Interdependent Value Auctions with Insider Bidders[*]

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November 2003

Abstract

I study auctions in which there exists an asymmetry in bidders' knowledge about their interdependent valuations. Bidders consist of two groups: the *insiders*, who are perfectly informed of their valuations, and the *outsiders*, who only observe one-dimensional signals, thus being partially informed of their valuations. When only one insider and one outsider exist, both English and second-price auctions allocate the good efficiently while a first-price auction fails to do so. When there are more than two bidders with at least one insider and one outsider, a second-price auction is no longer efficient. By contrast, under plausible conditions on bidders' valuations, an English auction implements the efficient allocation. I also study the revenue implication of the knowledge asymmetry, showing that the revenue generated from an English auction increases when an outsider is replaced by an insider with the same valuation (i.e., as more bidders become informed of their valuations). My results apply to a class of auctions in which bidders' valuations consist of common and private value components, with some bidders informed of the common component and others uninformed of it.

[*]I am grateful to Yeon-Koo Che, my advisor, for his continuing support and helpful comments and thank Phil Haile and Larry Samuelson for their helpful comments. My thanks also go to Phil Reny and Bill Sandholm for their kind suggestions. All remaining errors are mine.

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1 Introduction

In many auctions, some bidders have better knowledge than others about the value of the auctioned object. In the auctions of artworks, the experts are presumably better at appraising the value of those works than laymen. In a takeover auction, the bidders who own toeholds (existing shares of the target firm) may have the firm's inside information not available to others, and thus have better knowledge about the value of the target firm.[1] This knowledge asymmetry seems more pronounced when a current management team of the target firm participates in the bidding competition (management buyouts or MBO).[2] Another example of the knowledge asymmetry is the sale of drainage tracts in auctions for gas and oil leases (OCS auctions). Hendricks and Porter (1988) find that the firms who own neighboring tracts are better informed of the value of a lease than non-neighborhood firms. This paper presents a simple model of knowledge asymmetry in which the efficiency and revenue performance of the standard auctions such as first-price, second-price, and English auctions is explored. In my model, bidders consist of two types: the *insiders* who are perfectly informed of their valuations, and *outsiders* who observe one-dimensional signals containing only partial information about their valuations.

As is plausible with above examples, I assume that bidders' valuations are interdependent, which means that an outsider must infer other outsiders' and insiders' information to assess his valuation.[3] The standard information case without knowledge asymmetry is a special case of this model in which every bidder is an outsider. An important concern is whether standard auctions can allocate the object efficiently. There has been a growing interest in the allocative efficiency of auctions with auctions being increasingly used as a method of selling assets owned by the governments, whose primary objective is efficient allocation. Many of these auctions have interdependent value feature since, for instance, bidders in a spectrum right auction have only partial information about the future demand of a given PCS market, which is common to all bidders.

With interdependent values, the efficiency is not readily attainable even in the standard information case. To achieve efficiency, an auction format must provide bidders with incentives to reveal their information, thus enabling other bidders to assess their valuations accurately and to adjust their bidding strategies properly based on the revealed information.

[1] See Bulow, Huang, and Klemperer (1999) for an analysis of toeholders in auctions.

[2] Shleifer and Vishny (1988, p.96) argue that managers' special information about their companies is one reason for MBO's.

[3] In art auctions, for example, the appraisal of an auctioned work by experts helps laymen estimate its values to them.

What is crucial for the inference is the relationship between the bidders' equilibrium bidding behavior and underlying signals. In most existing models, the *precise* inference about signals is possible due to the assumption of one-dimensional signals, which can be revealed accurately if bidders adopt monotone bidding strategies. In my model, the presence of insiders poses a nontrivial inference problem since they essentially know the multidimensional profile of signal vectors, whose inference is not guaranteed by the monotonicity of bidding strategies. To illustrate, suppose that there are three bidders whose valuations are given by $v_i(s_1, s_2, s_3)$, $i = 1, 2, 3$. Assume that bidder 3 is an insider who knows the entire signal vector while an outsider $i = 1$ or 2 knows s_i only.[4] Consider an ascending-bid English auction in which the insider employs the weak dominant strategy of dropping out at his valuation v_3. The signals, s_1 and s_2, can be easily inferred from outsiders' drop-out prices, given that they are monotone with their one-dimensional signals. On the other hand, there is no guarantee that the insiders' signal, s_3, can be precisely inferred from his bidding behavior since v_3 is affected by the entire profile of signals. For instance, a higher value of v_3 does not necessarily mean a higher s_3. The focus of the current paper is whether a standard auction admits equilibrium strategies that overcome the inference problem caused by this multidimensionality of insiders' information.

The main finding of the current paper is that the English auction can implement the efficient allocation, given plausible restrictions on the valuation functions whereas the sealed-bid auctions, such as first-price or second-price, cannot. As mentioned above, it is important whether a given auction format allows outsiders to learn a sufficient amount of others' information and adjust their bidding strategies accordingly. The sealed-bid auctions, by their very nature, provide no new information in the bidding stage. By contrast, English auctions can induce bidders to reveal their information through the prices at which they drop out of the bidding competition. As is the case in the existing literature, the outsiders' signals are revealed through their adoption of monotone strategies. But, revealing insiders' signals requires more sophisticated inference procedure, exploiting the fact that the insiders' drop-out prices are equal to their valuations and thus convey a different kind of information from outsiders' drop-out prices. Unlike outsiders' signals which are revealed completely at the time they drop out, insiders' signals are never fully revealed in my construction, but rather updated continuously even after they drop out of the auction.

This paper is organized as follows. Section 2 introduces the model. In Section 3,

[4]In fact, to know the entire profile of signals is more than necessary for an insider to know his valuation. While my model is more general and encompasses this case, I take this case to highlight the multidimensional property of insiders' bidding behavior.

I analyze the standard auctions for the case of two bidders. Given the single-crossing property, the second-price auction, which is equivalent to the English auction in this case, can implement an efficient allocation while the first-price auction cannot do so in general.[5] Section 4 which is the main part of the current paper studies second-price and English auctions in the case of more than two bidders. It is first observed that the second-price auction is no longer efficient. Then, I construct an ex-post equilibrium of the English auction which implements an efficient allocation, provided that the valuation functions satisfy a condition called *single crossing on indifference curves* (SCIC). I also provide a sufficient condition which guarantees both SCIC and the existence of equilibrium.

Section 5 provides a revenue implication of the knowledge asymmetry by studying how the presence of insiders affects the revenue in English auctions. Specifically, I find that the revenue rises as an outsider is replaced by an insider with the same valuation, that is one more bidder becomes informed of his valuation. This result is similar in spirit to Milgrom and Weber (1982)'s linkage principle, but there is an important difference. In the current exercise, only one more bidder enjoys the better information, unlike Milgrom and Weber's case in which the additional information is made public to all bidders.[6] Also, the asymmetry in bidding strategies requires a completely different argument to establish the result, and my result does not depend on the affiliation among signals but holds even when the signals are independent. Furthermore, the revenue enhancement holds ex post, not merely ex ante as in Milgrom and Weber (1982). In Section 6, I show that all the previous results about efficiency and revenue hold in a class of auctions in which valuations are additively separable into the private and common value components, and insiders are informed of both common and private components while outsiders are only informed of private component.

The current paper is related to two branches of literature. Engelbrecht-Wiggans, Milgrom, and Weber (1982) study knowledge asymmetry in a special pure common value first-price auction model in which a single insider has proprietary information and other bidders have a public, or equivalently, no information. Hendricks and Porter (1988) and Hendricks, Porter, and Wilson (1994) apply and extend their analysis to the case of the drainage tracts in auctions for oil and gas leases. No efficiency issue arises in the pure common value case, however. By contrast, the interdependent value framework of the current study permits a meaningful study of efficiency. Further, I explores the revenue implication of knowledge asymmetry.

The second branch of literature concerns the allocative efficiency in the interdepen-

[5]Kim and Che (2002) show that first-price auction cannot implement the efficient allocation even in the private value case, when bidders have an asymmetric knowledge about rivals' types.

[6]For a treatment of the linkage principle, refer to Krishna (2002).

dent value environment without the knowledge asymmetry. Ausubel (1999), Dasgupta and Maskin (2000), and Perry and Reny (2002) construct abstract mechanisms which implement efficient allocation under a fairly weak condition, the single crossing condition. Chung and Ely (2002) and Mey-ter Vehn and Moldovanu (2002) provide the characterizations of ex-post implementable mechanisms in the interdependent value setup. Krishna (2001) is most closely related to this paper. He constructs an efficient ex-post equilibrium for English auctions in the standard information case, and also provides a sufficient condition, called the average crossing condition, that allows for such an equilibrium to exist.[7] In Krishna's equilibrium, a bidder drops out of an auction as soon as his *break-even* signal, which is calculated at each price level and strictly increases with it, equals his true one. Thus, each bidder's signal is precisely revealed as he drops out. While the same is true for outsiders' signals in my model, the inference of insiders' signals required a nontrivial change of outsiders' strategies and their inference procedures.

2 Model

A seller employs an auction to sell an indivisible object to one of n bidders. The value of the object to bidder i is a function of n-dimensional signal profile $\boldsymbol{s} \in \mathbb{R}^n$ and denoted by $v_i(\boldsymbol{s})$. It is assumed that v_i is twice continuously differentiable with regard to $\boldsymbol{s}$ and $v_i(\boldsymbol{0}) = 0$ for all i. At this point, I do not specify who observes what signals, which is a central part of the knowledge asymmetry modelling and will be done in the next paragraph. Still, I adopt the convention of calling i^{th} signal, s_i, bidder i's signal. Letting N denote the set of all bidders, let $\boldsymbol{s} = (s_k)_{k\in N}$, $\boldsymbol{s}_A = (s_k)_{k\in A}$ for $A \subset N$ and $\boldsymbol{s}_{-i} = (s_j)_{j\neq i}$.[8] The signal profile $\boldsymbol{s}$ is distributed according to $F : \mathbb{R}^n \to [0,1]$ with density f. I restrict the support of F to $[0,1]^n \subset \mathbb{R}^n$. The allocation in which the object goes to a bidder with the highest valuation for every realization of the signal profile is said to be *efficient*. The following single crossing condition is necessary to guarantee the efficient allocation:

$$\frac{\partial v_i}{\partial s_i}(\boldsymbol{s}) > \frac{\partial v_j}{\partial s_i}(\boldsymbol{s}) \text{ for all } \boldsymbol{s} \text{ and } i \neq j. \tag{1}$$

This is a requirement that the i^{th} signal affects bidder i's valuation more than other bidders' valuations. Furthermore, I assume that for all i and j

$$\frac{\partial v_i}{\partial s_j}(\boldsymbol{s}) \geq 0 \text{ for all } \boldsymbol{s},$$

[7] Krishna (2001) provides another sufficient condition, called *cyclical single crossing* condition.

[8] Throughout this paper, bold face letters are used to denote vectors.

with a strict inequality when $i = j$. I also assume that though the signal profile is realized only in $[0,1]^n$, there exist two signals $\underline{s}_k \le 0$ and $\overline{s}_k \ge 1$ for each k such that for any i and $\boldsymbol{s} \in \prod_{k=1}^{n} [\underline{s}_k, \overline{s}_k]$,

$$v_i(\boldsymbol{s}) > v_j(\boldsymbol{s}) \text{ if } s_i = \overline{s}_i \text{ and } s_j \le 1, \text{ and } v_i(\boldsymbol{s}) < 0 \text{ if } s_i = \underline{s}_k. \tag{2}$$

This is to ensure that the equilibrium constructed for English auction is well defined. In case some uncertainty remains after a signal profile is realized, i's valuation is better described as

$$v_i(\boldsymbol{s}) = \mathbb{E}[u_i|\boldsymbol{s}],$$

where u_i is the value of the object to bidder i with which $\boldsymbol{s}$ is affiliated. Then, the definition of efficiency changes so that the bidder with the highest expected value *conditional on a given signal profile* obtains the object. This change requires bidders to be risk-neutral.[9]

Now, I turn to modelling the knowledge asymmetry. In most previous literature, bidders are assumed to be symmetric in the sense that every bidder observes a one-dimensional signal, which we will henceforth refer to as the standard (information) case. I depart from those models by assuming that N is partitioned into two types of bidders, insiders and outsiders, who are heterogenous in the degrees to which they know about their valuations. Formally, an insider i knows the value of $v_i(\boldsymbol{s})$ for every realization $\boldsymbol{s}$ while an outsider j only knows s_j. I impose no restriction on the number of insiders or outsiders other than that there be at least one insider and one outsider. Importantly, the information structure described so far is common knowledge among bidders. In particular, outsiders know who insiders are.

I assume that each insider employs a weak dominant strategy of bidding (dropping out at) his valuation in a second-price (English) auction. Thus, the analysis of second-price and English auctions goes without further describing what information insiders have about the signal profile besides the knowledge about their valuations. For the analysis of the first-price auction, however, I consider two polar cases consistent with the insiders' knowledge:

Case 1 : An insider i knows the realized value of $v_i(\boldsymbol{s})$, but nothing else.

Case 2 : An insider i knows the realized signal profile $\boldsymbol{s}$.

An insider's information can be multidimensional in the sense that it encompasses an information structure such as Case 2.[10] While the two cases are identical in terms of an insider's

[9] Without risk-neutrality, risk sharing would be another factor to be considered in determining whether a given allocation is efficient.

[10] For example, this case can fit into a sale of an art work where an expert knows every aspect of the work

knowledge about his valuation, he has more refined knowledge about others' signals in Case 2 than in Case 1. Thus, these cases should be dealt with separately for the analysis of first-price auctions.[11,12]

3 Standard Auctions with Two Bidders

In this section, I study the standard auctions when there is only one insider. Suppose that bidder 1 is an insider. The following proposition establishes the efficiency of the second-price auction, which is equivalent to the English auction when there are only two bidders.[13]

Proposition 1 *In all undominated equilibria of the second-price (or English) auction, the object is efficiently allocated.*

Proof. Obviously, the only undominated strategy of bidder 1 is to bid $v_1(\boldsymbol{s})$ for every realization $\boldsymbol{s}$. For bidder 2's best response, suppose that $b_2(s_2)$ is an optimal bid for a fixed signal $s_2 \in [0,1]$. Then, the following must be true ;

$$\begin{aligned} b_2(s_2) &\leq v_2(0,s_2) \ \text{ if } \ v_1(0,s_2) > v_2(0,s_2) \\ &= v_2(\alpha,s_2) \ \text{ if } \ v_1(\alpha,s_2) = v_2(\alpha,s_2) \text{ for some } \alpha \\ &\geq v_2(1,s_2) \ \text{ if } \ v_1(1,s_2) < v_2(1,s_2). \end{aligned} \tag{3}$$

Assume first that $v_1(0,s_2) > v_2(0,s_2)$, which by the single crossing property, implies that $v_1(s_1,s_2) > v_2(s_1,s_2)$ for every $s_1 \in [0,1]$, so it is efficient for bidder 1 to obtain the object regardless of s_1. Given bidder 1's strategy, the winning surplus of bidder 2 is negative

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which is needed to determine its value, while a non-expert only knows one aspect, say its position in art history, which he is more interested in than other aspects. I thank Phil Reny for suggesting this example.

[11] In the first-price auction, a bidder wants to shade his bid below his valuation and the ability to do so is affected by the information he has about his rivals. For instance, Kim and Che (2002) show how the information about rivals' types can affect the performance of the first-price auction in the private value framework.

[12] Note that the existing literature—for example, Milgrom and Weber (1982) or Krishna (2001)—provides equilibria for second-price and English auctions in which each bidder i's strategy only depends on s_i. This strategy is not even feasible in Case 1 since each insider only knows his value and does not have an accurate information about any signal. In Case 2, while feasible, such a strategy is not plausible as an equilibrium since it would require insiders to employ a dominated strategy.

[13] The basic difference between English and second-price auctions is that in the former, a bidder can observe the prices at which others have dropped out, and make inferences about their signals, while in the latter, he cannot. When there are two bidders, however, no such information is available in either auction. Since the price and allocation are determined in the same manner, the two auctions are equivalent so that only the second-price auction is analyzed.

for any s_1. To avoid this loss, bidder 2 must bid no greater than $v_1(0, s_2)$. In fact, any $b \leq v_1(0, s_2)$ is optimal. A similar argument will establish that an optimal bid must be at least $v_2(1, s_2)$ if $v_1(1, s_2) < v_2(1, s_2)$, which results in efficient allocation. Lastly, assume that $v_1(\alpha, s_2) = v_2(\alpha, s_2)$ for some α. Note that this α is unique due to the single crossing. The optimal bid of bidder 2 must be $b_2(s_2) = v_1(\alpha, s_2) = v_2(\alpha, s_2)$, so bidder 2 wins if and only if $s_1 < \alpha$ or $v_1(s_1, s_2) < v_2(s_1, s_2)$ because of the single crossing, resulting in the efficient allocation. Bidder 2 has to submit a bid in the range $b \in [v_1(0, s_2), v_1(1, s_2)]$, otherwise it would be dominated by either $v_1(0, s_2)$ or $v_1(1, s_2)$. Then, there exists $\phi_1(b, s_2) \in [0, 1]$ such that $v_1(\phi_1(b, s_2), s_2) = b$. With such b, 2's payoff is

$$\int_0^{\phi_1(b,s_2)} (v_2(s_1, s_2) - v_1(s_1, s_2)) f_{S_1|S_2}(s_1|s_2) ds_1, \tag{4}$$

where $f_{S_1|S_2}(\cdot|\cdot)$ is the conditional density. The integrand is positive if and only if $s_1 < \alpha$. So, the payoff will be maximized when the integration is taken only in this range, which can be done by setting $b = v_1(\alpha, s_2) = v_2(\alpha, s_2)$. ***Q.E.D.***

Note that the equilibrium strategy of bidder 2 described in (3) is identical to that of Maskin (1992) in the standard case, where the bidder 1 employs the same type of equilibrium strategy. Now, bidder 1, an insider, switches to bidding his own valuation whereas the other bidder maintains the same strategy. It is important to note that by restricting attention to undominated equilibria, the equilibrium allocation is always efficient. This stands in contrast to the standard case where there exist a plethora of inefficient (undominated) equilibria for second-price auctions.[14,15]

Another widely used auction format is the first-price auction in which the highest bidder wins and pays his bid. While no characterization of equilibrium is available, it is possible

[14] Refer to Bikhchandani and Riley (1991) for the common value case and Chung and Ely (2001) for the interdependent value case.

[15] It is also worth mentioning the different roles the single crossing property plays for the existence of the equilibrium strategy in the standard case and in my case. In the standard case, the bidding strategy of Maskin(1992) is an equilibrium even with the exact opposite of the single crossing condition that one's valuation is more affected by the other's signal. The single crossing property is only needed to guarantee the efficiency of the equilibrium allocation. In my model, however, the single crossing property is indispensable for the strategy given in (3) to be an equilibrium. To see it, check the second order (necessary) condition of bidder 2's problem by differentiating (4) twice and evaluating the resulting expression at $\phi_1 = \alpha$ to get

$$\left(\frac{\partial \phi_1}{\partial b}\right)^2 \left(\frac{\partial v_2}{\partial s_1} - \frac{\partial v_1}{\partial s_1}\right) \leq 0.$$

Thus, the single crossing property is necessary for the existence of an undominated equilibrium in my model. As long as such an equilibrium exists, it leads to efficient allocation.

to show that efficient allocation is not available in either Case 1 or Case 2.

Proposition 2 *Suppose also that* $\frac{\partial v_1}{\partial s_2}, \frac{\partial v_2}{\partial s_1} > 0$ *(that valuations are strictly interdependent). Then, the following holds:*

1. *In Case 1, the first-price auction is not efficient.*
2. *In Case 2, the first-price auction is not efficient if the density is bounded such that for some finite* $R > 1$,

$$\frac{1}{R} < f(\boldsymbol{s}) < R \text{ for all } \boldsymbol{s}. \tag{5}$$

Proof. See Appendix C.

Part 1 of Proposition 2 can be readily established because bidder 1, who knows bidder 2's signal and thus his bid for certain, can outbid him whenever profitable. This makes bidder 2 suffer a loss, provided the equilibrium allocation is efficient. In Case 2 where bidder 1 only knows v_1 but not the entire signal profile, the inefficiency is not as readily established. The reason is that while certain about his own valuation, bidder 1 is now uncertain about the opponent's bidding strategy, which makes the above argument unavailable. Still, efficient allocation is not possible with a wide class of signal distributions.[16] This inefficiency of the first-price auction is expected to persist and become more severe as the number of bidders increases. From now on, hence, our analysis will focus on second-price and English auctions.

4 Second-price and English Auctions with More Than Two Bidders

This section explores second-price and English auctions in which there are more than two bidders including at least one insider and one outsider. The two auctions are no longer equivalent since the drop-out prices are observed by bidders in the English auction, which allows bidders to gather more information during the bidding process than the second-price

[16] The proof of this can be explained at an intuitive level as follows. Bidder 2 with any given signal can ignore the possibility that bidder 1 has a valuation close to 0 and submits a very low bid, which follows from the assumption of the bounded density given in (5). By contrast, bidder 1 with a valuation close to 0 should predict with a probability bounded away from 0 that bidder 2 observes a signal close to 0 and submits a very low bid. Thus, he will bid much less aggressively than does bidder 2 with a signal close to 0, causing an inefficiency.

auction does. This will turn out to make a significant difference in the capability of the two auctions to implement an efficient allocation.

I first prove that the efficiency of second-price auction does not go through in the $n \geq 3$ bidder case. The result is established with generic valuation functions. It is also shown that an inefficiency always arises in the symmetric case. The valuations are *symmetric* if it is true that

$$v_i(\boldsymbol{s}) = v(s_i; \boldsymbol{s}_{-i}), \tag{6}$$

where v is constant with any permutation of $\boldsymbol{s}_{-i}$. Note that along with symmetry, the single crossing property implies that whoever holds the highest signal has the highest valuation.

Theorem 1 *Assume that $\frac{\partial v_j}{\partial s_i}(\boldsymbol{s}) > 0$ for all $i \neq j$. Assume also that in the efficient allocation, insiders obtain the object with some positive probability less than one.*[17] *Then, for generic valuation functions, there does not exist any efficient equilibrium for the second-price auction. In particular, no symmetric valuation admits the efficient allocation.*

Proof. See Appendix C.

In a similar setup where outsiders are totally uninformed, Compte and Jehiel (2002) provide an example where no efficient equilibrium exists for the second-price auction. They assume that it is never efficient for an insider to obtain the good. By contrast, the above theorem establishes an inefficiency in case efficient allocation requires both outsiders and insiders to obtain the good with positive probabilities. In this sense, my result complements Compte and Jehiel's, and in fact, do so with much more general valuations while the valuation function in their example is linear. The main idea of the proof is simple and illustrated in the following example.

Example 1 *Suppose that there are three symmetric bidders with $v_i(\boldsymbol{s}) = as_i + \sum_{j \neq i} s_j$, where $a > 1$ satisfies the single crossing condition. Note that whoever has a higher signal has a higher value. Bidder 3 is assumed to be the only insider, who bids his value $v_3(\boldsymbol{s}) = as_3 + s_1 + s_2$. Suppose for a contradiction that an efficient equilibrium exists. Then, the equilibrium bid $b_2(s_2)$ of bidder 2 with any given $s_2 \in [0,1]$ has to satisfy*

$$\max_{\{(s_1', s_3') | s_2 \geq \max\{s_1', s_3'\}\}} v_3(s_1', s_2, s_3') \leq b_2(s_2) \leq \min_{\{(s_1', s_3') | s_3' \geq \max\{s_1', s_2\}\}} v_3(s_1', s_2, s_3') \tag{7}$$

[17] That is, in the efficient allocation, both insiders and outsider obtain the object with positive probabilities.

If the first inequality is violated, then bidder 2 cannot obtain the good when he has the highest value. Likewise, the second inequality is necessary for bidder 1 to get the good when he is an efficient bidder. However,

$$\max_{\{(s_1',s_3')|s_2\geq\max\{s_1',s_3'\}\}} v_3(s_1', s_2, s_3') = (a+2)s_2 > (a+1)s_2 = \min_{\{(s_1',s_3')|s_3'\geq\max\{s_1',s_2\}\}} v_3(s_1', s_2, s_3'),$$

whenever $s_2 > 0$.

The cause of this inefficiency is seen in equation (7) of the above example, where an outsider's bidding strategy that depends only on his own signal is not refined enough to efficiently match with the strategy of an insider, which varies with the entire signal profile.

From now on, I focus my attention on the English auction to first demonstrate its efficiency. Consider the Japanese format of English auctions, where bidders drop out of the auction as the price rises continuously starting from zero until only one bidder remains and is awarded the object at the last drop-out price. I first present the procedure to obtain an ex-post equilibrium and then explore what conditions are needed for such an equilibrium to exist and to be efficient. In the ex-post equilibrium I will construct, an outsider's signal is fully revealed via his drop-out price. The main difficulty of the analysis lies in the fact that an insider's signals cannot be fully revealed from his drop-out price since his drop-out price reflects all the signals. However, I show that this does not prevent the English auction from achieving the efficient allocation. Indeed, in the English auction, insiders' signals can be inferred, if not precisely, to such an extent that yields the efficient allocation.

Milgrom and Weber (1982) provide an efficient equilibrium for the English auction in the standard case with symmetric bidders.[18] Their result was extended by Krishna (2001) to the setup in which bidders have interdependent and asymmetric valuations. The logic underlying Krishna's equilibrium construction can be explained by the following procedure: (1) The signals of inactive bidders are fully revealed via their drop-out prices; (2) active bidders take the signals of inactive bidders as inferred in (1), and calculate the *break-even* signal profile of all active bidders as a function of the current price which would make each of them break even by immediately dropping out; and (3) each active bidder stays in (exits) the auction if his break-even signal at the current price is smaller (greater) than his true signal. Indeed, this procedure yields an efficient equilibrium when there is no insider. In the analysis that follows, I modify the above procedure to accommodate the presence of insiders. One significant difference arises from being unable to apply step (1) to the signals

[18] Bikhchandani, Haile, and Riley (2002) characterize the entire set of symmetric separating (thus, efficient) equilibria.

of inactive insiders. The reason, as already mentioned, is that an insider drops out at his valuation, which depends on the entire profile of signals.

In the equilibrium I present, step (1) still applies for the inference of inactive outsiders' signals, in that their drop-out prices are increasing with their signals. Thus, inferences on inactive outsiders' signals remain unchanged once revealed. The signals of inactive insiders remain unrevealed. The unrevealed signals of inactive outsiders and active bidders are calculated as the (unique) solution of a system of equations. As explained in the next paragraph, those equations extend Krishna (2001)'s system to incorporate any additional information that the insiders' drop-out prices convey. The drop-out decision of an outsider is then to follow the same rule (3). I show that the procedure just described produces an ex-post equilibrium.

To give a formal presentation of the above argument, I introduce a set of notations. A denotes the set of active bidders. O and I denote the sets of outsiders and insiders, respectively. Let p_i denotes the price at which bidder i drops out. Then, $\boldsymbol{p}_B = (p_i)_{i \in B}$ with B being an arbitrary set of bidders. The price profile $\boldsymbol{p}_{N \setminus A}$ can be identified with the history of the bidding game as $N \setminus A$ is the set of inactive bidders. Assuming that the signals of inactive outsiders have been (correctly) revealed to be $\boldsymbol{s}_{O \setminus A}$ in the history $\boldsymbol{p}_{N \setminus A}$, the break-even signal profile $(\boldsymbol{s}_A(p, \boldsymbol{p}_{N \setminus A}), \boldsymbol{s}_{I \setminus A}(p, \boldsymbol{p}_{N \setminus A}))$ at a current price p is defined to be a (unique) solution of the following system of equations,

$$v_k(\boldsymbol{s}_{O \setminus A}, \boldsymbol{s}_A(p, \boldsymbol{p}_{N \setminus A}), \boldsymbol{s}_{I \setminus A}(p, \boldsymbol{p}_{N \setminus A})) = \begin{cases} p & \text{for } k \in A \\ p_k & \text{for } k \in I \setminus A. \end{cases} \tag{8}$$

Suppose that the signal profile of active bidders and inactive insiders were equal to the break-even signal profile. Then, the first set of equalities in (8) says that the current active bidders break even by immediately dropping out at the current price p, and the second one that inactive insiders dropped out at their valuations. Put differently, given the other signals, $\boldsymbol{s}_A(p, \boldsymbol{p}_{N \setminus A})$ corresponds to the minimal level of active bidders' signals at which they do not suffer a loss by paying the current price. And, given the other signals, $\boldsymbol{s}_{I \setminus A}(p, \boldsymbol{p}_{N \setminus A})$ must have been the signal profile of inactive insiders since they have dropped out at their valuations. Note that since $\boldsymbol{s}_{I \setminus A}(p, \boldsymbol{p}_{N \setminus A})$, the inference about the inactive insiders' signal profile, is never accurate, it is being continuously updated as the price rises.

When setting up equation (8), I assumed that the signals of inactive outsiders are correctly revealed as $\boldsymbol{s}_{O \setminus A}$, which requires outsiders to adopt separating bidding strategies. In other words, a positive measure of an outsider' types should not drop out at the same price. In order to guarantee it, the break-even signals of active bidders must (strictly) increase with the price level, which is stated here as a condition. From now on, I will leave out the argument $\boldsymbol{p}_{N\setminus A}$ to simplify notations.

Condition EMS (Existence of Monotone Solution) *For every* $\boldsymbol{s}_{O\setminus A}$ *and* $\boldsymbol{p}_{N\setminus A}$*, there exists a unique set of differentiable functions* $\boldsymbol{s}_A : \mathbb{R}_+ \to \prod_{k\in A}[\underline{s}_k, \overline{s}_k]$ *and* $\boldsymbol{s}_{I\setminus A} : \mathbb{R}_+ \to \prod_{k\in I\setminus A}[\underline{s}_k, \overline{s}_k]$ *such that (1) equation (8) holds for all* $p \in [\max_{k\in N\setminus A} p_k, \min_{k\in O\cap A} s_k^{-1}(1)]$ *and (2)* $\boldsymbol{s}_A(p)$ *is strictly increasing.*

Later, I will provide more primitive condition imposed on bidders' valuation functions to ensure that EMS holds. The following proposition says that an ex-post equilibrium exists in which an active outsider drops out at the price level that makes his true signal equal to the break-even signal.

Theorem 2 *Suppose that EMS holds. Then, an ex-post equilibrium exists in which each* $k \in O \cap A$ *drops out at* p *if and only if* $s_k \leq s_k(p)$*.*

Proof. See Appendix A.

Example 1 demonstrates how the strategy described in Theorem 2 works.

Example 2 *Recall Example 1, where three bidders have symmetric linear valuations and bidder 3 is an insider. Calculate the break-even signals as a function of* p *in all possible history. Inverting the break-even signals will yield the drop-out prices for any given signal. When* $A = \{1, 2, 3\}$ *or when no one has yet dropped out,* $s_i(p) = \frac{p}{a+2}$*, implying that bidder* i *with* s_i *plans to drop out when the price reaches* $(a+2)s_i$*. After an outsider* $j \neq 3$ *with* s_j *drops out first and his signal is correctly inferred, the other (active) outsider* i *with* s_i *sets* $s_i(p) = \frac{1}{a+1}(p - s_j)$*, planning to drop out when the price reaches* $(a+1)s_i + s_j$*. The analysis so far is similar to the standard case. It differs if insider bidder 3 drops out first, in which case the break-even signals of the two remaining outsiders are the first two components of the solution of*

$$\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} p \\ p \\ p_3 \end{pmatrix}. \tag{9}$$

Some algebra shows that $s_i(p) = \frac{ap - p_3}{(a+2)(a-1)}$ *for* $i = 1, 2$ *and* $s_3(p) = \frac{(a+1)p_3 - 2p}{(a+2)(a-1)}$*. Thus,* i *drops out when the price reaches* $(a+1)s_i + \frac{1}{a}(p_3 - 2s_i)$*, which is increasing with* s_i *since* $a > 1$*. Noteworthy are two facts concerning* $s_3(p)$*, the inference about the insider's signal. First,* $s_3(p)$ *is decreasing with* p*. This is intuitive because staying active at a higher price implies higher signals of outsiders, which in turn implies a lower signal of an inactive insider since*

	Valuation Ranking	1st Drop-Out Price	2nd Drop-Out Price
(i) $s_3 > \max\{s_1, s_2\}$	$v_3 > \max\{v_1, v_2\}$	$(a+2)\min\{s_1, s_2\}$	$(a+1)\max\{s_1, s_2\} + \min\{s_1, s_2\}$
(ii) $s_1 > \max\{s_1, s_2\}$, $(a+1)s_2 > as_3 + s_1$	$v_1 > v_2 > v_3$	$p_3 = as_3 + s_1 + s_2$	$p_2 = (a+1)s_2 + \frac{1}{a}(p_3 - 2s_2)$
(iii) $s_1 > s_2 > s_3$, $(a+1)s_2 < as_3 + s_1$	$v_1 > v_2 > v_3$	$p_2 = (a+2)s_2$	$p_3 = as_3 + s_1 + s_2$
(iv) $s_1 > s_3 > s_2$, $(a+1)s_2 < as_3 + s_1$	$v_1 > v_3 > v_2$	$p_2 = (a+2)s_2$	$p_3 = as_3 + s_1 + s_2$

Table 1: The Outcome of The Equilibrium Bidding Strategy

the resulting valuation of an insider is set at its true level $p_3 = v_3$ as seen in (9). Second, $s_3(p) > s_3$ as long as p is less than the selling price, which implies that $s_3(p)$ overestimates the signal of an insider but gets more close to s_3 as the price increases.

The allocation resulting from this equilibrium strategy is efficient. Suppose that the equilibrium strategy and signal realization prescribe that after bidder 3 have first dropped out, bidder 1 outlasts bidders 2 to become a winner, which implies $s_1 > s_2$ because of the symmetry of their equilibrium strategies. Since whoever holds a higher signal has a higher valuation, $v_1(\boldsymbol{s}) > v_2(\boldsymbol{s})$. Efficiency also requires that $v_1(\boldsymbol{s}) > v_3(\boldsymbol{s})$. The fact that bidder 1 becomes a winner as 2 drops out at p_2 implies that $s_2(p_2) = s_2$ and $s_1(p_2) < s_1$, which, in turn, implies that $s_3(p_2) > s_3$ since

$$v_3(s_1(p_2), s_2(p_2), s_3(p_2)) = p_3 = v_3(s_1, s_2, s_3).$$

Also,

$$v_1(s_1(p_2), s_2, s_3(p_2)) = p_2 > p_3 = v_3(s_1(p_2), s_2, s_3(p_2)).$$

Thus, this inequality together with $s_1(p_2) < s_1$ and $s_3(p_2) > s_3$ implies $v_1(\boldsymbol{s}) > v_3(\boldsymbol{s})$ by the single crossing property. Note that the above comparison between the valuations of an outsider and an insider is made possible by the break-even condition that includes the third equation of (9), the equation that an insider has dropped at his true valuation.

In Table 1, I report what will be the outcomes if each bidder follows the above equilibrium strategy for every realized signal profile. Because of the symmetry between bidder 1 and 2, I omit the cases in which bidder 2 is a winner. In all cases, a bidder with the highest valuation obtains the object so that the allocation is efficient. It is interesting to see that orders of bidders' drop-outs and their valuations do not need to coincide with each other.

This, however, does not cause any inefficiency. In case (iii), bidder 2 has a higher valuation than bidder 3 even though the former drops out before the latter. As for the selling price, which is the second drop-out price, the ex-post selling price is strictly higher with 3 being an insider than with no insider, unless 3 is a winner, in which case it remains the same. In case (ii), for instance, the selling price with no insider is $(a+1)s_2 + s_3$*, which is smaller than* $(a+1)s_2 + \frac{1}{a}(p_3 - 2s_2)$ *since* $\frac{1}{a}(p_3 - 2s_2) - s_3 = \frac{1}{a}(s_1 - s_2) > 0$*, using* $p_3 = as_3 + s_1 + s_2$*.*

I will now investigate the efficiency of the equilibrium allocation in three cases.[19] First, the efficiency between insiders, which means that an insider winner has the highest valuation among insiders, is obvious from their truthful bidding strategy. Next, only one insider i and one outsider j remain, the object must go to whomever has a higher valuation. However, two distortions are possible: (1) $p_j < v_i(\boldsymbol{s}) < v_j(\boldsymbol{s})$ that is, outsider i who has a higher valuation drops out before insider j; (2) $v_j(\boldsymbol{s}) < v_i(\boldsymbol{s}) < p_j$ that is, outsider j who has a lower valuation plans to drop out later than insider i. However, (1) cannot happen in the ex-post equilibrium, otherwise j could profitably deviate to outlast i to win and pay $v_i(\boldsymbol{s})$, which is lower than j's valuation. Similarly, (2) cannot happen in the ex-post equilibrium. While this equilibrium argument could easily show that neither distortion arises, it is worth exploring what it implies about the outsider's strategy. Specifically, it requires that if $v_i(\boldsymbol{s}) = v_j(\boldsymbol{s})$, then $p_j = v_j(\boldsymbol{s})$, that is, when the two bidders have the same valuation, the outsider must drop out exactly at his valuation.[20] This is remarkable considering that the outsider cannot make the precise inference about some set of signals (or inactive insiders' signals) and may thus have much uncertainty about his valuation. An outsider can overcome this problem to avoid underbidding or overbidding his valuation by making the appropriate use of the information conveyed by insiders' drop-out prices, as reflected in the break-even equation.

Lastly, it is necessary to demonstrate that when two outsiders remain, a winner is the one who has a higher valuation. This easily holds for the symmetric and linear examples above. However, this may not be the case for general valuations. The main reason is that insiders' signals are not fully revealed even at the end of the auction. To see how this could prevent efficiency, take the example in which bidder 3 is the only insider among three bidders with possible non-linear and asymmetric valuations. Consider the case where bidder 3 first drops out at p_3 and bidder 1 becomes the winner when bidder 2 drops out at p_2. Then, the break-even condition at p_2 is

[19] Although the cases I present do not exhaust all possibilities, one can easily extend the argument to take account of other cases as in the proof of Theorem 3.

[20] Otherwise, a nearby signal profile for which (1) or (2) happens can easily be found.

$$v_k(s_1(p_2), s_2, s_3(p_2)) = \begin{cases} p_2 & \text{for } k = 1 \text{ or } 2 \\ p_3 & \text{for } k = 3, \end{cases} \tag{10}$$

since $s_2(p_2) = s_2$. Because $s_1 > s_1(p_2)$ from the fact that bidder 1 is the winner, the single crossing condition combined with the above equation implies

$$v_1(s_1, s_2, s_3(p_2)) > v_2(s_1, s_2, s_3(p_2)).$$

This inequality, however, does not ensure efficiency since $s_3(p_2)$ is not equal to bidder 3's true signal s_3. Indeed, replacing $s_3(p_2)$ by s_3 might reverse the above inequality so that it is more efficient to award the object to bidder 2 than 1. To avoid this kind of inefficiency, the following condition on the valuation functions is necessary.

Condition SCIC (Single Crossing on Indifference Curves) *Consider any signal profiles $\boldsymbol{s}$ and $\boldsymbol{s}'$ with $s'_i > s_i$ for $i \in O$ and $s'_j = s_j$ for all other $j \in O$, satisfying*

$$v_k(\boldsymbol{s}') = v_k(\boldsymbol{s}) \text{ for all } k \in I.$$

Then,

$$v_i(\boldsymbol{s}) = v_j(\boldsymbol{s}) \text{ implies } v_i(\boldsymbol{s}') > v_j(\boldsymbol{s}').$$

SCIC states that raising an outsider's signal along insiders' indifference curves makes his valuation increase faster than other outsiders' valuations. The above example can be used to show how this condition guarantees the efficiency of the equilibrium allocation. Recall that bidder 3 drops out at his valuation, so equation (10) implies

$$v_3(s_1(p_2), s_2, s_3(p_2)) = p_3 = v_3(s_1, s_2, s_3).$$

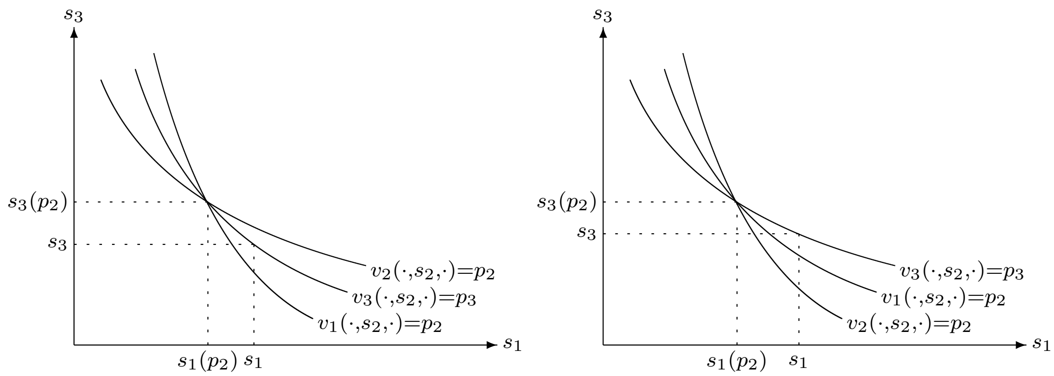
Thus, both signal profiles $(s_1(p_2), s_2, s_3(p_2))$ and (s_1, s_2, s_3) are on the same indifference curve of bidder 3. Then, letting $i = 1$ and $j = 2$ in the above definition of SCIC leads to

$$v_1(s_1, s_2, s_3) > v_2(s_1, s_2, s_3)$$

since

$$v_1(s_1(p_2), s_2, s_3(p_2)) = p_2 = v_2(s_1(p_2), s_2, s_3(p_2))$$

and $s_1 > s_1(p_2)$. Figure 1 gives further insight about SCIC requirements. Note that the signal of bidder 2 is fixed at its true level. By the single crossing property, the indifference



(i) SCIC holds : $v_1(\boldsymbol{s}) > v_2(\boldsymbol{s})$ (ii) SCIC might fail : $v_1(\boldsymbol{s}) \gtrless v_2(\boldsymbol{s})$

Figure 1: Single Crossing on Indifference Curves

curve of bidder 1 crosses that of bidder 3 from above as s_1 increases. In (i), bidder 1's valuation increases as his signal rises from $s_1(p_2)$ to s_1 along 3's indifference curve while bidder 2's valuation decreases. More importantly, 1's valuation grows faster than 2's valuation as 1's signal increases along 3's indifference curve. As seen in the figure, this feature is reinterpreted as

$$\mathrm{MRS}^1_{1\,3} := \frac{\partial v_1}{\partial s_1} \Bigg/ \frac{\partial v_1}{\partial s_3} > \frac{\partial v_2}{\partial s_1} \Bigg/ \frac{\partial v_2}{\partial s_3} =: \mathrm{MRS}^2_{1\,3}. \tag{11}$$

This inequality is reversed in (ii) of Figure 1 so that 2's valuation might be higher than 1's valuation at $\boldsymbol{s}$. One can ask when inequality (11) does hold. Given that $\frac{\partial v_1}{\partial s_1} > \frac{\partial v_2}{\partial s_1}$ by the single crossing condition, (11) requires that $\frac{\partial v_1}{\partial s_3}$ and $\frac{\partial v_2}{\partial s_3}$, the marginal effects of a third party's signal on two other bidders' valuations, should not be so different as to overturn the single crossing property. This was true for the symmetric linear valuation in Example 2 since the signal of a third party (bidder 3) has the same marginal affect on two other bidders' valuations. In Section 6, I will provide a class of valuation functions which fit this observation.

The following theorem generalizes the above argument to establish the efficiency of the ex-post equilibrium obtained in Theorem 2.

Theorem 3 *Suppose that SCIC holds. Then, the ex-post equilibrium of Theorem 2 leads to efficient allocation.*

Proof. See Appendix A.

When no insider exists, the single crossing condition was enough to guarantee the efficiency of the monotone equilibrium bidding strategy. The presence of insiders and their unrevealed signals requires strengthening the single crossing condition to SCIC. Proposition 2, Theorem 1, and Theorem 3 show how a dynamic auction differs from a static (sealed-bid) auction in terms of allocative efficiency. When the bidders are allowed to be heterogeneous in their knowledge about the valuations, a sealed-bid auction like a first-price or second-price auction fails to achieve efficient allocation even with symmetric valuations. On the other hand, the English auction can do so with some asymmetry on the valuations. The basic difference is the capability of the English auction to allow for heterogeneous bidders' bidding strategies to be coordinated by providing the less-informed bidders with the extra information during the bidding stage.

Now, I provide a sufficient condition for the valuation functions to satisfy both EMS and SCIC. Krishna (2001) suggested a condition, called average crossing, under which increasing break-even signals exist when there is no insider. Define the average valuation as

$$\overline{v}(\boldsymbol{s}) := \frac{1}{n} \sum_{k} v_k(\boldsymbol{s}). \tag{12}$$

The average crossing condition is the single crossing condition between the average valuation and one bidder's valuation with regard to another bidder's signal: for all $i, j \in N$ and $i \neq j$,

$$\frac{\partial \overline{v}(\boldsymbol{s})}{\partial s_i} > \frac{\partial v_j(\boldsymbol{s})}{\partial s_i}.$$

However, the average crossing condition is not sufficient to guarantee both EMS and SCIC when there are insiders, as shown in the following example.

Example 3 *Suppose that there are three bidders each of whom has the valuation* $v_i(\boldsymbol{s}) = \sum_{j=1}^{3} a_{ij} s_j$. *Let bidder* 3 *be the only insider. (12) requires that*

$$\frac{\sum_i a_{ij}}{3} > a_{kj} \text{ for } k \neq j. \tag{13}$$

However, it can be easily checked that EMS and SCIC are satisfied only when

$$a_{11} - a_{21} > \frac{a_{31}}{a_{33}}(a_{13} - a_{23}) \quad \text{and} \quad \frac{a_{32}}{a_{33}}(a_{13} - a_{23}) > a_{12} - a_{22}. \tag{14}$$

Clearly, (14) is not implied by (13).

In fact, (14) can rewritten as

$$\left.\frac{\partial \overline{v}}{\partial s_j}\right|_{v_3(\boldsymbol{s}) = \overline{v}_3} > \left.\frac{\partial v_k}{\partial s_j}\right|_{v_3(\boldsymbol{s}) = \overline{v}_3} \quad \text{for } k \neq j \text{ and } k, j \neq 3$$

That is, the average crossing needs to hold on the indifference curve of insider bidder 3. Therefore, I introduce the following condition to strengthen the average crossing condition proposed by Krishna (2001).

Condition ACIC (Average Crossing on Indifference Curves) *For any $B \subset I$, consider two arbitrary signal profiles $\boldsymbol{s}$ and $\boldsymbol{s}'$ with $s_i' > s_i$ for some $i \in N \backslash B$ and $s_j' = s_j$ for all other $j \in N \backslash B$, satisfying*

$$v_k(\boldsymbol{s}') = v_k(\boldsymbol{s}) \text{ for } k \in B.$$

Then, for any $j \neq i$,

$$\overline{v}(\boldsymbol{s}') - \overline{v}(\boldsymbol{s}) > \max\{v_j(\boldsymbol{s}') - v_j(\boldsymbol{s}), 0\}. \tag{15}$$

While holding constant the valuations of any given set B of insiders, raising the signal of a bidder outside of B makes the average value grow faster than the other bidders' valuations. Clearly, ACIC implies the average crossing condition, which is equivalent to requiring (15) with $B = \emptyset$. By investigating (14) of the above example, insights are gained as to the restrictions ACIC imposes on the valuation functions: (14) is likely to hold when $(a_{11} - a_{21})$ and $(a_{22} - a_{12})$ are relatively large, compared to $|a_{13} - a_{23}|$. So, ACIC requires that the effects of a third party's signal (s_3) on two other bidders' valuations (v_1 and v_2) are not too different relative to the magnitude of the single crossing property.

It is easy to see that ACIC implies SCIC. The following lemma shows that ACIC also implies EMS.

Lemma 1 *Suppose that ACIC is satisfied. Then, EMS holds.*

Proof. See Appendix C.

So, the following corollary results.

Corollary 1 *If the valuation functions satisfy ACIC, then the English auction has an ex-post equilibrium which leads to efficient allocation.*

According to Jehiel and Moldovanu (2001), when signals are multidimensional, an efficient allocation is not implementable in general. In my model, the insiders' information possesses the multidimensional property, but the inefficiency result of Jehiel and Moldovanu (2001) is not applicable. In their model, bidders are assumed to observe the disjoint pieces of multidimensional information. In my model, however, insiders' information overlaps with that of outsiders, so the dimensionality assumption of Jehiel and Moldovanu (2001) does not hold.[21]

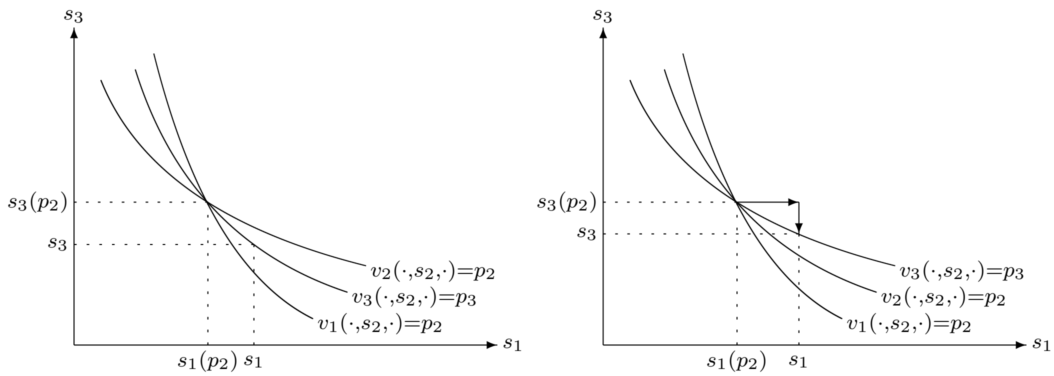
5 Revenue Implication in English Auctions

The linkage principle of Milgrom and Weber (1982) states that providing bidders with more information that is affiliated with that of the winning bidder increases revenues. In a similar spirit, I ask whether the English auction generates a higher revenue if more bidders know their valuations. Specifically, consider two English auctions, E and E', which only differ by a bidder who is an outsider in E but an insider in E'. I show that the ex-post selling price is at least as high in E' as in E. Suppose that a shift from E to E' occurs. While this operation is similar in spirit to Milgrom and Weber's in that the shift increases the information available to the bidders, an important difference exists. Unlike Milgrom and Weber's, the increased information is not public since only the outsider who switches to an insider becomes better informed of his valuation.[22] For this reason, there are two facts to check out: (1) how the bidding strategy of the switched bidder changes and (2) how it affects other outsiders' bidding strategies. To establish these two facts requires a completely different argument from Milgrom and Weber's because of the asymmetry in bidding strategies of two types. Milgrom and Weber's result is rather of a statistical nature since its derivation heavily depends on the assumption of affiliation among signals. My result does not rely on any statistical assumption and holds even when signals are independent. Also, Milgrom and Weber's result only address *ex-ante* revenue—revealing a public signal may not improve revenue ex-post, while the revenue increase holds ex-post in my case. My result is quite general and makes no assumption about how many bidders are insiders or outsiders initially.

To obtain the desired result, I first assume that efficient ex-post equilibria as obtained in the previous section, exist in both E and E'. This ensures that the same bidder wins in E and E'. I further require that higher signals of other bidders, in some sense, need to be 'good news' to a given bidder. To explain it, return to the previous example. Suppose that in E, bidder 3 is the only insider and, in E', 2 and 3 are insiders while 1 is still an outsider. Suppose also that with a given signal profile $\boldsymbol{s}$, 1 is the winner and pays 2's drop-out price in both E and E'. Hence, the selling prices are $v_2(s_1(p_2), s_2, s_3(p_2))$ in E and $v_2(\boldsymbol{s})$ in E'. Given this situation, Figure 2 shows two cases consistent with SCIC in which

[21] For instance, in Case 2 of Section 2, insiders also know outsiders' signal profile $\boldsymbol{s}_O$.

[22] In this sense, the enhanced information of the bidder switching from an outsider to an insider may be viewed as the result of the information gathering activity by the corresponding bidder rather than that of the public announcement by the seller.



(i) GNIC fails : $v_2(s_1(p_2), s_2, s_3(p_2)) > v_2(\boldsymbol{s})$ (ii) GNIC holds : $v_2(s_1(p_2), s_2, s_3(p_2)) < v_2(\boldsymbol{s})$

Figure 2: Signals as Good News on Indifference Curves

the revenue might or might not be increasing. Note that (i) of Figure 2 reproduces (i) of Figure 1. In case (i), it is bad news for bidder 2 to increase 1's signal along 3's indifference curve since $v_2(s_1(p_2), s_2, s_3(p_2)) > v_2(\boldsymbol{s})$ with $(s_1(p_2), s_2, s_3(p_2))$ and $\boldsymbol{s}$ being on the same indifference curve of bidder 3 and s_1 being greater than $s_1(p_2)$. So, bidder 2 drops out above his valuation in E and at his valuation in E'. Hence, bidder 2's switch to an insider hurts the seller. However, once case (i) is ruled out in Figure 2, it is possible to show that the ex-post revenue always increases with an extra insider. For instance, if the indifference curves are positioned as in (ii), then $v_2(s_1(p_2), s_2, s_3(p_2)) < v_2(\boldsymbol{s})$ so the revenue increases. Here, it is good news for both bidder 1 and 2 to increase 1's signal along 3's indifference curve. This observation is generalized in the following condition.

Condition GNIC (Signals as Good News on Indifference Curves) *For any set $B \subset N$, suppose two arbitrary signal profiles $\boldsymbol{s}$ and $\boldsymbol{s}'$ have $s_i' > s_i$ for any given $i \in N \backslash B$, $s_j' = s_j$ for all other $j \in N \backslash B$, and satisfy*

$$v_k(\boldsymbol{s}') = v_k(\boldsymbol{s}) \text{ for all } k \in B.$$

Then, for any $j \in N \backslash B$,

$$v_j(\boldsymbol{s}') - v_j(\boldsymbol{s}) \geq 0. \tag{16}$$

Also, (16) becomes strict with $j = i$.

This states that a higher signal of one bidder is better news to the bidders whose valuations are not fixed, when the valuations of the other bidders are held fixed. Of course, to increase one signal while holding some bidders' valuations fixed, causes some other signal(s) to decrease. For instance, in the above example, to increase bidder 1's signal and hold bidder 3's valuation and bidder 2's signal fixed will result in a decrease in bidder 3's signal, which would still increase bidder 2's valuation according to GNIC. This is in part due to the single crossing condition, which requires v_3 to decrease faster than v_2 in response to a decrease in s_3. So, if a decrease in s_3 offsets an increase in s_1 and keeps v_3 unchanged, then the same change would be likely to increase v_2. However, the single crossing alone does not warrant the above claim, which, in fact, would hardly hold if v_3 responds to the change in s_1 much more sensitively than v_2 does. For instance, in Example 3, for an increase in s_1 along 3's indifference curve to raise v_2, requires[23]

$$a_{21} - \frac{a_{23}}{a_{33}} a_{31} \geq 0. \tag{17}$$

This would not hold if $\frac{a_{31}}{a_{21}}$ (the difference in the marginal effects of bidder 1's signal on two other bidders' valuations) is too large relative to $\frac{a_{33}}{a_{23}}$ (i.e. the degree of the single crossing). This is similar to the observation that was needed for ACIC to be satisfied. Note that (17) holds for the symmetric linear valuation of Example 2.

Now, let $P(\boldsymbol{s})$ and $P'(\boldsymbol{s})$ denote the selling prices in, respectively, E and E' with $\boldsymbol{s}$ being the realized signal profile. Then, the following theorem results.

Theorem 4 *Suppose that efficient equilibria exist in both E and E' as described in Theorem 3. Suppose also that GNIC is satisfied. Then,*

$$P(\boldsymbol{s}) \leq P'(\boldsymbol{s}) \text{ for all } \boldsymbol{s}.$$

Proof. See Appendix B.

The result follows from two facts, for which GNIC plays a crucial role. First, each outsider drops out before the price reaches his valuation. Thus, bidder i drops out at a higher price in E' than in E. Second, i's higher drop-out price causes other active outsiders to also drop out at higher prices in E' than in E. At an intuitive level, this is mainly due to the overestimation of some signals. To make the point clear, consider the case in which E only consists of outsiders. Recall that the break-even signals of active outsiders are increasing but always below their true levels while inactive outsiders' signals are correctly inferred. Thus, every signal is (at least weakly) underestimated in E. In E', however, if every signal were underestimated, then insider i's drop-out price, which is set equal to his valuation

[23]Note that in view of the definition of GNIC, this corresponds to the case $B = \{3\}$, $i = 1$ and $j = 2$. There are similar requirements in the case $B = \{1\}$ or $\{2\}$.

calculated with those underestimated signals, would be less than his true valuation, thereby contradicting the insider's bidding strategy. That is, the underestimation of some signals requires the overestimation of some other signals. This can be seen in the three bidder example, where bidder 1 is a winner and pays bidder 2's drop-out price after bidder 3, the only insider, has first dropped out. The break-even equation at p_2 involves

$$v_3(s_1(p_2), s_2, s_3(p_2)) = p_3 = v_3(\boldsymbol{s}).$$

This equality implies that since bidder 1 being a winner implies $s_1 > s_1(p_2)$, $s_3 < s_3(p_2)$.[24] The underestimation of signals results from outsiders' attempting to avoid falling prey to the winner's curse by assuming as if the currently unknown signals are just high enough to avoid a loss at the current price. Thus, the detrimental effect of the winner's curse is alleviated by the presence of insiders. Therefore, having an extra insider is beneficial to the seller's revenue.

This result yields yet another implication of toeholds in takeover bidding. Bulow, Huang, and Klemperer (1999) show that the toeholds make a bidder bid more aggressively, thereby worsening the winner's curse for non-toeholders and making them bid less aggressively, which, in turn, makes the toeholder more aggressive, and so on. In fact, my model presents another reason for toeholders' aggressiveness. Toeholders are insiders with no uncertainty about their valuations, and thus bid aggressively or bid their true valuations since they are free from the winner's curse. According to my model, however, insiders' aggressive bidding also makes outsiders, or non-toeholders, aggressive, whereas Bulow, Huang, and Klemperer (1999) predict that non-toeholders will respond with less aggressive bidding.[25]

6 English Auctions with Private and Common Values

In this section, I provide a class of valuation functions with which all the previous assumptions hold. Suppose that bidder i's valuation is given by

$$v_i(\boldsymbol{s}) = h_i(s_i) + g(\boldsymbol{s}) \text{ with } h_i'(s_i) > 0 \text{ and } \frac{\partial g}{\partial s_k} \geq 0 \text{ for all } k. \tag{18}$$

[24]It can be easily shown in case of symmetric valuation that this overestimation leads to the revenue increase. Consider auctions E and E' in which bidder 3 switches while other bidders remain outsiders. Suppose that bidder 1 is a winner to pay bidder 2's drop-out prices in both E and E'. By the symmetry, when 2 drops out, the break-even signals of bidder 1 and 2 are the same, equal to s_2 in both auctions. Bidder 3's signal has already been revealed in E whereas it is overestimated at $s_2'(> s_2)$ in E' as seen above. Thus, the selling price in E', which is $v_2(s_2, s_2, s_3')$, is higher than that in E, which is $v_2(s_2, s_2, s_3)$.

[25]Note that without explicitly considering the toeholds as a part of the model, the analysis of this paper might not be directly applicable to the situation of Bulow, Huang, and Klemperer (1999).

Bidders' valuations are (additively) separable into two components, a private value $h_i(s_i)$ and a common value $g(\boldsymbol{s})$.[26] Now, an insider can be viewed as the bidder who is informed of the common value in addition to his private value, while an outsider is not.

Using a similar method to Example 1, the inefficiency of the second-price auction can easily be established. On the other hand, this model turns out to satisfy ACIC, yielding an efficient allocation for the English auction. I first observe that the equilibrium bidding strategy characterized in Section 4 has an interesting implication on the bidders' drop-out order when the valuation functions satisfy (18). And it is partly related to the efficiency of the resulting allocation. To see it, begin by verifying the following: For all i, j and $k \neq i$ or j,

$$\frac{\partial v_i}{\partial s_k}(\boldsymbol{s}) = \frac{\partial v_j}{\partial s_k}(\boldsymbol{s}) = \frac{\partial g}{\partial s_k}(\boldsymbol{s}) \text{ for every } \boldsymbol{s}. \tag{19}$$

Thus, one bidder's signal has the same marginal effect on the other two bidders' valuations. It then follows that the ranking of any two bidders' valuations be independent of the other bidders' signals.

Lemma 2 *Let two signal profiles $\boldsymbol{s}$ and $\boldsymbol{s}'$ be such that $s_i = s'_i$ and $s_j = s'_j$. Then, $v_i(\boldsymbol{s}) > v_j(\boldsymbol{s})$ if and only if $v_i(\boldsymbol{s}') > v_j(\boldsymbol{s}')$.*

Proof. See Appendix C.

This results in outsiders' dropping out in the order of their valuations, which needs not be true for all the valuation functions satisfying ACIC.

Proposition 3 *Given the equilibrium strategy described in Theorem 2, i drops out before j if and only if $v_i(\boldsymbol{s}) < v_j(\boldsymbol{s})$ for any outsiders i and j, and signal profile $\boldsymbol{s}$.*

Proof. See Appendix C.

Note that this result does not yield complete information about the order of drop-outs, for an outsider might drop out earlier (later) than does an insider who has a lower (higher) valuation than him, as shown in Example 2. Despite this possibility, an efficient allocation can still be achieved.

The efficiency is established by showing that ACIC is satisfied in this model. Recall from Example 3 that ACIC is likely to be satisfied if the marginal effects of a third party's signal

[26] There are some studies which adopt this valuation function in the standard information setup. For instance, see Wilson (1998) for a theoretical study and Hong and Shum (2002) for an empirical one.

on other bidders' valuations are not too different. Indeed, (19) meets this requirement in its extreme sense. The following lemma shows that both ACIC and GNIC are satisfied by the valuations in (18) since they satisfy (19).

Lemma 3 *The valuation functions for which (19) holds, satisfy ACIC. So do the valuation functions in (18). Also, the valuation functions in (18) satisfy GNIC.*

Proof. See Appendix C.

Hence, Corollary 1 and Theorem 4 may be used to conclude that the efficiency and revenue results in Section 4 and 5 apply to the current model.

7 Concluding Remarks

There have been previous attempts to formalize what it means for a bidder to be better informed.[27] This paper takes a simple approach by assuming that under the interdependent valuation setup, a bidder either fully knows his valuation or knows only a one-dimensional signal. This asymmetry has different effects on the performance of standard auctions in terms of their efficiency. I find that an asymmetry in bidding strategies resulting from more or less refined information about one's valuation, prevents sealed-bid auctions from achieving efficiency. By contrast, in the English auction, the information provided by drop-out prices enables poorly-informed bidders to gradually adjust their bidding strategies toward the efficient allocation. I also find that the presence of insiders has a revenue-enhancing effect in English auctions. One caveat is that introducing asymmetry requires a stronger sufficient condition to ensure the existence of an efficient equilibrium. Although this stronger sufficient condition might restrict the scope of applying our result, I find that the results hold where bidders' valuations consist of private and common value components.

An insider can be interpreted as a bidder who has become better-informed of his valuation via an information acquisition activity. Thus, one must ask if it is worthwhile for the bidder to acquire the information, given the cost it might entail. A rigorous analysis requires a characterization of equilibria before and after the acquisition. In equilibria characterized for the English auction in Section 4, a bidder who has switched to an insider, only changes the course of the auction when he drops out of the auction without being a winner. More specifically, by becoming an insider, a bidder only increases the price paid by the other

[27] For example, Persico (2000) and Parreiras (2002) adopt the statistical notion of 'accurateness' to define the informativeness of signals.

bidders without benefiting himself.[28] Thus, in view of what will happen in the subsequent auction stage, the information acquisition is not worth its cost. To conclude, while there is a beneficial effect involved in having more insiders, the English auction cannot provide enough incentive to induce bidders to become insiders. In this sense, sealed-bid auctions might be better at facilitating information acquisition.[29] Although sealed-bid auctions lack the efficiency property as seen in Proposition 2 and Theorem 1, some beneficial effect may be expected in terms of revenues. However, I have yet to characterize equilibria for these types of auctions.

Finally, I recognize the possibility that my revenue result depends crucially on the selection of equilibrium in the case multiple equilibria occur. This problem may not be simply ignored, given the fact that English auctions are susceptible to equilibrium multiplicity. Under the standard information setup with symmetric and interdependent values, Bikhchandani, Haile, and Riley (2002) observe that a continuum of symmetric separating equilibria exist and yield the same (ex-post) revenue for the seller. In fact, among these equilibria is the one employed in my equilibrium construction, given that there is no insider. Therefore, when the no insider case is compared to the one insider case, at least one equilibrium exists in the latter case which yields a higher revenue than all symmetric separating equilibria in the former case. The problem of equilibrium multiplicity (or uniqueness) must be explored in my alternative information setup before more can be said about the revenue result.

[28] This easily derives from the construction of the equilibrium strategy.

[29] Using the concept of accurate signals, Persico (2000) shows that the first-price auction induces bidders to acquire more accurate signals than the second-price auction.

Appendix A : Proofs of Theorem 2 and 3

Throughout the proof, we let $O = \{1, 2, \cdots, l\}$ and $I = \{l+1, l+2, \cdots, n\}$, without loss of generality. To simplify notations, $\boldsymbol{p}_{N\backslash A}$, as an argument of functions such as $\boldsymbol{s}_A(p, \boldsymbol{p}_{N\backslash A})$, will be dropped where there is no confusion. I say that two systems of equations are equivalent when they have the same form. To ease the reference, rewrite (8):

$$v_k(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p)) = \begin{cases} p & \text{for } k \in A \\ p_k & \text{for } k \in I\backslash A \end{cases} \tag{A.1}$$

Proof of Theorem 2. I only need to establish the optimality of the equilibrium strategy of an outsider, say, bidder l. Begin with the case in which following the equilibrium strategy, bidder l is supposed to win the object. A deviation in this situation means that l drops out at some price p^* and loses. This deviation can only be profitable if it enables l to avoid a loss resulting from winning at p^*. In order for l to win at p^*, all other active bidders need to drop out at p^*. Letting A be the set of active bidders just before the price reaches p^*, apply the equations (A.1) to obtain $\boldsymbol{s}_A(p)$ and $\boldsymbol{s}_{I\backslash A}(p)$. Since other active outsiders than l drop out at p^*,

$$s_k(p^*) = s_k \text{ for } k \in O \cap A \text{ and } k \neq l. \tag{A.2}$$

Also, since l is supposed to be active beyond p^*,

$$s_l(p^*) < s_l. \tag{A.3}$$

Now, consider the solution $(\hat{s}_l(p), \hat{\boldsymbol{s}}_I(p))$ of equations

$$v_k(\boldsymbol{s}_{O\backslash l}, \hat{s}_l(p), \hat{\boldsymbol{s}}_I(p)) = \begin{cases} p & \text{for } k = l \\ p^* & \text{for } k \in I \cap A \\ p_k & \text{for } k \in I\backslash A. \end{cases} \tag{A.4}$$

Due to (A.2), (A.1) and (A.4) are equivalent if $p = p^*$, so $\hat{s}_l(p^*) = s_l(p^*)$. The fact that insiders drop out at their valuations implies $\hat{s}_l(v_l(\boldsymbol{s})) = s_l$. Therefore, because of (A.3),

$$\hat{s}_l(v_l(\boldsymbol{s})) = s_l > s_l(p^*) = \hat{s}_l(p^*),$$

which implies $v_l(\boldsymbol{s}) > p^*$ since $\hat{s}_l(p)$ is increasing. Thus, bidder l doesn't suffer a loss by winning at p^*.

Now, I consider the case in which l is supposed not to win the object. The only profitable deviation of l is to outstay other active bidders and win the object. Assume w.l.o.g. that once the set of active bidders becomes $\{i, i+1, \cdots, j\}$ with $i \leq l-1$ or $j \geq l+1$, l's equilibrium strategy tells him to drop out at the current price p^* before any other bidder does so. It needs to be shown that l suffers a loss from deviating to be a winner. To ease the notational burden, let the outsiders drop out in the increasing order $i, i+1, \cdots, l-1$ and the insiders in the decreasing order $j, j-1, \cdots, l+1$, and write $\boldsymbol{s}^{i:j}$ to denote the signal profile of bidders from i to j, and similarly for other profiles. When the current price $p \geq p^*$, the break-even signal profile of active outsiders is the solution of

$$v_k(\boldsymbol{s}^{1:i-1}, \boldsymbol{s}^{i:j}(p), \boldsymbol{s}^{j+1:n}(p)) = \begin{cases} p & \text{for } \ k = i, \cdots, j \\ p_k & \text{for } \ k \geq j+1. \end{cases} \tag{A.5}$$

The equilibrium strategy according to which l is supposed to drop out at p^*, means

$$s_l^{i:j}(p) \geq s_l^{i:j}(p^*) = s_l \ \text{ for } \ p \geq p^*. \tag{A.6}$$

After an outsider i dropped out at p_i and his type was revealed to be s_i, the break-even signal profile of the active outsiders for $p \geq p_i$ is the solution of

$$v_k(\boldsymbol{s}^{1:i}, \boldsymbol{s}^{i+1:j}(p), \boldsymbol{s}^{j+1:n}(p)) = \begin{cases} p & \text{for } \ k = i+1, \cdots, j \\ p_k & \text{for } \ k \geq j+1. \end{cases} \tag{A.7}$$

Since $s_i^{i:j}(p_i) = s_i$, (A.5) and (A.7) are equivalent for $k \geq i+1$ if $p = p_i$ and, hence,

$$s_k^{i:j}(p_i) = s_k^{i+1:j}(p_i) \ \text{ for } \ k = i+1, \cdots, l. \tag{A.8}$$

After an insider j dropped out at p_j, the break-even signal profile for $p \geq p_j$ is the solution of

$$v_k(\boldsymbol{s}^{1:i-1}, \boldsymbol{s}^{i:j-1}(p), \boldsymbol{s}^{j:n}(p)) = \begin{cases} p & \text{for } \ k = i, \cdots, j-1 \\ p_k & \text{for } \ k \geq j. \end{cases} \tag{A.9}$$

If $p = p_j$, then (A.5) and (A.9) are equivalent so that

$$s_k^{i:j}(p_j) = s_k^{i:j-1}(p_j) \ \text{ for } \ k = i, \cdots, l. \tag{A.10}$$

Therefore, given (A.6) and, (A.8) and (A.10) with $k = l$,

$$\begin{aligned} s_l^{i+1:j}(p_{i+1}) \geq s_l^{i:j}(p_i) \geq s_l^{i:j}(p^*) = s_l \\ s_l^{i:j-1}(p_{j-1}) \geq s_l^{i:j}(p_j) \geq s_l^{i:j}(p^*) = s_l \end{aligned}$$

because $p_{i+1} \geq p_i$ and $p_{j-1} \geq p_j$, and the break-even signals are increasing with the price. Continuing this way until bidder l and only one other bidder remain, yields

$$\text{either } \ s_l \leq s_l^{l-1:l}(p_{l-1}) \ \text{ or } \ s_l \leq s_l^{l:l+1}(p_{l+1}), \tag{A.11}$$

according to whether l's last competitor is $l-1$ or $l+1$. I first deal with the case $l-1$ is the last competitor so the former inequality is true, while the other case can be analyzed by a parallel argument. Consider the solution of

$$v_k(\boldsymbol{s}^{1:l-1}, s_l(p), \boldsymbol{s}^{l+1:n}(p)) = \begin{cases} p & \text{for } k = l \\ p_k & \text{for } k \geq l+1. \end{cases} \tag{A.12}$$

Because $p_k = v_k(\boldsymbol{s})$ for $k \geq l+1$, we must have that when $p = v_l(\boldsymbol{s})$ in (A.12), $s_l(v_l(\boldsymbol{s})) = s_l$. The fact that $s_{l-1} = s_{l-1}^{l-1:l}(p_{l-1})$ implies that $s_l(p_{l-1}) = s_l^{l-1:l}(p_{l-1})$. Hence, using the first inequality of (A.11) yields

$$s_l(v_l(\boldsymbol{s})) = s_l \leq s_l^{l-1:l}(p_{l-1}) = s_l(p_{l-1}),$$

which implies $v_l(\boldsymbol{s}) \leq p_{l-1}$ because $s_l(p)$ increases with p. This inequality can be made strict if $p_{l-1} > p^*$ because, then, at least one of the inequalities in the above argument has to be strict. Thus, bidder l suffers a loss, paying more than his valuation.

In order to have the given strategy well-defined, the following is necessary: for any $i \in O \cap A$,

$$s_i^{-1}(s_i, \boldsymbol{p}_{N \setminus A}) > \max_{k \in N \setminus A} p_k.$$

Otherwise, the outsider i with s_i will drop out immediately after the price reaches $\max_{k \in N \setminus A} p_k$, making it impossible for others to infer his type. Let $p' = \max_{k \in N \setminus A} p_k$. Let B denote the set of active bidder before the price reaches p'. Then, as in (A.8) or (A.10), $s_i(p', \boldsymbol{p}_{N \setminus A}) = s_i(p', \boldsymbol{p}_{N \setminus B})$. Because i is active at p, it must be that $s_i(p', \boldsymbol{p}_{N \setminus B}) < s_i$ so $s_i(p', \boldsymbol{p}_{N \setminus A}) < s_i$. ***Q.E.D.***

Proof of Theorem 3. Assume wlog that outsiders drop out in the order $1, \cdots, l$, and n is the insider who has the highest valuation among insiders. I begin with the case in which l is the winner, which means that l remains active until the price gets past $v_n(\boldsymbol{s})$. He would suffer a loss if $v_l(\boldsymbol{s}) < v_n(\boldsymbol{s})$, and, thus, $v_l(\boldsymbol{s}) \geq v_n(\boldsymbol{s})$. Now, it suffices to show that l has the highest valuation among outsiders. I first show that $v_{l-1}(\boldsymbol{s}) \leq v_l(\boldsymbol{s})$. Suppose to the contrary that

$$v_{l-1}(\boldsymbol{s}) > v_l(\boldsymbol{s}) \geq v_n(\boldsymbol{s}). \tag{A.13}$$

Consider a deviation by $l-1$ that causes him to survive all the other bidders and win the object. Obviously, $l-1$'s last competitor is either l or n. It cannot be n because if so, the deviation would be profitable since $l-1$ could win the object at the price v_n, which, from

(A.13), is less than his valuation. So, $l-1$'s last competitor must be l. Assuming that l drops out at some p, the corresponding break-even condition is

$$v_k(\boldsymbol{s}_{O\setminus l-1}, s_{l-1}(p), \boldsymbol{s}_I(p)) = \begin{cases} p & \text{for } k = l, l-1 \\ p_k & \text{for } k \in I \end{cases} \tag{A.14}$$

since $s_l(p) = s_l$. Hence, from the fact that insiders drop out at their valuations, two signal profiles $\boldsymbol{s}$ and $(\boldsymbol{s}_{O\setminus l-1}, s_{l-1}(p), \boldsymbol{s}_I(p))$ lie on the same indifference curves of all insiders. Since the fact that $l-1$ has deviated implies that $s_{l-1}(p) > s_{l-1}$, to apply SCIC results in $v_l(\boldsymbol{s}) \geq v_{l-1}(\boldsymbol{s})$ because $p = v_l(\boldsymbol{s}_{O\setminus l-1}, s_{l-1}(p), \boldsymbol{s}_I(p)) = v_{l-1}(\boldsymbol{s}_{O\setminus l-1}, s_{l-1}(p), \boldsymbol{s}_I(p))$. This contradicts (A.13) and leads to the conclusion that $v_l(\boldsymbol{s}) \geq v_{l-1}(\boldsymbol{s})$. For the induction argument, suppose that

$$v_l(\boldsymbol{s}) \geq \max_{j+1 \leq k \leq l-1} v_k(\boldsymbol{s}).$$

It is to be proved that $v_l(\boldsymbol{s}) \geq v_j(\boldsymbol{s})$. Suppose to the contrary that

$$v_j(\boldsymbol{s}) > v_l(\boldsymbol{s}) \geq \max_{j+1 \leq k \leq l-1} v_k(\boldsymbol{s}). \tag{A.15}$$

Consider the deviation that causes j to survive all the other bidders and win. Then, the last competitor of j must be one of outsiders, $j+1, j+2 \cdots, l$. Let him be the bidder h with $j+1 \leq h \leq l$. A parallel argument to the above one can be used to yield $v_h(\boldsymbol{s}) \geq v_j(\boldsymbol{s})$, contradicting (A.15).

Now, consider the case in which n is the winner. If bidder l had a higher valuation than n, then he could have won the object and enjoyed some positive surplus by deviating to defeat n. Hence, it must be that $v_l(\boldsymbol{s}) \leq v_n(\boldsymbol{s})$. Turning to bidder $l-1$, if it were the case that $v_{l-1}(\boldsymbol{s}) > v_n$, then in case he deviates to win, his last competitor would have to be outsider l, as argued above. Again, using the above argument yields the contradiction that $v_{l-1}(\boldsymbol{s}) \leq v_l(\boldsymbol{s}) < v_n$. Thus, $v_{l-1}(\boldsymbol{s}) \leq v_n(\boldsymbol{s})$. The same inductive reasoning as above can be used to conclude that $v_k(\boldsymbol{s}) \leq v_n(\boldsymbol{s})$ for each $k \in O$. ***Q.E.D.***

Appendix B : Proof of Theorem 4

As a preparation for establishing Theorem 4, I prove three lammas. The first two lemmas apply to both E and E', while the last one focuses on the comparison between E and E'.

The first lemma shows that an outsider does not have an incentive to reenter the auction at any point after he dropped out, based on the information revealed up to that point.

Lemma B.1 *Consider any outsider j and price $p > p_j$. Then,*

$$v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p)) \leq p, \tag{B.1}$$

where A is the set of active bidders at p.

Proof. Suppose to the contrary that

$$v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p)) > p. \tag{B.2}$$

The break-even condition at p is

$$v_k(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p)) = \begin{cases} p & \text{for } k \in A \\ p_k & \text{for } k \in I\backslash A \end{cases} \tag{B.3}$$

Consider an alternative setup where bidders in $O\backslash A$ are outsiders and their signal profile is $\boldsymbol{s}_{O\backslash A}$ while the other bidders are insiders and their signal profile is $(\boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p))$. In this setup, j is still an outsider and his valuation is equal to $v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p))$. It is straightforward to see that the same history will unfold up to the price p. So, the bidders in $N\backslash A$ will drop out at the same prices as in the original setup. Bidders in A are all insiders and, thus, will be the winners at p, which, from (B.3), is the valuation of those bidders. However, from (B.2), bidder j has a higher valuation than p so that the allocation would be inefficient, contradicting Theorem 3. ***Q.E.D.***

Next lemma shows that at any point of the auction, the estimated valuation of an outsider, based on the information revealed up to that point, does not exceed his true valuation.

Lemma B.2 *For any price p and outsider j,*

$$v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p)) \leq v_j(\boldsymbol{s}),$$

where A is the set of active bidders at p. This implies that $p_j \leq v_j(\boldsymbol{s})$.

Proof. When j is active at p, the break-even conditions requires

$$v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p)) = p.$$

Thus, $v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p))$ is increasing with p. So, suppose that j is inactive at p. The signal profile $(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p))$ at p is on the indifference curves of the bidders $I\backslash A$. Also, $\boldsymbol{s}_A(p)$ is increasing with p while the signals of the bidders $O\backslash A$ remain the same. Applying GNIC, we get $v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p))$ increasing with p. Hence, $v_j(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\backslash A}(p))$ is maximized at the selling price. Now, let p denote the selling price and m the winner. I only analyze the case m is an outsider because the analysis is similar in the case m is an insider. Then, two signal profiles $(\boldsymbol{s}_{O\backslash m}, s_m(p), \boldsymbol{s}_I(p))$ and $\boldsymbol{s}$ are on the same indifference curves of all insiders. Furthermore, it must be that $s_m(p) < s_m$. Therefore, GNIC can be invoked again to conclude that $v_j(\boldsymbol{s}_{O\backslash m}, s_m(p), \boldsymbol{s}_I(p)) \leq v_j(\boldsymbol{s})$.

Q.E.D.

Lemma B.2 implies that bidder i drops out at a higher price in E' than in E.

Now, I prove that the higher drop-out price of bidder i causes other outsiders to stay active in E' as long as in E. From now on, for any variable x used in E, x' denotes its counterpart in E'. For instance, p'_k denotes the drop-out price of bidder k in E'. Assume wlog that in E, the set of outsiders in E is $O = \{1, 2, \cdots, l\}$ and outsiders drop out in the order, $1, \cdots, l$. Note that outsiders in E drop out at the same prices-their valuations-in E and E'.

Lemma B.3 *For any outsider j of E,*

$$p'_j \geq p_j. \tag{B.4}$$

Proof. From Lemma B.2, we know that $p_i \leq v_i(\boldsymbol{s}) = p'_i$. Clearly, E and E' follow the same history up to the price p_i since bidder i does not drop out in both E and E' until the price reaches p_i. Thus, for an outsider $j < i$, $p'_j = p_j$. If $i = l$, then the proof is done. So, assume $i < l$ and consider an outsider $j > i$.

Suppose by way of contradiction that there is some outsider who violates (B.4). Let j denote the first outsider to drop out violating (B.4) in E' so that $p_i < p'_j < p_j$. Let A and A' denote the set of active bidders in E and E', respectively, when the current price is p'_j. Counting j as an active bidder at p'_j in E', we must have $j \in A \subset A'$. Note that i might or might not be in A' while $i \notin A$. Note also that except for i, the sets A and A' only differ in their outsiders, so $I\backslash A = I\backslash A'$. Write down the break-even conditions at the price $p = p'_j$ in E and E'. In E, it is

$$v_k(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p'_j), \boldsymbol{s}_{I\backslash A}(p'_j)) = \begin{cases} p'_j & \text{for } \ k \in A \\ v_k(\boldsymbol{s}) & \text{for } \ k \in I\backslash A, \end{cases} \tag{B.5}$$

while, in E', it is

$$v_k(\boldsymbol{s}_{O\backslash A'}, \boldsymbol{s}'_{A'}(p'_j), s'_i(p'_j), \boldsymbol{s}'_{I\backslash A}(p'_j)) = \begin{cases} p'_j & \text{for } \ k \in A' \\ v_k(\boldsymbol{s}) & \text{for } \ k \in (I \cup \{i\})\backslash A'. \end{cases} \tag{B.6}$$

Note that the insiders' drop-out prices are equal to their valuations.

Let $\boldsymbol{s}(p'_j)$ and $\boldsymbol{s}'(p'_j)$ denote $(\boldsymbol{s}_{O\backslash A}, \boldsymbol{s}_A(p'_j), \boldsymbol{s}_{I\backslash A}(p'_j))$ and $(\boldsymbol{s}_{O\backslash A'}, \boldsymbol{s}'_{A'}(p'_j), s'_i(p'_j), \boldsymbol{s}'_{I\backslash A}(p'_j))$, respectively. The assumption that j drops out at p'_j in E' while being active in E, implies that

$$s_j(p'_j) < s_j = s'_j(p'_j), \tag{B.7}$$

of which I will obtain the contradiction. In order to do so, I first prove the following claim.

Claim 1 $s'_i(p'_j) \geq s_i$.

Proof. If $i \in A'$, then $s'_i(p'_j) > s'_i(p_i) = s_i(p_i) = s_i$, for $s'_i(\cdot)$ is increasing and $p'_j > p_i$. Hence, assume $i \notin A'$ and thus $p'_i = v_i(\boldsymbol{s}'(p'_j))$. Suppose to the contrary that $s'_i(p'_j) < s_i$. Since the bidders in $A'\backslash A$ are all active as outsiders in E', but not in E, it must be true that $\boldsymbol{s}'_{A'\backslash A}(p'_j) \ll \boldsymbol{s}_{A'\backslash A}$ and $p_k < p'_j$ for $k \in A'\backslash A$.[30] Observe that $\boldsymbol{s}(p'_j)$ and $\boldsymbol{s}'(p'_j)$ belong to the same indifference curves of the bidders $I \cup A$, and the signals of the bidders $(A'\backslash A) \cup \{i\}$ are higher with $\boldsymbol{s}(p'_j)$ than with $\boldsymbol{s}'(p'_j)$ while the signals of the bidders $O\backslash A'$ remain the same. GNIC can be applied to yield $p'_i = v_i(\boldsymbol{s}'(p'_j)) < v_i(\boldsymbol{s}(p'_j))$. However, Lemma B.2 requires $v_i(\boldsymbol{s}) = p'_i < v_i(\boldsymbol{s}(p'_j)) \leq v_i(\boldsymbol{s})$, a contradiction. ∎

Claim 1 says that the signal of insider i in E' is (weakly) overestimated at the price p'_j. Using this, I complete the proof by proving that the break-even signals of active bidders at p'_j in E are (weakly) greater in E than in E', which contradicts (B.7).

Claim 2 $\boldsymbol{s}'_A(p'_j) \leq \boldsymbol{s}_A(p'_j)$.

Proof. Define a new set of equations from (B.6) by replacing $s'_i(p'_j)$ with s_i and, then, dropping the equation for bidder i. Denote the resulting set of equations by $\hat{M}$ and its solution by $\hat{\boldsymbol{s}} = (\boldsymbol{s}_{O\backslash A'}, \hat{\boldsymbol{s}}_{A'}, s_i, \hat{\boldsymbol{s}}_{I\backslash A})$. Since $s_i \leq s'_i(p'_j)$ from Claim 1, it must be true that

$$\hat{\boldsymbol{s}}_{A'} \geq \boldsymbol{s}'_{A'}(p'_j). \tag{B.8}$$

Otherwise, or if $\hat{s}_k < s'_k(p'_j)$ for some $k \in A'$, then it can easily be shown using GNIC that $v_k(\hat{\boldsymbol{s}}) < p'_j$, which cannot be true by the definition of $\hat{\boldsymbol{s}}$.[31] Also,

$$\hat{s}_k \geq s_k \text{ for at least one } k \in A'\backslash A. \tag{B.9}$$

[30] It might be that $A'\backslash A = \emptyset$.

[31] The argument to show this goes as follow. Two signal profiles $\boldsymbol{s}'(p'_j)$ and $\hat{\boldsymbol{s}}$ are on the same indifference curves of the bidders $I \cup A' - \{k, i\}$. Also, the signals of i and k decrease from $s'_i(p'_j)$ and $s'_k(p'_j)$ to s_i and $\hat{s}_k$, respectively. Thus, GNIC implies that $v_k(\hat{\boldsymbol{s}}) < p'_j$ since $v_k(\boldsymbol{s}(p'_j)) = p'_j$.

Otherwise, $\hat{\boldsymbol{s}}_{A'\backslash A} \ll \boldsymbol{s}_{A'\backslash A}$ holds so that GNIC requires $p'_j = v_k(\hat{\boldsymbol{s}}) < v_k(\boldsymbol{s}(p'_j))$ for all $k \in A'\backslash A$ since both $\hat{\boldsymbol{s}}$ and $\boldsymbol{s}(p'_j)$ are on the same indifference curves of bidders $I \cup A$ and the signals of bidders $A'\backslash A$ are higher with $\boldsymbol{s}(p'_j)$ than with $\hat{\boldsymbol{s}}$ while other signals remain the same. This, however, contradicts Lemma B.1, which requires $v_k(\boldsymbol{s}(p'_j)) \leq p'_j$ because $p'_j > p_k$ for $k \in A'\backslash A$. Now, pick any outsider $k \in A'\backslash A$ for whom $\hat{s}_k \geq s_k$, and modify $\hat{M}$ by replacing $\hat{s}_k$ with s_k and dropping the equation for bidder k. Denote the resulting set of equations by $\hat{\hat{M}}$ and its solution by $\hat{\hat{\boldsymbol{s}}} = (\boldsymbol{s}_{O\backslash A'}, \hat{\hat{\boldsymbol{s}}}_{A'\backslash k}, s_k, s_i, \hat{\hat{\boldsymbol{s}}}_{I\backslash A})$. Since $s_k \geq \hat{s}_k$, it must hold that

$$\hat{\hat{\boldsymbol{s}}}_{A'\backslash k} \geq \hat{\boldsymbol{s}}_{A'\backslash k}, \text{ and } \hat{\hat{s}}_{k'} \geq s_{k'} \text{ for at least one } k' \in A'\backslash A - \{k\}, \tag{B.10}$$

for the same reason as (B.8) and (B.9) hold. This replacement can be performed one at a time until the signals of entire bidders in $A'\backslash A$ are replaced by $\boldsymbol{s}_{A'\backslash A}$ their true signals. During this process, the signals of bidders A keep (weakly) increasing as in (B.8) and (B.10), and finally become equal to $\boldsymbol{s}_A(p'_j)$, thus completing the proof.[32] ■

Q.E.D.

Finally, I prove Theorem 4, using Lemma B.2 and B.3.

Proof of Theorem 4. Since the equilibria are efficient, E and E' have the same winner. Thus, in the case i is a winner in E, he is also a winner in E'. Hence, all bidders but i will drop out at the same prices in E and E' so that the selling price does not change. Suppose alternatively that in E, i is not a winner, but a runner-up, causing the winner to pay p_i. In this case, i is also a runner-up in E' because i now drops out at his valuation, which is greater than p_i from Lemma B.2, and hence, the bidder who drops out before i in E, also drops out before i in E'. So, the selling price (weakly) increases from p_i to $v_i(\boldsymbol{s})$ as i becomes an insider. Lastly, suppose that i is neither a winner nor a runner-up in E. We can then combine Lemma B.2 and B.3 to obtain

$$P(\boldsymbol{s}) = \max_{k\in I}\left\{\max_{j\in O}\{p_j\}, v_k(\boldsymbol{s})\right\} \leq \max_{k\in I}\left\{\max_{j\in O}\{p'_j\}, v_k(\boldsymbol{s})\right\} = P'(\boldsymbol{s}).$$

Q.E.D.

Appendix C : Other Proofs

Proof of Proposition 2. A few notations are introduced first. Define a function α as in the proof of Proposition 1, that is $\alpha : [0, \overline{s}_2] \to [0, 1]$ satisfies $v_1(\alpha(s_2), s_2) = v_2(\alpha(s_2), s_2))$,

[32] After all the replacements, the resulting set of equations becomes (B.5), which is why the signals of A become equal to $\boldsymbol{s}_A(p'_j)$, the solution of (B.5).

where $\overline{s}_2$ is the highest s_2 at which $\alpha(s_2)$ is well-defined. It is easy to verify that $\alpha(0) = 0$, $\alpha(\cdot)$ is (strictly) increasing with s_2, and $\overline{s}_2 > 0$. For notational convenience, denote $v_s = v_1(\alpha(s), s) = v_2(\alpha(s), s)$ for $s \in [0, \overline{s}_2]$, and $\overline{v} = v_1(\alpha(\overline{s}_2), \overline{s}_2)$. $F_{S_2|V_1}(\cdot|v)$ and $f_{S_2|V_1}(\cdot|v)$ denote the marginal distribution of S_2 on conditional $V_1 = v$, and its density, respectively. $F_{S_1|S_2}$ and $f_{S_1|S_2}$ are similarly defined.

Part 1: Suppose to the contrary that there exists an efficient equilibrium, which requires that bidder 2 of type $s \in [0, \overline{s}_2]$ wins if and only if $s_1 < \alpha(s)$. Then, I claim that bidder 2 must bid v_s. Any bid $b > v_s$ would defeat bidder 1 of such type $s_1 > \alpha(s)$ that $b > v_1(s_1, s) > v_s$, which causes an inefficiency. Likewise, any bid $b < v_s$ would be defeated by bidder 1 of such type $s_1 < \alpha(s)$ that $b < v_1(s_1, s) < v_s$, which also causes an inefficiency. Thus, I conclude that bidder 2 of type s wins and pays v_s if and only if $s_1 < \alpha(s)$. However,

$$v_2(s_1, s_2) < v_s = v_2(\alpha(s), s) = v_1(\alpha(s), s),$$

which would yield a loss to bidder 2.

Part 2: The following claim is needed.[33]

Claim 1 *There exists an efficient equilibrium only if*

$$\frac{F_{S_2|V_1}(s|v_s)}{f_{S_2|V_1}(s|v_s)} = \frac{F_{S_1|S_2}(\alpha(s)|s)}{\alpha'(s) f_{S_1|S_2}(\alpha(s)|s)} \text{ for all } s \in [0, \overline{s}_2]. \tag{C.1}$$

Proof. Let $b_1 : v \in [0, \overline{v}_1] \mapsto b_1(v) \in \mathbb{R}_+$ and $b_2 : s \in [0,1] \mapsto b_2(s) \in \mathbb{R}_+$ denote the bidding strategies of bidder 1 and 2, respectively, where $\overline{v}_1 = \max_{\boldsymbol{s} \in [0,1]^2} v_1(\boldsymbol{s})$. Suppose that these bidding functions lead to the efficient allocation, which means that bidder 1 wins if and only if $\alpha(s_1) \geq s_2$. It follows that b_1 and b_2 are strictly increasing and thus differentiable almost everywhere in $[0, \overline{v}]$ and $[0, \overline{s}_2]$, respectively. Also, it must be that

$$b_1(v_s) = b_2(s) \text{ for (almost every) } s \in [0, \overline{s}_2]. \tag{C.2}$$

In equilibrium, bidder 1 with $v_s \in (0, \overline{v})$ chooses

$$v_s \in \arg\max_{v \in [0, \overline{v}]} (v_s - b_1(v)) F_{S_2|V_1}(b_2^{-1}(b_1(v))|v_s).$$

Differentiating the objective function with v and setting $v = v_s$ give

$$-b_1'(v_s) F_{S_2|V_1}(s|v_s) + (v_s - b_1(v_s)) f_{S_2|V_1}(s|v_s) \frac{b_1'(v_s)}{b_2'(s)} = 0,$$

[33] This claim itself almost suffices to yield an inefficiency for the generic distribution and valuation functions. Suppose for instance that (C.1) holds for a given distribution and given valuation functions. If either the distribution or the valuations are slightly perturbed, then (C.1) will fail.

since $b_2^{-1}(b_1(v_s)) = s$ due to (C.2). Some rearrangement yields

$$\frac{F_{S_2|V_1}(s|v_s)}{f_{S_2|V_1}(s|v_s)} = \frac{(v_s - b_1(v_s))}{b_2'(s)}. \tag{C.3}$$

As for bidder 2 with $s \in [0, \overline{s}]$, he has to choose

$$\begin{aligned} s \in \arg\max_{s' \in [0,\overline{s}]} & \left(\mathbb{E}[v_2(S_1, s) \mid S_1 \leq \alpha(s')] - b_2(s')\right) F_{S_1|S_2}(\alpha(s')|s) \\ &= \int_0^{\alpha(s')} v_2(t, s) dF_{S_1|S_2}(t|s) - F_{S_1|S_2}(\alpha(s')|s) b_2(s'). \end{aligned}$$

Differentiating the objective function with s' and setting $s' = s$ yield after some rearrangement

$$\begin{aligned} \alpha'(s) f_{S_1|S_2}(\alpha(s)|s)(v(\alpha(s), s) - b_2(s)) &= F_{S_1|S_2}(\alpha(s)|s) b_2'(s) \\ \text{or } \frac{F_{S_1|S_2}(\alpha(s)|s)}{\alpha'(s) f_{S_1|S_2}(\alpha(s)|s)} &= \frac{(v_s - b_1(v_s))}{b_2'(s)}, \end{aligned} \tag{C.4}$$

since $v(\alpha(s), s) = v_s$ and $b_2(s) = b_1(v_s)$ from (C.2). By combining (C.3) and (C.4), (C.1) is obtained. ■

Now, it would complete the proof to show that $F_{S_2|V_1}(s|v_s)$, $f_{S_2|V_1}(s|v_s)$, $f_{S_1|S_2}(\alpha(s)|s)$, and $\alpha'(s)$ are all bounded away from zero for any s while $F_{S_1|S_2}(\alpha(s)|s)$ becomes arbitrarily small with sufficiently small s, which contradicts (C.1). Start with a few observations. The first one is that the twice differentiable valuation functions on the compact set and their positive partial derivatives imply that for some finite $K > 1$,

$$\frac{1}{K} < \frac{\partial v_j}{\partial s_i}(\boldsymbol{s}) < K \text{ for all } i, j \text{ and } \boldsymbol{s}. \tag{C.5}$$

Combined with this, the single crossing condition implies that for some finite $L > 1$,

$$\frac{1}{L} < \alpha'(s) = \frac{\frac{\partial v_2}{\partial s_2} - \frac{\partial v_1}{\partial s_2}}{\frac{\partial v_1}{\partial s_1} - \frac{\partial v_2}{\partial s_1}} < L \text{ for all } s \in [0, \overline{s}_2]. \tag{C.6}$$

From (5), the last observation is that for all $(s_1, s_2) \in [0, 1]^2$ and $s \in [0, \overline{s}_2]$,

$$\frac{1}{R^2} < f_{S_1|S_2}(s_1|s_2), f_{S_2|V_1}(s|v_s) < R^2 \tag{C.7}$$

So, $f_{S_2|V_1}(s|v_s)$, $f_{S_1|S_2}(\alpha(s)|s)$, and $\alpha'(s)$ are bounded away from zero. Now, we show that $F_{S_2|V_1}(s|v_s)$ is also bounded away from zero. Consider an arbitrary $s \in [0, \overline{s}_2]$ and define

$A = \{\boldsymbol{s} \mid s_2 \le s \text{ and } v(\boldsymbol{s}) = v_s\}$ and $B = \{\boldsymbol{s} \mid s_2 \ge s \text{ and } v(\boldsymbol{s}) = v_s\}$ Then, with (C.5) and (C.6), it is not hard to show that for some finite M,[34]

$$\int_{\boldsymbol{s}\in B} 1 < M \int_{\boldsymbol{s}\in A} 1. \tag{C.8}$$

We thus have

$$\begin{aligned} F_{S_2|V_1}(s|v_s) = \frac{\int_A f(\boldsymbol{s})}{\int_A f(\boldsymbol{s}) + \int_B f(\boldsymbol{s})} &= \frac{1}{1 + \frac{\int_B f(\boldsymbol{s})}{\int_A f(\boldsymbol{s})}} \\ &> \frac{1}{1 + \frac{R^2 \int_B 1}{\int_A 1}} > \frac{1}{1 + MR^2}, \end{aligned}$$

where the first inequality follows from (5) and the second one from (C.8). Therefore, $F_{S_2|V_1}(s|v_s)$, $f_{S_2|V_1}(s|v_s)$, $f_{S_1|S_2}(\alpha(s)|s)$, and $\alpha'(s)$ are all bounded away from zero for any s. That $F_{S_1|S_2}(\alpha(s)|s)$ can be made arbitrarily small follows from the facts that for small s, $\alpha(s)$ is also small, and that the conditional density $f_{S_1|S_2}$ is uniformly bounded above by K^2. ***Q.E.D.***

Proof of Theorem 1. I provide the proof when $n = 3$. The general proof follows the same line.

Suppose to the contrary that there exists an efficient equilibrium. For a given bidder i, let's define

$$E_i := \{\boldsymbol{s} \in [0,1]^n \mid \ v_i(\boldsymbol{s}) \ge v_j(\boldsymbol{s}) \ \text{ for } \ j \ne i\}$$

to be the set of other bidders' signals for which bidder i is an efficient bidder. Because of the assumption that in the efficient allocation, insiders obtain the good with some positive probability less than one, there must be some outsider, say bidder 1, and some insider, say bidder 2, such that $E_1 \cap E_2$ is 2-dimensional manifold. Now, for a given s_1, define

$$E_{12}(s_1) = \{\boldsymbol{s}' | s_1' = s_1 \text{ and } \boldsymbol{s}' \in E_1 \cap E_2\}.$$

Then, it must be that for a non-zero measure set of s_1, $E_{12}(s_1)$ has a positive length. Fix any such s_1. Given the efficient allocation and truthful bidding of bidder 2, the bid $b_1(s_1)$ of bidder 1 with s_1 has to satisfy

$$\max_{\{\boldsymbol{s}'\in E_1|s_1'=s_1\}} v_2(\boldsymbol{s}') \le b_1(s_1) \le \min_{\{s'\in E_2|s_1'=s_1\}} v_2(\boldsymbol{s}'). \tag{C.9}$$

[34]Noth that each integral measures the length of the sets A and B. The proof of this inequality follows from the fact that the slope of $\alpha(\cdot)$ and the slope of the indifference curve $v(\boldsymbol{s}) = v_s$ are both bounded away from zero and bounded above.

If the first inequality is violated, then the good sometimes does not go to bidder 1 even though he has the highest valuation. The second inequality is necessary for bidder 2 to obtain the good when he is an efficient bidder. Equation (C.9) requires $v_2(\boldsymbol{s})$ to be a constant, say k, for all $\boldsymbol{s} \in E_{12}(s_1)$. Otherwise, the maximum in LHS of (C.9) would be greater than the minimum in RHS. So, it must be that $v_1(\boldsymbol{s}) = v_2(\boldsymbol{s}) = k$ for all $\boldsymbol{s} \in E_{12}(s_1)$, which cannot be satisfied by generic valuation functions.

The second part can be derived directly from (C.9), which, by the symmetry, becomes

$$\begin{aligned}\max_{\{(s_2',s_3')|s_1\geq\max\{s_2',s_3'\}\}} v(s_1;s_2',s_3') &= v(s_1;s_1,s_1)\\ &\leq v(s_1;s_1,0) = \min_{\{(s_2',s_3')|s_2'\geq\max\{s_1,s_3'\}\}} v_2(s_1,s_2',s_3'),\end{aligned}$$

a contradiction for $s_1 > 0$. ***Q.E.D.***

I borrow a lemma from Krishna (2001) to prove Lemma 1. Say that an $m \times m$ matrix $\boldsymbol{A}$ satisfies the *dominant average* condition if

$$\frac{1}{m}\sum_{k=1}^{m} a_{kj} > \ a_{ij}, \ \text{ for all } \ i \neq j \tag{C.10}$$

and

$$\sum_{k=1}^{m} a_{kj} > 0, \ \text{ for all } \ j. \tag{C.11}$$

Krishna (2001) establishes the following lemma.

Lemma C.0 *Suppose $\boldsymbol{A}$ is an $m \times m$ matrix that satisfies the dominant average condition. Then, $\boldsymbol{A}$ is invertible. Also, there exists a unique $\boldsymbol{x} \gg 0$ such that*

$$\boldsymbol{A}\boldsymbol{x} = \mathbf{1},$$

where $\mathbf{1}$ is a column vector of m 1's.

Before giving the proof of Lemma 1, I introduce some notations. First,

$$\boldsymbol{v}'_{A \cdot B}(\boldsymbol{s}) \equiv (v'_{ij}(\boldsymbol{s}))_{i\in A, j\in B},$$

where $v'_{ij}(\boldsymbol{s}) = \frac{\partial v_i(\boldsymbol{s})}{\partial s_j}$. Let $\mathbf{0}_A$ and $\mathbf{1}_A$ denote column vectors of $|A|$ 0's and 1's, respectively.

Proof of Lemma 1. The proof is similar to that of Lemma 2 in Krishna (2001). The main difference comes from the addition of the second set of equations in (8). Start with the case $A = N$ in which no one has dropped out yet. Differentiating (8) with p yields

$$\boldsymbol{v}'_{N\cdot N}(\boldsymbol{s}')\boldsymbol{s}'(p) = \mathbf{1}_N.$$

Therefore, unique increasing solution of (8) exists if and only if there exists unique increasing solution of the following differential equation :

$$\begin{aligned} \boldsymbol{v}'_{N\cdot N}(\boldsymbol{s}(p))\boldsymbol{s}'(p) &= \mathbf{1}_N \\ \boldsymbol{s}(0) &= \mathbf{0}_N. \end{aligned}$$

Since valuations functions are twice continuously differentiable, and $\boldsymbol{v}'_{N\cdot N}$ is invertible and satisfies the dominant average condition, Peano's theorem and Lemma C.0 imply that there exists a unique increasing solution of the above differential equation for $p \leq \min_{k\in O} s_k^{-1}(1)$.[35]

Now, suppose that the set of active bidders is A and the unique solution of (8) exists up to the price $p' = \max_{k\in N\setminus A} p_k$. I extend the solution to the price $p \in [p', \min_{k\in O\cap A} s_j^{-1}(1)]$. Given $\boldsymbol{s}_{O\setminus A}$ has been revealed, let $(\underline{\boldsymbol{s}}_A, \underline{\boldsymbol{s}}_{I\setminus A})$ denote the solution of (8) at p'. Differentiate both sides of (8) with p to get

$$\begin{aligned} \begin{pmatrix} \boldsymbol{v}'_{A\cdot A}(\boldsymbol{s}^A(p)) & \boldsymbol{v}'_{A\cdot I\setminus A}(\boldsymbol{s}^A(p)) \\ \boldsymbol{v}'_{I\setminus A\cdot A}(\boldsymbol{s}^A(p)) & \boldsymbol{v}'_{I\setminus A\cdot I\setminus A}(\boldsymbol{s}^A(p)) \end{pmatrix} \begin{pmatrix} \boldsymbol{s}'_A(p) \\ \boldsymbol{s}'_{I\setminus A}(p) \end{pmatrix} &= \begin{pmatrix} \mathbf{1}_A \\ \mathbf{0}_{I\setminus A} \end{pmatrix} \\ (\boldsymbol{s}_A(p'), \boldsymbol{s}_{I\setminus A}(p')) &= (\underline{\boldsymbol{s}}_A, \underline{\boldsymbol{s}}_{I\setminus A}), \end{aligned} \tag{C.13}$$

where $\boldsymbol{s}^A(p) = (\boldsymbol{s}_{O\setminus A}, \boldsymbol{s}_A(p), \boldsymbol{s}_{I\setminus A}(p))$. Indeed, a unique solution of (8) exists if and only if the unique solution $(\boldsymbol{s}_A(p), \boldsymbol{s}_{I\setminus A}(p))$ of this differential equation exists for $p \in [p', \min_{k\in O\cap A} s_j^{-1}(1)]$, which can be shown using a parallel argument to the one in the last paragraph. I have yet to show that $\boldsymbol{s}_A(p)$ is increasing. To begin with, dropping the arguments and rearranging, write the last $|I\setminus A|$ lines of (C.13) as

$$\boldsymbol{v}'_{I\setminus A\cdot I\setminus A}\boldsymbol{s}'_{I\setminus A} = -\boldsymbol{v}'_{I\setminus A\cdot A}\boldsymbol{s}'_A$$

or

$$\boldsymbol{s}'_{I\setminus A} = -(\boldsymbol{v}'_{I\setminus A\cdot I\setminus A})^{-1}\boldsymbol{v}'_{I\setminus A\cdot A}\boldsymbol{s}'_A.$$

Substitute this into the first $|A|$ lines of (C.13) to obtain after some rearrangement

$$\left(\boldsymbol{v}'_{A\cdot A} - \boldsymbol{v}'_{A\cdot I\setminus A}(\boldsymbol{v}'_{I\setminus A\cdot I\setminus A})^{-1}\boldsymbol{v}'_{I\setminus A\cdot A}\right)\boldsymbol{s}'_A = \mathbf{1}_A. \tag{C.14}$$

Let's define

$$\boldsymbol{V} \equiv \boldsymbol{v}'_{A\cdot A} - \boldsymbol{v}'_{A\cdot I\setminus A}(\boldsymbol{v}'_{I\setminus A\cdot I\setminus A})^{-1}\boldsymbol{v}'_{I\setminus A\cdot A}. \tag{C.15}$$

[35] In fact, this can be true only in the case

$$\min_{k\in O} s_k^{-1}(1) \leq \min_{k\in I} s_k^{-1}(\bar{s}_k), \tag{C.12}$$

since otherwise, for p such that $s_j^{-1}(\bar{s}_j) < p \leq \min_{k\in O} s_k^{-1}(1)$ with some $j \in I$, the solution $s_j(p)$ has to go beyond $\bar{s}_j$, which cannot happen. However, it is easy to verify that (2) guarantees (C.12).

It is straightforward to see that ACIC implies each column of $\boldsymbol{V}$ must satisfy (C.10) and (C.11). Therefore, $\boldsymbol{V}$ satisfies the dominant average condition. From Lemma C.0 and (C.14), we have $\boldsymbol{s}'_A \gg 0$. ***Q.E.D.***

Proof of Lemma 2. Wlog, let $i = 1$ and $j = 2$. Suppose that

$$v_1(\boldsymbol{s}) > v_2(\boldsymbol{s}). \tag{C.16}$$

Consider such a signal profile $\boldsymbol{s}''$ that $s_3{}'' = s_3{}'$ and $s_j{}'' = s_j$ for $j \neq 3$. Then,

$$v_1(\boldsymbol{s}'') - v_1(\boldsymbol{s}) = \int_{s_3}^{s_3{}'} \frac{\partial v_1(t, \boldsymbol{s}_{-3})}{\partial s_3} dt = \int_{s_3}^{s_3{}'} \frac{\partial v_2(t, \boldsymbol{s}_{-3})}{\partial s_3} dt = v_2(\boldsymbol{s}'') - v_2(\boldsymbol{s}),$$

where the second equality follows from (19). Combining this equality with (C.16) yields

$$v_1(\boldsymbol{s}'') > v_2(\boldsymbol{s}''). \tag{C.17}$$

Construct another signal vector $\boldsymbol{s}'''$ for which $s_4{}''' = s_4{}'$ and $s_j{}''' = s_j{}''$ for $j \neq 4$. A similar derivation shows that $v_1(\boldsymbol{s}''') - v_1(\boldsymbol{s}'') = v_2(\boldsymbol{s}''') - v_2(\boldsymbol{s}'')$. Combining this with (C.17) yields $v_1(\boldsymbol{s}''') > v_2(\boldsymbol{s}''')$. Changing the elements of $\boldsymbol{s}$ one at a time leads to the conclusion that $v_1(\boldsymbol{s}') > v_2(\boldsymbol{s}')$.

Q.E.D.

Proof of Proposition 3. Let A be the set of active bidders just before i drops out. If i drops out before j, the break-even condition at p_i is

$$v_i(\boldsymbol{s}_{O\setminus A}, s_i, \boldsymbol{s}_{A\setminus i}(p_i), \boldsymbol{s}_{I\setminus A}(p_i)) = p_i = v_j(\boldsymbol{s}_{O\setminus A}, s_i, \boldsymbol{s}_{A\setminus i}(p_i), \boldsymbol{s}_{I\setminus A}(p_i)).$$

Since $s_j > s_j(p_i)$, the single-crossing property implies

$$(\boldsymbol{s}_{O\setminus A}, s_i, s_j, \boldsymbol{s}_{A\setminus\{i,j\}}(p_i), \boldsymbol{s}_{I\setminus A}(p_i)) < v_j(\boldsymbol{s}_{O\setminus A}, s_i, s_j, \boldsymbol{s}_{A\setminus\{i,j\}}(p_i), \boldsymbol{s}_{I\setminus A}(p_i)),$$

which is equivalent to $v_i(\boldsymbol{s}) < v_j(\boldsymbol{s})$, according to Lemma 2. ***Q.E.D.***

Proof of Lemma 3. I prove this lemma by showing that if v_i's satisfy (19), then the matrix $\boldsymbol{V}$ in (C.15) satisfies (C.10) and (C.11). Let $g'_k = \frac{\partial g}{\partial s_k}$ and $\boldsymbol{g}'_A = (g'_k)_{k\in A}$, where $\boldsymbol{g}'_A$ is a column vector. $\boldsymbol{D}_A$ denotes the diagonal matrix whose diagonal entry is $v'_{kk} - g'_k$ for $k \in A$. Because of (19), any given column of $\boldsymbol{v}'_{A\cdot A}$ has the identical non-diagonal entries. Also, any given column of $\boldsymbol{v}'_{A\cdot B}$ for $A \cap B = \emptyset$ has the identical entries. Thus, for any $A, B \subset N$,

$$\boldsymbol{v}'_{A\cdot B} = \begin{cases} \boldsymbol{D}_A + \mathbf{1}_A(\boldsymbol{g}'_A)^t & \text{if } A = B \\ \mathbf{1}_A(\boldsymbol{g}'_B)^t & \text{if } A \cap B = \emptyset, \end{cases}$$

where $(\cdot)^t$ denotes the transpose of a matrix. Now, rewrite $\boldsymbol{V}$ in (C.14) as

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{D}_A + \mathbf{1}_A(\boldsymbol{g}'_A)^t - \mathbf{1}_A(\boldsymbol{g}'_{I\setminus A})^t \left(\boldsymbol{D}_{I\setminus A} + \mathbf{1}_{I\setminus A}(\boldsymbol{g}'_{I\setminus A})^t\right)^{-1} \mathbf{1}_{I\setminus A}(\boldsymbol{g}'_A)^t \\
&= \boldsymbol{D}_A + \mathbf{1}_A(\boldsymbol{g}'_A)^t - x\mathbf{1}_A(\boldsymbol{g}'_A)^t = \boldsymbol{D}_A + (1-x)\mathbf{1}_A(\boldsymbol{g}'_A)^t, \qquad \text{(C.18)}
\end{aligned}$$

where $x = (\boldsymbol{g}'_{I\setminus A})^t \left(\boldsymbol{D}_{I\setminus A} + \mathbf{1}_{I\setminus A}(\boldsymbol{g}'_{I\setminus A})^t\right)^{-1} \mathbf{1}_{I\setminus A}$. All the entries in any given column of the matrix $\mathbf{1}_A(\boldsymbol{g}'_A)^t$ are identical and the diagonal entries of $\boldsymbol{D}_A$ are positive. Thus, it is easily verified that $\boldsymbol{V}$ satisfies (C.10). Since $\boldsymbol{g}'_A \geq 0$, if $x < 1$ in (C.18), then all entries of $\boldsymbol{V}$ are non-negative and the diagonal ones are strictly positive, which will imply that (C.11) holds and thus both ACIC and GNIC are satisfied. Hence, the proof is completed by showing $x < 1$:

$$\begin{aligned}
x &= (\boldsymbol{g}'_{I\setminus A})^t \left(\boldsymbol{D}_{I\setminus A} + \mathbf{1}_{I\setminus A}(\boldsymbol{g}'_{I\setminus A})^t\right)^{-1} \mathbf{1}_{I\setminus A} \\
&= (\boldsymbol{g}'_{I\setminus A})^t \left(\boldsymbol{D}^{-1}_{I\setminus A} - \left(\frac{1}{1 + (\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A}}\right) \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A}(\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\right) \mathbf{1}_{I\setminus A} \\
&= (\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A} - \frac{\left((\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A}\right)^2}{1 + (\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A}} \\
&= \frac{(\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A}}{1 + (\boldsymbol{g}'_{I\setminus A})^t \boldsymbol{D}^{-1}_{I\setminus A}\mathbf{1}_{I\setminus A}} = \frac{\sum_{k\in I\setminus A} g'_k/(v'_{kk} - g'_k)}{1 + \sum_{k\in I\setminus A} g'_k/(v'_{kk} - g'_k)} < 1
\end{aligned}$$

since $g'_k \geq 0$ and $v'_{kk} > g'_k$, where the second equality was derived using the formula for an inverse matrix,

$$\left(\boldsymbol{A} + \boldsymbol{b}\boldsymbol{c}^t\right)^{-1} = \boldsymbol{A}^{-1} - \left(\frac{1}{1 + \boldsymbol{c}^t\boldsymbol{A}^{-1}\boldsymbol{b}}\right)\boldsymbol{A}^{-1}\boldsymbol{b}\boldsymbol{c}^t\boldsymbol{A}^{-1},$$

with $\boldsymbol{A} = \boldsymbol{D}_{I\setminus A}$, $\boldsymbol{b} = \mathbf{1}_{I\setminus A}$, and $\boldsymbol{c} = \boldsymbol{g}'_{I\setminus A}$, and it is routine to verify the last equality.

Q.E.D.

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