MBS TECHNICAL REPORT 17-02

On a meaningful axiomatic derivation of some relativistic equations

Jean-Claude Falmagne*
University of California, Irvine

Abstract

The mathematical expression of a scientific or geometric law typically does not depend on the units of measurement. This makes sense because measurement units have no representation in nature. Any mathematical model or law whose form would be fundamentally altered by a change of units would be a poor representation of the empirical world. This paper formalizes this invariance of the form of the laws as a *meaningfulness* axiom. In the context of this axiom, relatively weak, intuitive constraints may suffice to generate standard scientific or geometric formulas, possibly up to some numerical parameters. We give several example of such constructions, with a focus of some relativistic formulas.

When properly formalized, the invariance of the mathematical form of a scientific or geometric law under changes of units becomes a powerful '*meaningfulness*' axiom. Combining thismeaningfulness axiom with abstract, intuitive, 'gedanken' properties such as associativity, permutability, bisymmetry, or other conditions in the same vein, enables the derivation of scientific or geometrical laws (possibly up to some parameter values). In the last part of this paper, I will show how, in the context of meaningfulness, the axiom

$$L(L(\ell, v), w) = L(\ell, v \oplus w) \tag{1}$$

yields specific numerical expressions for the function L and the operation $\oplus$.

Equation (1) is an *abstract axiom* representing the mechanisms possibly involved in the Lorentz-FitzGerald Contraction (Feynman, Leighton, and Sands, 1963, Vol. 1, 15-3) or the Doppler effect. The operation $\oplus$ represents the relativistic addition of velocities. The left hand side of Equation (1) formalizes an iteration of the function L. The equation states that such an iteration amounts to adding a velocity via the relativistic addition of velocities operation.

A. Motivating the meaningfulness condition

The trouble with an equation such as

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2}, \tag{2}$$

*I thank Chris Doble, Jean-Paul Doignon, and Louis Narens for their collaboration on various part of my work in this area.

representing the Lorentz-FitzGerald Contraction is its ambiguity: the units of ℓ, which denotes the length of an object, and of v and c, for the speed of the observer and the speed of light, are not specified. Writing $L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to 70 meters and 3 kilometers per second, respectively. While such a parenthetical reference is standard in a scientific context, it is not instrumental for our purpose, which is to express, formally, an invariance with respect to any change in the units. To rectify the ambiguity, I propose to interpret

$$L(\ell, v) \qquad \text{as a shorthand notation for} \qquad L_{1,1}(\ell, v),$$

in which ℓ and L on the one hand, and v on the other hand, are measured in terms of two particular *initial* or *anchor* units fixed arbitrarily. Such units could be m (meter) and km/sec, if one wishes. The '$1,1$' index of $L_{1,1}$ signifies these initial units.

Describing the phenomenon in terms of other units means that we multiply ℓ and v in any pair (ℓ, v) by some positive constants α and β, respectively. At the same time, L also gets to be multiplied by α, and the speed of light c by β. Doing so defines a new function $L_{\alpha,\beta}$, which is different from $L = L_{1,1}$ if either $\alpha \neq 1$ or $\beta \neq 1$ (or both).

But, from an empirical standpoint, $L_{\alpha,\beta}$ carries exactly the same information as $L_{1,1}$. For example, if our new units are km and m/sec, then the two expressions

$$L_{10^{-3},10^{3}}(.07, 3000) \quad \text{and} \quad L(70, 3) = L_{1,1}(70, 3),$$

while numerically not equal, describe the same empirical situation.

This points to the appropriate definition of $L_{\alpha,\beta}$ in the case of the Lorentz-FitzGerald Contraction. It turns out (c.f. Definition 3) that we should write:

$$L_{\alpha,\beta}(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{\beta c}\right)^2}.$$

The connection between L and $L_{\alpha,\beta}$ is actually:

$$\frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = \left(\frac{1}{\alpha}\right)\alpha\ell\sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2} = L(\ell, v).$$

This implies, for any α, β, ν and μ in $\mathbb{R}_{++}$,

$$\frac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu}L_{\nu,\mu}(\nu\ell, \mu v), \tag{3}$$

which is a special case of the invariance equation we were looking for, in the particular case of the Lorentz-FitzGerald Contraction Equation (or in the cases of the Doppler Effect or Beer's Law).

1 Remark. Looking at Equation (3), one might object that going in that direction would render the scientific or geometric notation very complicate. But the complication is only temporary. When we have extracted all the useful consequences from the meaningfulness axiom, we can go back to the usual notation. In fact, we already have the equation permitting to go back to our usual notation. Indeed, Equation (3) implies

$$\tfrac{1}{\alpha}L_{\alpha,\beta}(\alpha\ell, \beta v) = L_{1,1}(\ell, v) = L(\ell, v).$$

Note that the concept of meaningfulness is of course related to standard physical concepts such as dimensional analysis. I will not deal with this issue here, but see Narens (1981, 1988, 2002, 2007).

B. Defining meaningfulness

Our example of the Lorentz-FitzGerald equation makes clear that the concept of meaningfulness must apply to a **collection** of scientific or geometric functions (we call them *codes*), and not to a particular function.

2 Definition. We write $\mathbb{R}_{++} =]0, \infty[$ for the set of positive real numbers, and $\mathbb{R}_+$ for the non negative reals. Let J_1, J_2, and J_3 be three non-negative real intervals, and suppose that

$$\mathcal{F} = \{F_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{R}_{++}\}$$

is a collection of *codes*, with for the *initial code* F:

$$F = F_{1,1} : J_1 \times J_2 \xrightarrow{\text{onto}} J_3 \,.$$

Each of α and β indexing a code $F_{\alpha,\beta}$ in $\mathcal{F}$ represents a change of the unit of one of the two measurement scales[1].

Let δ_1 and δ_2 be two of rational numbers. The collection of codes $\mathcal{F}$ defined above is (δ_1, δ_2)-*meaningful* if for any

$$(x_1, x_2) \in J_1 \times J_2 \quad \text{and} \quad (\alpha, \beta), (\mu, \nu) \in \mathbb{R}^2_{++}, \quad \text{we have}$$

$$\frac{1}{\alpha^{\delta_1}\beta^{\delta_2}} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu^{\delta_1}\nu^{\delta_2}} F_{\mu,\nu}(\mu x_1, \nu x_2) = F_{1,1}(x_1, x_2)$$

which yields

$$F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \alpha^{\delta_1}\beta^{\delta_2} F_{1,1}(x_1, x_2) = \alpha^{\delta_1}\beta^{\delta_2} F(x_1, x_2).$$

The role of δ_1 and δ_2 is to specify the measurement scale of the function $F_{\alpha,\beta}$ relative to those of its two variables. In the case of the Lorentz-FitzGerald and similar equations, the measurement scale of the code is the same as that of the first variable. The relevant definition is given below.

3 Definition. A meaningful collection of codes, with $\mathcal{F} = \{F_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{R}_{++}\}$ as in the previous definition, is called $(1, 0)$-*meaningful* if it is (δ_1, δ_2)-meaningful with $\delta_1 = 1$ and $\delta_2 = 0$. We have then, for any $(x_1, x_2) \in J_1 \times J_2$ and $(\alpha, \beta), (\mu, \nu) \in \mathbb{R}^2_{++}$,

$$\frac{1}{\alpha^1\beta^0} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu^1\nu^0} F_{\mu,\nu}(\mu x_1, \nu x_2),$$

$$\Longleftrightarrow$$

$$\frac{1}{\alpha} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu} F_{\mu,\nu}(\mu x_1, \nu x_2), \qquad (\alpha^1\beta^0 = \alpha,\ \mu^1\nu^0 = \mu)$$

$$= F_{1,1}(x_1, x_2)$$

which yields $\quad F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \alpha F_{1,1}(x_1, x_2).$

Such collections are also called *ST-meaningful*, with *ST* standing for *self transforming*.

Many scientific or geometric laws are self transforming. We give several examples in this paper.

[1] In this paper, we only deal with scientific or geometric functions in two variables, and with ratio measurement scales.

C. As an introduction: the Pythagorean Theorem

One example of an abstract axiom is the *associativity equation:*

$$F(F(x,y),z) = F(x,F(y,z))$$

which can be shown to hold for right triangles, with each of

$$F(x,y), \quad F(x,z), \quad F(F(x,y),z) \quad \text{and} \quad F(x,F(y,z))$$

denoting the measures of the hypothenuses of a right triangle. In the figure below, $F(x,y)$ denotes the length of the hypothenuse of the right triangle $\triangle ABC$, with sides lengths x and y, while $F(y,z)$ denotes the length of the hypothenuse of the right triangle $\triangle BCD$.

The two remaining triangles: $\triangle ABD$, with sides lengths x and $F(y,z)$, and $\triangle ACD$, with sides lengths z and $F(x,y)$, have the common hypothenuse AD. Its length is

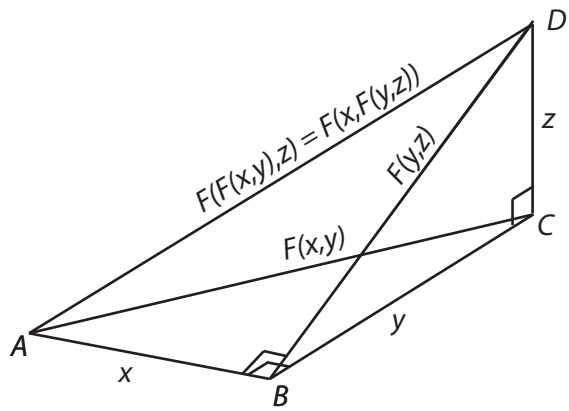
$$F(F(x,y)z) = F(x,F(y,z). \tag{4}$$

This shows that the hypothenuse of a right triangle is an associative function of (the lengths of) its two sides.

Using functional equations arguments (Aczél, 1966, Chapter X), we can prove that the associativity equation (4) has a representation

$$F(x,y) = f^{-1}(f(x)+f(y))$$

which is an equation generalizing the Pythagorean Theorem.



Under meaningfulness, and in the context of reasonable background conditions, we can prove that the function F has the form

$$F(x,y) = (x^\eta + y^\eta)^{\frac{1}{\eta}},$$

see Theorem 5 below. The exact statement requires recalling some conditions.

4 Definition. A code $F : \mathbb{R}_{++} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ is

symmetric if $\qquad F(x,y) = F(y,x) \qquad$ for $x, y \in \mathbb{R}_{++}$.

Such a code is

homogeneous if $\qquad F(\theta x, \theta y) = \theta F(y,x) \qquad$ for $x, y, \theta \in \mathbb{R}_{++}$.

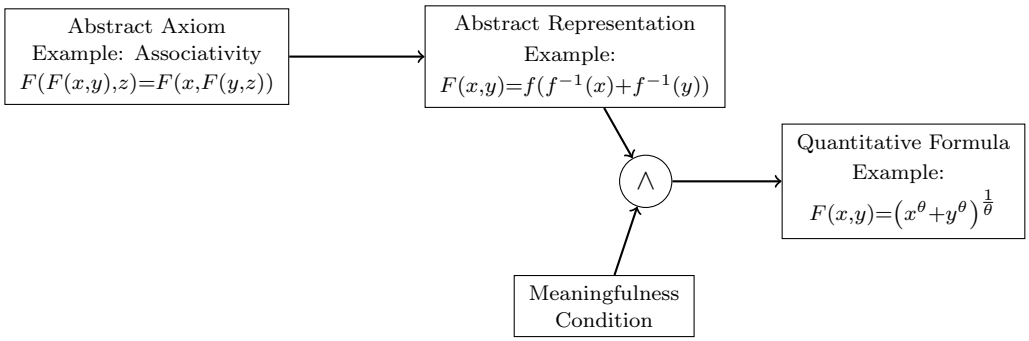
5 Theorem. *Suppose that $\mathcal{F} = \{F_\alpha \,|\, \alpha \in \mathbb{R}_{++}\}$ is a $(\frac{1}{2},\frac{1}{2})$-ST-meaningful collection of codes, with $F_\alpha : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{onto} \mathbb{R}_{++}$ for all α in $\mathbb{R}_{++}$. If one of these codes is symmetric, homogeneous and associative, then any code $F_\alpha \in \mathcal{F}$ must have the form*

$$F_\alpha(x,y) = \left(x^\theta + y^\theta\right)^{\frac{1}{\theta}} = F(x,y)\,,$$

for some constant $\theta \in \mathbb{R}_{++}$.

For a proof, see Falmagne and Doble (2015a, Theorem 7.1.1, page 85). The fact that we must have $\theta = 2$ can be derived from the Area of the Square Postulate and a couple of other intuitively obvious postulates of geometry.

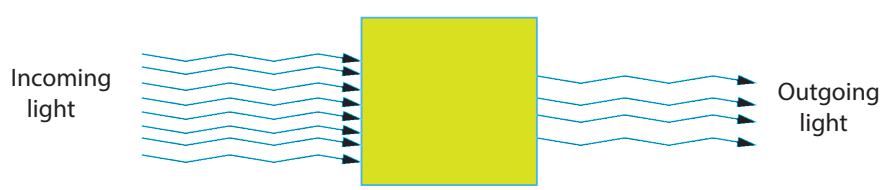
The proofs of Theorem 5 and a couple of other results given in this paper follow the schema illustrated by the next graph.



Proofs schema: An abstract axiom yields an abstract representation. The latter, paired with a meaningfulness condition leads, via functional equation arguments, to one or a couple of potential scientific laws specified up to the value(s) of numerical parameter(s).

D. Another example: The Translation Equation for Beer's law

The Beer-Lambert law, also known as Beer's law, the Lambert-Beer law, or the Beer-Lambert-Bouguer law is an equation describing the attenuation of light resulting from the properties of the material through which the light is traveling. (See the figure below.)



Following the guidelines of the Proof Schema, we first formulate the abstract axiom.

6 Definition. Let J and J' be two non-negative real intervals. A code $F : J \times J' \to J$ is *translatable*, or equivalently, satisfies the *translation equation*[2] if

$$F(F(x,y),z) = F(x,y+z) \qquad (x \in J,\ y,z,y+z \in J')\,. \tag{5}$$

[2]See Aczél (1966, page 245) for this concept and for the proof of Lemma 7.

An example of a translatable code is Beer's Law:

$$I(x,y) = x\,\mathrm{e}^{-\frac{y}{c}}\,. \tag{6}$$

Indeed, we have

$$I(I(x,y),z) = I(x,y)\,\mathrm{e}^{-\frac{z}{c}} = x\,\mathrm{e}^{-\frac{y}{c}}\,\mathrm{e}^{-\frac{z}{c}} = x\,\mathrm{e}^{-\frac{y+z}{c}} = I(x,y+z).$$

Next, we need the abstract representation in this case. It is formulated in the next lemma.

7 Lemma. *Let $F : J \times J' \to H$ be a code such that $J' =]d,\infty[$ for some $d \in \mathbb{R}_+$, and suppose that, for some $a \in \mathbb{R}_+$, and for some $b \in \mathbb{R}_+$ or $b = \infty$, we have $J =]a,b]$ or $J =]a,b[$ with $F(x,y)$ strictly decreasing in y.*

Then, the code $F : J \times J' \to H$ is translatable if and only if there exists a function f satisfying the equation

$$F(x,y) = f(f^{-1}(x) + y)\,.$$

Injecting now the meaningfulness condition, we obtain our quantitative formula.

8 Theorem. *Let $\mathcal{F} = \{F_{\mu,\nu} \,|\, \mu,\nu \in \mathbb{R}_{++}\}$ be a $(1,0)$-meaningful ST-collection of codes, with $F_{\mu,\nu} : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{onto} \mathbb{R}_{++}$. Suppose that one of these codes, say the code $F_{\mu,\nu}$, is strictly decreasing in the second variable, translatable, and left homogeneous of degree one, that is: for any a in $\mathbb{R}_{++}$, we have $F_{\mu,\nu}(ax,y) = aF_{\mu,\nu}(x,y)$. Then there is a positive constant c such that the initial code F has the form*

$$F(x,y) = x\,\mathrm{e}^{-\frac{y}{c}};$$

so for any code $F_{\alpha,\beta} \in \mathcal{F}$

$$F_{\alpha,\beta}(x,y) = x\,\mathrm{e}^{-\frac{y}{\beta c}}.$$

We summarize below the proof contained in Falmagne and Doble (see the proof of Theorem 7.4.1, page 98, 2015a).

Sketch of proof. We first show that, if one of the codes in the collection $\mathcal{F}$ is translatable, then by the meaningfulness condition, the translatability condition *propagates* to all the codes in the collection. Without loss of generality, we suppose that the initial code $F = F_{1,1}$ is translatable.
Successively, we have for any code $F_{\alpha,\beta}$ in $\mathcal{F}$:

$$\begin{aligned}
F_{\alpha,\beta}(F_{\alpha,\beta}(x,y),z) &= \alpha F\left(\frac{F_{\alpha,\beta}(x,y)}{\alpha}, \frac{z}{\beta}\right) && \text{(by ST-meaningfulness)}\\
&= \alpha F\left(F\left(\frac{x}{\alpha}, \frac{y}{\beta}\right), \frac{z}{\beta}\right) && \text{(by ST-meaningfulness)}\\
&= \alpha F\left(\frac{x}{\alpha}, \frac{y}{\beta} + \frac{z}{\beta}\right) && \text{(by the translatability of } F\text{)}\\
&= F_{\alpha,\beta}(x,y+z) && \text{(by ST-meaningfulness)}\,.
\end{aligned}$$

So, $F_{\alpha,\beta}$ is translatable. By meaningfulness, we can also show that left homogeneity of degree one propagates to all the codes in the collection $\mathcal{F}$. (We omit this part of the proof.)

Because $F_{\alpha,\beta}$ is translatable, Lemma 7 implies that there exists a strictly decreasing function $f_{\alpha,\beta} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that

$$\begin{aligned} F_{\alpha,\beta}(ax, y) &= f_{\alpha,\beta}(f^{-1}_{\alpha,\beta}(ax) + y) \\ &= a f_{\alpha,\beta}(f^{-1}_{\alpha,\beta}(x) + y) = a F_{\alpha,\beta}(x, y) \qquad \left(\begin{array}{c}\text{by left homogeneity}\\ \text{of } F_{\alpha,\beta}\end{array}\right). \end{aligned}$$

Set $f^{-1}_{\alpha,\beta}(x) = w$, and so $f_{\alpha,\beta}(w) = x$. Applying $f^{-1}_{\alpha,\beta}$ on both sides of the second equation above, we get

$$(f^{-1}_{\alpha,\beta} \circ a f_{\alpha,\beta})(w) + y = (f^{-1}_{\alpha,\beta} \circ a f_{\alpha,\beta})(w + y),$$

or with $g_{a,\alpha,\beta} = (f^{-1}_{\alpha,\beta} \circ a f_{\alpha,\beta})$,

$$g_{a,\alpha,\beta}(w) + y = g_{a,\alpha,\beta}(w + y),$$

a Pexider equation in the variables w and y. So,the function $g_{a,\alpha,\beta}$ is of the form

$$g_{a,\alpha,\beta}(w) = w + B(a, \alpha, \beta).$$

for some function $B(a, \alpha, \beta)$ which must be decreasing in a. Rewriting the last equation in terms of the function $f_{\alpha,\beta}$ yields

$$(f^{-1}_{\alpha,\beta} \circ a f_{\alpha,\beta})(w) = w + B(a, \alpha, \beta)$$

or equivalently, with $x = f_{\alpha,\beta}(w)$, we get

$$f^{-1}_{\alpha,\beta}(ax) = f^{-1}_{\alpha,\beta}(x) + B(a, \alpha, \beta),$$

another Pexider equation (c.f. Aczél, 1966, page 141) that is, an equation of the form: $h(ax) = h(x) + g(a)$. By functional equations arguments, the equation

$$f^{-1}_{\alpha,\beta}(ax) = f^{-1}_{\alpha,\beta}(x) + B(a, \alpha, \beta),$$

implies for some constants $k(\alpha, \beta) > 0$ and $b(\alpha, \beta)$,

$$f^{-1}_{\alpha,\beta}(x) = -k(\alpha, \beta) \ln x + b(\alpha, \beta)$$

which gives us, with $t = f^{-1}_{\alpha,\beta}(x)$,

$$f_{\alpha,\beta}(t) = \mathrm{e}^{\frac{t - b(\alpha,\beta)}{-k(\alpha,\beta)}}.$$

So, we get

$$F_{\alpha,\beta}(x, y) = f_{\alpha,\beta}(f^{-1}_{\alpha,\beta}(x) + y) = x\, \mathrm{e}^{-\frac{y}{k(\alpha,\beta)}}$$

after some manipulation. By the left homogeneity of $F_{\alpha,\beta}$ and the ST-meaningfulness of the family $\mathcal{F}$, we must have

$$\tfrac{1}{\alpha} F_{\alpha,\beta}(\alpha x, \beta y) = F_{\alpha,\beta}(x, \beta y) = x\, e^{-\frac{\beta y}{k(\alpha,\beta)}} = F(x, y).$$

The last equation shows that $\frac{\beta}{k(\alpha,\beta)}$ does not depend upon α or β.
Defining $c = \frac{k(\alpha,\beta)}{\beta}$, we finally obtain

$$F(x, y) = x\, \mathrm{e}^{-\frac{y}{c}}.$$

Accordingly, we obtain for any code $F_{\mu,\nu} \in \mathcal{F}$, using left homogeneity of degree 1 in the second equation below

$$F_{\mu,\nu}(x, y) = \mu F\left(\tfrac{x}{\mu}, \tfrac{y}{\nu}\right) = F\left(x, \tfrac{y}{\nu}\right) = x\, \mathrm{e}^{-\frac{y}{\nu c}}. \qquad \square$$

Various other results in the same vein are reported in Falmagne and Doble (2015a) (see also Falmagne, 2015b). The last two lines of the table below summarizes some of these results. The functional equations results mentioned in the second (abstract representation) column of the table may be found, together with a considerable list of other results and extended references, in Janos Azcél's classic volume (Aczél, 1966).

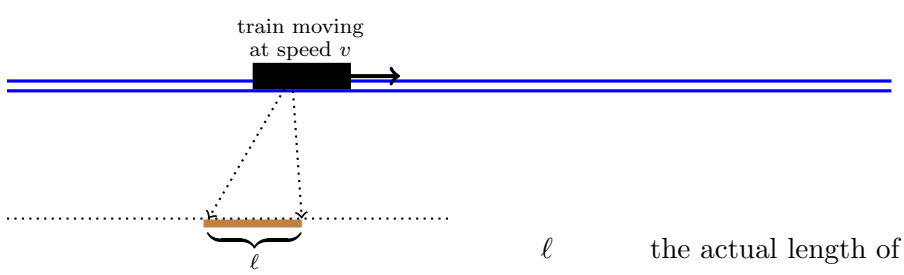
Name and formula of abstract axiom	Abstract representation: $\exists$ functions f, m, g, etc.	Resulting possible scientific laws[2]
Associativity $F(F(x,y),z)=F(x,F(y,z))$	$F(x,y)=f(f^{-1}(x)+f^{-1}(y))$	$F(x,y)=(y^\eta+x^\eta)^{\frac{1}{\eta}}$
Translatability $F(F(x,y),z)=F(x,y+z)$	$F(x,y)=f(f^{-1}(x)+y)$	$F(x,y)=x\mathrm{e}^{-\frac{y}{c}}$
Quasi-permutability $F(G(x,y),z)=F(G(x,z),y)$	$F(x,y)=m(f(x)+g(y))$	$F(x,y)=(x^\eta+\lambda y^\eta+\theta)^{\frac{1}{\eta}}$ or $F(x,y)=\phi x y^\gamma$ or $(x^\eta+y^\eta)^{\frac{1}{\eta}}$
Bisymmetry $F(F(x,y),F(z,w))=F(F(x,z),F(y,w))$	$F(x,y)=f\big((1-q)f^{-1}(x)+qf^{-1}(y)\big)$	$F(x,y)=(1-q)x^\eta+qy^\eta)^{\frac{1}{\eta}}$ or $F(x,y)=x^{1-q}y^q$

E. Meaningful derivation of relativistic laws

The figure below illustrates, in the case of the Lorentz-FitzGerald Contraction, a possible empirical context of the equation:

$$L(L(\ell, v), w) = L(\ell, v \oplus w). \tag{7}$$

Imagine that an observer is moving very fast along on a straight line, for example in a bullet train. Suppose that this observer measures the length of a rod placed along a line parallel to the train track. If the train moves very fast, at a sizable fraction of the speed of light, the length of the rod will appear smaller than its actual length.



ℓ	the actual length of the rod at rest
v	the speed of the train
$L(\ell, v)$	$\left\{\begin{array}{c}\text{the length of the rod measured}\\ \text{from the moving train}\end{array}\right\}$

It may not be obvious why Equation (7) is consistent with situation evoked by the figure. However we will see in Theorem 10 that Equation (7) is equivalent to the formula

$$L(\ell, v) \leq L(\ell', v') \iff L(\ell, v \oplus w) \leq L(\ell', v' \oplus w), \tag{8}$$

which may seem more immediately congruous with that situation.

However, at this stage, we cannot assume that taking Equations (7) or (8) as our abstract axiom and combining it with meaningfulness will deliver the exact functional form of the Lorentz-FitzGerald Contraction, that is, Equation (2). This is the reason why, in our next definition, we give the pair $(L, \oplus)$ the uncommitled name of *abstract LFD-pair* (for Lorentz-FitzGerald-Dopper).

9 Definition. Let $L : \mathbb{R}_{++} \times [0, c[\to \mathbb{R}_{++}$ be a *code*, with $c > 0$ a constant representing the speed of light. The code L is a *Lorentz-FitzGerald-Doppler Function*, or a *LFD function* for short, if there is a binary operator $\oplus : [0, c[\times [0, c[\to [0, c[$ such that the pair $(L, \oplus)$ satisfies the following five conditions:

1. The function L is strictly increasing in the first variable, strictly decreasing in the second variable, continuous in both variables, and for all $\ell, \ell' \in \mathbb{R}_+$ and $v, v' \in [0, c]$, and for any $a > 0$, we have

$$L(\ell, v) \leq L(\ell', v') \quad \iff \quad L(a\ell, v) \leq L(a\ell', v').$$

2. $L(\ell, 0) = \ell$ for all $\ell \in \mathbb{R}_+$.
3. $\lim_{v \to c} L(\ell, v) = 0$.
4. The operation $\oplus$ is continuous, strictly increasing in both variables, and has 0 as an identity element.
5. Either Axiom [R] or Axiom [M] below is satisfied for $\ell, \ell' > 0$, and v, v', $w \in [0, c[$:

   [R] $L(L(\ell, v), w) = L(\ell, v \oplus w)$;

   [M] $L(\ell, v) \leq L(\ell', v') \iff L(\ell, v \oplus w) \leq L(\ell', v' \oplus w)$.

When these five conditions are satisfied, the pair $(L, \oplus)$ is called an *abstract LFD-pair*.

In words, Axioms [R] and [M] state the following ideas.

> Axiom [R]: *One iteration of the function L involving two velocities v and w has the same effect on the perceived length as adding v and w via the operation $\oplus$.*
>
> Axiom [M]: *Adding a velocity via the operation $\oplus$ preserves the order of the function L.*

10 Theorem. *Suppose that $(L, \oplus)$ is an abstract LFD-pair. Then the following equivalences hold:*

$$[\mathrm{R}] \quad \iff \quad \left([\mathrm{DE}^\dagger] \ \& \ [\mathrm{AV}^\dagger]\right) \quad \iff \quad [M],$$

with for some strictly increasing function u and some positive constant ξ:

$$[\mathrm{DE}^\dagger] \qquad L(\ell, v) = \ell \left(\frac{c - u(v)}{c + u(v)}\right)^{\xi};$$

$$[\mathrm{AV}^\dagger] \qquad v \oplus w = u^{-1}\left(\frac{u(v) + u(w)}{1 + \frac{u(v)u(w)}{c^2}}\right).$$

(For a proof, see Falmagne and Doignon, 2010).

We now have the representation formulas for the abstract axioms [R] and [M]. The next definition introduces the meaningful collection with initial pair $(L, \oplus)$.

11 Definition. Let $\mathcal{L} = \{L_{\mu,\nu} \,|\, \mu, \nu \in \mathbb{R}_{++}\}$ be a ST-meaningful collection of codes, with $L_{\mu,\nu} : \mathbb{R}_{++} \times [0, \nu c[\xrightarrow{\text{onto}} \mathbb{R}_{++}$ and $c \in \mathbb{R}_{++}$. Let $\mathcal{O} = \{\oplus_\nu \,|\, \nu \in \mathbb{R}_{++}\}$ be a $(\frac{1}{2}, \frac{1}{2})$-meaningful collection of operators, with

$$\oplus_\nu : [0, \nu c[\times [0, \nu c[\xrightarrow{\text{onto}} [0, \nu c[\quad \text{and} \quad v \oplus_\nu w = \nu \left(\frac{v}{\nu} \oplus \frac{w}{\nu} \right) \qquad (v, w \in [0, \nu c[\,).$$

Suppose that each code $L_{\mu,\nu} \in \mathcal{L}$ is paired with a binary operation $\oplus_\nu \in \mathcal{O}$, forming an ordered pair $(L_{\mu,\nu}, \oplus_\nu)$, with the initial ordered pair $(L_{1,1}, \oplus_1) = (L, \oplus)$. Then the pair of collections $(\mathcal{L}, \mathcal{O})$ is called a *meaningful LFD-system*.

Note that the measurement scale of the operation $\oplus_\nu$ is the same as that of the second variable of the function $L_{\mu,\nu}$.

12 Remark. There is an equivocation in the definition of the function L as defined on the Cartesian product $\mathbb{R}_+ \times [0, c[$. In the proof of the next lemma, we have as the first equation

$$L_{\alpha,\beta}(\ell, v) = \alpha L \left(\frac{\ell}{\alpha}, \frac{v}{\beta} \right) \tag{9}$$

which is equivalent to

$$L_{\alpha,\beta}(\alpha\ell, \beta v) = \alpha L(\ell, v). \tag{10}$$

By definition, the domain of the function L in Equation (10) is $\mathbb{R}_+ \times [0, c[$ with $v \in [0, c[$. But the domain of L in (9) cannot be $\mathbb{R}_+ \times [0, c[$: we cannot have $\frac{v}{\beta} \in [0, c[$. Indeed, this would imply $v \in [0, \beta c[$ with possibly, if $1 < \beta$ with $c < v < \beta c$, contradicting $v \in [0, c[$.

This means that, while the two functions L in (9) and (10) are identical, they don't have the same domain. To deal with this ambiguity, we should define in general the domain of the function L by the statement

$$L : \mathbb{R}_+ \times [0, \theta c[: (\ell, \theta v) \mapsto L(\ell, \theta v) \qquad (c, \theta \in \mathbb{R}_{++}),$$

which makes clear that the function L is the same in (9) and (10) but with different domains: with $\theta = 1$ in (10) and $\theta = \frac{1}{\beta}$ in (9).

13 Propagation lemma for abstract LFD-pairs. *Suppose that one ordered pair $(L_{\mu,\nu}, \oplus_\nu)$ from a meaningful LFD-system $(\mathcal{L}, \mathcal{O})$ is an abstract LFD-pair, that is, $(L_{\mu,\nu}, \oplus_\nu)$ satisfies Conditions* 1-5 *of the definition of an abstract LFD-pair. Then any ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$, with $L_{\alpha,\beta} \in \mathcal{L}$ and $\oplus_\beta \in \mathcal{O}$, is also such an abstract LFD-pair.*

So, meaningfulness enables the propagation of all five conditions to all the ordered pairs $(L_{\alpha,\beta}, \oplus_\beta)$ in a meaningful LFD-system $(\mathcal{L}, \mathcal{O})$.

Sketch of proof. Without loss of generality, we can assume that the ordered pair $(L, \oplus)$ of initial code L is an abstract LFD-pair. That is, $(L, \oplus)$ satisfies the five conditions of Definition 9.

Since by meaningfulness, we have:

$$L_{\alpha,\beta}(\ell, v) = \alpha L\left(\tfrac{\ell}{\alpha}, \tfrac{v}{\beta}\right) \quad \text{and} \quad v \oplus_\beta w = \beta\left(\tfrac{v}{\beta} \oplus \tfrac{w}{\beta}\right),$$

Conditions 1 to 4 follow almost immediately. For example, Condition 1 holds because, successively:

$$\begin{aligned}
&L_{\alpha,\beta}(\ell, v) \le L_{\alpha,\beta}(\ell', v') \\
&\qquad \Longleftrightarrow \alpha L\left(\frac{\ell}{\alpha}, \frac{v}{\beta}\right) \le \alpha L\left(\frac{\ell'}{\alpha}, \frac{v'}{\beta}\right) && \text{(by ST-meaningfulness)} \\
&\qquad \Longleftrightarrow \alpha L\left(a\,\frac{\ell}{\alpha}, \frac{v}{\beta}\right) \le \alpha L\left(a\,\frac{\ell'}{\alpha}, \frac{v'}{\beta}\right) && \text{(by Condition 1 for } (L, \oplus)) \\
&\qquad \Longleftrightarrow L_{\alpha,\beta}(a\ell, v) \le L_{\alpha,\beta}(a\ell', v') && \text{(by ST-meaningfulness).}
\end{aligned}$$

For Condition 3, we have $\lim_{v\to\beta c} L_{\alpha,\beta}(\ell, v) = \alpha \lim_{v\to\beta c} L\left(\frac{\ell}{\alpha}, \frac{v}{\beta}\right) = 0$.

We turn to Condition 5. Since Axioms [R] and [M] are equivalent by Theorem 10, it suffices to prove that the ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$ satisfies Axiom [R].

By the ST-meaningfulness of $\mathcal{L}$,

$$L_{\alpha,\beta}(L_{\alpha,\beta}(\ell, v), w) = \alpha L\left(\frac{L_{\alpha,\beta}(\ell, v)}{\alpha}, \frac{w}{\beta}\right) = \alpha L\left(\frac{\alpha L\left(\frac{\ell}{\alpha}, \frac{v}{\beta}\right)}{\alpha}, \frac{w}{\beta}\right).$$

Canceling the α's in the fraction inside the parentheses in the r.h.s. gives

$$\begin{aligned}
L_{\alpha,\beta}(L_{\alpha,\beta}(\ell, v), w) &= \alpha L\left(L\left(\frac{\ell}{\alpha}, \frac{v}{\beta}\right), \frac{w}{\beta}\right) \\
&= \alpha L\left(\frac{\ell}{\alpha}, \frac{v}{\beta} \oplus \frac{w}{\beta}\right) && \left(\begin{smallmatrix}\text{by Axiom [R]}\\ \text{applied to } L\end{smallmatrix}\right) \\
&= \alpha L\left(\frac{\ell}{\alpha}, \frac{1}{\beta}(v \oplus_\beta w)\right) && \text{(by the meaningfulness of } \mathcal{O}) \\
&= L_{\alpha,\beta}\left(\ell, v \oplus_\beta w\right) && \text{(by the ST-meaningfulness of } \mathcal{L}).
\end{aligned}$$

□

14 Representation Theorem. *Suppose that one ordered pair $(L_{\mu,\nu}, \oplus_\nu)$ from a meaningful LFD-system $(\mathcal{L}, \mathcal{O})$ is an abstract LFD-pair, that is, $(L_{\mu,\nu}, \oplus_\nu)$ satisfies Conditions* 1-5 *of Definition 9. Then, Axioms* [DE†] and [AV†] of Theorem 10 *become for the initial code L:*

$$L(\ell, v) = \ell\left(\frac{c - v}{c + v}\right)^\xi \qquad \text{(for some positive constant } \xi \text{ and } v \in [0, c[\,) \tag{11}$$

and for the operation $\oplus$:

$$v \oplus w = \frac{v + w}{1 + \frac{vw}{c^2}} \qquad \text{(with } v, w \in [0, c[\,). \tag{12}$$

Both of these equations are of course well-known in special relativity. For example, with $\xi = \frac{1}{2}$, Equation (11) is related to the *red shift* phenomenon, and Equation (12) is the standard formula for the relativistic addition of velocities.

Proof. Without loss of generality, we can assume that the initial code of the meaningful LFD-system $(\mathcal{L}, \mathcal{O})$ is an abstract LFD-pair. That is, the function L satisfies the

five conditions of an abstract LFD-pair. From meaningulness, we have for any code $L_{\alpha,\beta}$:

$$L_{\alpha,\beta}(\ell,v) = \alpha L\left(\frac{\ell}{\alpha},\frac{v}{\beta}\right) = \alpha\left(\frac{\ell}{\alpha}\right)\left(\frac{\frac{c}{\beta}-u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta}+u\left(\frac{v}{\beta}\right)}\right)^{\xi} \qquad \left(\text{with } \frac{v}{\beta}\in\left[0,\frac{c}{\beta}\right[\right)$$

by the representation theorem for abstract LFD pairs. So, we have

$$L_{\alpha,\beta}(\ell,v) = \alpha L\left(\frac{\ell}{\alpha},\frac{v}{\beta}\right) = \ell\left(\frac{\frac{c}{\beta}-u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta}+u\left(\frac{v}{\beta}\right)}\right)^{\xi} \qquad \left(\text{with } \frac{v}{\beta}\in\left[0,\frac{c}{\beta}\right[\right). \tag{13}$$

The fact that the last equation requires that $0\le\frac{v}{\beta}<\frac{c}{\beta}$ rather than $0\le\frac{v}{\beta}<c$ has been discussed in Remark 12.

Because $L_{\alpha,\beta}(\ell,v)$ in the l.h.s. of (13) does not depend upon β, (which is just an indication of the unit) the r.h.s. cannot depend upon β either.

As the ratio

$$\frac{\frac{c}{\beta}-u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta}+u\left(\frac{v}{\beta}\right)} \qquad \left\{\begin{array}{c}\text{is a function of } v \text{ only}\\ \text{independent of } \beta,\\ \text{we must have}\end{array}\right\} \qquad g(v)=\frac{\frac{c}{\beta}-u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta}+u\left(\frac{v}{\beta}\right)}$$

for some function $g:[0,c[\,\to\,[0,c[$. Setting $\frac{1}{\beta}=z$ and rearranging, we get

$$u(vz) = zc\left(\frac{1-g(v)}{1+g(v)}\right)$$

and with $h(v)=c\frac{1-g(v)}{1+g(v)}$, we get the Pexider equation $u(vz)=zh(v)$, whose solution for the function u is, for some constant θ:

$$u(v)=\theta v \qquad \text{for all } v\in]0,c[.$$

Using the representation [DE†] from Theorem 10, we get

$$L(\ell,v)=\ell\left(\frac{c-u(v)}{c+u(v)}\right)^{\xi}=\ell\left(\frac{c-\theta v}{c+\theta v}\right)^{\xi}.$$

But the code L must satisfy Condition 3 of an abstract LFD-pair (Definition 9), which requires that $\lim_{v\to c}L(\ell,v)=0$. This implies

$$\lim_{v\to c}\ell\left(\frac{c-\theta v}{c+\theta v}\right)^{\xi}=\ell\left(\frac{c-\theta c}{c+\theta c}\right)^{\xi}=\ell\left(\frac{1-\theta}{1+\theta}\right)^{\xi}=0 \qquad \text{which holds only if } \theta=1.$$

We conclude that the function u must be the identity function: $u(v)=v$.

With the function u being the identity function, the two equations obtained in Theorem 10 from the representation of abstract LFD-pairs

[DE†] $\qquad L(\ell,v)=\ell\left(\frac{c-u(v)}{c+u(v)}\right)^{\xi}$;

[AV†] $\qquad v\oplus w=u^{-1}\left(\frac{u(v)+u(w)}{1+\frac{u(v)u(w)}{c^2}}\right)$.

become:

$$L(\ell, v) = \ell \left(\frac{c - v}{c + v} \right)^{\xi}$$
$$v \oplus w = \frac{v + w}{1 + \frac{vw}{c^2}}. \qquad \square$$

I was a bit puzzled by this result, whose importance is debatable. However, it could certainly be taken as one more example of the power of meaningfulness in the derivation of scientific or geometrical formulas.

Note that I did obtain a representation theorem for the Lorentz-FitzGerald Equation, which was using a different kind of meaningfulness constraints based on the concept of *meaningful transformations* (see Falmagne, 2004).

References

Aczél, J. *Lectures on Functional Equations and their Applications*. Academic Press, New York and San Diego, 1966. Paperback edition, Dover, 2006.

Falmagne, J.-Cl. Meaningfulness and order invariance: two fundamental principles for scientific laws. *Foundations of Physics*, 9:1341–1384, 2004.

Falmagne, J.-Cl. Deriving meaningful scientific laws from abstract, "gedanken" type, axioms: three examples. *Aequationes Mathematicae*, 89: 393-435, 2015.

Falmagne, J.-Cl. and Doble, C.W. On meaningful scientific laws. Springer: Berlin, Heidelberg, 2015.

Falmagne, J.-Cl. and Doignon, J.-P. Axiomatic derivation of the Doppler factor and related relativistic laws. *Aequationes Mathematicae*, 80 (1):85–99, 2010.

Feynman, R.P., Leighton, R.B., and Sands, M. *The Feynman lectures on physics*. Addison-Weisley, Reading, Mass, 1963.

Narens, L. A general theory of ratio scalability with remarks about the measurement-theoretic concept of meaningfulness. *Theory and Decision*, 13:1–70, 1981a.

Narens, L. Meaningfulness and the Erlanger program of Felix Klein. *Mathématiques Informatique et Sciences Humaines*, 101:61–72, 1988.

Narens, L. *Theories of Meaningfulness*. Lawrence Erlbaum Associates, New Jersey and London, 2002.

Narens, L. *Introduction to the Theories of Measurement and Meaningfulness and the Use of Symmetry in Science*. Lawrence Erlbaum Associates, Mahwah, New Jersey and London, 2007.