mu_1 \gg 1$ is the tax underpayment coefficient

$\mu_2 \in (0, \epsilon), \epsilon \ll 1$ is the tax overpayment coefficient

Y_t is the amount of tax paid (in terms of good m) in period t.

$\tau_t = 1$ if $t = \nu$, 0 otherwise.

This form of the utility function would imply that:

(i) household $[m, n, \nu]$ will trade its endowment good $m, (1 \leq m \leq N, m \neq n)$ for good n given positive prices for both goods.

(ii) household $[m, n, \nu]$ would prefer to spread consumption over all three periods rather than concentrate all consumption in one period.

(iii) Each household would pay exactly the amount due in taxes during the period in which it receives its endowment. That the household precisely pays its tax bill reflects the values of μ_1 and μ_2. The former is high enough that all taxes will surely be paid if possible, the latter low enough that they will not be overpaid.

Any of the goods including futures contracts (for goods deliverable in subsequent periods) and bonds can be bought and sold in any period.

The model posits a population of households, each household with an endowment of one good, m, in one period, desiring another good n spread over the three lifetime periods. Each household has an endowment of quantity $r + 1$ of good $m, 1 \leq m \leq N$. We consider both the case of absence and of presence of a double coincidence of wants. It is a consequence of the nonconvex transaction technologies posited that the equilibrium results are similar. In both cases, scale economies focus the trading practices of the economy on monetary trade.

2.3 Households Population

Let $H^\kappa = \{[m, m \oplus i, \nu] | m = 1, 2, ...N; i = 1, 2, ..., \kappa; \nu \in \{1, 2, 3\}\}$ represent a population of households comprising the economy, where $[m, n, \nu]$ is a households that is endowed with good m in period ν and has a positive utility for good n. κ is the population parameter.

We define the relation $\oplus$ as follows:

$$m \oplus k = m + k, m + k \leq N$$

$$m \oplus k = m + k - N, m + k > N$$

We have three possible cases to deal with, based on the value of the population parameter, κ.

Case 1: If $2 \leq \kappa < \frac{N}{2}$, H^κ represents a set of κN households without double coincidence of wants.

One way to visualize H^κ's situation is to think of the households arrayed in a circle clockwise, each household's position designated by endowment. They can arrange a Pareto-improving redistribution by each taking his endowment and sending it i places counterclockwise. However, reflecting the absence of double coincidence of wants, if each of the households in H^κ goes to the trading post where his endowment is traded against his desired good, he finds himself alone and is dealing on a thin market.

Case 2: When $\kappa = N - 1$, on the other hand, it represents a population of households with double coincidence of wants. For each pair of goods $\{i, j\}$, we have a subpopulation of households $H_i^{N-1} = \{[i, j, \nu], k = 1, \ldots, N \text{ and } k \neq i\}$ that are endowed with good i in some period and that have a positive demand for good j, as well as a complementary subpopulation H_j^{N-1}, similarly defined, each of whose elements is endowed with j in some period.

Case 3: When $N/2 \leq \kappa < N - 1$, we have a population that has a double coincidence of wants for some pairs of goods, but other pairs of goods do not have such a double coincidence. As we consider market pricing, those households experiencing a double coincidence would find the prices on the post where they are likely to trade table 3 (the table for those trading posts with a double coincidence of wants) and those households not experiencing double coincidence would find the prices for the posts where they are likely to trade in table 1 (the table for those trading posts without double coincidence).

2.4 Trading Posts

The function of a trading post is to facilitate trade between a pair of goods. A trading post denoted as $\{i, j\}$ would buy as well as sell goods i and j. A household that wants to exchange good i for good j or vice-versa could typically come to the $\{i, j\}$ trading post and make the exchange at prevailing prices.

Each trading post incurs costs to maintain the operation of the trading post. To keep the accounting and market clearing conditions simple, we assume that good 0 is the only input required to ensure operation of any trading post. It is simplest to interpret good 0 as the labor required to run the post available a price of unity in exchange for either good $N + 1$ or the other good traded at the post. Alternatively, the operator of the trading post may be endowed with good 0 supplying it willingly at a ratio of 1 to 1 for the goods traded at the post.

Prices are quoted as a rate of exchange between the two goods traded at the post. $q_i^{\{i,j\}}$ is the bid price of good i (expressed in units of good j) at the trading post where they are traded. By definition, the ask price of j, $p_j^{\{i,j\}}$ is the inverse of the bid price of i; $p_j^{\{i,j\}} = \frac{1}{\mathrm{q_i^{\{i,j\}}}}$. The trading post buys at the bid price and sells at the ask price of the goods it trades in, and covers its operating costs by the difference between the bid and ask prices. This is indicated in the transaction cost structure specified below. In equilibrium, we assume a market among the trading posts that results in zero profits for each of the posts.

We assume the following transaction cost structure on the goods (ordinary, money, and bonds) for pairwise trading occurring at trading posts. Each trading post has costs to operate it, and this cost is specified to accrue in good 0. We argued that the uniqueness of money as a medium of exchange comes from scale economies in transaction cost — once money is the high volume good, it is also the low cost good. The low transaction cost is reflected in narrow bid/ask spreads at monetary trading posts, inducing trade to concentrate on the monetary posts, hence confirming the low cost and narrow bid/ask spread. We formalize the notion of scale economy in transaction cost in the following way. At low trading volumes, the cost in terms of good 0 of trading in good i is linear in trading volume at a unit cost of δ^i. As trading volume increases, transaction costs hit a ceiling of γ^i. This is the scale economy; declining marginal costs, declining average costs after a threshold.

In the presence of scale economies, marginal cost pricing will not lead to a meaningful market equilibrium. We assume instead that the trading posts practice average cost pricing and we seek an average cost pricing market clearing equilibrium. Since the trading post represents a mix of actual business activities (retailers, wholesalers, etc.) and non-marketed activities (personal efforts, unpriced search time, ...) this pricing structure is thought to be an appropriate representation of the mix of market and non-market resources going into a trade.

We use the following notation to denote the volume of goods bought and sold by the trading post in a given time period t:

$y_{t,i}^{\{i,j\}B}$ is the quantity of good i bought by the trading post during period t

$y_{t,i}^{\{i,j\}S}$ is the quantity of good i sold by the trading post during period t

$y_{t,0}^{\{i,j\}S}$ is the quantity of good 0 bought by the trading post during period t to cover its operating costs

So, the cost of operating the trading post $\{i,j\}$ under these trade volumes is:
$C_t^{\{i,j\}} = y_{t,0}^{\{i,j\}B} = \min[\delta^i y_{t,i}^{\{i,j\}B}, \gamma^i] + \min[\delta^j y_{t,j}^{\{i,j\}B}, \gamma^j]$

This cost function reflects precisely the scale economy discussed above: linear unit costs at the rate δ^i at low volume of trade in i, a ceiling of γ^i on costs at high volume. The cost of operating the trading posts is recovered from the trades made at the posts, and this is built into the difference between the prevailing bid and ask prices at the trading post.

The price structure posited below prices exchange at trading posts including all possible combinations of goods traded: spot versus futures, futures versus futures, money versus spot, money versus futures, money versus bonds, bonds versus spot and futures. These are market clearing equilibrium prices. However most markets are inactive: this is a corner solution, most trading posts produce no trade in equilibrium. In a monetary equilibrium the active trading posts are those trading money for spot goods. Other posts could host trade — they are priced so that the post would willingly host active trade — but buyers and sellers respond to these unattractive prices — wide bid/ask spreads — by trading elsewhere, leaving the non-monetary posts inactive.

3 Equilibrium Price Structure

Consider the following set of prices. We prove that, at these prices, the trading posts for trade in pairs of money, bonds, ordinary goods $1, 2, \ldots, N$, and their futures markets clear. Most of the trading posts operate at zero volume; the active posts we concentrate on are the posts for money versus ordinary goods and money versus bonds. Nevertheless, all the trading posts are priced at prices reflecting prevailing costs (an average cost pricing equilibrium) and the trades clear. Thus, we prove that these prices are sustainable in equilibrium. This is not necessarily the only possible equilibrium. We only claim to demonstrate one such example where prices ensure a sustainable equilibrium.

Table 1: Equilibrium prices at the $\{i,j\}(j = i \oplus l, 1 \le l \le \kappa)$ trading post for the case $1 \le \kappa < \frac{N}{2}$

Bid Price	Ask Price
$q_i^{\{i,j\}} = \frac{1-\delta^i}{1+\delta^j}$	$p_i^{\{i,j\}} = 1$
$q_j^{\{i,j\}} = 1$	$p_j^{\{i,j\}} = \frac{1+\delta^j}{1-\delta^i}$

Denoting q as the bid price and p as the ask price at the transaction posts that the goods are traded for, we have the price structure is given by tables 1, 5, 6 and 3. This pricing structure is valid for all periods.

The price structure at the trading posts comes about as the result of the transaction costs of operating the trading posts. The monetary trading posts have lower transaction costs at higher trading volumes (economies of scale) and thus can post lower differences between

Table 2: Equilibrium prices during period $t(T > t)$ at the $\{i, j(T)\}(j = i \oplus l, 1 \leq l \leq \kappa)$ trading post for the case $1 \leq \kappa < \frac{N}{2}$

Bid Price	Ask Price
$q_{t,i}^{\{i,j(T)\}} = \frac{1-\delta^i}{1+\delta^j}$	$p_{t,i}^{\{i,j(T)\}} = 1$
$q_{t,j(T)}^{\{i,j\}} = 1$	$p_{t,j(T)}^{\{i,j(T)\}} = \frac{1+\delta^j}{1-\delta^i}$

Table 3: Equilibrium price at the $\{i, j\}(j = i \oplus l, 1 \leq l \leq \kappa)$ trading post for the case $\kappa = N - 1$

Bid Price	Ask Price
$q_i^{\{i,j\}} = 1 - \delta^i$	$p_i^{\{i,j\}} = \frac{1}{1-\delta^i}$

Table 4: Equilibrium price during period $t(T > t)$at the $\{i, j(T)\}$ $(j = i \oplus l, 1 \leq l \leq \kappa)$ trading post (for the case $\kappa = N - 1$)

Bid Price	Ask Price
$q_{t,i}^{\{i,j(T)\}} = 1 - \delta^i$	$p_i^{\{i,j(T)\}} = \frac{1}{1-\delta^i}$

Table 5: Equilibrium price for goods i and $N + 1$ at the $\{i, N + 1\}$ trading post

Bid Price	Ask Price
$q_i^{\{i,N+1\}} = max(\frac{1-\delta^i}{1+\delta^{N+1}}, 1 - \frac{\gamma^i+\gamma^{N+1}}{\kappa\cdot(r+1)})$	$p_i^{\{i,N+1\}} = 1$
$q_{N+1}^{\{i,N+1\}} = 1$	$p_{N+1}^{\{i,N+1\}} = min(\frac{1+\delta^{N+1}}{1-\delta^i}, \frac{1}{1-\frac{\gamma^i+\gamma^{N+1}}{\kappa\cdot(r+1)}})$

Table 6: Equilibrium price of bonds at the {N+1, N+2} trading post

Period	Bid Price	Ask Price
T1	α	θ
T2	α	1
T3	1	1

bid and ask prices. This makes the monetary posts more attractive to transactors and thus leads to further increase in the volume of trade through the monetary trading posts, thus confirming the lower prices. The prices at the monetary trading posts are given by table 5.

The barter markets (trading posts $\{i, j\}, 1 \leq i, j \leq N$) have a higher posted price difference between bid and ask prices, which are listed in table 1, when the population parameter is described by $1 < \kappa < \frac{N}{2}$, and by table 3 when the population parameter is $\kappa = N - 1$. . This results in a low trading volume, and this confirms the higher prices until we find that all the trade takes place only through the monetary trading posts (trading posts $\{i, N + 1\}, 1 \leq i \leq N$).

Table 7: Equilibrium prices at the $\{i, N+2\}$ trading post at period t

Bid Price	Ask Price
$q_{t,i}^{\{i,N+2\}} = \frac{1-\delta^i}{1+\delta^{t,N+2}}$	$p_{t,i}^{\{i,N+2\}} = 1$
$q_{t,N+2}^{\{i,N+2\}} = 1$	$p_{t,N+2}^{\{i,N+2\}} = \frac{1+\delta^{t,N+2}}{1-\delta^i}$

4 Households' Response to Equilibrium Prices

We now consider how the households respond to the posted equilibrium prices. The households face choices of trades to make and will pick those choices that maximize their utility over their three-period lives.

During each period, household $h_\nu \equiv [m, n, \nu]$ can make the following trades:

1) good m for good n (barter with preferred good)

2) good m for good $N+1$ (monetary trade)

3) good m for bonds (good $N+2$) (barter with bonds)

4) good m for good i where $i \notin \{n, N+1, N+2\}$ (barter with intermediate good)

5) good $N+1$ for bonds (good $N+2$) (monetary trade with bonds)

6) bonds (good $N+2$) for good $N+1$ (monetary trade with bonds)

7) bonds (good $N+2$) for good n (barter with bonds)

8) good $N+1$ for good n (monetary trade)

9) good $i \notin \{n, N+1, N+2\}$ with any of the goods in $\{n, N+1, N+2\}$ (barter intermediate goods)

At the posted equilibrium prices, money is used as the medium of exchange and has low transaction cost due to the high volume of trade and rules out barter when the following condition holds:

$\frac{1-\delta^i}{1+\delta^j} < 1 - \frac{\gamma^i+\gamma^{N+1}}{\kappa\cdot(r+1)} \quad \forall i, j \in 1, 2, \ldots, N$

This occurs when κ is sufficiently large or when the γ values are small compared to the δ values, the households have large quantities of endowments to trade or some combination of the above.

Also, at the posted equilibrium prices, if the following condition holds for all time periods t, we can rule out trade at the bond-good trading post:

$\frac{1-\frac{\gamma_i+\gamma_{N+1}}{\kappa\cdot(r+1)}}{\theta} > \frac{1-\delta^i}{1+\delta^{t,N+2}}$

Give that two conditions specified above hold, we make the following observations:

(a) Given goods i and j, using money (good $N+1$) as a medium of exchange is preferable to direct barter.This is because such a transaction is less costly and therefore provides a higher return to the household. This holds true even if there is a double coincidence of wants. This rules out trades of type (1).

(b) Similar reasoning shows that with bonds, trading bonds (good $N+2$) with good j would be less expensive if done through the monetary markets.
This rules out trades of type (3) and (7).

(c) Trading an unwanted good m for another unwanted good i is definitely *not* preferable to trading m with money (good $N+1$), using the $\{m, N+1\}$ trading posts, which are less expensive and post lower differences between the bid and ask prices.
This rules out trades of type (4).
Since a household does not make trade type (4), by symmetry it cannot have transaction of type (9).

(d) Consider a household that has an endowment in period ν. For purchases that such a household wishes to make in period $\nu+1$, it will always carry money over from period ν to period $\nu+1$.

Consider the alternatives. We have already proven that given these prices, trade through the monetary route is better for households than any kind of barter. The alternative is to buy bonds in period ν and sell them in period $\nu+1$ to finance consumption in the latter period. If $\nu=1$, this yields $(\alpha-\theta)$, and if $\nu=2$, this yields 0. These are the only possibilities.

In either case, this yield is negative or zero (since $\alpha<\theta$), and so the household will prefer to hold money over one period to finance such purchases. In the case with zero yield for bonds, we posit that the household is indifferent between buying bonds and holding money, it will not buy bonds.

(e) Consider holding money over two periods. If the household instead bought bonds in period ν and sold them in period $\nu+2$, it would yield $1-\theta$, which is a positive value. In our three-period model, this can only happen when $\nu=1$. Carrying money over from period 1 to period 3 on the other hand yields 0. So, the household will prefer to hold bonds to finance consumption two periods in the future.

(f) Finally, we wish to point out that the households that receive endowments in periods 2 and 3 voluntarily sell bonds in the first period to finance consumption in that period, because that decision is the optimal one to make, from their point of view.

Our monetary equilibrium has the property that only the trades represented above by types (2), (5), (6) and (8) occur. This happens because, according to the posted prices, trading using money as a medium of exchange has lower transaction costs than trading by barter. Also, we have proven that trade types 1, 3, 4 and 7 will *not* occur in our monetary equilibrium.

There may be other non-monetary equilibria where these trades occur. It is easy to verify that trade type 1 will be the only active type in a barter equilibrium.

We consider in turn, the maximizing behavior of each of the three types of households as a response to the posted prices given above. We consider a household $h_\nu \equiv [m, n, \nu]$.

$X_{t,i}$ denotes the quantity of good i consumed in period t

$b_{t,i}$ denotes the quantity of good i bought in period t

$s_{t,i}$ denotes the quantity of good i sold in period t

$M_{t,v}$ denotes the quantity of money (good $N+1$) held between periods t and v.

We also note that a household's consumption in each period would then be equal to the quantity of good n it buys in that period. It would not wait to consume until a later period, since the utility function is discounted for later periods. Therefore, for all types of households and for $t = \{1, 2, 3\}$, we have
$X_{t,n} = b_{t,n}$

Each household of type $[m, n, \nu]$ is endowed with quantity $r + 1$ of good m in period ν. After paying a tax equivalent to 1 unit of good m, the household is left with effectively r units of endowment. Having paid its tax, its optimization is to arrange its trades to maximize the non-tax portion of its utility function (financed by the remaining endowment or the proceeds of the sale of the remaining endowent).

4.1 Type-1 household

A type-1 household $h_1 \equiv [m, n, 1]$ is endowed with a quantity r of good m in period 1. Intuitively, we know that h_1 would want to trade good m for good n, given its preferences.

$h^{[m,n,1]}$ would trade its endowment of good m, for money (good $N+1$) in the first period.
$q_m \cdot (r + 1) = b_{1,N+1}^{[m,n,1]}$

Some of money (good $N+1$) is traded at the $n, N+1$ trading post for good n in period 1. $h^{[m,n,1]}$ also buys some quantity of bonds (good $N+2$) (bonds) in the first period, at the ask price of θ with the intention of selling it in period 3 at a higher price, and use the proceeds to buy good n in period 3. The remaining quantity of money (good $N+1$) is carried over to period 2 ($M_{1,2}^{[m,n,1]}$).

$$b_{1,N+1}^{[m,n,1]} = 1 \cdot b_{1,n}^{[m,n,1]} + \theta \cdot b_{1,N+2}^{[m,n,1]} + M_{1,2}^{[m,n,1]}$$

In period 2, $h^{[m,n,1]}$ does not receive any endowment. It can, however, trade the quantity of money (good $N+1$) carried over from period 1, $M_{1,2}^{[m,n,1]}$ in exchange for good n.

$$M_{1,2}^{[m,n,1]} = 1 \cdot X_{2,n}^{[m,n,1]}$$

In period 3, $h^{[m,n,1]}$ would trade the bonds (good $N+2$) it bought in period 1 with money (good $N+1$), and then trade money (good $N+1$) with good n. It can sell at most the same quantity of bonds (good $N+2$) as was bought in period 1.
$s_{3,N+2}^{[m,n,1]} = b_{1,N+1}^{[m,n,1]}$
$s_{3,N+1}^{[m,n,1]} = 1 \cdot b_{3,n}^{[m,n,1]}$
Subject to these constraints, we consider the behavior of our type-1 household, which maximizes its utility

$U^{[m,n,1]}(X_{1,n}^{[m,n,1]}, X_{2,n}^{[m,n,1]}, X_{3,n}^{[m,n,1]}) = \sqrt{X_{1,n}^{[m,n,1]}} + \beta \cdot \sqrt{X_{2,n}^{[m,n,1]}} + \beta^2 \cdot \sqrt{X_{3,n}^{[m,n,1]}}$.

Solving this, we get:

$$X_{1,n}^{[m,n,1]} = \frac{q_m \cdot (r+1)\theta}{\theta + \theta\beta^2 + \beta^4}$$
$$X_{2,n}^{[m,n,1]} = \frac{q_m \cdot (r+1)\beta^2\theta}{\theta + \theta\beta^2 + \beta^4}$$
$$X_{3,n}^{[m,n,1]} = \frac{q_m \cdot (r+1)\beta^4}{\theta(\theta + \theta\beta^2 + \beta^4)}$$

The quantity of bonds (good $N+2$) that household $h^{[m,n,1]}$ buys in period 1 and subsequently sells in period 3 is given by
$b_{1,N+2}^{[m,n,1]} = \frac{q_m \cdot (r+1)\beta^4}{\theta(\theta+\theta\beta^2+\beta^4)}$

4.2 Type-2 household

A type-2 household $h^{[m,n,2]} \equiv [m,n,2]$ is endowed with a quantity r of good m in period 2.

Such a household would sell bonds (good $N+2$) in period 1 to finance its purchase of good n. $h^{[m,n,2]}$ would trade bonds (good $N+2$) for money (good $N+1$), and then trade all of its money (good $N+1$) for good n in period 1.
$\alpha \cdot s_{1,N+2}^{[m,n,2]} = b_{1,N+1}^{[m,n,2]}$
$b_{1,N+1}^{[m,n,2]} = 1 \cdot b_{1,n}^{[m,n,2]}$

In the second period, $h^{[m,n,2]}$ receives its endowment in terms of good m, and so it trades the good m it receives for money (good $N+1$), and then trades part of that received quantity of money (good $N+1$) for good n, and carries over some of money (good $N+1$) to the third period ($M_{2,3}^{[m,n,2]}$).
$q_m \cdot (r+1) = b_{2,N+1}^{[m,n,2]}$
$b_{2,N+1}^{[m,n,2]} = M_{2,3}^{[m,n,2]} + 1 \cdot b_{2,n}^{[m,n,2]}$

In the third period, $h^{[m,n,2]}$ is required to buy back the quantity of bonds (good $N+2$) that it had sold in period 1. In addition, $h^{[m,n,2]}$ trades part of money (good $N+1$) that was carried over to buy good n.
$M_{2,3}^{[m,n,2]} = b_{3,N+2}^{[m,n,2]} + b_{3,n}^{[m,n,2]}$
$b_{3,N+2}^{[m,n,2]} = s_{1,N+2}^{[m,n,2]}$

Subject to these constraints, our type-2 household maximizes its utility, or
$U^{[m,n,2]}(X_{1,n}^{[m,n,2]}, X_{2,n}^{[m,n,2]}, X_{3,n}^{[m,n,2]}) = \sqrt{X_{1,n}^{[m,n,2]}} + \beta \cdot \sqrt{X_{2,n}^{[m,n,2]}} + \beta^2 \cdot \sqrt{X_{3,n}^{[m,n,2]}}$.

Solving this, we get:

$$X_{1,n}^{[m,n,2]} = \frac{q_m \cdot (r+1)\alpha}{\alpha + \beta^2 + \beta^4}$$
$$X_{2,n}^{[m,n,2]} = \frac{q_m \cdot (r+1)\beta^2}{\alpha(\alpha + \beta^2 + \beta^4)}$$

$X_{3,n}^{[m,n,2]} = \frac{q_m \cdot (r+1)\beta^4}{\alpha(\alpha+\beta^2+\beta^4)}$

The quantity of bonds (good $N+2$) that the household $h^{[m,n,2]}$ sells in period 1 is $s_{1,N+2}^{[m,n,2]} = \frac{q_m \cdot (r+1)}{\alpha+\beta^2+\beta^4}$

4.3 Type-3 household

A type-3 household $h^{[m,n,3]} \equiv [m,n,3]$ is endowed with good m in period 3.

In the first period, as was the case with $h^{[m,n,2]}$, household $h^{[m,n,3]}$ would sell bonds (good $N+2$) to finance its purchases. But it would need to sell a larger quantity of bonds (good $N+2$) as compared to $h^{[m,n,2]}$, because it receives its endowment later than $h^{[m,n,2]}$ does.

$\alpha \cdot s_{1,N+2}^{[m,n,3]} = b_{1,N+1}^{[m,n,3]}$

Some of this quantity of money is carried over to later periods, and part of it is used to buy good n in period 1.

$b_{1,N+1}^{[m,n,3]} = M_{1,2}^{[m,n,3]} + b_{1,n}^{[m,n,3]}$

Household $h^{[m,n,3]}$ needs to carry money only to finance purchases in period 2, and not in period 3, since it receives its endowment in period 3. So, all the money carried over from period 1 to period 2 is used to buy good n in period 2.

$M_{1,2}^{[m,n,3]} = X_{2,n}^{[m,n,3]}$

In period 3, household $h^{[m,n,3]}$ receives an endowment of good m in the quantity r. This is exchanged for money (good $N+1$), which is used to buy back the quantity of bonds (good $N+2$) that $h^{[m,n,3]}$ sold in period 1, and also buy good n in period 3.

$q_m \cdot (r+1) = b_{3,N+1}^{[m,n,3]}$

$b_{3,N+2}^{[m,n,3]} = b_{3,N+2}^{[m,n,3]} + b_{3,n}^{[m,n,3]}$

Since $h^{[m,n,3]}$ needs to buy back the quantity of bonds (good $N+2$) in period 3 that it sold in period 1, we have

$b_{3,N+2}^{[m,n,3]} = s_{1,N+2}^{[m,n,3]}$

Subject to these constraints, our type-2 household maximizes its utility, or

$U^{[m,n,3]}(X_{1,n}^{[m,n,3]}, X_{2,n}^{[m,n,3]}, X_{3,n}^{[m,n,3]}) = \sqrt{X_{1,n}^{[m,n,3]}} + \beta \cdot \sqrt{X_{2,n}^{[m,n,3]}} + \beta^2 \cdot \sqrt{X_{3,n}^{[m,n,3]}}$.

Solving this, we get:

$X_{1,n}^{[m,n,3]} = \frac{q_m \cdot (r+1)\alpha^2}{\alpha+\alpha\beta^2+\beta^4}$

$X_{2,n}^{[m,n,3]} = \frac{q_m \cdot (r+1)\alpha^2\beta^2}{\alpha+\alpha\beta^2+\beta^4}$

$X_{3,n}^{[m,n,3]} = \frac{q_m \cdot (r+1)\beta^4}{\alpha+\alpha\beta^2+\beta^4}$

The quantity of bonds (good $N+2$) that household $h^{[m,n,3]}$ sells in period 1 and buys back in period 3 is given by

$s_{1,N+2}^{[m,n,3]} = \frac{q_m \cdot (r+1)\alpha^2(1+\beta^2)}{\alpha+\alpha\beta^2+\beta^4}$

4.4 Market Clearing for bonds (good $N+2$)

The trade in bonds (good $N+2$) takes place in periods 1 and 3. In period 1, type-1 households buy bonds (good $N+2$) and type-2 and type-3 households sell them. In period 3, the roles are reversed, and type-2 and type-3 households buy bonds (good $N+2$), while type-1 households sell them.

We only have to prove market-clearing for bonds (good $N+2$) in period 1 and that would imply that the market for bonds (good $N+2$) clears in period 3 as well. This follows from the requirement that each household must buy and sell bonds (good $N+2$) in equal quantities over their three-period lives.

The condition for market clearing in period 1 for the bond market would be given by $b_{1,N+2}^{[m,n,1]} = s_{1,N+2}^{[m,n,2]} + s_{1,N+2}^{[m,n,3]}$

This would imply that

$\frac{\beta^4}{\theta(\theta+\theta\beta^2+\beta^4)} = \frac{1}{\alpha+\beta^2+\beta^4} + \frac{\alpha^2(1+\beta^2)}{\alpha+\alpha\beta^2+\beta^4}$

4.5 Market Clearing for goods $1, ...N$

For each of the goods 1 to N, for market clearing we require that the demand match the supply at the equilibrium prices. The market clearing conditions for goods for each period are as follows:

Consider the market for good k in period t. We have aggregate demand for good k in period t being
$D_{t,k} = \sum_{i=1,i\neq k}^{i=N} X_{t,k}^{[i,k,1]} + X_{t,k}^{[i,k,2]} + X_{t,k}^{[i,k,3]}$
The supply for good k in period 1 is:
$S_{t,k} = \sum_{i=1,i\neq k}^{i=N} r_k^{[k,i,t]}$
For market clearing during period 1 in good k, we have the following condition:
$D_{t,k} = S_{t,k}$

For each period t, we have market clearing, which gives us the above equation for demand and supply. Thus, we have three equations. We have four unknowns, α, β, θ and q_i, which is the price of good i.

We also have the equation for market clearing in the bond (good N+2) market, and so we have four equations and four unknowns. We can solve this to get the values of the parameters α, β and θ which would result in a market equilibrium.

4.6 Market clearing for money (good $N+1$)

We have markets in $N+2$ goods. We have demonstrated market clearing in $N+1$ markets. Therefore, we must have market clearing also in the money market(good $N+1$).

5 Conclusion

Part of the agenda Gerard Debreu set for general equilibrium theory in Theory of Value (1959) is an "important and difficult question... not answered by the approach taken here: the integration of money in the theory of value..." Modifying the Walras-Arrow-Debreu model to include fiat money, market segmentation, and transaction costs displaying scale economies allows derivation of the use of fiat money as the unique medium of exchange in monetary trade as an outcome of the general equilibrium. The existence and use of a common medium of exchange is the result of a fully articulated price system, not an additional assumption. As a consequence of endogenously determined transaction costs priced in bid and ask prices, money is held as a stock to transfer purchasing power over short periods of time, despite the yield dominance of bonds; bonds are held over longer periods. The model allows simultaneous existence in equilibrium of positive holdings of money and of higher yielding money-denominated bonds.

6 References

Baumol, W. (1952)
Debreu, G. (1959)
Hahn, F. (1971)
Heller, W. P. and R. M. Starr (1976)
Hicks, J. (1935)
Tobin, J. (1956)
Wallace, N. (1981)
Wallace, N. and Zhou (2003)