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Preference, Utility, and Subjective Probability[1,2]

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Of the major areas into which experimental psychology has been traditionally partitioned, motivation is the least well understood and systematized. This is true whether we consider theory, experimental paradigms, or experimental results.

> The psychology of motivation, compared with that of learning or of sensory and perceptual processes, is peculiarly retarded and confused. Much of what passes for discussion of motivation is pieced together out of fragments from physiological psychology, learning, personality, social psychology, and psychopathology. These patches often look violently out of place; worse still, they conceal the underlying cloth so well that one may doubt whether it exists at all. (Irwin, 1958, p. 152)

Nevertheless, as Irwin goes on to point out, at least one motivational concept; *preference*, is just as basic to psychological theory as, for example, are the concepts of discrimination and instrumental conditioning that have arisen in the other more developed branches of psychology. Moreover, of the various notions usually considered to be primarily motivational, preference is the only one that mathematical psychologists have attempted to analyze with any care: there are almost no satisfactory formal theories concerning, for example, drive and incentive, and those that exist are best discussed as aspects of learning. So this chapter on mathematical theories of motivation is limited to a study of preference and to the closely related constructs of utility and subjective probability.

1. GENERAL REMARKS ON THE STUDY OF PREFERENCE

1.1 Origins of the Mathematical Theories of Preference

Although we are correct in viewing formal theories of preference as a part of mathematical psychology, it would badly wrench history to suggest that the research was mostly carried out by people who classify themselves as psychologists. Much of the theory, and certainly the best of it, was worked out by economists and statisticians who needed psychological underpinnings for their decision theories. Only in the last half dozen years have psychologists begun to isolate for separate study these inherently psychological theories of preference. While being elaborated as distinct and testable psychological theories, the theories of preference have begun to acquire a richness and complexity—hopefully reflecting a true richness and complexity of behavior—that renders them largely useless as bases for economic and statistical theories. Perhaps we may ultimately find simple, yet reasonably accurate, approximations to the more exact descriptions of behavior that can serve as psychological foundations for other theoretical developments, but at the moment this is not the main trend.

The psychological reader should be warned that statisticians and economists have long employed a technique of nonempirical, rationalistic argument which is totally foreign to many psychologists, who tend to the other extreme of automatically rejecting plausible hypotheses unless they have powerful experimental support. Psychologists sometimes seem slightly naive in their reverence for what are held to be pure empirical facts, when actually most experimental inferences depend on some more or less implicit theoretical position, often partially embodied in a statistical model. Be that as it may, many theories of preference were formulated and evaluated initially in terms of what a "rational" person having unlimited computational abilities and resources ought to do. Frequently these debates have a somewhat tenuous quality for psychologists, especially since there does not exist any really satisfactory comprehensive definition of rationality and only a few of its properties are generally agreed on.

Some psychologists have simply rejected work of this genre as sterile, but those who have taken an active interest in it have found two research paths open. First, can these so-called normative theories also be adequate as descriptions of behavior? Attempts to answer this question have led to laboratory tests of the normative theories—with, however, mostly ambiguous results. Second, can theories be devised that are frankly descriptive in intent, that try to encompass explicitly some of the phenomena that seem to be found in the laboratory? Several attempts are discussed.

Our attitude in this chapter is primarily psychological: we ask whether a theory seems to describe behavior, not whether it characterizes a rational man; we report experiments, although admittedly not enough experimental work has yet been done to provide us with either completely satisfactory designs or highly reliable phenomena; and we explore some of the relations between theories of preference and other psychological theories. At the same time, we try to recount the normative considerations that led originally to the theories and to cite the more important rationalistic criticisms that have been leveled against them.

1.2 Preference, Discrimination, and Bias

Irwin (1958, p. 152) developed in detail the widely accepted thesis that, "... preference is exactly as fundamental as discrimination and that, indeed, the two are so intimately related in behavior that if the organism exhibits a discrimination, it must also exhibit a preference, and conversely." Without reproducing the supporting argument in detail, we may suggest the main point. Suppose that outcome x_1 results either when response r_1 is made to stimulus presentation s_1 or when r_2 is made to s_2; whereas x_2 results when r_2 is made to s_1 or when r_1 is made to s_2. Before we can expect a subject to respond differentially and thus to show his ability to discriminate between s_1 and s_2, it must matter to him which outcome occurs—he must prefer one outcome to the other. Equally well, he can evidence a preference between x_1 and x_2 only if he is capable of discriminating the discriminative stimuli, for only then can he know (or learn) which response is appropriate on each trial to achieve the preferred outcome.

A careful examination of Irwin's discussion suggests that this view is correct: the notions of discrimination and preference are both fundamental to an understanding of behavior and they are profoundly intertwined in parallel roles. If so, it appears perverse, if not worse, to divide psychology—in particular, that portion based upon choice experiments—into a part concerned primarily with the discrimination of stimuli and a part concerned primarily with preferences among outcomes. We are saved from total chaos, however, by the experiments that in principle we should perform, but rarely do because we are certain of the results on the basis either of the informal experimentation of experience or of studies deeply buried in the past of psychology. In a psychophysical or learning experiment we assume that we know what a subject's preferences are among the outcomes. We are confident that hungry animals prefer receiving food pellets to not receiving them, that a student prefers winning five cents to losing three, etc. Equally well, in (human) preference experiments, we always attempt to select discriminative stimuli, for example, labels to identify the outcomes, that we are sure from our knowledge of people are perfectly discriminable. We could perform the necessary auxiliary experiments to prove these assumptions, but usually we do not because we are so certain of the results. This means, for example, that when a subject exhibits some inconsistency in his choices in a preference experiment, we automatically attribute it to an ambivalence about the worth of the outcomes, not to his inability to tell which outcomes are associated with which stimulus-response pairs nor to his inability to discriminate among the relevant stimuli.

Although the accuracy of these assumptions can always be questioned, there is little doubt that serious attempts have been made to realize them in the preference experiments we shall discuss here. Therefore, throughout the chapter, we take it for granted that the subject has no difficulty in discriminating the alternatives presented to him; our focus is entirely on his preferences. This is not to deny that any ultimately satisfactory theory of choice behavior will have to encompass discrimination, preference, and learning. The current partitioning is a convenient way to subdivide a complex problem into somewhat more manageable chunks.

A second point emphasized by Irwin (1958) is more deeply confounding in practice than the interlocking of preference and discrimination. Subjects often—almost always, in fact—exhibit "preferences" among the responses as well as among the outcomes. To keep the terminology straight, we follow Irwin by speaking of these as *biases* among the responses. Perhaps the most striking example of response biases is the frequently noticed position habits of animals that seem to occur whenever the discrimination is at all difficult or whenever the differences among the outcomes are slight. Furthermore, it should be recalled that response biases are ubiquitous in psychophysics (see Chapters 3, 4, and 5 of Vol. I) and that the more recent psychophysical theories explicitly provide for them.

Unfortunately, the same cannot be said for the mathematical theories of preference. None of them acknowledges response biases. To some extent, theoreticians either have not been particularly sensitive to the phenomenon or they have felt that the biases arise from relatively trivial experimental errors that can be overcome by appropriate techniques, such as randomization. Actually, however, response biases seem to be a fairly deep problem, and no one knows how to eliminate them experimentally. Randomization is no solution, despite its widespread acceptance among experimentalists, because it only buries the biases in increased variability; it does not eliminate them. We badly need theories of preference that explicitly provide for response biases.

1.3 A Classification of Theories of Preference

As in psychophysical research, preference studies have been largely restricted to steady-state (asymptotic) behavior. Behavioral transients, which are the province of learning, are sufficiently complicated to analyze that only the simplest learning experiments have been dealt with in any detail. These learning experiments are too simple to provide answers to many of the questions that we wish to ask about preferences, and so we are forced to confine our attention to asymptotic behavior. Unfortunately, some recent studies (Edwards, 1961a; Lindman & Edwards, 1961) suggest that true experimental asymptotes may be harder to come by than one would wish.

Because the focus is on equilibrium behavior, many of the theories of preference are simply static in character, including no explicit mechanism for temporal changes. A few recent theories, however, derive asymptotic mean predictions from well-specified stochastic learning processes. As we shall see (Sec. 7.3), the predictions of these two types of theories are not very compatible, but the problem does not seem to be simply to choose between them; rather, some sort of fusion is needed. The static theories embody, admittedly roughly, something of a subject's cognitive or rational analysis of the choice problem; whereas the asymptotic learning theories encompass the fine-grain adjustments made by subjects to their recent experiences. One can hardly doubt that both mechanisms exist in people.

Several distinctions that are important both substantively and mathematically can be made among these equilibrium theories of preference. We make three and use them to organize the chapter.

The first is between what may be called *algebraic* and *probabilistic theories*. Let us suppose that the responses that a subject makes to stimulus presentations are governed by probability mechanisms. Then the theory can be viewed as algebraic if all such probabilities are either 0, $\frac{1}{2}$, or 1. Although this definition is strictly correct, it misleadingly suggests that the algebraic theories are simply special cases of the probabilistic ones. They are not. The algebraic ones employ mathematical tools, namely, algebraic tools, that are different from those used in the probabilistic theories, and so it is appropriate to develop them separately. Historically, the algebraic theories were studied first, and they have been used in economics and statistics almost exclusively. The probabilistic ones are largely the product of psychological thought, forced upon us by the data we collect in the laboratory.

Some authors have chosen to refer to the probabilistic theories of preference as stochastic theories, and this usage is gaining in popularity. In our opinion, it is unfortunate because it blurs a valuable distinction. A probability process that includes a time parameter, continuous or discrete, has come to be called a *stochastic process* (see Chapter 20). One that does not involve transitions over time should be called something else. Thus, for example, an asymptotic learning theory for a preference experiment is properly described as a stochastic theory of preference, but theories that simply postulate a probabilistic choice mechanism without a means of changing the probabilities in time (for example, with experience) we shall speak of as probabilistic theories.

The second distinction, an experimental one, is between certain and uncertain outcomes. It is simply a question of whether the outcome is prescribed by the stimulus presentation plus the response or whether these only determine a probability distribution over the outcomes. In terms of the notation of Chapter 2 (p. 86), if s is the presentation and r the response, the conditional outcome schedule $\omega = \phi(r, s)$ is a random variable with the trials as its domain and the set A of outcomes as its range. If the outcome schedules have the special property that for all $x \in A$ and all $\omega \in \Omega$, where Ω is the set of conditional outcome schedules, the conditional probability $\pi(x \mid \omega)$ is either 0 or 1, then we shall say that the experiment has *certain outcomes*. Otherwise, the outcomes are *uncertain*.

Theories for certain outcomes are usually stated in such a way that, in principle, they can be applied to uncertain outcomes; however, in that context they usually seem to be very weak theories because they fail to take into account the complex structure of the uncertain outcomes. On the other hand, theories for uncertain outcomes that explicitly include the probability distribution over the set of outcomes can always be specialized to the certain case, but again the effect is to produce weak theories. Presumably, this is a temporary problem and ultimately the specialization of a theory for uncertain outcomes will yield a satisfactory theory for certain ones.

Statisticians sometimes attempt to make further distinctions among what we are calling uncertain outcomes. When the subject knows the probability distribution over the outcomes, they speak of risky choices, and they reserve the word "uncertain" either for all cases that are neither risky nor certain or for those in which the subject has no information at all about the distribution. As these distinctions are difficult to make precise and they do not seem particularly useful for our purposes, we shall not bother with them.

To add further to the terminological confusion, information theorists use the word "uncertain" to describe situations in which the probability distributions are known, that is, to refer to situations that decision theorists describe as pure risk. Their use is not inconsistent with ours, but it is less broad.

The third and final distinction is between *simple choice experiments* and *ranking experiments*. Strictly speaking, both are choice experiments as that term is used in Chapter 2. The question is whether the subject is asked to select among several outcomes or whether he is asked to rank order them, that is, to select from the set of rankings. It is generally believed that there must be some regular relation between his behavior in these two kinds of experiments when the same outcomes are involved, and the ranking theories attempt to describe these relations.

With three binary distinctions, there ought to be eight theoretical sections to the chapter; actually there are only five. The reasons are that, at present, the algebraic ranking theories are trivial extensions of the choice theories, and so do not warrant separate sections, and that no special probabilistic ranking theories have yet been developed for uncertain outcomes.

1.4 Previous Surveys of the Literature

In the last decade several excellent surveys of various aspects of the subject matter of this chapter have appeared. Edwards (1954d) and (1961b) are particularly well known to psychologists. Adams (1960) provides an excellent survey of Bernouillian utility theory (a topic discussed in Sec. 3 of this chapter) for both economists and psychologists. Arrow (1951b, 1964) and Majumdar (1958) are mainly addressed to economists, but both articles contain many useful remarks that will help psychologists who seek a general theoretical orientation in the field. Material of a similar sort is also to be found in Luce and Raiffa (1957), which is addressed to a general social science audience.

We emphasize that some of the work described and analyzed in these sources is not covered in this chapter, and the interested reader is urged to refer to them.

2. GENERAL ALGEBRAIC CHOICE THEORIES

As was noted previously, algebraic choice theories for certain outcomes have a long history in classical economics, beginning at least as early as the work of Jeremy Bentham in the eighteenth century. His definition of utility, given in the first chapter of his famous *The Principles of Morals and Legislation* (1789), still provides a good starting point for a discussion of certain theories.

> By utility is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good, or happiness (all this in the present case comes to the same thing), or (what comes again to the same thing) to prevent the happening of mischief, pain, evil, or unhappiness to the party whose interest is considered: if that party be the community in general, then the happiness of the community; if a particular individual, then the happiness of that individual.

Bentham interpreted his maxim of "the greatest good for the greatest number" as meaning that utility is maximized, and he spent considerable effort in formulating a program for measuring utility.

In the classical economic literature on utility, the most important distinction is between ordinal and cardinal utility. The term *ordinal* refers to the assumption of order only, whereas *cardinal* refers to the assumption of additivity or, if not that, at least uniqueness of numerical assignment up to a linear transformation (see Chapter 1, Vol. I, p. 12). During the nineteenth century it was commonly believed that little economic theory could be developed solely on the basis of ordinal utility functions. One of Pareto's (1906) great achievements was to show that much could be done with purely ordinal assumptions. It is fair to say that Pareto was the first main contributor to the theory of ordinal utility functions, to which we now turn.

2.1 Ordinal Utility Functions

Because of its structural simplicity, the theory of ordinal utility functions is a good place to begin for logical as well as historical reasons. Essentially the theory deals just with the qualitative preference for one alternative over another. To apply mathematical analysis in this setting, the primary thing that is needed is the representation of preference in the form of a numerical utility function. Once such a numerical function is available it may be used in subsequent theoretical developments, with the possibility open of applying standard mathematical tools to problems of behavior. For this reason, most of this section is devoted to a sequence of increasingly general theorems on the existence of numerical utility functions that reflect the qualitative preference structure.

Although psychologists find much of the economic literature on utility rather far removed from direct experimental or behavioral questions, the theory of ordinal utility functions was, in fact, developed to answer some rather specific questions about consumer behavior. It is appropriate to begin by sketching the setting of these questions. Most of what we say here is drawn from Wold and Jureen (1953) and Uzawa (1960).

Suppose that a consumer has income M at time t_0 (in the sequel we consider only the static case of t_0, so we drop explicit reference to time). With his income the consumer may purchase a bundle of commodities—traditionally one of these commodities may be savings. We can describe such a bundle by a real n-dimensional vector $x = (x_1, \ldots, x_n)$, where the ith component x_i specifies the amount of commodity i to be consumed. Following standard notation, we say that bundle x is *greater than* x', in symbols, $x > x'$, if for every i, $x_i \geq x_i'$ and for some i, $x_i > x_i'$. As the foundation of the theory of consumer demand, a preference relation P on commodity bundles is introduced. The relation xPy is read: commodity bundle x is preferred to commodity bundle y. The consumer's income M, the current market prices p_i, and the structure of his preference relation P determine his choice of commodities.

In addition to the relation P, it is also necessary to introduce a relation I of indifference, for it is not reasonable that of two distinct bundles one is necessarily strictly preferred to the other. Technical economy of formulation is achieved at many points by replacing P and I by the weak preference relation R. The relation R stands to P in the same way that the numerical relation $\geq$ stands to $>$. The obvious equivalences are:

$$\begin{array}{llll} xRy & \text{if and only if} & xPy & \text{or} \quad xIy, \\ xIy & \text{if and only if} & xRy & \text{and} \quad yRx, \\ xPy & \text{if and only if} & xRy & \text{and not} \quad yRx, \end{array}$$

and we shall assume them without comment. The following postulates on the structure of the (weak) preference relation R are reasonably intuitive, and they guarantee the existence of a utility function.

Definition 1. *A relation R on the set of all commodity bundles (n-dimensional vectors) is a* preference relation *if the following axioms are satisfied for any bundles x, y, and z:*

1. Transitivity: *if xRy and yRz, then xRz;*
2. Connectivity: *xRy or yRx;*
3. Nonsatiety: *if $x > y$, then xPy;*
4. Continuity: *if xRy and yRz, then there is a real number λ such that $0 \leq \lambda \leq 1$ and $[\lambda x + (1 - \lambda)z]Iy$.*

Of these four postulates, the first two are of a very general nature; their formulation does not depend in any way on x, y, and z being n-dimensional vectors. When 1 and 2 are satisfied, the relation R is called a *weak ordering*. It should be understood that the *or* of Axiom 2 is inclusive, that is, both xRy and yRx may hold.

Axioms 3 and 4 are of a more special character. Axiom 3 is called the axiom of nonsatiety because it postulates that the consumer, *ceteris paribus*, always prefers more of any commodity to less. This is obviously unrealistic when all possible commodity bundles are considered; it becomes more realistic, however, if finite bounds are imposed on each commodity.

It might be thought that Axiom 4 is not really necessary to prove the existence of a utility function u such that for all bundles x and y

$$u(x) \geq u(y) \quad \text{if and only if} \quad xRy. \tag{1}$$

There is, however, a classical and important counterexample that shows that an arbitrary weak ordering cannot be represented by a numerical function.

Counterexample. Consider the *lexicographic ordering* of the plane: $(x_1, x_2)R(y_1, y_2)$ if and only if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Suppose that there exists a real-valued function u satisfying Eq. 1. We fix x_2 and y_2 with $x_2 < y_2$ and define for each x_1:

$$u'(x_1) = u(x_1, x_2)$$

$$u''(x_1) = u(x_1, y_2).$$

In terms of these functions define the following function f from real numbers to intervals:

$$f(x_1) = [u'(x_1), u''(x_1)].$$

On the assumption that the ordering is lexicographic, f must be 1-1 since two distinct numbers are mapped into two disjoint intervals. For instance, if $x_1 > x_1'$, then $u'(x_1) = u(x_1, x_2) > u(x_1', y_2) = u''(x_1')$. But it is well known that it is impossible for a one-to-one correspondence to hold between the uncountable set of real numbers and a countable set of nondegenerate disjoint intervals. Thus no such function f can exist, and a fortiori there can be no function u satisfying Eq. 1 for the lexicographic ordering.

Note that Axiom 4 is not satisfied by the counterexample because, for example, the point (4, 2) is between (2, 2) and (4, 4) in the lexicographic ordering, but there is no number λ such that $\lambda(2, 2) + (1 - \lambda)(4, 4) = (4, 2)$.

We now show that the postulates of Def. 1 do guarantee the existence of a utility function.

Theorem 1. *Let R be a preference relation on the set of commodity bundles, in the sense of Def.* 1. *Then there exists a utility function u that satisfies* Eq. 1.

PROOF. We first define the function u for the "diagonal" bundles x, which are defined by the property that $x_i = x_1$ for all i, by $u(x) = x_1$. Since any two indifferent vectors must have the same utility, we may extend u to any vector by constructing a diagonal vector to which it is indifferent. Let y be any vector, let y^* be the diagonal vector with all of its components equal to the smallest component in y, and let y^{**} be the diagonal vector with all its components equal to the largest component in y. Obviously, $y^{**} \geq y \geq y^*$, and therefore by the axiom of nonsatiety, $y^{**}RyRy^*$. By the axiom of continuity, there is a λ such that $yI[\lambda y^{**} + (1 - \lambda)y^*]$, and λ can easily be shown to be unique whenever y is not a diagonal vector. Since $\lambda y^{**} + (1 - \lambda)y^*$ is a diagonal vector, from the definition of u we have in terms of the first components of y^* and y^{**},

$$u(y) = \lambda y_1^{**} + (1 - \lambda)y_1^*.$$

Let z be any other vector. We must show that

$$u(y) \geq u(z) \quad \text{if and only if} \quad yRz.$$

First, assume that yRz and $u(y) < u(z)$. Then

$$\lambda' z^{**} + (1 - \lambda')z^* > \lambda y^{**} + (1 - \lambda)y^*,$$

and therefore xRz by the nonsatiety axiom, and thus yIz. Therefore $u(y) = u(z)$, contrary to the supposition $u(y) < u(z)$, so $u(y) \geq u(z)$. Conversely, assume that $u(y) \geq u(z)$. Then by definition

$$\lambda y^{**} + (1 - \lambda)y^* \geq \lambda' z^{**} + (1 - \lambda')z^*,$$

and therefore by the axiom of nonsatiety

$$[\lambda y^{**} + (1 - \lambda)y^*]R[\lambda' z^{**} + (1 - \lambda')z^*].$$

Thus, because the ordering of R is preserved by substitution of I-indifferent vectors, it follows that yRz, which completes the proof.

Given prices $p_1, \ldots, p_n$, a consumer with income M can afford to buy any commodity bundle $x = (x_1, \ldots, x_n)$ such that

$$p_1x_1 + \ldots + p_nx_n \leq M.$$

The behavioral prediction of the theory is that he will select a bundle that maximizes utility, subject to this income restraint.

The problem of the numerical representation of preferences is not completely solved by proving the existence of a numerical utility function. In order to know the extent to which numerical methods of analysis may then be applied, it is also necessary to know how unique the obtained utility function is. It is a simple matter to prove the following theorem (see Theorem 8, Chapter 1, p. 29; also see Chapter 1 for a more extensive discussion of questions of uniqueness).

Theorem 2. *Let R be a preference relation in the sense of Def.* 1. *Then any two utility functions satisfying* Eq. 1 *are related by an increasing monotone transformation.*

The restriction of the domain of utility functions to n-dimensional commodity spaces is not desirable even in economics, and it is certainly unacceptable in psychological investigations of preference and choice. Fortunately, the most general circumstances under which an ordinal utility function exists can be rather easily characterized. First, we generalize Def. 1 to say that a relation R on an arbitrary set A is a *preference relation* for A if it is a weak ordering of A, that is, if it satisfies

Axioms 1 and 2 of Def. 1. Second, let B be a subset of A. Then we say that B is *R-order-dense* in A if and only if for every x and y in A, but not in B, such that xPy there is a z in B such that xRz and zRy. Note that the denumerable set of rational numbers is order-dense with respect to the natural numerical ordering $\geq$ in the nondenumerable set of all real numbers. This relationship between the denumerable rational numbers and all real numbers is just the one that is necessary and sufficient for the existence of a utility function satisfying Eq. 1. With respect to a preference relation that is a weak ordering a minor complication arises in applying the denumerability condition, namely, the elements of the order-dense subset must not be indifferent. This additional condition is made precise in the statement of the theorem.

Theorem 3. *Let A be an infinite set and let R be a preference relation on A. Then a necessary and sufficient condition that there exist a utility function satisfying Eq. 1 is that there is a denumerable subset B of A such that* (i) *B is R-order-dense in A and* (ii) *no two elements of B stand in the relation I, that is, for any distinct x and y in B either xPy or yPx.*

PROOF. [The proof is related to the classical ordinal characterization of the continuum by Cantor (1895). We do not give all details here, but sketch the main outlines; for some additional details and related theorems, see Sierpinski (1958, Chapter 11) and Birkhoff (1948, pp. 31–32).]

To prove the sufficiency of the condition, let B be a denumerable subset with properties (i) and (ii). Moreover, if A has endpoints with respect to the ordering of R, we may without loss of generality include them in B. First, we know that there exists a utility function u for B, just because B is denumerable (this is Theorem 6, Chapter 1, p. 26). Now by the R-order-dense condition on B, each element y of A that is not in B defines a *cut* in B, that is, the partition of B into two sets, $X = \{x \mid x \in B \;\&\; xRy\}$ and $Z = \{z \mid z \in B \;\&\; yRz\}$. Now let

$$r_1 = \operatorname*{g.l.b.}_{x \in X} u(x)$$

and

$$r_2 = \operatorname*{l.u.b.}_{z \in Z} u(z).$$

We then extend u to y by defining

$$u(y) = \frac{r_1 + r_2}{2}.$$

It is easy to show that the utility function u thus extended from B to A satisfies Eq. 1. For example, if w_1 and w_2 are in A but not in B, then if $w_1 R w_2$, there is a z in B such that $w_1 R z R w_2$ and thus

$$u(w_1) \geq u(z) \geq u(w_2).$$

The remaining details of this part of the argument may be supplied by the reader.

To prove the necessity of (i) and (ii), we assume that we have a function u satisfying Eq. 1. The set of nonempty intervals I_i of the real numbers with rational endpoints is a denumerable set because of the denumerability of the rational numbers. We next construct corresponding intervals J_i of A by taking the inverse image under u of each interval I_i. (Note that not every point in I_i need correspond to a point in A.) From each J_i that is not empty we select an element x_i. Since the set of intervals is denumerable, the set X of elements x_i of A is denumerable. Now, let $\mathscr{R}$ be the set of real numbers r such that for some y in A, $u(y) = r$ and

$$u(y) - \underset{x \in Y}{\text{l.u.b.}}\, u(x) > 0$$

where

$$Y = \{x \mid x \in A \;\&\; xRy\}.$$

Because this set $\mathscr{R}$ defines a set of nonoverlapping intervals of real numbers it is at most denumerable (of course, $\mathscr{R}$ can, and in some cases would, be empty). Let X' be the inverse image under u of $\mathscr{R}$. Then $B = X \cup X'$ is denumerable. To show that B is order-dense in A, let t_1 and t_2 be two elements in A but not in B such that $t_1 P t_2$. Now if there are no elements of A between t_1 and t_2, then t_1 is in X', contrary to hypothesis. On the other hand, if there are elements between t_1 and t_2, then at least one, say z, will be such that $u(z)$ lies in an interval with rational endpoints which is nested in the interval $[u(t_2), u(t_1)]$, and thus B is order-dense in A. This completes the proof.

It is, of course, possible to generalize Theorem 3 by no longer requiring the utility function to be real-valued. General theorems for any preference relation, that is, for any weak ordering of a set, are given in Chipman (1960a). He shows that if for the range of the utility function we replace the real numbers by transfinite sequences of real numbers under their natural lexicographic order, then such a utility function exists for any preference relation. Such generalizations are not pursued here, for they are of dubious interest for psychological theory.

2.2 Topological Assumptions[3]

It is easily seen that, although the utility function we constructed in the proof of Theorem 1 is continuous, the construction may be modified to obtain a discontinuous function satisfying Eq. 1. For example, for a

[3] This section can be omitted without loss of continuity.

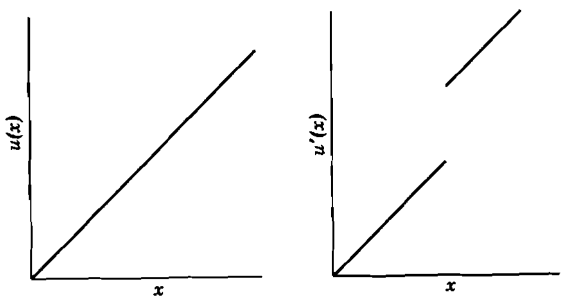


Fig. 1. Graphs of the continuous function u and the ordinally equivalent discontinuous one u'.

diagonal x let

$$u'(x) = \begin{cases} x_1 & \text{if} \quad x_1 \leq 1 \\ x_1 + 1 & \text{if} \quad x_1 > 1. \end{cases}$$

Thus for diagonal bundles, we have the two graphs of u and u' shown in Fig. 1. In Theorem 3 it is clear that no claims of continuity can be made for the utility function considered.

For further theoretical or practical developments of the theory of preference it is almost always desirable to confine our attention to well-behaved functions, for which continuity is often a desirable if not always a sufficient restriction. Two questions naturally arise. First, once we drop the restrictive assumption of commodity spaces of Def. 1, what shall we mean by continuity and what postulates guarantee the existence of at least one continuous utility function? Second, how can we guarantee that all, not merely one, of the possible utility functions are continuous?

To answer these questions requires a brief excursion into topology. Our treatment of these matters is not extensive, even though topological considerations have been prominent in the recent economic literature on utility and preference, because it is doubtful if they are of direct significance for the research of psychologists. Our discussion is intended only to give the reader some feeling for the generality of topological methods without pretending to provide a really adequate systematic introduction to the subject. We assume no prior knowledge of topology, but we do suppose that the reader is familiar with the notion of continuity that is defined in textbooks on the calculus.

Intuitively, a function from real numbers to real numbers is continuous if its graph has no jumps or gaps in it. The rigorous definition that makes precise this intuition is easily extended to functions from n-dimensional vectors of real numbers to the real numbers. However, when we consider functions defined over arbitrary sets, there is no immediate method of defining continuity; indeed, unless some additional structure is imposed on the set, nothing can be done. The type of structure that permits us to formulate a general definition of continuity is known as a topology.

The most important property of continuous functions is that they map points that are "near" one another into points that are again "near" one another. Put another way, the function does not create too much scrambling of points that are reasonably close to each other. In talking about functions of real numbers, this idea is adequately captured by requiring that intervals be mapped *into* intervals. Let us restrict ourselves to the open intervals, that is, to intervals that do not include their endpoints. Then the natural topology of the real line is the family of open intervals together with the sets that are formed from arbitrary unions and finite intersections of open intervals. Members of this family of sets are called open sets.

It is these last ideas that are generalized to define topologies for arbitrary sets.

Definition 2. *A pair* $(X, \mathscr{T})$ *is a* topological space *if* $\mathscr{T}$ *is a family of subsets of* X, *called* open sets, *such that*

(i) *the empty set is in* $\mathscr{T}$;
(ii) X *is in* $\mathscr{T}$;
(iii) *the union of arbitrarily many sets in* $\mathscr{T}$ *is also in* $\mathscr{T}$;
(iv) *the intersection of any finite number of sets in* $\mathscr{T}$ *is in* $\mathscr{T}$.

We also say that $\mathscr{T}$ *is a* topology for X.

Correspondingly, the ordinary ε-δ, that is, interval, definition of continuous functions is generalized to a condition in terms of open sets.

Definition 3. *A function from one topological space into another is* continuous *if the inverse image of every open set is open. More precisely, let* $(X, \mathscr{T})$ *and* $(Y, \mathscr{U})$ *be topological spaces, and let* f *be a function from* X *into* Y. *Then* f *is* $\mathscr{T}$-$\mathscr{U}$ continuous *if whenever* $U \in \mathscr{U}$, *then* $f^{-1}(U) \in \mathscr{T}$.

Finally, we define the notion of separability of a topological space, for which it is useful first to introduce the notion of a base.

Definition 4. *A* base *for the topology* $\mathscr{T}$ *is a class* $\mathscr{B}$ *of open sets (that is, members of* $\mathscr{T}$*) such that, for every* x *in* X *and every open set* T *containing* x, *there is a set* B *in* $\mathscr{B}$ *such that* $x \in B$ *and* $B \subseteq T$.

The usual topology of the real line defined above is determined by the requirement that the class of all open intervals be a base.

Definition 5. *A space* $(X, \mathscr{T})$ *is called* separable *if* $\mathscr{T}$ *has a countable base.*

It is, of course, easy to construct a countable base for the real line and, in fact, for any n-dimensional Euclidean space.

In a sense the theory of ordinal utility functions would be flawed if the search for continuous functions required a topology on the set of objects or outcomes that is extraneous to the preference relation itself. Fortunately, this is not the case. Any relation on a set generates a topology on the set in precisely the same way that the natural topology of the real line is generated from the open intervals. For a set A with relation R, we define a *R-open interval* to be any subset of A of the following form:

$$(x, y) = \{z \mid z \in A \ \& \ xPzPy\},$$

where P is defined in terms of R as indicated earlier. The *R-order topology* for A is then the collection of sets generated from R-open intervals by arbitrary unions and finite intersections. It is apparent that the order topology on the set of real numbers generated by the relation $\geq$ is the natural topology already defined in terms of open intervals.

Theorem 4. *Let R be a preference relation for the infinite set A, and let the R-*order *topology for A be separable. Then there exists on A a continuous, real-valued, order-preserving utility function.*

The proof of this theorem is too long to include here. A proof of essentially the same theorem is given by Debreu (1954); as far as we know, his paper contains the first discussion of the continuity of utility functions from a topological standpoint. For additional discussion see Chipman (1960a), Debreu (1963), Murakami (1961), and Newman and Read (1961).

Theorem 4 settles positively the first of the two questions raised at the beginning of this section. The second question concerned conditions that would guarantee the continuity of all the utility functions on A. It should be apparent that some condition beyond Eq. 1, that is, beyond the order-preserving property, must be imposed in order to obtain this result. The matter does not seem to have been discussed in the literature as extensively as is desirable, but additional sufficient conditions can be formulated by requiring that the utility function preserve the structure of the R-order topology for the set A.

2.3 Additivity Assumptions

In the latter part of the nineteenth century, the economists Jevons, Walras, and, to a certain extent, also Marshall made the assumption that the utility of a commodity bundle $(x_1, \ldots, x_n)$ is just the sum of the utilities of the individual components, that is,

$$u(x_1, \ldots, x_n) = u_1(x_1) + \ldots + u_n(x_n). \tag{2}$$

This assumption was made in the context of the development of consumer behavior until Edgeworth, in his *Mathematical Psychics* (1881), noticed that for the purposes of the economic theory of the time there was no need to assume Eq. 2 as well as little justification for doing so. A few years later Pareto (1906) took the next step and observed that only ordinal assumptions were required. The upshot of this evolution of ideas in economics is that additivity assumptions have not been much investigated or used in the economic literature.

On the other hand, the additivity assumption represented by Eq. 2 does seem to hold promise for several kinds of psychological investigations of preference and choice. The only directly pertinent studies known to us are Adams and Fagot (1959), Debreu (1960a), Edwards (1954c), Fagot (1956), and Luce and Tukey (1964). Most of the things we shall discuss in this section are to be found in the works of Adams and Fagot, Debreu, and Luce and Tukey.

To begin with, an additive utility model may be a reasonable candidate for application in any choice situation in which a selection among multicomponent alternatives is required. For example, the model might apply to an attitude scaling task in which subjects are asked to choose between political candidates varying on several dimensions, say, tax policy, foreign policy, and civil liberties legislation. It should be realized that even for small sets of alternatives, such as three or four political candidates, the additivity assumption is not automatically satisfied, and thus its assumption has immediate behavioral implications. As an example, let $A_1 = \{a, b\}$, $A_2 = \{x, y\}$, and let R be a preference relation on the Cartesian product $A_1 \times A_2$. Suppose that the strict preference ordering is

$$(a, x)P(a, y)P(b, y)P(b, x).$$

For this preference ordering it is apparent at once that there are no numerical functions u_1 and u_2 such that

$$u_1(a) + u_2(x) > u_1(a) + u_2(y)$$
$$u_1(b) + u_2(y) > u_1(b) + u_2(x),$$

for the first inequality implies $u_2(x) > u_2(y)$ and the second, $u_2(x) < u_2(y)$, which is impossible.

This simple counterexample to the existence of an additive utility function raises the problem of formulating a set of conditions on the observable choice relation R that will guarantee the existence of such a function. For reasons that are presented in more detail in Sec. 2.4, this problem seems to be incapable of a completely satisfactory general solution. It is perhaps desirable to be a little more explicit about what we mean by a general solution. Given the data from a particular experiment it is, of course, possible to decide whether or not there exists an additive utility function that represents the data in the sense of Eq. 2 and that is order-preserving, because the problem for any particular case is just one of solving a finite set of linear homogeneous inequalities. Adams and Fagot, in fact, give one constructive procedure for solving the inequalities.

The general problem, on the other hand, is to impose structural conditions on the relation R which guarantee a solution. Transitivity is an example of a necessary but not sufficient structural condition. Another necessary structural condition that eliminates the counter-example given previously is the condition of *independence:*

$$\begin{gathered}\text{If}\quad (a, x)R(a, y), \quad \text{then}\quad (b, x)R(b, y),\\ \text{and}\\ \text{if}\quad (a, x)R(b, x), \quad \text{then}\quad (a, y)R(b, y).\end{gathered} \tag{3}$$

It is easy to see that Eq. 3 follows from Eqs. 1 and 2. This is called the condition of independence because interaction between the components is ruled out. In the language of economists, the two components must be neither competitive nor complementary. Adams and Fagot give a counterexample in which each set A_1 and A_2 has three elements to show that the addition of Eq. 3 is not sufficient to guarantee the existence of an additive utility function.

The structural conditions we have just discussed are *open* conditions in the sense that they apply to all sets of alternatives for which additivity holds. They do not postulate the existence of any special alternatives nor do they impose on the whole pattern of alternatives a condition that is sufficient but not necessary for the existence of an additive utility function. As we have already remarked, and as we shall indicate in more detail in Sec. 2.4, there are deep reasons for the difficulty, if not impossibility, of giving a fixed finite list of open structural conditions that are sufficient to guarantee the existence of an additive utility function, even for two-component alternatives that are finite in number. However, Scott (1964) has given an infinite list of open conditions, in the form of a fairly simple scheme, which are jointly necessary and sufficient. His results are discussed in a slightly different context in Sec. 2.4.

Once *open* conditions are abandoned, a relatively simple set of sufficient conditions for the existence of two-dimensional additive utility functions can be given by the axioms of conjoint measurement formulated by Luce and Tukey (1964). Moreover, they can be readily generalized to handle any finite number of components. For simplicity, however, we consider only the two-component case.

We begin with a preference relation R on $A_1 \times A_2$, where A_1 and A_2 are (not necessarily disjoint) sets. The first two axioms are necessary open conditions, namely,

1. Weak ordering. *R is a* weak ordering of $A_1 \times A_2$.
2. Cancellation. *For all* $a, b, f \in A_1$ *and* $p, x, y \in A_2$, *if* $(a, p)R(f, y)$ *and* $(f, x)R(b, p)$, *then* $(a, x)R(b, y)$.

The second axiom can be cast in an apparently simpler form by defining a new relation D on $A_1 \times A_2$ which corresponds to a comparison of differences instead of sums of components (see Sec. 2.4), namely,

$$(a, x)D(b, y) \quad \text{if and only if} \quad (a, y)R(b, x).$$

Then the cancellation axiom is simply the assertion that D is transitive.
The third axiom is not open in that it postulates that there are adequate elements in A_1 and A_2 so that certain equations can always be solved.

3. Solution of equations. *For any* $a, b \in A_1$ *and* $x, y \in A_2$, *there exist* $f \in A_1$ *and* $p \in A_2$ *such that*

$$(a, x)I(f, y) \quad \text{and} \quad (a, x)I(b, p).$$

From these three assumptions it is not difficult to show that the condition of independence, Eq. 3, holds, and so the following induced orderings R_1 on A_1 and R_2 on A_2 are well defined:

$$aR_1b \quad \text{if and only if, for} \quad x \in A_2, \quad (a, x)R(b, x),$$
$$xR_2y \quad \text{if and only if, for} \quad a \in A_1, \quad (a, x)R(a, y).$$

With these definitions, we may introduce the main constructive device used by Luce and Tukey in their proof and in terms of which they introduced their final axiom. A nontrivial *dual standard sequence* (dss) is any set of elements of the form $\{(a_i, x_i) \mid i$ any positive or negative integer or $0\}$ for which

(i) if $i \neq j$, then not $a_iI_1a_j$ and not $x_iI_2x_j$;
(ii) for all i, $(a_i\, x_i)I(a_{i+1}, x_{i-1})$;
(iii) for all i, $(a_{i+1}, x_i)I(a_i, x_{i+1})$.

The final axiom is

4. Archimedean. *For any nontrivial dss* $\{(a_i, x_i)\}$ *and for any* $(b, y) \in A_1 \times A_2$, *there exist integers m and n such that*

$$(a_n, x_n)R(b, y)R(a_m, x_m).$$

With these four axioms, the following result can be proved.

Theorem 5. *If* $\mathscr{A} = \langle A_1 \times A_2, R\rangle$ *satisfies Axioms* 1 *to* 4, *then there are real-valued functions* u *on* $A_1 \times A_2$, u_1 *on* A_1, *and* u_2 *on* A_2 *such that for all* $a, b \in A_1$ *and* $x, y \in A_2$,

(i) $(a, x)R(b, y)$ *if and only if* $u(a, x) \geq u(b, y)$;
(ii) aR_1b *if and only if* $u_1(a) \geq u_1(b)$;
(iii) xR_2y *if and only if* $u_2(x) \geq u_2(y)$;
(iv) $u(a, x) = u_1(a) + u_2(x)$.

Moreover, any other functions u', u_1', *and* u_2' *satisfying* (i) to (iv) *are related to* u, u_1, *and* u_2 *by linear transformations of the following form:*

$$u_1' = \alpha u_1 + \beta,$$
$$u_2' = \alpha u_2 + \gamma,$$
$$u' = \alpha u + \beta + \gamma,$$

where $\alpha > 0$.

The utility representation of a dss is particularly simple in that successive elements in the dss are equally spaced in utility, that is, if $\{(a_i, x_i)\}$ is a nontrivial dss, then for all integers n

$$u(a_n, x_n) - u(a_0, x_0) = n[u(a_1, x_1) - u(a_0, x_0)],$$
$$u_1(a_n) - u_1(a_0) = n[u_1(a_1) - u_1(a_0)],$$
$$u_2(x_n) - u_2(x_0) = n[u_2(x_1) - u_2(x_0)].$$

Much the same representation was arrived at by Debreu (1960a) except that some of his assumptions are topological (see Sec. 2.2) rather than entirely algebraic. Specifically, in the two-component case he assumed that:

(1) *R is a weak ordering of $A_1 \times A_2$.*

(2) *A_1 and A_2 are both connected and separable topological spaces.* (The notion of a separable space was given in Def. 5; a topological space is *connected* if it cannot be partitioned into two nonempty, disjoint, open subsets.)

(3) *All sets of the form*

$$\{(a, x) \in A_1 \times A_2 \mid (a, x)P(b, y)\}$$

and

$$\{(a, x) \in A_1 \times A_2 \mid (b, y)P(a, x)\}$$

are open for all $(b, y) \in A_1 \times A_2$.

(4) *R is an independent relation in the sense of Eq.* 3*; and*

(5) *The equivalence relations I_1 on A_1 and I_2 on A_2 defined in terms of I with, respectively, the second and the first component fixed,* (by 4, I_1 and I_2 are unique, independent of the choice of the fixed element) *have at least two equivalence classes.*

Under these five assumptions, Debreu showed that there exist real-valued continuous functions u_i on A_i such that for $a,\ b \in A_1$ and $x, y \in A_2$,

$$(a, x)R(b, y) \quad \text{if and only if} \quad u_1(a) + u_2(x) \geq u_1(b) + u_2(y).$$

Moreover, $u_1 + u_2$ is unique up to positive linear transformations. By suitably generalizing these assumptions (only 4 is the least bit tricky), he was able to show that the obvious generalized representation holds for any finite number of components. The proof is lengthy and is not included here.

Note added in proof. J. Aczél has pointed out to us that there are results in the mathematical literature that are closely related to the question of the existence of additive utility functions. These are classed under the theory of nets or, as it was called somewhat earlier, the theory of webs (in German, the language of many of the publications, Gewebe). Roughly, the problem treated there is this: given three families of (abstract) curves in a plane for which (1) each point belongs to exactly one curve in each family (and so two different curves of the same family do not intersect) and (2) curves from two different families have exactly one point in common, when is this configuration equivalent (in either an algebraic or a topological sense) to three families of parallel straight lines? The intersection assumptions, (1) and (2), are very similar to, but stronger than, the assumption that equations can be solved, Axiom 3 above. A necessary and, with these intersection assumptions, an essentially sufficient condition for such a mapping to exist is the cancellation axiom (Axiom 2 above) with equalities replacing the inequalities. In the mathematical literature, this important property is known as the Thomsen condition. For a detailed discussion of such results, and for further references to the literature, see Aczél (1964).

2.4 Higher Ordered Metrics

Coombs (1950) introduced the term *ordered metric* to designate those scales that generate an ordering on the set of alternatives and, in addition, a partial or stronger ordering on the differences between alternatives (utility intervals). In suggesting these ordered metric scales as something stronger than ordinal scales, but not as strong as interval scales, Coombs was not thinking of the classical theory of demand, where nothing stronger than ordinal scales is needed, but rather he had in mind problems of attitude scaling and in particular the development of methods for representing the attitudes of a number of different individuals on a common scale. Further development of ordered metric scales was given in Coombs (1952), and a few years later direct applications to choice behavior were made by Adams and Fagot (1959), Fagot (1959), Hurst and Siegel (1956), Siegel (1956), and others.

If we speak in terms of the utility difference, or the difference in preference, between pairs of alternatives, then the classical objection of economists is that choices between alternatives do not yield behavioral evidence on these differences. One can, of course, follow the classical introspective methodology of attitude tests and simply ask subjects to make judgments of the differences. Although the method of introspection has no doubt been unduly deprecated in some of the modern literature on choice behavior, it is nonetheless natural to ask if its difficulties can be overcome by designing operational procedures that will elicit relatively direct behavioral evidence on judgments of utility differences. In Sec. 3 we discuss several methods for doing this in the context of uncertain outcomes, with choices expressed between gambles. Without introducing any element of risk, however, a technique can be used that is very close to the one underlying the additivity assumptions just discussed. This is simply to present two pairs of objects and to ask for a choice between the pairs. To translate the response into an expression of utility differences, we make the following obvious transformation. If

$$u(a) + u(b) \geq u(c) + u(d),$$

then clearly

$$u(a) - u(d) \geq u(c) - u(b),$$

and this last inequality is just a way of expressing utility differences. This parallels the correspondence between the R and D relations discussed in connection with the cancellation axiom in Sec. 2.3.

Two other methods for obtaining direct behavioral evidence on utility differences which do not require uncertain outcomes are described by Suppes and Winet (1955). One interpretation, which depends on equating utility differences with different amounts of money, depends only on the assumption that amount of money is a monotonic increasing function of utility difference. A linear relationship is not required. Of course, this method is restricted to alternatives that can be thought of as objects or commodities. Suppose that the subject prefers Commodity x to Commodity y and also prefers Commodity p to Commodity q. Suppose further that he has in his possession Commodities y and q. We then present him with the opportunity of paying money to replace y by x and q by p. The utility difference between x and y is at least as great as that between p and q if and only if he will pay at least as much money to replace y by x as to replace q by p. Let it be emphasized that there is nothing special about the use of money in this method. Any sort of object or activity that may be represented on a continuum is suitable. For example, an alternative operational definition of utility differences which has exactly the same form can be given in terms of work. In economic terms, what is needed is simply a commodity that, like money, is flexible enough to serve in different situations and such that its marginal utility is either always positive or always negative in the situations under consideration.

A second method, which eliminates the need for a commodity or activity external to the set of alternatives, is a rather slight modification of the above procedure. Following an example of Suppes and Winet, suppose that we confront a housewife with six household appliances of approximately the same monetary value and that she does not have: a mixer, a toaster, an electrical broiler, a blender, a waffle iron, and a waxer. Two of the appliances are selected at random and presented to her. Suppose that they are the toaster and the waxer. She is then confronted with the choice of trading either the toaster for the waffle iron or the waxer for the blender. Presumably, she will exchange the toaster for the waffle iron if and only if the utility difference between the waffle iron and the toaster is at least as great as the difference between the blender and the waxer (due account being taken of the algebraic sign of the difference). A sequence of such exchanges, possibly with repetitions, can be easily devised in such a way that every utility difference is compared to every other. Unfortunately, however, we know of no experiments that have attempted to apply either of these methods to the measurement of utility differences.

The formal characterization of various kinds of higher ordered metric scales is a somewhat complicated matter, and we shall not present a complete survey here. Fortunately, much of the discussion of these matters has already been covered in Chapter 1 of Vol. I, particularly in Secs. 3.3 and 3.4.

Coombs' original concept of an ordered metric scale as a simple ordering of alternatives and a partial ordering of differences between alternatives is formulated precisely by the following axioms. It is, however, convenient to replace the concept of a partial ordering by that of a quasi-ordering on the differences of alternatives (a quasi-ordering is a transitive and reflexive relation, as expressed in the first two axioms).

Definition 6. *A quaternary relation D on a set A is a* weak higher ordered metric *if and only if the following five axioms are satisfied for every a, b, c and d in A:*

1. *if $abDcd$ and $cdDef$, then $abDef$;*
2. *$abDab$;*
3. *if $abDcd$, then $acDbd$;*
4. *if $abDcd$, then $dcDba$;*
5. *$abDbb$ or $baDbb$.*

The intended interpretation of the quaternary relation D is that $abDcd$ if and only if $u(a) - u(b) \geq u(c) - u(d)$. In terms of this intended numerical interpretation, the meanings of Axioms 3 and 4 should be clear. Axiom 3 expresses what has been called in the literature the quadruple condition, initially formulated by Marschak (first stated in print, as far as we know, in Davidson and Marschak, 1959; also see Sec. 5.4), and Axiom 4 expresses a natural condition on the sign of the differences. Axiom 5 states a restricted connectivity condition needed to obtain a weak ordering on alternatives as opposed to differences between alternatives. To show how a complete ordering may be obtained from these axioms on alternatives, we define the weak preference relation R as follows.

Definition 7. *aRb if and only if $abDbb$.*

The intended interpretation, of course, is that aRb just when a is weakly preferred to b. In terms of the desired numerical interpretation, aRb if and only if $u(a) \geq u(b)$. We now prove

Theorem 6. *R is a weak ordering.*

PROOF. (1) *If aRb and bRc, then aRc.*

By hypothesis aRb and bRc. In terms of the definition, this means that $abDbb$ and $bcDcc$; from Axiom 2 we have $bcDbc$ and so, by Axiom 3, $bbDcc$; from Axiom 4 and one of the hypotheses we have $ccDcb$; and thus by transitivity (Axiom 1) $abDcb$. However, by Axiom 3 this means that $acDbb$. Once again using the fact that $bbDcc$ and transitivity, we obtain $acDcc$ which, by virtue of the definition of R, is equivalent to aRc, as desired.

(2) *aRb or bRa.*

From Axiom 5 we have $abDbb$ or $baDbb$. If $abDbb$, then aRb from the definition of R. On the other hand, if $baDbb$, then from Axioms 2 and 3 we also have $bbDaa$ and thus by transitivity $baDaa$, and therefore by definition bRa. This concludes the proof.

Additional elementary theorems can be proved, but it is not possible to go on to prove that there exists a real-valued function u defined on the set A such that

$$abDcd \quad \text{if and only if} \quad u(a) - u(b) \geq u(c) - u(d). \tag{4}$$

An immediate reason is that if such a function u were to exist, then we would necessarily have a weak ordering of differences between alternatives, but our axioms only guarantee a quasi-ordering. The implication of not having such a function u is that we cannot use numerical techniques in working with a weak ordered metric. It is necessary to restrict ourselves to elementary properties of the quaternary difference relation D and of the ordering relation R, thereby severely limiting the possibility of connecting a weak higher ordered metric to other parts of an analysis of choice behavior.

It seems a natural move, therefore, to go from a weak higher ordered metric to a strong higher ordered metric by requiring that the quaternary difference relation itself be connected. This is expressed in the following definition.

Definition 8. *Let D be a weak higher ordered metric on the set A. Then D is a* strong higher ordered metric *if for every a, b, c, and d in A, the axiom of strong connectivity holds, that is, $abDcd$ or $cdDab$.*

This last higher ordered metric is essentially the one applied by Hurst and Siegel (1956) and Siegel (1956). These experiments are discussed in Sec. 4.1.

Unfortunately, even with its strong conditions, it is not possible to show that there exists a numerical representing function u preserving the structure of the difference relation D of a higher ordered metric. An example showing this was given by Scott and Suppes (1958). Moreover, by extending their initial example, they showed that no finite list of axioms that involve only open conditions (as defined in Sec. 2.3), that is, that do not involve existential assertions, are sufficient to guarantee the existence of a numerical representing function. From a psychological standpoint, this means that the structural conditions on behavior required to strengthen a strong higher ordered metric sufficiently to obtain a numerical representation and yet not sufficiently strong to guarantee an interval scale are quite complicated in character.

Because of the close connection between additivity assumptions and conditions on a difference relation, it should also be apparent that the same negative results apply a fortiori to the imposition of a fixed finite number of open structural conditions to guarantee the existence of an additive utility function.

Scott (1964) has, however, shown that an open schema that represents a bundle of axioms whose number increase with the cardinality of the set of alternatives is adequate.

Definition 9. *A quaternary relation D on a set A is a* strict higher ordered metric *if and only if the following axioms are satisfied:*

1. *If $abDcd$, then $acDbd$;*
2. *$abDcd$ or $cdDab$;*
3. *for all sequences of elements $a_0, \ldots, a_n$, $b_0, \ldots, b_n \in A$, and all permutations π, σ of $\{0, 1, \ldots, n\}$,* if *$a_ib_iDa_{\pi(i)}b_{\sigma(i)}$ for $0 \leq i < n$, then $a_{\pi(n)}b_{\sigma(n)}Da_nb_n$.*

To make the meaning of Axiom Scheme 3 clearer, we may derive the transitivity of D. To apply Condition 3, it is convenient to formulate transitivity as follows: if $a_0b_0Da_1b_1$ and $a_1b_1Da_2b_2$, then $a_0b_0Da_2b_2$. Now let π and σ be the permutation of $\{0, 1, 2\}$ such that $\pi(0) = \sigma(0) = 1$, $\pi(1) = \sigma(1) = 2$ and $\pi(2) = \sigma(2) = 0$. Application of Condition 3 for this π and σ yields the desired result at once.

What Axiom Scheme 3 really comprises is all possible cancellation laws, some simple examples of which were discussed in Sec. 2.3. The difficulty with this axiom from a psychological standpoint is that there seems to be no simple way of summarizing what it says about choice behavior, but this we take to be an inherent complexity of the structural relations that must hold between elements of any finite set in order to guarantee the existence of a utility function that preserves the order of utility differences.

Scott established the following theorem.

Theorem 7. *A necessary and sufficient condition that a quaternary relation D on a finite set A has a representing utility function in the sense of* Eq. 4 *is that D be a strict higher ordered metric on A.*

The relation between strict and strong higher ordered metrics immediately follows from Theorem 7.

Theorem 8. *Every strict higher ordered metric D on a finite set A is also a strong higher ordered metric on A.*

The many conveniences that follow from having a numerical representation have led to the investigation of several methods of extending strong higher ordered metric scales to guarantee the existence of a numerical representing function One procedure, that of Suppes and Winet (1955), is to add strong conditions that require the set of alternatives to be infinite. This amounts to adding to the axioms for strong higher ordered metrics those for infinite difference systems (Axioms 5, 6, and 7) as stated in Chapter 1 of Vol. I, p. 35. Because a detailed discussion of infinite difference structures is given there, we do not pursue them further here.

A second alternative is to build up on the concept of a finite equal difference system (Def. 16, Chapter 1, Vol. I, p. 39). The intuitive idea of such an equal difference system is to keep the set of alternatives finite, but to impose the strong structural condition that alternatives be equally spaced in terms of utility differences. This means that the total structure is analogous to a single dual standard sequence in the sense of Luce and Tukey. The assumption that alternatives are equally spaced is, in general, much too special and restrictive for applications, but if we can assume the availability of a standard sequence of equally spaced alternatives, we can then use them to provide a scale for the approximate measurement of the utility of other alternatives.

The axioms for the equally spaced alternatives are extremely simple, namely, those for strong higher ordered metrics plus the assumption needed to guarantee equal spacing. Because of their simplicity, we include them here. In order to formulate the additional axiom for the standard sequence, we need the notion that one element in the standard sequence is an immediate R-successor of another. Denoting by S the subset that is the standard sequence of alternatives, the definition of the immediate successor relation J for S is just the following.

Definition 10. *aJb if and only if aPb and for all c in S aPc implies bRc.*

To define a finite difference system with a standard sequence, we impose only the additional assumption that any alternative is bounded from above and below by members of the standard sequence.

Definition 11. *A* finite difference system with a standard sequence *is a triple $\mathscr{A} = \langle A, S, D\rangle$ in which the following axioms are satisfied:*

1. *the set A is finite and S is a subset of A;*
2. *the quaternary relation D is a strong higher ordered metric on A;*
3. *for a, b, c, and d in the standard sequence S, if aJb and cJd, then $abDcd$ and $cdDab$;*
4. *for every b in A, there are alternatives a and c in the standard sequence S such that aRb and bRc.*

The full proof of the following theorem, although rather long, is entirely elementary, and therefore we shall omit it (see Suppes, 1957, pp. 267–274).

Theorem 9. *If $\mathscr{A} = \langle A, S, D\rangle$ is a finite difference system with a standard sequence, then there is a real-valued function u defined on A such that*

(i) *for all a, b in A,*

$$aRb \quad \text{if and only if} \quad u(a) \geq u(b);$$

(ii) *for all a, b, c, and d in the standard sequence S,*

$$abDcd \quad \text{if and only if} \quad u(a) - u(b) \geq u(c) - u(d).$$

Moreover, any function u' satisfying (i) *and* (ii) *is related to u by a linear transformation on S. Let u' be related to u on the set S by the transformation $u'(a) = \alpha u(a) + \beta$, for every a in S, and for any a and c in S such that aJc define the u-unit by*

$$u(a) - u(c) = \Delta_u.$$

Then for any b in A, u and u' are related as follows:

$$\left| u(b) - \frac{u'(b) - \beta}{\alpha} \right| < \Delta_u.$$

This theorem shows that the utility of alternatives in the standard sequence S are measured exactly on an interval scale. For alternatives not in the standard sequence, measurements are made to within the accuracy of the unit interval Δ_u between adjacent members of the standard sequence. Approximations in terms of a standard sequence, but based on a somewhat different rationale, were used experimentally by Davidson, Suppes, and Siegel (1957); however, the alternatives were bets involving uncertain outcomes. The discussion of these results is given in Sec. 4.2.

We began this section on ordered metrics by mentioning Coombs' 1950 article, but as yet we have not mentioned a second important idea contained in that article. This is Coomb's concept of an unfolding technique which is designed to place both individuals and alternatives on the same scale. We may think of the position of the individual on the scale as his ideal utility point. A good example of the usefulness of this kind of model is in the problem faced by a voter when choosing between less than perfect candidates. In many situations he finds that some of the candidates are politically to the left and the others to the right of his own "ideal" position. Presumably he should vote for the one nearest to his own position, whether to the left or right of his ideal.

To sketch how these ideas may be formalized, let x be an individual and a and b two alternatives, and let $T(x, a, b)$ be the relation of x preferring a to b. We then want a utility function u to satisfy the following condition:

$$T(x, a, b) \quad \text{if and only if} \quad |u(x) - u(a)| \leq |u(x) - u(b)|.$$

The problem of formulating conditions on the relation T to guarantee the existence of such a utility function u is closely connected to the corresponding problem for weak, strong, and strict higher ordered metrics, and so we do not pursue it further here. Some additional remarks on the formal problem can be found in Sec. 3.6 of Chapter 1.

2.5 JND Assumptions

The algebraic choice theories discussed thus far all assume that the individual makes such a clear and definite judgment of preference that the relation of indifference is transitive, and so is an equivalence relation. The challenge to this assumption is familiar from psychophysics (see, for example, the discussion in Chapters 3 and 4, Vol. I). Economists have offered similar objections to classical utility theory. A relatively early discussion of these matters is found in Armstrong (1939, 1948, and 1951) who argues, plausibly enough, that a given alternative a may be indifferent to a second alternative b, b may be indifferent to a third alternative c, and yet a is preferred to c.

From a logical standpoint, the first and most natural question to ask about nontransitive indifference relations concerns suitable axiomatic generalizations of simple orderings. This was the concern in Luce (1956), where the concept of a semiorder was introduced as a natural generalization of the familiar concept of a simple ordering. Luce originally stated the axioms in terms of two binary relations, one of preference and one of indifference. In Scott and Suppes (1958), the axioms were simplified and only a single binary relation of preference was used. Their axioms are stated in the following definition.

Definition 12. *A* semiorder *is a binary relation P on a set A that satisfies the following three axioms for all a, b, c, and d in A:*

1. *not aPa;*
2. *if aPb and cPd, then either aPd or cPb;*
3. *if aPb and bPc, then either aPd or dPc.*

In Luce (1956) a jnd function is introduced which varies with the individual elements of A, that is, the jnd function is defined on A. Intuitively it would be desirable if Luce's results were the best possible for semiorders. Unfortunately, they may be strengthened (see Scott and Suppes, 1958) to show that a numerical interpretation of P can be found which has as a consequence that the jnd function is constant for all elements of A. In particular, the following theorem may be proved (see p. 32, Chapter 1, Vol. I for the proof):

Theorem 10. *Let the binary relation P be a semiorder on the finite set A. Then there exists a real-valued function u on A such that for every a and b in A*

$$aPb \quad \textit{if and only if} \quad u(a) > u(b) + 1.$$

It should be noted that this theorem is restricted to finite sets of alternatives. As is evident from our earlier discussion, additional axioms are needed in order to prove such a representation theorem for infinite sets. As far as we know, no substantial investigation of semiorders for infinite sets has been made.

Matters become much more complicated if we admit subliminal differences and at the same time attempt to obtain a numerical representation stronger than an ordinal scale. Such a set of axioms was given by Gerlach (1957). To obtain more powerful numerical results she followed a proposal that originated with Wiener (1921) and introduced a four-place relation that has the following interpretation. The relation $abLcd$ holds whenever the subjective difference between a and b is algebraically sufficiently greater than the subjective difference between c and d. Gerlach's axioms on L are sufficiently strong to permit her to prove that there is a real-valued function u defined on the set of alternatives and a jnd measure Δ (that is, Δ is a real number) such that $abLcd$ holds if and only if either

(i) $|u(c) - u(d)| < \Delta$ and $u(a) - u(b) \geq \Delta$; or
(ii) $|u(a) - u(b)| < \Delta$ and $u(d) - u(c) \geq \Delta$; or
(iii) $|u(a) - u(b)| \geq \Delta$ and $|u(c) - u(d)| \geq \Delta$ and $[u(a) - u(b)] - [u(c) - u(d)] \geq \Delta.$

Note that Condition (i) corresponds to the case when c and d are separated by less than a jnd, and a and b are separated by at least a jnd; Condition (ii) reverses these relations; and Condition (iii) covers the case when a and b as well as c and d are separated by at least a jnd. Moreover, Mrs. Gerlach proved that the numerical function u is unique up to a linear transformation, with, of course, the just noticeable difference Δ being transformed not by the entire linear transformation, but only by the multiplicative part of the transformation. We do not give here the rather complicated axioms formulated by Mrs. Gerlach.

A recent extensive study of jnd structures, building on the earlier work of Luce (1956), can be found in Adams (1963). Roughly speaking, Adams added to Luce's ordinal concept of a semiorder an operation of combination, with particular reference to weighing with a balance. Although most of his detailed results are restricted to this additive case, his methods of attack are of more general interest and could be applied to the higher ordered metrics discussed in Sec. 2.4. Adams showed that no matter how insensitive a balance is, it is always possible, on the basis of a finite number of comparisons on the pans of the balance, to determine the weight of any object to an arbitrary degree of accuracy. Of course, for a fixed set of objects and a fixed number of observations the accuracy has a fixed limit. The primary limitation in applying his methods to psychological studies of preference is the absence of any probabilistic considerations of the sort surveyed in Secs. 5 to 8 of this chapter.

3. ALGEBRAIC CHOICE THEORIES FOR UNCERTAIN OUTCOMES

3.1 The Expected Utility Hypothesis

The ordinal theory of Pareto, which dominated the economic theory of utility from the beginning of this century until the publication of von Neumann and Morgenstern's treatise on the theory of games in 1944, rested squarely on the assumption that the individual in choosing among alternatives has no uncertainty about the consequences of these alternatives. Once uncertainty in the consequences is admitted, no ordinal theory of choice can be satisfactory. The simplest sorts of examples suffice to make this fact clear. Consider an individual who has five dollars that he is thinking about betting on the outcome of a throw of a die. Suppose that if he makes the bet and the die comes up either one or three, he will receive nine dollars; whereas, if the die comes up two, four, five, or six, he will lose his initial five dollars. His problem is to decide whether to bet the five dollars or not. Granting the ordinal assumption that the individual surely prefers more money to less—in this case, that he prefers nine to five to zero dollars—does not carry the individual very far in deciding whether or not to place the bet. Only slight reflection makes it clear that this rather artificial example of deciding whether or not to place a bet is but one of many examples, some of which are quite serious, in which decisions must be made in the face of inherently risky outcomes. One of the most common examples, one that almost all (middle and upper class) individuals in our society face, is the variety of decisions about what and how much insurance to carry. A typical middle-class member of our society now has insurance coverage for automobile collision and liability, his own death, destruction of his house by fire, loss of possessions by theft, medical and hospital insurance, and possibly accident and income insurance. Every decision to invest in such an insurance policy involves a choice in the face of uncertain outcomes, for the individual does not know what the present state of affairs will lead to in the way of future consequences for him. Insurance is a way of taking a decision against suffering a financial disaster, either to himself or to his family, in case a personal catastrophe does occur.

The expected utility hypothesis, which probably was first clearly formulated by Daniel Bernoulli (1738), is the most important approach that has yet been suggested for making decisions in the context of uncertain outcomes. The fundamental idea is exceedingly simple. The individual must make a decision from among several possible alternatives. The possible decisions may have a variety of consequences, and ordinarily the consequences are not simply determined by the decision taken but are also affected by the present state of affairs (also called the *state of nature*). We suppose that the subject has a utility function on the possible consequences and that he has a probability function on the possible states of nature. According to the expected utility hypothesis, the wise decision maker selects a decision or course of action that maximizes his expectation.

It is perhaps useful to illustrate this important complex of ideas by considering a simple example such as the decision of whether or not to go to a football game in uncertain weather. Let the set S of states of nature have as members the two possible states of raining, s_1, or not raining, s_2, during the game. Let the set C of possible consequences be those of being at the game and not being rained on, c_1, staying home, c_2, and being at the game and being rained on, c_3. The two decisions are going to the game, d_1, and not going to the game, d_2. Formally, d_1 and d_2 are functions from S to C such that $d_1(s_1) = c_3$, $d_1(s_2) = c_1$, $d_2(s_1) = d_2(s_2) = c_2$. Suppose now that the individual assigns a subjective probability of $\frac{1}{3}$ to s_1 and $\frac{2}{3}$ to s_2, and that he prefers consequence c_1 to c_2 to c_3. It should be evident, as we remarked earlier, that such merely ordinal preferences are insufficient to lead to a rational decision between d_1 and d_2. Let him, however, also assign numerical values to the consequences; in particular, let his utility function u be such that $u(c_1) = 12$, $u(c_2) = 6$, $u(c_3) = -9$ (and we suppose u is unique up to a choice of unit and zero). Then the expected utility hypothesis invokes him to compute the expectation in the ordinary sense of random variables for both d_1 and d_2, using the numerical utility function to define the values of the random variables, and then to choose the decision that has the greater expectation. He finds that

$$E(d_1) = \tfrac{1}{3}(-9) + \tfrac{2}{3}(12) = 5$$
$$E(d_2) = \tfrac{1}{3}(6) + \tfrac{2}{3}(6) = 6,$$

and so he should elect not to go to the game, d_2.

A central problem for normative or descriptive behavior theory is to state axioms on behavior that lead to a numerical representation of utility and probability so that decisions are based on a maximization of expected utility. In the next subsection, 3.2, we consider axiom systems of this sort which assume in their statement the existence of numerical probabilities. In Sec. 3.3 we widen the context by considering axiom systems that also impose behavioral assumptions on probability, that, in essence, treat probability as a subjective degree of belief. In Sec. 3.4 we survey briefly various decision principles that have been proposed as alternatives to the expected utility hypothesis.

Before we turn to detailed analyses of behavioral assumptions that yield the expected utility hypothesis, it is natural to ask just how much beyond ordinal requirements are needed to sustain the expected utility hypothesis analysis. From the remarks made in Sec. 2.4 about representation problems for higher ordered metrics, it should be evident that the situation is not simple. Suppose, for simplicity at the moment, that we have only a finite number of states of nature. We denote by s_i the numerical probability (objective or subjective) of state i and by u_{di} the numerical utility of the consequence that results from taking decision d when state i is the true state of nature. We then express the notion that the expectation of decision d is greater than that of decision e by the following inequality:

$$\sum_i s_i u_{di} > \sum_i s_i u_{ei}.$$

It is simple enough to construct counterexamples to show that probability and utility must be measured on scales stronger than ordinal ones in order to preserve this inequality under admissible transformations. On the other hand, when the set of states of nature is finite and no additional assumptions are made, we cannot prove that for given decisions d and e utility must be measured on an interval scale, even if we assume that numerical probability measures are already given. For present purposes, there are really not any natural groups of transformations lying between the group of monotone increasing transformations (ordinal scales) and the group of positive linear transformations (interval scales) that permit a precise analysis of the uniqueness aspect of the expected utility hypothesis. Consequently, as we shall see, the various axiom systems developed to represent the expected utility hypothesis end up with the (unnecessarily restrictive) result that utility is measured on an interval scale. This is done, of course, by imposing rather strong structural conditions of an existential character on either the set of states of nature or on the set of consequences, for quite strong assumptions are necessary in order to show that utility *must* be measured on an interval scale.

3.2 Axiom Systems That Assume Numerical Probability

In this title we have deliberately said "numerical probability" instead of "objective probability," because the axiom systems we shall consider can be interpreted in terms of subjective probability provided that we assume a prior numerical measurement of subjective probability that satisfies the ordinary probability axioms. The scheme for measuring utility by using numerical probability originates with von Neumann and Morgenstern (1944). Because of its considerable historical interest and its essential simplicity, we state with only trivial modifications the original von Neumann-Morgenstern axiom system. Numerical probabilities enter into the statement of the axioms in the following essential way. If we have two alternatives, x and y, then we may consider the new alternative consisting of alternative x with probability α and alternative y with probability $1 - \alpha$. We denote this new alternative by $x\alpha y$, which expression is often called the α mixture of x and y. Once such mixtures are available, we may ask the subject for his preferences not only among pure alternatives but also among probability mixtures of alternatives. Thus we might ask if he prefers the pure alternative z to an α mixture of x and y, that is, to receiving x with probability α and y with probability $1 - \alpha$. It is important to realize, of course, that x and y are not numbers and that the juxtaposition "$x\alpha y$" does not signify multiplication. The expression denotes a single ternary operation, say h, and postulates are then required to express its properties. To make this point quite explicit, we could write in place of "$x\alpha y$" a more explicit functional notation such as $h(x, \alpha, y)$. In terms of this notation, the following Axioms 3 and 4 would be expressed as

$$h(x, \alpha, y) = h(y, 1 - \alpha, x),$$

$$h[h(x, \alpha, y), \beta, y] = h(x, \alpha\beta, y).$$

The von Neumann-Morgenstern axioms are incorporated in the following definition.

Definition 13. *A triple* $\mathscr{A} = \langle A, R, h\rangle$ *is a* von Neumann-Morgenstern system of utility *if and only if the following axioms are satisfied for every* x, y *and* z *in* A *and every* α *and* β *in the open interval* (0, 1), *where the operation* h *is denoted by juxtaposition:*

1. *R is a weak ordering of A;*
2. *$x\alpha y$ is in A;*
3. *$x\alpha y = y(1 - \alpha)x$;*
4. *$(x\alpha y)\beta y = x\alpha\beta y$;*
5. *if xIy, then $x\alpha zIy\alpha z$;*
6. *if xPy, then $xPx\alpha y$ and $x\alpha yPy$;*
7. *if xPy and yPz, then there is a γ in* (0, 1) *such that $yPx\gamma z$;*
8. *if xPy and yPz, then there is a γ in* (0, 1) *such that $x\gamma zPy$.*

Axiom 1 just requires the familiar ordinal restriction that R be a weak ordering of A. The remaining axioms together impose much stronger requirements, which are difficult to satisfy exactly in practice. This difficulty, has in fact, been much of the source of inspiration for the subsequent developments considered in Secs. 5 to 8. The second axiom is simply a closure axiom requiring that if two alternatives are in the set A, then any probability mixture of them is also in the set of alternatives. It follows from this postulate, from the existence of a, $b \in A$ such that aPb, and from Axiom 1 that the set A is infinite. (In discussions of this sort we always implicitly require that A be nonempty and, here, we also require that there be at least two alternatives, one of which is strictly preferred to another.) The third and fourth axioms state simple assumptions about mixtures and their combinations. In more recent discussions, the properties expressed by Axioms 2, 3, and 4 are central to the characterization of what are called mixture-spaces (see discussion later of the Herstein and Milnor axiomatization of utility). Notice that both Axioms 3 and 4 are trivially true if we interpret x and y as numbers. The fifth axiom asserts that if an individual is indifferent between x and y, then he is also indifferent between any probability mixture of x and z and the same mixture of y and z. The sixth axiom asserts that if alternative x is strictly preferred to alternative y, then x is strictly preferred to any probability mixture of x and y, and any probability mixture of x and y is strictly preferred to y. The last two axioms state what amount to a continuity condition on preferences. If y is between x and z in strict preference, then Axiom 7 asserts that there is a probability mixture of x and z that is preferred to y, and Axiom 8 asserts that there is a probability mixture of x and z to which y is preferred.

The proof that these eight axioms are sufficient to guarantee the existence of a numerical utility function, unique up to a linear transformation, is rather tedious and is not given here. Readers are referred to the quite detailed proof in von Neumann and Morgenstern (1947). The precise theorem they established is the following:

Theorem 11. *Let $\mathscr{A} = \langle A, R, h\rangle$ be a von Neumann-Morgenstern system of utility. Then there exists a real-valued function u defined on A such that for every x and y in A and α in* (0, 1)

(i) *xRy if and only if $u(x) \geq u(y)$,*
(ii) $u(x\alpha y) = \alpha u(x) + (1 - \alpha)u(y)$.

Moreover, if u' is any other function satisfying (i) *and* (ii), *then u' is related to u by a positive linear transformation.*

The theorem just stated shows that the axioms are sufficient for the representation; it is also interesting to observe that they are necessary as well, once the concept of a mixture-space is introduced. That is, once we require Axioms 2, 3, and 4, then the remaining axioms express necessary conditions as well as sufficient ones on any numerical utility function having the properties expressed in the theorem.

Since the publication of the von Neumann and Morgenstern axiomatization, there have been a number of modifications suggested in the literature and, more important, considerable simplifications of the proof originally given by von Neumann and Morgenstern. The intensive examination of this axiomatization is undoubtedly due to the considerable importance that has been attached to it by economists and also by mathematical statisticians. One of the first searching examinations was by Marschak (1950). Perhaps the most discussed subsequent axiomatization is that given by Herstein and Milnor (1953). The notation of Herstein and Milnor has been changed slightly to conform as much as possible to that used earlier in defining a von Neumann-Morgenstern system of utility. Their axioms are given in the following definition:

Definition 14. *A triple $\mathscr{A} = \langle A, R, h\rangle$ is a* Herstein-Milnor system of utility *if and only if the following axioms are satisfied for every x, y, and z*

in A and every α and β in the closed interval [0, 1], *where the operation h is denoted by juxtaposition:*

1. *R is a weak ordering of A;*
2. *$x\alpha y$ is in A;*
3. *$x\alpha y = y(1-\alpha)x$;*
4. *$(x\alpha y)\beta y = x\alpha\beta y$;*
5. *$x1y = x$;*
6. *if xIy, then $x\frac{1}{2}zIy\frac{1}{2}z$;*
7. *the sets $\{\alpha \mid x\alpha yRz\}$ and $\{\alpha \mid zRx\alpha y\}$ are closed.*

The first four axioms of Herstein and Milnor, as formulated here, are precisely the same as the first four axioms of von Neumann and Morgenstern given earlier. Herstein and Milnor's fifth axiom differs because they permit probabilities to lie in the closed interval [0, 1], rather than just in the open interval (0, 1). A set A and a ternary operation satisfying Axioms 2 to 5 are said to constitute a *mixture-space*. The sixth axiom of Herstein and Milnor is just a weakening of von Neumann and Morgenstern's Axiom 5: from their sixth axiom one can prove Axiom 5 of the earlier definition. Finally, their seventh axiom, formulated in terms of closed sets, just imposes a condition of continuity, and it has essentially the same consequences as Axioms 7 and 8 of the earlier definition. It is pretty much a matter of mathematical taste whether one prefers the kind of elementary continuity formulation contained in the earlier definition or the very simple topological assumption made in Axiom 7 by Herstein and Milnor. Probably the main advantage of the topological version is that the resulting proof of the existence of a numerical utility function is more elegant.

Although we shall not give the complete Herstein-Milnor proof that there exists a numerical utility function for any system satisfying their axioms, we list their series of theorems and for the conceptually more important ones indicate the nature of the proof.

Theorem 12. *If x, y, and z are in A and xRy and yRz, then there exists a probability α such that $yIx\alpha z$.*

PROOF. Let $T = \{\alpha \mid x\alpha zRy\}$. By virtue of Axiom 7, T is a closed subset of the unit interval [0, 1]. Moreover, since xRy by hypothesis and $x1z = x$ by Axiom 5, 1 is in T and thus T is not empty. Similarly, $W = \{\beta \mid yRx\beta z\}$ is closed and nonempty. By virtue of Axioms 1 and 2, every probability, that is, every number in the closed interval [0, 1], is in either T or W, and therefore $T \cup W = [0, 1]$. Since the unit interval is topologically connected, that is, it cannot be decomposed into a union of closed, disjoint sets, it follows that $T \cap W$ is not empty; any element α in the intersection satisfies the requirement of the theorem.

Theorem 13. *If x and y are in A and xIy, then $x\alpha zIy\alpha z$.*

Note that Theorem 13 is Axiom 5 of Def. 13. This theorem shows that the weakened Axiom 6 of Def. 14 is adequate.

Theorem 14. *If xPy and if $0 < \alpha < 1$, then $x\alpha yPy$.*

This theorem is just the second part of Axiom 6 of Def. 13.

Theorem 15. *If xPy, then $x\alpha yPx\beta y$ if and only if $\alpha > \beta$.*

Theorem 16. *All alternatives in A are indifferent or there are an infinite number of alternatives differing in preference.*

PROOF. If for some x and y in A, xPy, then by Theorem 15 and the density property of the real numbers there is an infinite number of preferentially distinct alternatives.

Theorem 12 is next strengthened to a uniqueness condition.

Theorem 17. *If xPy and yPz, then there is a unique α such that $yIx\alpha z$.*

PROOF. Use Theorem 15.

The following elementary theorem is not explicitly stated by Herstein and Milnor, but it is useful in establishing the "linearity" of the utility function, that is, property (ii) of Theorem 11.

Theorem 18. $(x\beta y)\alpha(x\gamma y) = x[\alpha\beta + (1-\alpha)\gamma]y$.

PROOF.
$$
\begin{aligned}
x[\alpha\beta + (1-\alpha)\gamma]y &= x\left[\frac{\alpha\beta}{\gamma} + (1-\alpha)\right]\gamma y \\
&= (x\gamma y)\left[\frac{\alpha\beta}{\gamma} + (1-\alpha)\right]y && \text{(by Axiom 4)} \\
&= y\left(1 - \frac{\alpha\beta}{\gamma} - 1 + \alpha\right)(x\gamma y) && \text{(by Axiom 3)} \\
&= y\alpha\left(1 - \frac{\beta}{\gamma}\right)(x\gamma y) \\
&= \left[y\left(1 - \frac{\beta}{\gamma}\right)(x\gamma y)\right]\alpha(x\gamma y) && \text{(by Axiom 4)} \\
&= \left[(x\gamma y)\frac{\beta}{\gamma}y\right]\alpha(x\gamma y) && \text{(by Axiom 3)} \\
&= (x\beta y)\alpha(x\gamma y) && \text{(by Axiom 4)}
\end{aligned}
$$

Theorem 19. *Suppose xPy and define $S_{xy} = \{z \mid xRzRy\}$. Let f and g be two linear, order-preserving functions defined on S_{xy} such that for r_0 and r_1 in S_{xy} with r_1Pr_0, we have $f(r_0) = g(r_0)$ and $f(r_1) = g(r_1)$. Then f is identical with g on S_{xy}.*

PROOF. To say that f is linear and order-preserving on S_{xy} is just to say that for s and t in S_{xy}

$$f(s\alpha t) = \alpha f(s) + (1-\alpha)f(t)$$

and

$$sRt \quad \text{if and only if} \quad f(s) \geq f(t).$$

First, suppose $r_1 R z R r_0$. Then $z I r_1 \alpha r_0$ for some α by virtue of Theorem 12. Thus

$$\begin{aligned} f(z) &= \alpha f(r_1) + (1 - \alpha) f(r_0) \\ &= \alpha g(r_1) + (1 - \alpha) g(r_0) \qquad \text{(by hypothesis)} \\ &= g(r_1 \alpha r_0) \qquad \text{(by linearity)} \\ &= g(z). \end{aligned}$$

Suppose now that $z P r_1$. Then again by virtue of Theorem 12, for some α, $r_1 I z \alpha r_0$. Therefore

$$\begin{aligned} f(r_1) &= \alpha f(z) + (1 - \alpha) f(r_0) \\ &= g(r_1) \\ &= \alpha g(z) + (1 - \alpha) g(r_0), \end{aligned}$$

and since $z P r_1$, $\alpha > 0$ and therefore $f(z) = g(z)$. A similar proof holds for the remaining case of $r_0 P z$.

Theorem 20. *There exists a linear, order-preserving utility function on A, and this utility function is unique up to a positive linear transformation.*

PROOF. Suppose xPy and define as in Theorem 19

$$S_{xy} = \{z \mid xRzRy\}.$$

By virtue of Theorem 17, for any z in S_{xy} there is a unique $\alpha_{xy}(z)$ such that

$$zIx\alpha_{xy}(z)y.$$

We choose now r_0 and r_1 with $r_1 P r_0$, and we keep them fixed for the rest of the proof. We consider only pairs x, y such that

$$r_0, r_1 \in S_{xy},$$

and for any $z \in S_{xy}$, we define

$$u(z) = \frac{\alpha_{xy}(z) - \alpha_{xy}(r_0)}{\alpha_{xy}(r_1) - \alpha_{xy}(r_0)}.$$

Now by virtue of Theorem 15, $\alpha_{xy}(z) > \alpha_{xy}(w)$ if and only if zPw, and thus

$$u(z) > u(w) \quad \text{if and only if} \quad zPw.$$

The linearity of u is established by the following argument. Consider $z\beta w$. Then by Theorem 17

$$zIx\alpha(z)y$$

$$wIx\alpha(w)y,$$

and therefore

$$z\beta w I[x\alpha(z)y]\beta[x\alpha(w)y],$$

and thus by Theorem 18

$$z\beta wIx[\beta\alpha(z) + (1 - \beta)\alpha(w)]y.$$

Now by definition,

$$\begin{aligned}
\beta u(z) + (1 - \beta)u(w) &= \beta\left[\frac{\alpha(z) - \alpha(r_0)}{\alpha(r_1) - \alpha(r_0)}\right] + (1 - \beta)\left[\frac{\alpha(w) - \alpha(r_0)}{\alpha(r_1) - \alpha(r_0)}\right] \\
&= \frac{[\beta\alpha(z) + (1 - \beta)\alpha(w)] - \alpha(r_0)}{\alpha(r_1) - \alpha(r_0)} \\
&= u\{x[\beta\alpha(z) + (1 - \beta)\alpha(w)]y\} \qquad \text{(by definition)} \\
&= u[z\beta w] \qquad \text{(by above equivalence).}
\end{aligned}$$

Obviously $u(r_1) = 1$ and $u(r_0) = 0$, and in view of Theorem 19, the numerical value of u for any $z \in S_{xy}$ is independent of x and y. Thus u may be extended without ambiguity or inconsistency over the whole of A.

The proof that u is unique up to a positive linear transformation we leave as an exercise.

Generalizations of the von Neumann-Morgenstern approach to utility have been made in several directions. Hausner (1954) dropped the Archimedean properties, expressed by Axioms 7 and 8 of Def. 13, and he obtained a representation in terms of vectors rather than real numbers. (His axioms are not precisely the remaining Axioms, 1 to 6, of Def. 13, but they are essentially equivalent.)

A different generalization has been pursued by Aumann (1962), who investigated the consequences of dropping the connectivity axiom which demands that any two alternatives be comparable in preference, that is, that one must be weakly preferred to the other. Aumann weakened Eq. 1 of Sec. 2.1 to the implication:

$$\text{if} \quad xRy, \quad \text{then} \quad u(x) \geq u(y). \tag{5}$$

In addition to the axioms on a mixture space (Axioms 2 to 5 of Def. 14), he postulated that

(i) R is transitive and reflexive on the mixture space;
(ii) if $0 < \alpha < 1$, then xRy if and only if, for all z, $x\alpha zRy\alpha z$;
(iii) if $x\alpha yPz$ for all $\alpha > 0$, then not zPy.

Without assuming connectivity he proved that there exists a linear utility function satisfying Implication 5.

An earlier result of the same sort, but in the context of a finite set of alternatives for which subjective probability is also derived from behavioral assumptions, was given by Davidson, Suppes, and Siegel (1957, Chapter 4).

Still a third sort of generalization has been pursued by Pfanzagl (1959a, b). He introduced the general concept of a "metric" operation in an ordered set and developed the consequences of having two such operations. Roughly speaking, a binary operation $\circ$ is a metric operation in Pfanzagl's sense if it is monotonic, continuous, and bisymmetric in the ordering relation [it is bisymmetric if $(x \circ y) \circ (z \circ w) I (x \circ z) \circ (y \circ w)$]. Moreover, he showed that two metric operations $\circ$ and $*$ on the same ordered set lead to scales identical up to linear transformations if the isometry-relation

$$(x \circ y) * (z \circ w) I (x * z) \circ (y * w)$$

holds for any elements x, y, z and w in the set. The mixture space operation $x\alpha y$ is a special case of his metric operation, and two distinct probabilities α and β satisfy the isometry-relation, that is,

$$(x\alpha y)\beta(z\alpha w) I (x\beta z)\alpha(y\beta w).$$

Among other things, to construct a utility scale unique up to a linear transformation it is sufficient to consider only mixtures with a fixed probability α. The reader is referred to Pfanzagl's monograph (1959a) for full details on these matters.

3.3 Axiom Systems for Subjective Probability

In this subsection we consider the most important direction in which the von Neumann-Morgenstern approach has been generalized, namely, the derivation of a numerical subjective probability function, as well as a utility function, from qualitative postulates on preferences among choices. In many situations in which an individual must decide or choose among alternatives, there is no quantitative measure of probability at hand, ready to be used to evaluate the expected utility of the various alternatives. Thus it is natural to try to extend the axiom systems considered in the previous section to include axioms on decisions that yield measures of both utility and probability, thereby extending the domain of applicability of the expected utility hypothesis. Because it is intended that such enlarged axiom systems should apply when a relative frequency characterization of probability is either not available or meaningful, it is customary in the literature to refer to the probability that is derived as *subjective probability* in contrast to the objective probability defined in terms of relative frequency. We shall say more about the subjective character of probability as we consider various proposals that have been made.

A little reflection on the problem of jointly axiomatizing structural conditions that are sufficient to yield measurement of both utility and subjective probability suggests two different ways to proceed. One is to attempt to state axioms in such a way that we obtain first a measure of utility, which is then used to obtain a measure of subjective probability. The other approach proceeds in the reverse order: we state axioms that permit us first to obtain a measure of subjective probability, which is then used to measure utility along the line of argument described in the preceding subsection. The earliest approach (Ramsey, 1931)[4] followed the first tack, that is, utility is measured first. Ramsey's essential idea was to find a chance event with subjective probability of one-half, then to use this event to determine the utilities of outcomes, and, finally, to apply the constructed utility function to measure the subjective probabilities of the states of nature.

Because this approach has been used extensively in choice experiments by a number of people (its experimental application originates with Davidson, Suppes, & Siegel, 1957), we describe it in greater detail. The first thing to make clear is that one can determine if a chance event has subjective probability $\frac{1}{2}$ without first having a quantitative measure of probability. Suppose a subject is choosing between the two options shown in the following matrix:

$$\begin{array}{c c} & \begin{array}{cc} \text{Option 1} & \text{Option 2} \end{array} \\ \begin{array}{c} E \\ \tilde{E} \end{array} & \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \end{array}$$

If the subject chooses Option 1 and if event E happens, then he receives outcome a. If, however, the complementary event $\tilde{E}$ happens, he receives outcome b. On the other hand, if he chooses Option 2 and event E occurs, then he receives outcome c; whereas, if $\tilde{E}$ occurs, he receives d. If the subject chooses Option 1 over Option 2, and if we had a subjective probability function s and a utility function u with the (subjective) expected utility property, then we would express his choice by the following inequality:

$$s(E)u(a) + s(\tilde{E})u(b) \geq s(E)u(c) + s(\tilde{E})u(d). \tag{6}$$

If the subject were indifferent between the options, the inequality would become the following equality:

$$s(E)u(a) + s(\tilde{E})u(b) = s(E)u(c) + s(\tilde{E})u(d). \tag{7}$$

The notion of indifference is critical to Ramsey's theory.

We now attempt to find an E^* such that for every pair of alternatives a and b we have

$$s(E^*)u(a) + s(\tilde{E}^*)u(b) = s(E^*)u(b) + s(\tilde{E}^*)u(a). \tag{8}$$

[4] In actual fact, Ramsey's two essays on this matter were written in 1926 and 1928, but they were not published until after his death in 1931.

In terms of the preceding matrix this last equation means that the outcomes of the two options may now be expressed as follows:

$$\begin{array}{c c} & \begin{array}{cc} \text{Option 1} & \text{Option 2} \end{array} \\ \begin{array}{c} E^* \\ \bar{E}^* \end{array} & \begin{bmatrix} a & b \\ b & a \end{bmatrix} \end{array}.$$

If $u(a) \neq u(b)$, it follows immediately from Eq. 8 that

$$s(E^*) = s(\bar{E}^*). \tag{9}$$

On the assumption that subjective probabilities add up to one, then $s(E^*) = \frac{1}{2}$. In this section we assume subjective probabilities satisfy the usual axioms of probability. Thus, we talk about the subjective probability measure of an event, where an event is given the usual formal interpretation as a subset of the sample space X. The whole space X is the certain event, and the empty set ϕ is the impossible event.

A *probability measure* P on the subsets of X is required to satisfy the following axioms for every A and B that are subsets of X:

1. $P(A) \geq 0$;
2. $P(X) = 1$;
3. if $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$.

(Some additional conditions are often imposed when the set X is infinite, but since we shall mainly consider finite sets in this subsection we do not state these additional requirements.) Hereafter, when we refer to a subjective probability measure, we assume these three axioms are satisfied, even though Edwards (1962) and some other writers have challenged the assumption that subjective probabilities do or should satisfy the additivity postulate (Axiom 3). In any case, by putting E^* in Eq. 7 and using Eq. 9 we obtain: $u(a) + u(b) = u(c) + u(d)$, and from this equation we derive at once the equality of utility differences: $u(a) - u(c) = u(d) - u(b)$. Following the Ramsey approach, we then try to impose axioms on utility differences so as to guarantee the existence of the function u. How this is to be done from a behavioral standpoint without initially assuming the existence of any numerical measures is only sketched in Ramsey's original article. For finite sets of alternatives these ideas were worked out in detail in Davidson and Suppes (1956), and for infinite sets of alternatives in Suppes and Winet (1955); also see Sec. 2.4.

Once the measure of utility is obtained, other simple axioms are added to justify, in terms of Eq. 7, the following measure of the subjective probability of the event E:

$$s(E) = \frac{u(d) - u(b)}{u(a) - u(c) + u(d) - u(b)}. \tag{10}$$

We shall not go more deeply into the formal aspects of the Ramsey approach, partly because the axioms are very similar in spirit to those discussed in Sec. 2.4 dealing with utility differences. Readers are referred to Davidson and Suppes (1956) and Suppes (1956).

The approach that begins with a consideration of probability rather than utility originated with de Finetti; a comprehensive statement of this viewpoint is found in de Finetti (1937), which includes extensive references to his earlier works on the foundations of probability. The most important recent work on these matters is Savage's (1954) book, which extended de Finetti's ideas, in particular by paying greater attention to the behavioral aspects of decisions, although from the standpoint of experimental psychology Savage's approach is still very far from a thorough-going behavioral one. Six relevant articles, including an English translation of de Finetti (1937), have been reprinted in Kyburg and Smokler (1964).

We do not discuss here the many important ideas of de Finetti and Savage concerning the foundations of probability and statistics, but we do want to emphasize those ideas that seem especially important for the psychological theory of preference and choice.

Perhaps the best place to begin is with de Finetti's axioms for qualitative probability. In spirit, these axioms are similar to those we considered earlier for higher ordered metrics. Suppose we ask our subject to tell us for a variety of pairs of events which of each pair he believes to be the more probable. The question then arises: how complicated must the conditions be on the qualitative relation *more probable than* in order to obtain a numerical probability measure over events?

The question to be asked of our subject has been formulated in a nonbehavioral fashion; however, it is easy to give a behavioral method that leads to the kinds of responses we wish to obtain. Suppose we want behavioral evidence as to whether the subject thinks event E is more or less probable than event F. Consider two options with outcomes a, b, or c and assume that a is preferred to b. For example, outcome a might be winning five cents, b losing five cents, and c winning or losing nothing. The matrix of the game we present him has the following form:

$$\begin{array}{r c}
 & \begin{array}{cc}\text{Option 1} & \text{Option 2}\end{array} \\
\begin{array}{r} E \\ F \\ \widetilde{E \cup F} \end{array} & \begin{bmatrix} a & b \\ b & a \\ c & c \end{bmatrix}
\end{array}$$

where $\widetilde{E \cup F}$ is the event that neither E nor F occurs. Thus our three events are exhaustive, although not necessarily mutually exclusive.

Suppose that the subject chooses Option 1. Under the hypothesis that he is maximizing subjective utility we have the following inequality:

$$s(E)u(a) + s(F)u(b) + s(\widetilde{E \cup F})u(c) \geq s(E)u(b) + s(F)u(a) + s(\widetilde{E \cup F})u(c) \quad (11)$$

On the assumption that a is preferred to b, that is, that $u(a) > u(b)$, it follows immediately from Eq. 11 that

$$s(E) \geq s(F).$$

Thus from behavioral data on choices between options and the hypothesis that subjects are choosing so as to maximize expected utility, we obtain inequalities on subjective probability that are not based on introspective data.

A discussion of experiments whose objective has been the measurement of subjective probability is reserved until Sec. 4.3.

Corresponding to the analysis of various higher ordered metrics in Sec. 2.4, it is natural to ask what formal requirements must be placed on the qualitative relation *more probable than* in order to guarantee the existence of a probability measure that reflects the order structure of the relation.

Let $\geq$ be the relation of (*weakly*) *more probable than*. For the formal statement of the axioms, it is convenient to assume that the relation $\geq$ holds between events that are subsets of a given sample space X. In other words, we use the usual set-theoretical notions for representing events.

Definition 15. *A pair* $(X, \geq)$ *is a* qualitative probability structure *if the following axioms are satisfied for all subsets* A, B, *and* C *of* X:

1. *if* $A \geq B$ *and* $B \geq C$, *then* $A \geq C$;
2. $A \geq B$ *or* $B \geq A$;
3. *if* $A \cap C = \phi$ *and* $B \cap C = \phi$, *then* $A \geq B$ *if and only if* $A \cup C \geq B \cup C$;
4. $A \geq \phi$;
5. *not* $\phi \geq X$.

The first two axioms just assert that $\geq$ is a weak ordering of the subsets of X. The third axiom formulates in qualitative terms the important and essential principle of additivity of mutually exclusive events. The fourth axiom says that any event is (weakly) more probable than the impossible event, and the fifth that the certain event is strictly more probable than the impossible event. Defining the strict relation $>$ in the customary fashion,

$$A > B \quad \text{if and only if not} \quad B \geq A,$$

we may state the fifth axiom as $X > \phi$. It is a simple matter to interpret behaviorally each of the axioms in terms of the option scheme already described.

To give a somewhat deeper sense of the structure imposed by the axioms, we state some of the intuitively desirable and expected consequences of the axioms. It is convenient in the statement of theorems to use the (weakly) less probable relation, defined in the expected manner:

$$A \leq B \quad \text{if and only if} \quad B \geq A.$$

The first theorem says that $\leq$ is an extension of the subset relation.

Theorem 21. *If $A \subseteq B$, then $A \leq B$.*

PROOF. Suppose, on the contrary, that not $A \leq B$, that is, that $A > B$. By hypothesis $A \subseteq B$, so there is a set C disjoint from A such that $A \cup C = B$. Then, because $A \cup \phi = A$, we have at once

$$A \cup \phi = A > B = A \cup C,$$

and therefore by contraposition of Axiom 3, $\phi > C$, which contradicts Axiom 4.

Theorem 22. *If $\phi < A$ and $A \cap B = \phi$, then $B < A \cup B$.*

Theorem 23. *If $A \geq B$, then $\tilde{B} \geq \tilde{A}$.*

Theorem 24. *If $A \geq B$, $C \geq D$, and $A \cap C = \phi$, then $A \cup C \geq B \cup D$.*

Theorem 25. *If $A \cup B \geq C \cup D$ and $C \cap D = \phi$, then $A \geq C$ or $B \geq D$.*

Theorem 26. *If $B \geq \tilde{B}$ and $\tilde{C} \geq C$, then $B \geq C$.*

Because it is relatively easy to prove that a qualitative probability structure has many of the expected properties, as reflected in the preceding theorems, it is natural to go on and ask the deeper question whether or not it has all of the properties necessary to guarantee the existence of a numerical probability measure P such that for any subsets A and B of X

$$P(A) \geq P(B) \quad \text{if and only if} \quad A \geq B. \tag{12}$$

If X is an infinite set, it is moderately easy to show that the axioms of Def. 15 are not strong enough to guarantee the existence of such a probability measure. General arguments from the logical theory of models in terms of infinite models of arbitrary cardinality suffice; a particular counterexample is given in Savage (1954, p. 41). de Finetti (1951) stressed the desirability of obtaining an answer in the finite case. Kraft, Pratt, and Seidenberg (1959) showed that the answer is also negative when X is finite; in fact, they found a counterexample for a set X having five elements (and thus, 32 subsets), but it is too complicated to present here.

It is, of course, apparent that by adding special structural assumptions to the axioms of Def. 15 it is possible to guarantee the existence of a probability measure satisfying Eq. 12. In the finite case, for example, we can demand that all the atomic events be equiprobable, although this is admittedly a very strong requirement to impose.

Fortunately, a simple, general solution of the finite case has recently been found by Scott (1964). (Necessary and sufficient conditions for the existence of a probability measure in the finite case were formulated by Kraft et al., but their multiplicative conditions are difficult to understand. Scott's formulation represents a real gain in clarity and simplicity.) The central idea of Scott's formulation is to impose an algebraic condition on the characteristic functions of the events. Recall that the characteristic function of a set is just the function that assigns the value 1 to elements of the set and the value 0 to all elements outside the set. For simplicity of notation, if A is a set we denote its characteristic function by A^c. Scott's conditions are then embodied in the following theorem, whose proof we do not give.

Theorem 27. *Let X be a finite set and $\geq$ a binary relation on the subsets of X. Necessary and sufficient conditions that there exist a probability measure P on X satisfying Eq. 12 are the following: for all subsets A and B of X,*

1. $A \geq B$ *or* $B \geq A$;
2. $A \geq \phi$;
3. $X > \phi$;
4. *for all subsets* $A_0, \ldots, A_n$, $B_0, \ldots, B_n$ *of* X, *if* $A_i \geq B_i$ *for* $0 \leq i < n$, *and*

$$A_0{}^c + \ldots + A_n{}^c = B_0{}^c + \ldots B_n{}^c,$$

then $A_n \leq B_n$.

To illustrate the force of Scott's Condition 4, we may see how it implies transitivity. First, for any three characteristic functions it is necessary that

$$A^c + B^c + C^c = B^c + C^c + A^c,$$

that is, for all elements x

$$A^c(x) + B^c(x) + C^c(x) = B^c(x) + C^c(x) + A^c(x).$$

By hypothesis, $A \geq B$ and $B \geq C$, and therefore by virtue of Condition 4, $C \leq A$, and thus, by definition, $A \geq C$, as desired. The algebraic equation of Condition 4 just requires that any element of X, that is, any atomic event, belongs to exactly the same number of A_i and B_i, for $0 \leq i \leq n$. Obviously, this algebraic condition cannot be formulated in the simple set language of Def. 15, and thus it represents quite a strong condition. It is, however, an open condition in the sense of Sec. 2.3.

Where X is infinite, a number of strong structural conditions have been shown to be sufficient but not necessary. For example, de Finetti (1937) and independently Koopman (1940a, 1940b, 1941) used an axiom to the effect that there exist partitions of X into arbitrarily many events equivalent in probability. This axiom, together with those of Def. 15, is sufficient to prove the existence of a numerical probability measure. Other related conditions of a similar existential sort are discussed in Savage (1954). Extending the methods used to prove Theorem 27, Scott has improved on these earlier results for the infinite case and has found properties that are both necessary and sufficient, but we do not state his somewhat complicated conditions here.

Thus far we have not explicitly considered sets of axioms that characterize utility and subjective probability together, although we began this subsection with a discussion of the alternative approaches that originate with Ramsey and de Finetti. We conclude with a sketch of the best known set of axioms, namely, those of Savage (1954). Savage has a single primitive relation $\geq$ of weak preference on decisions or acts, and decisions are themselves functions from the set S of states of nature to the set A of consequences. In terms of $\geq$ it is fairly straightforward to define a corresponding relation of preference on consequences and one of comparative probability on the states of nature. Assuming these two additional relations, Savage's seven postulates may be formulated verbally as follows.

1. The relation $\geq$ is a weak ordering of the set D of decisions.
2. Given two decisions restricted to a subset of the states, then one is weakly preferred to the other.
3. Given two decisions, each of which has a constant outcome or consequence on a subset X of states of nature, then one decision is weakly preferred to the other given the set X if and only if the constant outcome of the first decision is weakly preferred to that of the second decision.
4. For any two sets of states of nature, X and Y, one is (weakly) more probable than the other, that is, $X \geq Y$ or $Y \geq X$.
5. All consequences are not equally preferred.
6. If decision f is strictly preferred to g, then there is a partition of S so fine that if f' agrees with f and g' agrees with g, except on one element of the partition, then f is strictly preferred to g' and f' is strictly preferred to g.
7. If every possible consequence of the decision f is at least as attractive as the decision g considered as a whole, then f is weakly preferred to g.

Postulate 7 formulates the sure-thing principle about which we shall have more to say in the next subsection.

Because the formal statement of these seven axioms requires several rather complex definitions and to avoid overlong and conceptually opaque qualifying conditions, we do not give a technical formulation here, but refer the reader to Savage's book or to the summary in Luce and Raiffa (1957). On the basis of these axioms, Savage proves the following.

Theorem 28. *If $\mathscr{S} = \langle S, A, D, \geq \rangle$ is a system satisfying Axioms 1 to 7, then there exists a subjective probability function s on subsets of S and there exists a utility function u on A such that for any finite partition $X_1, \ldots, X_n$ of S and any functions f and g of D constant on elements of the partition*

$$f \geq g$$

if and only if

$$\sum_i s(X_i)u[f(X_i)] \geq \sum_i s(X_i)u[g(X_i)].$$

From a behavioral standpoint the most serious weakness of Savage's system, and all those similar to it, is the essential use of the constant decisions, that is, those decisions that have a constant outcome independent of the state of nature. In actual practice it is rare indeed for such decisions to be realizable; each outcome cannot be realized for some possible state of nature. Unfortunately, there seems to be no simple method of getting around their use. In Savage's own technical discussion, the constant decisions are used both to define the ordering on consequences and the one on states. (For additional discussion of these matters, see Suppes, 1956.)

3.4 Other Decision Principles

A great deal of the modern theoretical literature about decisions made in uncertain situations has concentrated on decision principles other than the maximization of expected utility. The basic motivation for this work is the recognition that, in general, a decision maker does not have information adequate to assign probabilities to the uncertain events that, in part, determine the outcome that ultimately results. This is especially a problem when the uncertainty arises not only from random factors in the environment but also from the more or less rational decisions that are made by other people or organizations. If a decision maker is unable, or unwilling, to act as if he knows the probability of each event occurring, then he must invoke some weaker decision principle than the maximization of expected utility—some principle that depends on incomplete information about the relative likelihoods of the several events. The formulation and mathematical exploration of these different principles are the main foci of the theory of games and of much of statistical decision theory. For more extensive treatments of these ideas than will be presented here, see Blackwell and Girshick (1954), Luce and Raiffa (1957), Raiffa and Schlaifer (1961), and von Neumann and Morgenstern (1944, 1947, 1953).

Beyond any doubt, the simplest and least controversial principle of rational behavior—one that does not presume any knowledge at all about the relative likelihoods of the relevent events—is the *sure-thing principle*. It asserts that if two strategies (that is, possible decisions or acts available to the decision maker) a and a' are such that for each possible event (that is, state of nature or strategies of other decision makers), the outcome that results from the choice of a is at least as desirable as the one that results from a' and that for at least one event the outcome from a is strictly preferred to that from a', then strategy a is *better than* strategy a'. A rational person is assumed to choose a over a'.

The main weakness of the sure-thing principle as a guide to, or description of, behavior is that it can so rarely be applied; in general, of two strategies neither is better than the other, in the preceding sense. To show, however, that it is not totally without force, consider the famous game known as the prisoner's dilemma. There are two players (as decision makers are called in game theory), A and B, each with two strategies, and the payoffs (in money or utility units, as the case may be) are given by

$$\begin{array}{ccc} & & \text{Player B} \\ & & \begin{array}{cc} \quad b_1 \quad & \quad b_2 \quad \end{array} \\ \text{Player } A & \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} (5, 5) & (-10, 10) \\ (10, -10) & (-1, -1) \end{bmatrix}, \end{array}$$

where player A receives the first payoff in the cell selected and player B the second one. Now, looking at the situation from A's point of view, if B chooses b_1, then clearly a_2 is better than a_1 since, by definition, 10 is preferred to 5; and equally well, if b_2 is chosen, a_2 is better than a_1 since -1 is preferred to -10. Hence, by the sure-thing principle, A should choose a_2. An exactly parallel argument leads B to choose b_2, and so the resulting payoff is $(-1, -1)$.

This is an odd, somewhat disturbing result from a social standpoint since it is evident that the strategy choice $\langle a_1, b_1\rangle$, which leads to the outcome (5, 5), is preferable to both players. Indeed, we can formulate a social version of the sure-thing principle that dictates this choice. A strategy pair $\langle a, b\rangle$ is *better than* the pair $\langle a', b'\rangle$ provided that the outcome from $\langle a, b\rangle$ is at least as good for both players and is strictly preferred by one of them to the outcome from $\langle a', b'\rangle$. Any strategy pair such that no other pair is better in this sense is said to be *Pareto optimal*. It is easy to see in the prisoner's dilemma that $\langle a_1, b_1\rangle$ is Pareto optimal. A rational social decision principle should, it is generally agreed, result in Pareto optimal strategies. (This notion can be generalized in the obvious way to games involving three or more players; we do not spell out the details.)

The prisoner's dilemma makes it quite clear that the sure-thing principle for individuals is not generally consistent with the sure-thing principle (Pareto optimality) for social conflicts. Since it is not usually possible to enforce social principles without a good deal of machinery beyond the decision situation itself and since the existing empirical data (Deutsch, 1958, 1960; Lutzker, 1960; Rapoport, 1963, pp. 558–561; Scodel, Minas, Ratoosh, & Lipetz, 1959) strongly suggests that under normal circumstances the individual principle overrides the social one, it seems appropriate to attempt to generalize the individual sure-thing principle to cover decision situations to which it cannot itself be applied.

One idea is to search for strategies, one for each player, such that no person acting alone can benefit himself by a change of strategy. We make this precise for two players; the generalization to any number is obvious. If $\langle a, b\rangle$ is a pair of strategies, let $u_i(a, b)$ denote the corresponding outcome to player i. The strategy pair $\langle a, b\rangle$ is said to be *in equilibrium* if for any other strategy a' of player A,

$$u_A(a, b) \geq u_A(a', b),$$

and if for any other strategy b' of player B,

$$u_B(a, b) \geq u_B(a, b').$$

In the prisoner's dilemma, $\langle a_2, b_2\rangle$ is in equilibrium. In the game

$$\begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} (2, 1) & (-1, -1) \\ (-1, -1) & (1, 2) \end{bmatrix} \end{array},$$

the sure-thing principle is powerless, but it is easy to see that both $\langle a_1, b_1\rangle$ and $\langle a_2, b_2\rangle$ are in equilibrium. Thus the notion of an equilibrium pair extends the class of games to which considerations of individual rationality can be applied. Nevertheless, it has two failings. First, as the preceding example shows, it does not tell an individual player what to do to get one of the desired outcomes: if player A were to choose a_1 in an attempt to get the (2, 1) outcome and player B were to choose b_2 in an attempt to get (1, 2), then the nonequilibrium pair $\langle a_1, b_2\rangle$ would result, and the players would receive the nondesired outcome $(-1, -1)$. Second, the equilibrium notion is incapable of resolving all games since, for example,

$$\begin{bmatrix} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{bmatrix}$$

has no pair of strategies that are in equilibrium, as is easily verified. We take up the second problem first.

The major resolution that has been suggested does not involve a further weakening of the decision principle, but rather the given discrete game is extended to a continuous one by enriching greatly the strategies available to each player. von Neumann (1928; see also von Neumann and Morgenstern, 1944, 1947, 1953) pointed out that whenever a set of strategies is available, then so are all the probability distributions over it. A player can generate any distribution he chooses by using suitable auxiliary devices. Thus, if his preferences satisfy the axioms for expected utility when the probabilities are known, as they are when he generates them himself, then the expected payoff can be calculated for each probability distribution over the given strategies, that is, for each *mixed strategy*, as such a distribution over strategies is called. This was, in fact, the main purpose for which von Neumann and Morgenstern first developed their theory of expected utility. Within the context of mixed strategies, it can be shown that every game has at least one pair of (mixed) strategies that are in equilibrium (Nash, 1950). This by no means trivial result is a partial generalization of the famous minimax theorem of von Neumann (1928) which not only established the existence of mixed strategy equilibrium pairs in those two-person games for which the payoff to one player is always the negative of that to the other (the so-called zero-sum games), but also showed that if $\langle a, b\rangle$ and $\langle a', b'\rangle$ are in equilibrium, then so are $\langle a, b'\rangle$ and $\langle a', b\rangle$, and that the payoffs to a player are the same for all pairs of strategies that are in equilibrium. Neither of these last two statements is generally true for nonzero-sum games, as can be seen in the second game, nor for games with more than two players. Thus, although the first problem mentioned about equilibrium strategies, namely, that they do not prescribe rational behavior for the individual, is not a problem when we restrict our attention to two-person zero-sum games, it is for other classes of games.

An alternative approach to solving the two weaknesses of the equilibrium notion is suggested by an important property of equilibrium strategies in the two-person zero-sum case. For each of a player's strategies, suppose that we determine what is the worst (minimum) outcome that can occur, and then find those strategies for which this worst outcome is as good as possible (that is, find the strategy having the maximum value of the minimum outcome). Such a strategy is called, for obvious reasons, a *maximin strategy*. For two-person zero-sum games, it can be shown that a pair $\langle a, b\rangle$ of mixed strategies is in equilibrium if and only if a and b are both maximin mixed strategies. Observe that the notion of a maximin strategy is defined for any game and that such a strategy always exists whether we restrict ourselves to pure strategies or whether mixed strategies are admitted; however, except for two-person zero-sum games, the notion of maximin strategies does not have any simple relation to that of equilibrium strategies. The principle underlying maximin strategies is conservative in the extreme: concern is focused exclusively on the worst possible outcome from any course of action, no matter how improbable that outcome may be.

Savage (1951) suggested that this last notion should be applied not to the given matrix of payoffs, or to its mixed strategy extension, but rather to what he called the *regret matrix*, which is constructed as follows. For a given player, replace the outcome in a given cell by the following number: find the best outcome (or expected outcome, as the case may be) that he could have achieved by using any of his (mixed) strategies, on the assumption that the choices of chance and other players are fixed, and from it subtract the actual value in that cell. For the prisoner's dilemma, this yields the transformation

$$\begin{bmatrix} (5, 5) & (-10, 10) \\ (10, -10) & (-1, -1) \end{bmatrix} \rightarrow \begin{bmatrix} (5, 5) & (9, 0) \\ (0, 9) & (0, 0) \end{bmatrix}.$$

For example, the -10 in the $\langle a_1, b_2 \rangle$ cell becomes 9 because, with b_2 fixed, the best player A could have done is -1 by choosing a_2, and that minus the -10 of the entry yields 9, which is his "regret" in having chosen a_1 rather than a_2 when B chooses b_2. The 10 in the same cell for player 2 becomes 0 because the best he could do with a_1 fixed is 10, which minus the 10 of the cell yields 0.

The proposed decision rule is to choose any strategy that minimizes the maximum regret. In the prisoner's dilemma this rule again dictates $\langle a_2, b_2 \rangle$, the solution previously obtained by the sure-thing principle. In the following game

$$\begin{bmatrix} (2, 1) & (-1, -1) \\ (-1, -1) & (1, 2) \end{bmatrix} \rightarrow \begin{bmatrix} (0, 0) & (2, 2) \\ (3, 3) & (0, 0) \end{bmatrix},$$

the rule yields $\langle a_1, b_2 \rangle$, which is not one of the two equilibrium pairs of the game, but rather what one would expect if each player attempted to get his preferred equilibrium pair. Thus, at least in this case, the minimum regret criterion seems more descriptive than the equilibrium notion.

Savage's proposal, although intuitively reasonable, suffers from the defect that no theory of utility now in existence is adequate to justify calculating the differences required for the regret matrix and, at the same time, to justify the expected utility property which is needed if mixed strategies are to be used.

Before turning to other ideas, it should be mentioned that several authors, most notably Milnor (1954), have attempted to gain a better understanding of these various decision criteria by means of an axiomatic approach. They have listed various "elementary" axioms that a decision principle might satisfy, and they have shown which sets of these axioms are necessary and sufficient to characterize a given principle. These results give some insight into the various concepts of rationality that are embodied and excluded from the different decision principles. For a summary of these results, see Luce and Raiffa (1957, pp. 286–309).

The decision principles that we have just discussed all differ sharply from the principle of expected utility maximization in that they presume absolutely no information about the relative likelihoods of the events that affect the outcome, whereas the latter principle can be applied only when the decision maker is willing to commit himself in advance to relative likelihoods. Neither extreme seems to capture the reality of most decision problems, which characteristically involve some qualitative information about the relative likelihoods of events, but not enough to justify the assignment of a unique probability measure to them. Little, if any, satisfactory axiomatic theory yet exists to deal with decisions in the presence of some but incomplete information; however, some relevant ideas have been discussed, and so we now turn our attention to them.

One framework of discussion centers about the concept of *level of aspiration*, initially introduced by Dembo (1931) and now familiar to most psychologists. The first direct comparison of it with a utility analysis of choice behavior was made by Simon (1955, 1956). Simon illustrated his ideas by the familiar example of an individual selling his house. Each day he sets an acceptance price based on various factors: bids he has received in the past, rumors about prevailing market conditions, his urgency in consummating a sale, etc. If during the day he receives one or more offers above this price, he accepts the highest. If not, he sets a new acceptance price the next day and awaits further offers. The acceptance price he sets each day is his level of aspiration (for selling the house) at that moment in time. Simon contends that this kind of model, including the mechanisms for changing the level of aspiration with experience, is much closer than the expected utility model to the decision process actually used when the alternatives are complex and information about them is far from complete. Simon has given a certain amount of theoretical development in the two articles cited, but at present the formal aspects, especially the mechanism for processing information, are far from fully stated.

Some possible theoretical connections between utility theory and level of aspiration were postulated by Siegel (1957). For him, the level of aspiration is a point on the individual's utility curve. All outcomes below this point have negative utility and all above it, positive utility. When the number of outcomes is finite, Siegel defines the level of aspiration to be "that goal which has the largest difference in utility between it and the next lower goal." However, the rationale for this definition is far from clear to us. In any case, some experimental applications of these ideas are given in Siegel and Fouraker's (1960) book on experiments in bilateral monopoly.

Although the concept of level of aspiration has great face validity and probably must ultimately be incorporated in some form into any adequate theory of decision making, it is fair to say that so far it has not been developed in any very thorough theoretical fashion or tested extensively—certainly not to the extent that the concepts previously mentioned in this subsection have been.

A quite different concept for dealing with uncertainty is Shackle's (1949, 1955) *potential surprise*. But again, these ideas neither have been stated with any great formal clarity nor have they been applied to any experiments; they were developed to handle various difficulties that arise in connection with expectations in economic theory. In spite of this separation from most psychological work on choice behavior, Shackle's writings include a number of psychologically insightful and original remarks about choice behavior.

To each possible future event he assumes that there can be assigned a degree of potential surprise, and rules are given for combining potential surprises. These differ from the usual rules for dealing with probabilities mainly in being nonadditive. Specifically, if $\eta(E)$ denotes the potential surprise of event E, then the combining rules are essentially the following:

(i) if $E_1 \cap E_2 = \phi$, then $\eta(E_1 \cup E_2) = \min [\eta(E_1), \eta(E_2)]$;

(ii) $\eta(E \cap F) = \max [\eta(E \mid F), \eta(F)]$, where $\eta(E \mid F)$ is the potential surprise of E given that F has occurred.

Thus, when E and F independent, that is, $\eta(E \mid F) = \eta(E)$, rule (ii) reduces to

$$\eta(E \cap F) = \max [\eta(E), \eta(F)].$$

Arrow (1951a, b, 1964) pointed out that it is a consequence of these rules that "there is no law of large numbers or elimination of risks by consolidation of independent events, for the potential surprise attached to a sequence of unfavorable outcomes is as large as to any one." Just how strange such a calculus can be is illustrated by a simple example. Ten successive heads in as many flips of a fair coin seems, intuitively, to have positive potential surprise; yet in Shackle's calculus this event is assigned a potential surprise of zero. We can see this as follows. Consider first one flip, and let H be the event of a head and T the event of a tail. Since $H \cup T$ is certain, $\eta(H \cup T) = 0$; because $H \cap T = \phi$, rule (i) yields

$$\eta(H \cup T) = \min [\eta(H), \eta(T)].$$

Since surely $\eta(H) = \eta(T)$, we have

$$\eta(H) = 0.$$

Now, making the usual assumption of independence,

$$\eta(H_1 \cap \ldots \cap H_{10}) = \max [\eta(H_1), \ldots, \eta(H_{10})] = 0.$$

Clearly, 10 plays no role in this argument: any finite number of successive heads has zero potential surprise!

Shackle has a good deal to say in favor of his rules in spite of these consequences. Essentially, his defense rests on the claim that in real life situations we are never in a position to make the probability computations of textbook examples: such computations are possible only for divisible and repeatable experiments. Indeed, he has argued against the general applicability of either a subjective or frequency theory of probability, but his remarks were mainly directed toward the frequency theory and he has not dealt in an explicit and technical way with the many persuasive arguments put forward by subjectivists, such as de Finetti.

4. EXPERIMENTAL TESTS OF ALGEBRAIC MODELS

Over the past fifteen years a number of experimental studies have been performed that are more or less relevant to the algebraic theories of preference discussed in Secs. 2 and 3, but unfortunately few bear directly on the empirical validity of the various theories discussed. There are several reasons for this. The most important is that the algebraic theories of preference are special cases of algebraic theories of measurement, and as such do not readily lend themselves to an exact statistical analysis of their relation to experimental data. For example, the axioms for the various ordered metrics are formulated in such a fashion that it is natural simply to ask whether or not a set of data satisfies the axioms, not to attempt a statistical goodness-of-fit evaluation. In fact, it is not entirely clear how the theory of goodness-of-fit should be stated for the various kinds of higher ordered metrics. In models similar to those of von Neumann and Morgenstern the basic closure postulate requires an infinity of objects, and again it is not clear when we should be willing to regard a set of data as satisfying or definitely not satisfying the axioms. We would not expect, of course, any finite sample of data to satisfy the axioms exactly, and the difficult question is how to formulate an appropriate criterion.

In view of such difficulties, it is not surprising that the bulk of the experimental literature is concerned with studies designed to test certain limited aspects of choice behavior. Many of the main points that have been established are rather far removed from the theoretical issues we have discussed in the preceding sections. In the interest of completeness we describe the studies of greatest theoretical import, but because the accumulated literature is relatively large we do not attempt a really complete survey. The reader interested in additional references should consult the survey articles mentioned in Sec. 1.4.

Our summary of these experimental studies is organized in terms of the major headings of Secs. 2 and 3, not in strict chronology. Under a given heading, however, the order is primarily chronological. The first section deals with higher ordered metrics, the second with the measurement of utility, the third with the measurement of subjective probability, and the fourth with other models that have been proposed as a result of experimental conjectures or results, but which we have not discussed with any thoroughness in the preceding sections.

4.1 Higher Ordered Metrics

Coombs' (1950) concept of a higher ordered metric has been applied to experimental studies of preference by Coombs and his collaborators in a number of studies and also by Siegel (1956) and Hurst and Siegel (1956).

Coombs (1954) applied his ideas about ordered metrics and the unfolding technique to judgments of the esthetic quality of a series of isosceles triangles, all of which had a base of 1 in. and an altitude that varied from 0.25 to 2.5 in. in steps of 0.25 in. Sets of three triangles were presented to subjects who judged both the most and least preferred in each triad. All 120 possible sets of three triangles formed from a set of 10 were presented. (It should be emphasized that this use of the method of triads is not essential to the application of Coombs' ordered metric concepts.)

In analyzing the data, it was assumed that the most preferred triangle was located nearest the subject's own "ideal" triangle and that the least preferred triangle was located furthest away. The data may be described as follows. The unfolded data from 17 of the 31 subjects satisfied a common interval scale for the 10 stimulus triangles. The responses of 8 additional subjects satisfied the ordinal properties of this common scale, but they required different metric relations. The 6 remaining subjects failed to satisfy the ordinal properties of the common scale. It is to be emphasized that the imposition of the possibility of unfolding the data to get a common ordering of stimuli and subjects is, of course, a much stronger assumption than simply assuming that each subject has his own weak higher ordered metric. Considering the subjectivity usually associated with such esthetic preferences, it is somewhat surprising that the data from 17 of the subjects could be interpreted in terms of a common strict higher ordered metric.

A useful discussion of some of the problems and mathematical consequences of determining an ordered metric from experimental data was given by Coombs and Beardslee (1954), but the only data they reported are from a pilot experiment with one subject, so we do not review the results here.

In moving from a strong higher ordered metric to a strict higher ordered metric, in the sense of the definitions given in Sec. 2.4, it is necessary but not sufficient for adjacent utility intervals to be additive. Coombs and Komorita (1958) investigated the extent to which additivity is satisfied in an experimental ordered metric for the utility of money. Only three graduate student subjects participated in the experiment, but a fairly large number of preferences were required from each. A method of triads was used that was similar to the one just described. The experimental data quite strongly supported the additivity hypothesis. In particular, of 30 predictions tested, 29 were confirmed. It should be mentioned that one can expect the results to be best when the outcomes are money, because the natural order of preference is very strongly entrenched in all subjects. One never expects a subject to prefer less to more money, but inconsistencies of simple preference are common with nonmonetary outcomes.

Two other papers by Coombs (1958, 1959) are also concerned with the problems of applying the unfolding technique to obtain an ordered metric, but because the preferences were mostly probabilistic, the summary of this experiment is reserved until Sec. 8.2.

Siegel (1956) described a method for obtaining a strong ordered metric scale in the sense of Def. 8 of Sec. 2.4. He used simple gambles constructed from events with subjective probabilities of one-half, following the procedure suggested in Davidson, Suppes, and Siegel (1957), to obtain not only an ordering of alternatives but also a complete ordering of the differences between alternatives. The ordinal comparison of utility intervals in Siegel's method is based on the kinds of options and the resulting equations discussed in Sec. 3.3 (see, in particular, Inequality 6.) Because the original article presented only a pilot study based on one subject, we do not consider it further. However, Hurst and Siegel (1956), using the same methodology, ran 30 prison inmates with cigarettes as outcomes rather than money. Of the 30 subjects, 29 completed the experiment. The following is a typical offer:

$$\begin{array}{c c} & \begin{array}{cc} \text{Option 1} & \text{Option 2} \end{array} \\ \begin{array}{c} QUJ \\ QUG \end{array} & \begin{bmatrix} +30 & +5 \\ -10 & +5 \end{bmatrix} \end{array}$$

where QUJ is a nonsense syllable printed on three sides of a die and QUG is a nonsense syllable printed on the other three sides. Sufficient options were used in order to construct an ordered metric (in the sense of Def. 8, Sec. 2.4) for each subject, and from this metric a number of choices was predicted between additional options.

The main hypothesis tested was the accuracy of these predictions as compared with those based on the model in which it is assumed the subject maximizes the expected number of cigarettes. For 6 subjects the two models gave identical predictions, which means that their subjective utility scales were essentially linear in the number of cigarettes. For 15 subjects the ordered metric model yielded better predictions than the simple expectation model, but data for 8 subjects fell in the opposite direction. For an over-all statistical test, the total number of erroneous predictions for each of the two models was determined for each subject, and the differences between these two totals were ranked. A Wilcoxon matched-pairs, signed-ranks test indicated that the observed rank sums differed significantly from what would be expected under the null hypothesis and favored the ordered metric model ($P < 0.025$).

Because of the close relation between the additivity assumptions discussed in Sec. 2.3 and the ordered metric models, we mention at this point an experiment by Fagot (1956), reported in Adams and Fagot (1959), which was designed to test additivity assumptions. Each of 24 subjects was instructed to take the role of a personnel manager for a corporation and choose among hypothetical applicants, two at a time, for an executive position of a specific nature. Applicants were described in terms of just two characteristics, intelligence and ability to handle people, with four levels of ability admitted for each characteristic. Each choice of a subject was represented as an inequality in utility values as described in Sec. 2.3. The additive model is satisfied if and only if the entire set of 77 inequalities has a solution (43 of the 120 applicant pairs were omitted because one weakly dominated the other on both characteristics). Analysis of the data showed that 6 subjects satisfied perfectly the additive model. The remaining 18 subjects who did not satisfy the additive model surprisingly enough did not satisfy the simple ordinal model, that is, each of the 18 violated the transitivity axiom at least once. More detailed analysis suggested that the deviations from either the ordinal or additive model were due mainly to occasional inattention, for 80% of the subjects had at most three such "errors" or deviations, that is, at least 74 of the 77 observed choices satisfied the additive model. Only two of the subjects had more than four such errors.

4.2 Measurement of Utility

Although discussions of the measurement of utility have had a relatively long history both in economics and psychology, and the proposal to extend classical psychophysical methods to the measurement of utility goes back at least to Thurstone (1945), the first real experimental attempt to use any of these ideas in an actual measurement of utility was by Mosteller and Nogee (1951). Their experiment was designed to test the empirical validity of the von Neumann and Morgenstern axiomatization as applied to alternatives consisting of the gain and loss of small amounts of money, together with probability combinations of such outcomes. The subject was presented with bets which he could accept or refuse (the bets were presented to four or five subjects at a time, but each subject made his own decisions). If he refused the bet, no money changed hands; if he accepted it, he either lost 5 cents or won some amount x of money, which one depending on whether or not a specified chance event E occurred. When the subject was indifferent between the two options, that is, when he accepted the bet approximately half the time, then the following equation was used to calculate the relative utilities of the outcomes:

$$u(0¢) = pu(-5¢) + (1 - p)u(x),$$

where p is the probability of E occurring. Since u is an interval scale, $u(0¢)$ and $u(-5¢)$ can be fixed. By setting them to be 0 and -1, respectively, $u(x) = p/(1 - p)$. Note that p is an objective probability, not a measured subjective one. Thus, by selecting an event with probability p and varying the amount x of money until the subject was indifferent between the options, it was possible to find the amounts of money corresponding to any fixed utility $u = p/(1 - p)$. Nine utility points were determined for each subject ranging from $u(-5¢) = -1$ to 101. This constructed utility function was then used to predict the choices of each subject when he was faced with somewhat more complex options.

Fourteen subjects completed the experiment, of whom nine were Harvard undergraduates and five were from the Massachusetts National Guard. Each subject was given \$1.00 at the beginning of each hour of play with which to gamble. Testing took place over a period of about four months.

The bets were constructed from possible hands of poker dice. After three initial sessions, each group of subjects was instructed in how to calculate the true odds for any hand, and in addition, each was given a sheet with the true odds for all the poker dice hands used and was, moreover, "required to keep this sheet in front of him in all future sessions." Additional information was provided to the subjects about the number of times a particular hand had been previously played and the number of times players had won when they selected it during the initial "learning" sessions.

In order to obtain utility curves for individual subjects, the first step in the analysis requires that the point of indifference be determined for each subject for each hand of poker dice used. This means that the amount of money must be found for which the subject is indifferent as to whether he accepts or rejects the bet. Mosteller and Nogee (1951, p. 383) described in clear fashion their procedure for finding this point.

> For each hand a range of offers had been made. The proportion of times the subject elected to play each offer was calculated, and these points were plotted on ordinary arithmetic graph paper with vertical axis as per cent participation and horizontal axis as amount of offer in cents. A freehand curve or a broken-line curve was then fitted to the points. The abscissa value of the point where this curve crossed the 50 per cent participation line gave in cents the subject's indifference offer for that hand. In other words for that hand this calculation yielded an interpolated offer which the subject would be equally likely to accept or reject if given the opportunity.
>
> Since the probability p of winning is fixed for a given hand, the utility of the indifference offer is just $(1 - p)/p$.

Figure 2 shows how this indifference point was constructed for subject B-1 and the poker dice hand 55221. (Amounts of money in cents are plotted on the abscissa.) Note that Fig. 2 approximates rather closely the perfect step function that is required theoretically by the von Neumann-Morgenstern axiomatization and, in fact, by all of the standard algebraic theories for choice among uncertain outcomes.

Utility curves constructed in this manner for two of the student subjects are shown in Fig. 3. Both these curves support the classical assumption of decreasing marginal utility, but, as Mosteller and Nogee pointed out, such apparent support for this economic idea must be interpreted cautiously. For one thing, the range of bets for the students ran only from 5 cents to about \$5.50, and only up to about \$1.00 for the Guardsmen. (For obvious financial reasons, this restriction of the range is not peculiar to the Mosteller and Nogee experiment, but is true of all the experiments

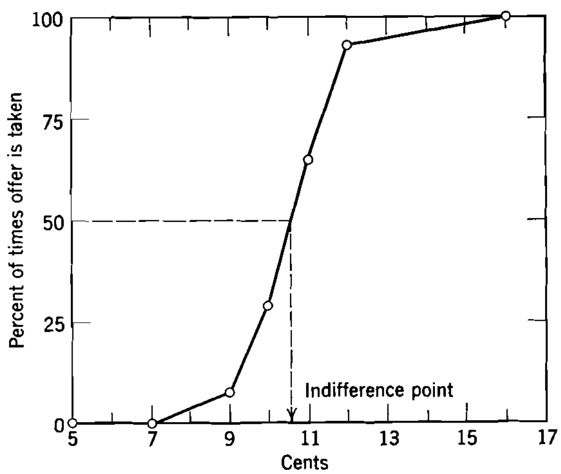
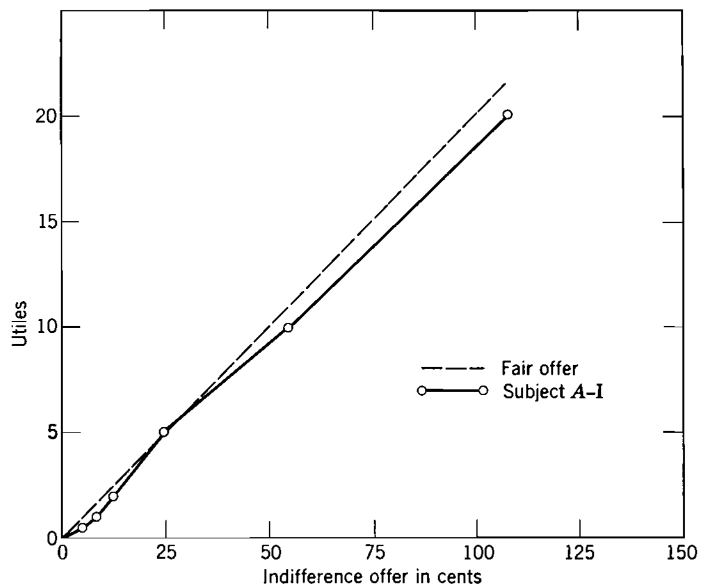


Fig. 2. The data for one subject and hand 55221 are plotted to show how the indifference point is actually obtained. Adapted by permission from Mosteller & Nogee (1951, p. 385).

that have been performed.) For another, the utility curves of the Guardsmen generally supported the assumption of increasing marginal utility in the range of bets considered.

Mosteller and Nogee gave the subjects additional offers of a more complicated sort which they called doublet hands. In a doublet hand the subject is presented with two hands of poker dice: if he beats the low hand, he gets a certain reward; if he beats the high hand, he receives an additional reward as well; and if he fails to beat either hand, he loses 5 cents. As the authors remarked, the introduction of these doublet hands caused considerable confusion in the betting patterns of the subjects. The most definite evidence of this confusion was the increase in what the authors called the zone of inconsistency, that is, in the zone in which the subject did not consistently choose to accept or refuse a bet. The presence of a fairly wide zone of inconsistency, both for the original bets and for the doublet hands, of course, is not compatible in a literal sense with the von Neumann-Morgenstern axiomatization. This incompatibility is particularly reflected by Theorem 17 of Sec. 3.2, which asserts that there is a unique probability combination of alternatives that is indifferent to a given alternative. As was already remarked, the existence of zones of inconsistency is precisely the reason for the development of the probabilistic theories discussed in Secs. 5 to 8 of this chapter. Apart from the existence of a larger zone of inconsistency, the constructed utility curves were fairly successful in predicting behavior with respect to the doublet hands. We do



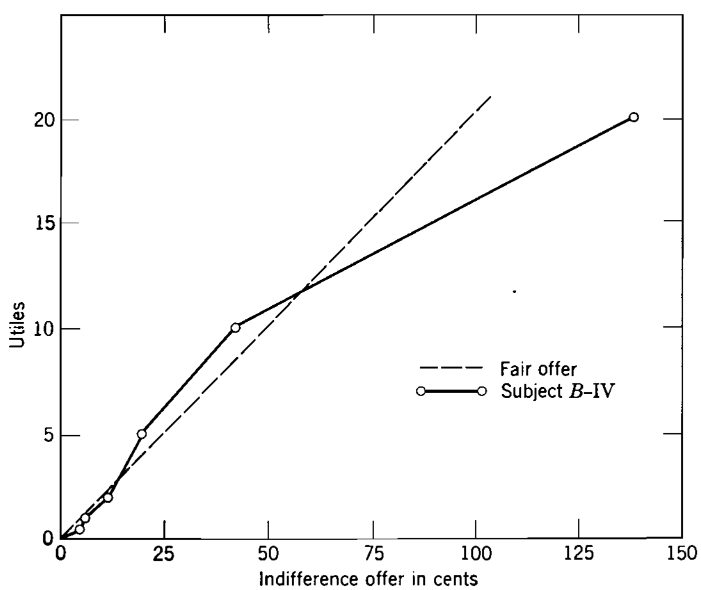


Fig. 3. Utility curves for two student subjects and straight line showing mathematically fair offer. Offer for 101 utiles is not shown. Adapted by permission from Mosteller & Nogee (1951, p. 387).

not attempt a quantitative summary of the data here. The reader is urged to refer to the original Mosteller and Nogee article, which contains a large number of interesting side remarks and insights into the behavior of the subjects, as well as a good deal of additional systematic material we have not attempted to summarize. Mosteller and Nogee summarize (p. 403) their experiment in the following fashion:

1. that it is feasible to measure utility experimentally;
2. that the notion that people behave in such a way as to maximize their expected utility is not unreasonable;
3. that on the basis of empirical curves it is possible to estimate future behavior in comparable but more complicated risk-taking situations.

We next turn to the experimental studies of Davidson, Suppes, and Siegel (1957), which in part were undertaken to meet the following three important criticisms of the Mosteller and Nogee experiment. First, Mosteller and Nogee failed to check systematically the adequacy of their utility curves in terms of simple bets of the same type as those used to construct them. Such checks using simple hands would have provided a better test of the von Neumann-Morgenstern axioms than those based on the more complicated doublet hands which may introduce complications not embodied in the axiom system. Because of this lacuna, it is somewhat difficult to evaluate precisely whether or not the Mosteller and Nogee experiment substantiates a claim that utility can be measured on an interval scale; for some recent remarks on this point, see Becker, DeGroot, and Marschak (1964).

The second criticism is that the choices used by Mosteller and Nogee to estimate the utility functions involved either accepting or rejecting a gamble. One option always involved playing, and therefore taking a risk, whereas the other resulted in no play. Thus, if participation itself involves either a negative or a positive utility, the experiment was so designed that this special feature could produce maximum distortion.

The third criticism is that their analysis, which to some degree was necessitated by the experimental design, assumed that subjective probability is equal to objective probability. In spite of the fact that they went to some pains to brief subjects on the probability considerations involved in the offers with which they were faced, no clear evidence exists to show that they were wholly successful in equating the two probabilities from the subject's standpoint.

In designing a study to meet these three criticisms, Davidson, Suppes, and Siegel (1957) essentially followed the Ramsey (1931) line of development for measuring utility and subjective probability. For the moment we restrict ourselves to their efforts to measure utility.

To begin with, it was necessary to find an event whose subjective probability is $\frac{1}{2}$ (see Sec. 3.3, particularly Eqs. 8 and 9). Initially they tried a coin, but most subjects in a pilot study showed a preference for either heads or tails. An event that was found to satisfy to a close approximation the conditions expressed in Eq. 8 of Sec. 3.3 was produced by a specially made die that had the nonsense syllable *ZOJ* engraved on three faces and *ZEJ* on the other three faces. Two similar dice were also used with *WUH* and *XEQ* in one case, and *QUG* and *QUJ* in the other. The syllables selected are ones that, according to Glaze (1928) and others, have essentially no associative value. Because their method of generating chance events that have a subjective probability $\frac{1}{2}$ has been used in several studies, some additional remarks may be desirable concerning exactly in what sense Eq. 8 of Sec. 3.3 has been verified. A problem exists because the subject must choose one option or the other, and so there can be no direct evidence of indifference. Using outcomes that varied by amounts of 1 cent, Davidson et al. used a method of approximation that was found to hold within the 1 cent limit. Consider the following two pairs of options:

$$\begin{array}{cc} & \begin{array}{cc}\text{Option 1} & \text{Option 2}\end{array} \\ \begin{array}{c} ZOJ \\ ZEJ \end{array} & \begin{bmatrix} +(x+1) & -x \\ -x & +x \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc}\text{Option 1} & \text{Option 2}\end{array} \\ \begin{array}{c} ZOJ \\ ZEJ \end{array} & \begin{bmatrix} +x & -x \\ -x & +(x+1) \end{bmatrix} \end{array}.$$

If Option 1 is selected in the left matrix and Option 2 in the right one for all choices of x, then it is reasonable to infer that the subject is satisfying Eq. 8 approximately and, thus, that the subjective probability of one nonsense syllable occurring is approximately equal to that of the other occurring.

Having found an event with subjective probability approximately equal to $\frac{1}{2}$, the next step was to find a similar approximation to axioms for utility differences (Def. 11 of Sec. 2.3). Roughly speaking, the method used consisted of finding an upper and lower bound for each point of an equally spaced utility function. Tests of the interval-scale property may then be formulated in terms of these bounds.

A summary of the data from Chapter 2 of Davidson et al., showing the bounds in cents for fixed points on the utility scale, is given in Table 1. The 15 subjects for whom data are shown were male students hired through the Stanford University Employment Service. Four other subjects were run for whom the data on the subjective event of probability $\frac{1}{2}$ were satisfactory, but they did not satisfy the checks required in order to substantiate a claim of interval measurement and, therefore, their utility scales are not shown in Table 1.

Eight of the subjects were twice rerun after periods varying from a few days to several weeks, and two of these were run a fourth time. Bounds for these remeasured utility curves were, in general, quite similar to those determined in the initial session. This evidence is encouraging, particularly in the light of the concern expressed by several investigators that experimentally determined utility curves are temporally unstable.

It is natural to ask whether the upper and lower bounds found for the utility points f, c, d, and g in Table 1 enclose the actuarial value; that is, to the accuracy of the measurement scheme, is the utility function simply

Table 1 Summary of Data Determining Bounds in Cents for Fixed Points on the Utility Scale with $u(-4¢) = -1$ and $u(6¢) = 1$. (Davidson, Suppes, & Siegel, 1957, p. 62)

Subject	Bounds for f Where $u(f) = -5$	Bounds for c Where $u(c) = -3$	Bounds for d Where $u(d) = 3$	Bounds for g Where $u(g) = 5$
1	−18 to −15¢	−11 to −10¢	11 to 12¢	14 to 18¢
2	−34 to −30	−12 to −11	12 to 18	31 to 36
3	−18 to −11	−8 to −7	10 to 13	14 to 22
4	−29 to −24	−15 to −14	14 to 17	25 to 31
5	−21 to −14	−10 to −9	10 to 12	16 to 24
6	−25 to −21	−14 to −13	13 to 15	19 to 23
7	−18 to −7	−7 to −6	7 to 14	10 to 23
8	−25 to −21	−14 to −13	14 to 17	23 to 28
9	−35 to −29	−12 to −11	16 to 18	43 to 50
10	−26 to −20	−15 to −14	14 to 15	20 to 27
11	−22 to −19	−14 to −13	11 to 13	18 to 22
12	−21 to −13	−12 to −11	8 to 12	11 to 15
13	−34 to −23	−14 to −13	13 to 17	23 to 32
14	−16 to −13	−10 to −9	12 to 15	20 to 24
15	−12 to −8	−8 to −7	8 to 10	11 to 15

linear in money? Of the 60 pairs of bounds shown in Table 1, 19 of the intervals are entirely above the linear money value, 21 are entirely below it, and 20 include it.

Finally, to avoid the second criticism of the Mosteller and Nogee experiment, choices in the experiment of Davidson et al. were primarily between pairs of options, both of which involved uncertainty. It is interesting, therefore, to note the strong similarity between the shape of the utility curves obtained and the quality of predictions made in the two experiments. This similarity is, in fact, rather promising for experimental results in this area.

Davidson et al. (pp. 75–79) stated two important criticisms of their experiment. One is that the method is restricted to objects that are equally spaced in utility, which severely limits its applicability. It is suitable for essentially continuous commodities, such as money, but it is quite unsatisfactory for many interesting alternatives that are essentially not divisible. The other criticism is that the method of approximation used is cumbersome, and it becomes especially unsatisfactory when a large number of utility points are sought because the width of the bounds tends to spread as the number of points is increased.

In the next chapter of their book, Davidson et al. attempted to meet these two criticisms by using a linear programming model to measure utility. As was already remarked in Sec. 3.3, when a chance event E^* of subjective probability $\frac{1}{2}$ is used, a choice of Option 1 over Option 2 in

$$\begin{array}{cc} & \begin{array}{cc}\text{Option 1} & \text{Option 2}\end{array} \\ \begin{array}{c} E^* \\ \bar{E}^* \end{array} & \begin{bmatrix} a & \qquad c \\ b & \qquad d \end{bmatrix} \end{array}$$

may be represented by the inequality that is based on the maximization of expected utility

$$u(a) + u(b) \geq u(c) + u(d). \tag{13}$$

The essential idea is to apply linear programming methods to solve a set of inequalities of this form. In the experiment, options were built up from all possible pairs of six different outcomes, but excluding those for which a prediction follows from ordinal preferences alone. The subject's choices in the remaining 35 pairs of options therefore determined 35 inequalities in six variables. Only rarely does this set of inequalities have a solution. Therefore, Inequality 13 was replaced by

$$u(a) + u(b) + \theta \geq u(c) + u(d), \tag{14}$$

where θ is a constant to be determined.

Using linear programming methods, a solution is subject to a set of inequalities of this form such that θ is a minimum and such that the normalizing restriction is met that the smallest interval be of length one. Intuitively, θ may be thought of as the threshold of preference: if the utility difference between a and c differs by more than θ from the difference between d and b, then the larger interval must be chosen, that is, Option (a, b) is chosen over (c, d). Some additional probabilistic postulates for the behavior when the difference between two utility intervals is less than θ were also formulated by these authors, but we do not consider them here.

It must be emphasized that the utility function obtained by this linear programming approach is not unique: the minimum θ is compatible with a convex polyhedron of solutions. Because of limited computational facilities, Davidson et al. reported only the first solution reached, although, as they remark, the best single choice is probably the centroid of the polyhedron.

To test this linear programming model, the following three-session experiment was performed with ten music students as subjects and long-playing records as outcomes. In the first session a utility curve for six records was determined by the methods just described. In the second session a utility curve was found for another six records, two of which were also in the first set, thus permitting the construction of a joint utility curve. This joint curve could then be used to predict choices

Table 2 Summary of Predictions of Linear Programming and Ordinal Models (Davidson, Suppes, & Siegel, 1957, p. 92)

	Linear Programming Model				Ordinal Model		
Subject	Clear Predictions Correct	Clear Predictions Wrong	Predictions Within θ	Total	Predictions Correct	Predictions Wrong	Total
1	30	1	24	55	6	0	6
2	24	7	24	55	18	2	20
3	21	0	34	55	11	2	13
4	23	15	17	55	21	11	32
5	21	19	15	55	27	19	46
6	4	0	51	55	0	0	0
7	10	13	32	55	7	12	19
Total	133	55	197	385	90	46	136

between untested combinations of the full set of ten records used in the first two sessions; these predictions were tested in the third session.

Of the 10 subjects, only 7 were able to complete the entire experiment. The predictions for these 7 subjects are summarized in Table 2. The first column gives the number of predictions that were clearly correct—"clearly" because the utility differences were greater than the estimated θ. The second column gives the number of predictions that were clearly wrong, and the third, those that fell within θ and therefore need a more detailed probabilistic analysis. The right-hand side of the table gives the accuracy of the predictions that follow from the simple ordinal model that is obtained in the obvious way from the rankings given by the subjects. Unfortunately, a rather large proportion of predictions fell within θ, thus making it difficult to evaluate the adequacy of the model. If θ is ignored and one simply predicts the option having the larger of two utility intervals, then 254 are correct and 126 are wrong. In only five cases is no prediction made because of exact equality of the intervals.

Perhaps the major difficulty encountered in applying the linear programming model to data from this experiment is the fact that only one subject satisfied the purely ordinal axiom of transitivity, that is, the ordinal predictions of how the long-playing records would be ordered in preference in Session 3 was completely verified for only one subject. Lack of transitivity accounts for the large θ's and the resulting insensitivity of the model which is reflected in Table 2.

Davidson et al. pointed out two undesirable features of the model itself. In the first place, when the inequalities have integer coefficients, as these do, the application of linear programming methods almost always yields very simple values for the utility intervals; in fact, there is a strong tendency for the utility intervals to become identical, a tendency which seems psychologically unrealistic. For example, under the normalizing assumption stated previously that the minimum interval be of length one, all utility intervals of adjacent alternatives in Session 1 were either 1, $\frac{3}{2}$, 2, or 3. For five atomic intervals and seven subjects, that is, for a total of 35 intervals to be measured, this restricted set of numerical results certainly seems far too simple.

The second criticism is that the model leads to simple violations of the sure-thing principle when it is applied to monetary outcomes; specifically, it predicts that a larger interval will not be chosen with probability one when both utility differences are less than θ. Thus, to use their example, when the subject is presented with the following pair of options:

$$\begin{array}{c c} & \begin{array}{cc} \text{Option 1} & \text{Option 2} \end{array} \\ \begin{array}{c} E^* \\ \bar{E}^* \end{array} & \begin{bmatrix} 50¢ & 45¢ \\ -48¢ & -50¢ \end{bmatrix} \end{array}$$

the sure-thing principle and common sense choose Option 1. However, for a θ of any size that choice is not predicted with probability one.

This failing was eliminated in a closely related nonlinear programming model proposed by Suppes and Walsh (1959). In their setup inequalities of the form of Eq. 14 were replaced by nonlinear inequalities of the following sort

$$\eta[u(a) + u(b)] \geq u(c) + u(d),$$

and the problem was to minimize the threshold parameter η subject to the restriction that $\eta \geq 1$. Their experimental procedure was very similar to that of Davidson et al., just described, except that the outcomes were monetary. Their subjects were eight sailors from an air base. Without going into their experimental results in detail, suffice it to say that they did find some predictive superiority of the nonlinear utility model over the actuarial model. A rather thorough critique of their interpretation of their results was given by DeGroot (1963), who makes a good case for claiming that their model shows little real superiority over the actuarial model.

A probabilistic version of the linear programming model has recently been developed and applied by Dolbear (1963). His results are discussed in Sec. 8.4.

In an early experiment—the next following the classic Mosteller and Nogee study—Edwards (1955) was concerned with the confounding of utility and subjective probability, and he proposed to test a subjective expected utility model. Rather than apply one of the rather elaborate methods of measurement discussed in Sec. 3, he invoked a convenient but highly simplifying assumption, which, in the notation of Sec. 3.2, may be expressed as follows. There is an integer N such that

$$u(y\tfrac{1}{2}x_0) \approx Nu(\epsilon\tfrac{1}{2}x_0), \tag{15}$$

where y is a reasonably large amount of money like \$5.50, ϵ is a small amount of money like 10 cents, and x_0 is the outcome of neither winning nor losing any money. Naturally we do not expect exact equality to hold but a good approximation is feasible. Applying now the expected utility property to Eq. 15 and choosing the scale so that $u(x_0) = 0$, we obtain

$$u(y) \approx Nu(\epsilon). \tag{16}$$

If ϵ is fixed throughout the experiment, it is natural to set $u(\epsilon) = 1$. Then the utility of the alternative y is uniquely determined under the standard assumption of an interval scale. Unfortunately, as Edwards (1961b) has subsequently pointed out himself, if the N-bets assumption is taken in full generality, it may be shown to imply that utility is linear in money.

The subjects were five male undergraduates, each of whom was run for a total of 50 sessions. The first 12 sessions were devoted to the N-bets method of utility measurement; the technique used for determining indifference was similar to that of Mosteller and Nogee. The most outstanding thing about the resulting utility curves is their relatively high degree of linearity, which is much greater than that found by Mosteller and Nogee and by Davidson, Suppes, and Siegel.

We reserve consideration of the rest of this experiment, in particular, the number of correct and incorrect predictions based on the subjective expected utility model, until we have discussed various methods for measuring subjective probability (Sec. 4.3).

In concluding this subsection on the measurement of utility, we wish once again to emphasize to those readers mainly oriented in psychology and not overly familiar with economics that an extensive and in many points a very penetrating literature exists on the measurement of utility within economics. [See the bibliography of Majumdar (1958) for a guide to these papers and books.] However, much of this literature, in spite of its acuity, differs considerably from the usual psychological article. On the one hand, no systematic experimental or empirical results are presented, and, on the other hand, no systematic formal methods or results are set forth. A high proportion of these articles consists of acute observations on what should be the case in the behavior of any reasonable person, and, as might be expected, much of the discussion is devoted to normative rather than descriptive concerns. It is nonetheless a trifle startling to find that Majumdar's entire book, which is devoted to the measurement of utility, contains no reference to any experiments aside from a very casual one to Mosteller and Nogee.

4.3 Measurement of Subjective Probability

The bifurcation of these measurement problems into utility and subjective probability is artificial for several of the studies that we are summarizing because both have been measured within a single study, sometimes even within the same experimental session, and predictions have been made from both measures. On the other hand, several of the studies about subjective probability evidenced no concern with utility whatsoever, and so we have been led to make a simple distinction between the measurement of utility and subjective probability. Moreover, as far as we know, no experimental studies exist in which extensive measurements of subjective probability were made first, following the development of Savage (1954) which was discussed in Sec. 3.3, and then followed by the measurement of utility. In all the studies with which we are familiar the order has been reversed: utility measurement has preceded subjective probability measurement. (In making this statement we do not count as a quantitative measurement of probability the determination of an event with subjective probability $\frac{1}{2}$.)

The first attempt to measure subjective probability experimentally was apparently made by Preston and Baratta (1948). Subjects were run in groups of two or more, and they used play money to bid for gambles in a simple auction game. The successful bidder was permitted to roll a set of dice after the bidding was completed. The probability of winning with the dice corresponded exactly to the probability stated on the card presented for auction. For example, on a given play subjects might bid for a prize of 250 points with probability 0.25 of winning. If for this gamble the average successful bid was 50, then the authors computed the

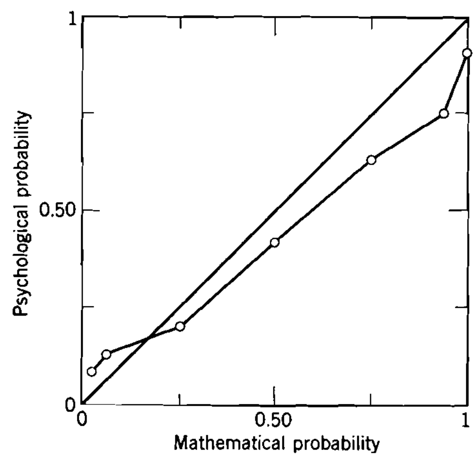


Fig. 4. Functional relationship between psychological and mathematical probability Adapted with permission from Preston & Baratta (1948, p. 188).

psychological probability to be 50/250 = 0.20. From our earlier discussion, it is clear that this method of computing subjective probability assumes that the utility of play money is linear with the value assigned to the money. There is little evidence to support this assumption, but, in spite of this, the experimental results are of considerable interest.

Using this method of computation, perhaps their most interesting conclusion was that objective probabilities less than 0.20 are systematically overestimated and objective probabilities greater than 0.20 are systematically underestimated. The mathematical or objective probabilities of the gambles they offered and of the psychological probabilities computed from the bid data are shown in Fig. 4. The point of intersection of the subjective and objective probability curves is about 0.20. Other related results are referred to shortly.

Two other conclusions are noteworthy. First, by comparing the data of sophisticated subjects with those of subjects who were relatively naive about the facts of probability, they concluded that the underestimating and overestimating effects just described exist in both kinds of subjects. Second, an increase in the number of players, and therefore in the competition among them, tended to increase the underestimation of high probabilities and the overestimation of low ones, which is a somewhat surprising result.

A readymade, real life comparison of subjective and objective probabilities is provided by the betting on horses under the pari-mutuel system as compared with the objective probabilities of winning as determined a posteriori after the races. The total amount of money bet on a horse divided by the net total bet on all horses in a race determines the odds and thus the collective subjective probability that the horse will win. Griffith (1949) examined data from 1386 races run in 1947; his results are in remarkable qualitative agreement with those obtained by Preston and Baratta. Low objective probabilities are systematically overestimated by the subjective probabilities, and high objective probabilities are systematically underestimated. For Griffith's data, objective and subjective probability are equal at 0.16, which is close to the 0.20 value obtained by Preston and Baratta. Griffith remarks in a final footnote that the same computations were carried through for all races run at the same tracks in August 1934, and essentially the same results were obtained. In this case the indifference point fell at 0.18 rather than at 0.16. The invariance of the results from the depression economy of 1934 to the relatively affluent one of 1947 increases their significance.

A still more extensive study, which is very similar to Griffith's, was made by McGlothlin (1956) of 9605 races run from 1947 to 1953, mostly on California tracks. As one would expect, the general character of his results agree closely with those of the earlier study. The objective and subjective probability curves intersect between 0.15 and 0.22. McGlothlin also found some interesting tendencies during the course of the usual eight races on a given day. First, there was an increasing tendency to overestimate low probabilities, and this phenomenon was particularly striking for the final race of the day. There seems to be evidence that on the last race of the day many bettors are uninterested in odds that are not sufficient to recoup their losses from the earlier races. This phenomenon is so striking that bets placed on low odds horses had a net positive expectation even after subtracting the tracks' take, which is 13% in California. This kind of dynamic phenomena is not easily incorporated into an expected utility model. There was also some evidence that losing bettors increased the size of their wagers on the next race more than did winning bettors.

As has already been remarked, Mosteller and Nogee could and did interpret their experimental results in terms of subjective probability rather than utility. Their comparison of objective and subjective probabilities for their two groups of students, as well as the data from the Preston and Baratta study, are shown in Table 3. These results are not in qualitative agreement with those of Preston and Baratta and of Griffith. The Harvard students systematically underestimated objective probabilities over the entire range of values. (Naturally, this could not happen if subjective probabilities were required to add up to 1, but this constraint was not built into the studies we have been considering.) The Guardsmen, on the other hand, equated subjective and objective probability at about

$p = 0.50$, which is considerably higher than the corresponding points obtained by Preston and Baratta, Griffith, and McGlothlin. We join Mosteller and Nogee in being at a loss to explain the differences in these results.

Table 3 Comparison of True and Psychological Probabilities from Preston and Baratta (1948) and for the Student and Guardsmen Groups of the Mosteller and Nogee Experiment (1951, p. 397)

Approximate True Probability	P & B	Students $N = 10$	Guardsmen $N = 5$
0.667 . . .	0.55	0.54	0.56
0.498 . . .	0.42	0.47	0.50
0.332 . . .	0.26	0.30	0.36
0.167 . . .	0.15	0.16	0.28
0.090 . . .	0.14	0.081	0.18
0.047 . . .	0.12	0.038	0.083
0.010 . . .	0.07	0.0085	0.052

We noted in Sec. 4.2 that Edwards (1955) measured subjective probability by using the utility curves constructed by his N-bets method and the familiar equation for indifference between a certain gain of a fixed amount and a probability p of winning another amount or obtaining nothing. Under the utility normalization chosen (see p. 320), the subjective probability of the event having objective probability p is just the obvious ratio of the two utilities. He devoted sessions 13 to 18 to subjective probability measurements, using both negative and positive outcomes. Edwards' results were different from any previously mentioned. When the outcomes were positive, the five subjects almost uniformly overestimated objective probabilities in the probability range tested, namely, from $\frac{1}{8}$ to $\frac{7}{8}$. When the outcomes of the bets were negative (losses), the subjective values were extremely close to the objective ones.

Edwards used the constructed utility and subjective probability functions to predict choices in Sessions 19 to 22. The subjective expected utility model he applied, which was described on p. 320, predicted significantly better than chance the choice behavior of subjects in these four sessions. One hundred and twelve offers were made of each of four bets at each of five levels of expected value as determined by their monetary outcomes and objective probabilities. Of the resulting 2240 offers, the subjective expected utility (SEU) model correctly predicted the choice made in 1629 cases. Edwards compared these predictions with those obtained from the expected utility (EU) model that utilizes objective probability and measured utility and with the subjective expected monetary (SEM) model that utilizes monetary value and subjective probability. The results for the EU model were slightly worse than chance, and therefore very unsatisfactory. The SEM model, on the other hand, did slightly better than the SEU model, but not at a level that is statistically significant. These results argue that the difference between subjective and objective probability is of considerably greater importance than that between utility and monetary outcome. Some extensions of Edwards' findings are reported by Becker (1962).

Davidson, Suppes, and Siegel (1957, Chapter 2) used their constructed utility curves (Sec. 4.2) and Eq. 10 of Sec. 3.3 to measure subjective probability. For seven of their subjects they made three different determinations of the subjective probability of an event whose objective probability was 0.25. This event was generated by a four-sided die—two opposite ends were rounded so that it always landed on one of the other four sides. Using the method of approximation described in Sec. 4.2, they found upper and lower bounds for each subject and for each combination of utilities used. For five of the seven subjects the three sets of lower and upper bounds had a nonempty intersection, and for two they did not. For four of the five subjects with a nonempty intersection, the upper bound was below 0.25 and for the remaining subject the lower bound was below and the upper bound above 0.25. These results agree with those of Preston and Baratta, Griffith, and McGlothlin concerning the underestimation of probabilities above 0.20.

Among the experimental studies devoted to the measurement of subjective probability, this is the only one in which the procedure used could admit the conclusion that subjective probability cannot be consistently measured, although a number of experimental studies are cited in Sec. 4.4 which cast doubt on the possibility of such measures. The point is that a claim of fundamental measurement can only really be made when it is clear that enough data have been collected so that it is possible to reject the model. The systematic pursuit of the cross-checks implied by any model of fundamental measurement seems to have been unduly neglected in most experimental studies of subjective probability.

Toda (1951, 1958) has proposed a two-person game method for measuring subjective probability which is very similar to the auction procedure of Preston and Baratta. One of its more interesting applications was made by Shuford (1959) to obtain extensive measurements of subjective probability of both elementary and compound events. Subjects rolled a

20-face die (with the integers 1 to 20 on the faces) twice to select the row and column of a 20×20 matrix of vertical and horizontal bars. The subjects, 64 airmen in training at an airbase, were run in pairs. The sequence of events on a given trial was as follows. The matrix of horizontal and vertical bars was projected on a screen. Subject A wrote down his bid x that a horizontal bar, say, would be selected ($0 \leq x \leq 10$), subject B decided to "buy" or "sell" A's bid, the die was rolled by the experimenter to decide the bet, that is, which element of the projected matrix was selected, and finally the subjects scored themselves for the play. It can be shown that A's minimax strategy in this game is to set x equal to 10 times his subjective probability of the favorable outcome.

Two games were played with each pair of subjects. In the first game the payoff was determined by the occurrence of a horizontal or vertical bar, as the case might be, in the position of the matrix selected by the two rolls of the 20-face die. In the second game, the bet paid off if two successive selections of elements of the matrix made with the die resulted in two bars of the same type.

Shuford presented the results for individual subjects, but as these are too complicated to give here, we content ourselves with some summary observations. Confirming the earlier findings of Preston and Baratta, Griffith, and McGlothlin, a fairly large fraction of the subjects overestimated low probabilities and underestimated high ones. This was true for both the elementary and the compound events. On the other hand, Shuford found that the subjective probability estimates of a number of the subjects were fit quite well by a linear function of objective probability, although the slope and intercept of this function varied from one subject to another. His findings about the estimation of compound events are particularly interesting, for, to our knowledge, this is the only experiment explicitly concerned with this issue. A majority of the subjects approximated the correct rule, that is, they estimated the probability of the compound event as approximately the square of the probability of the elementary event. That the application of the correct rule was so common is surprising, because when the subjects were asked at the end of the series of trials what rule they had used, only two stated the correct one. Those who did not approximate the correct rule came very close to approximating the probability of a single elementary event. For at least two subjects, however, no simple rule seemed to account for their estimates of the probability of compound events. As Shuford remarked, the investigation of the compounding rules used in other experimental situations is of considerable importance.

Finally, we should like merely to mention a series of studies that, although not explicitly concerned with the measurement of subjective probability, bear a rather close relation to it. The common thread in this work is a scale of proportion that is based on the responses of subjects to randomly composed arrays of two or more types of elements, usually in a relatively difficult perceptual situation. The specific references are Philip (1947), Shuford (1961a, b), Shuford and Wiesen (1959), Stevens and Galanter (1957), Wiesen (1962), and Wiesen and Shuford (1961). The three papers involving Wiesen are of particular interest because of the emphasis on Bayesian estimation schemes.

4.4 Experimental Studies of Variance-Preference, Utility-of-Gambling, and Other Models

A number of investigators have attempted either to show that models of choice behavior based only on the concepts of subjective probability and utility cannot possibly work, because, for example, of an interaction between utility and probability, or to establish that alternative models, in fact, do a better job of prediction. We review briefly the studies most relevant to the central concerns of this chapter.

Probably the most extensive set of empirical studies that have attempted to isolate factors in gambling that are not easily accounted for by an expected utility model were performed by Edwards. In the first paper of the series (1953), probability preferences in gambling were studied using 12 undergraduates as subjects who selected between bets of equal expected monetary value. The chance events were generated by a pinball machine; the objective probabilities varied by $\frac{1}{8}$ steps from $\frac{1}{8}$ to $\frac{7}{8}$. Edwards found that two main factors determined choices between bets of equal expected value. First, there was a general tendency either to prefer or to avoid long shots, that is, bets with a low probability of winning or losing; and second, there were personal preferences for specific probabilities, the two most important being a definite preference for the $\frac{4}{8}$ probability and an aversion to the $\frac{6}{8}$ probability of winning. The existence of specific probability preferences certainly raises difficulties for any axiomatization of utility and subjective probability along the lines discussed in Sec. 3. In Edwards (1954a) the methods of the earlier study were extended to bets having different expected monetary values. The results tended to show that subjects' choices were influenced by preferences for bets with higher expected values (or lower negative expected values) as well as by preferences among the probabilities. Edwards (1954b) replicated and extended slightly the findings of the two earlier studies. In particular, one experiment showed a similar pattern of preferences among probabilities in a nongambling game that involved uncertainty. The exact details of the game are too complicated to describe here; suffice it to say that it was conceptualized to the subjects as a strategic game of military warfare in which they were to act as the head of an amphibious task force whose mission was to destroy ammunition stored at one of eight ports.

Edwards (1954c) asked whether or not variance preferences exist in gambling. Using the pinball game mentioned, some preferences for variances were exhibited when the probabilities were held constant, but on the whole these seemed to be secondary in importance to probability preferences. The same pattern of probability preferences found in Edwards' earlier work was again evidenced.

These studies of Edwards are essentially negative in character in the sense that no well-formulated alternative models are set forth to account for the experimental findings. Moreover, most of the experiments involved the assumption either of objective probabilities or of utilities linear with money, and, therefore, a more detailed critique is necessary to refute the subjective expected utility model. On the other hand, some of Edwards' data on probability preferences are so striking that it seems clear that no simple SEU model can adequately explain his findings.

Let us turn in more detail to the question of variance preferences. The suggestion is an old one (Fisher, 1906): do individuals make choices on the basis not only of expectation but also on the dispersion of the possible outcomes? That variance preferences are a significant variable has been argued especially by Allais (1953) in his criticisms of the "American school" that advocates the analysis of choice behavior in the presence of uncertainty in terms of expected utility. The issues here are thoroughly tangled. Although a number of possible classifications can be devised for analyzing the problem, such as the variance of the objective distribution, the variance of the subjective distribution, a linear function of the mean and variance of the objective distribution, a linear function of the mean and variance of the subjective distribution, etc., perhaps the most constructive approach is to postulate preferences in terms of the objective probabilities and to compare that with the expected-utility model. This tack was followed by Coombs and Pruitt (1960) who suggested, as an alternative to the expected utility model, that a gamble be characterized in terms of the expectation, variance, and skewness of its objective probability distribution over money. They make the valid point that although a variance preference usually can be interpreted alternatively in terms of a utility function for money, the number of admissible preference orderings is much reduced when a gamble is characterized as a function of expectation, variance, and skewness than when arbitrary monotonic utility functions are permitted. However, this criticism of utility theory is faulted when parametric cardinal utility functions are used.

For example, a polynomial utility function with three degrees of freedom would make a natural comparison to their function of expectation, variance, and skewness.

They conducted a study with 99 undergraduates who chose among bets having a constant expectation but different skewnesses and variances. A typical choice was between

Bet A: $\frac{1}{3}$ to win \$1.40, $\frac{2}{3}$ to lose 70¢,

Bet B: $\frac{1}{2}$ to win \$1.00, $\frac{1}{2}$ to lose \$1.00.

All bets had an expected monetary value of 0. From the analysis of their data, Coombs and Pruitt drew two principal conclusions of relevance here. First, variance preference orderings were limited almost exclusively to those that could be generated by unfolding the natural ordering of the variance scale, using the unfolding technique described at the end of Sec. 2.4. Approximately one-third of the subjects preferred low variance, one-third high variance, and one-third intermediate degrees of variance. Second, the great majority of subjects had their ideal positions at one end or the other of the skewness scale.

Pruitt (1962) extended the Coombs and Pruitt approach to a pattern and level of risk (PLR) model. According to Pruitt's definition, two bets have the same pattern if one may be obtained from the other by multiplying the outcomes by a positive constant. Thus, the following two bets have the same pattern:

A: $\frac{1}{2}$ chance to win \$1.00
$\frac{1}{3}$ chance to lose 75¢
$\frac{1}{6}$ chance to lose 60¢

B: $\frac{1}{2}$ chance to win \$3.00
$\frac{1}{3}$ chance to lose \$2.25
$\frac{1}{6}$ chance to lose \$1.80

His definition of the level of risk of a bet is the sum of its negative outcomes weighted by their respective probabilities of occurrence. For instance, the level of risk of Bet A is 35 cents; and of Bet B, \$1.05. Pruitt argued that these two aspects of bets are the ones that people usually perceive, and he was lead to define the utility, $u(A)$, of an Alternative or Bet A by the following equation:

$$u(A) = r(A) \cdot \rho(A) + g(A),$$

where $r(A)$ is the level of risk of A, $\rho(A)$ is the utility of the pattern of A, and $g(A)$ is the utility of risk of A. Pruitt did not attempt to derive this somewhat arbitrary equation from simple behavioral assumptions analogous to those considered in Sec. 3. Rather, he derived several consequences of this equation combined with some Coombsian assumptions about the individual possessing an "ideal" level of risk. Typical examples are the following two propositions which he tested on the Coombs and Pruitt (1960) data mentioned previously and on the data from a similar experiment with 39 undergraduates as subjects. The first proposition asserts that the order of preference among patterns is independent of the level of risk. The second proposition asserts that the more preferred a pattern of risk, the higher will be the ideal level of risk for that pattern. Both propositions are firmly supported by the data from the two experiments. Pruitt's article is also recommended to the reader because of its useful and incisive review of the strengths and weaknesses of various expected utility models, all of which have already been mentioned in this chapter.

A specific utility-of-gambling model has been proposed by Royden, Suppes, and Walsh (1959). The behavioral postulate of the model is that individuals choose among options so as to maximize the sum of the expected monetary value and the utility of gambling. For bets or options with two outcomes and with subjective probability $\frac{1}{2}$ for each outcome, each bet is uniquely determined by its mean and the absolute difference between the two outcomes; the absolute difference is itself twice the standard deviation. The utility of gambling of a bet was defined then as a function of the mean and the difference, and the two-dimensional surface corresponding to the function was experimentally determined by an experimental procedure similar to that of Davidson, Suppes, and Siegel (1957). The utility of gambling surfaces obtained for individual subjects turned out to be sufficiently complex to cast doubt on the possible adequacy of any simple theory of probability and variance preferences as complete determiners of choices.

The subjects were eight sailors from a naval airbase and eight university undergraduates. The utility of gambling surface constructed for each subject was used to predict choices in an additional experimental session. In comparison with the actuarial model—maximization of expected monetary value—the utility-of-gambling model was significantly better in predicting the choices of the sailors and significantly worse with the students as measured by a chi-square test that combined individual significance levels.

Finally, we mention the empirical studies of risk and gambling by John Cohen and his associates at the University of Manchester [for an overview see either the book by Cohen and Hansel (1956) or the book by Cohen (1960)]. Although some interesting findings are reported, the experiments are not quantitatively oriented and so fall outside the main concern of this chapter.

5. GENERAL PROBABILISTIC CHOICE THEORIES[5]

Paralleling our presentation of the algebraic theories of preference, we first examine those probabilistic theories that impose no mathematical structure (other than the set structure) over the outcome set. Following this, in Sec. 6, we take up the general probabilistic theories that attempt to describe how subjects rank order sets of outcomes. In Sec. 7, we discuss theories that deal explicitly with outcomes that are probability distributions over a set of "elementary" outcomes. And finally, in Sec. 8, we examine the experiments performed to test these models.

5.1 Response Probabilities

As we emphasized in Sec. 1, theorists are currently trying to cope with the class of preference experiments for which the subject has no discrimination problems. In such experiments his responses simply inform the experimenter which of the several offered outcomes the subject prefers at that instant. That being so, it has generally been taken for granted that one can suppress all the notational apparatus for the stimuli and responses and act as if what is chosen is an element from the unordered set X of presented outcomes, where X is a subset of the set A of all outcomes used or potentially available in the experiment. Actually, this oversimplifies matters a great deal by ignoring the possibility of response biases, but because all of the existing theories are of this character we follow the convention that we need only consider the set X of outcomes offered and the one, $x \in X$, chosen by the subject.

We shall suppose in these sections that the subject's response to the same choice situation is governed by a probability mechanism, and so in general he exhibits inconsistencies. Indeed, it was the empirical observation of inconsistencies (Sec. 4.2) that led to the study of probability models in the first place. The assumption of a simple probability mechanism to explain behavioral inconsistency is certainly not the only possibility. For example, a number of authors believe that inconsistencies arise from

[5] Many of the ideas and most of the theorems that we discuss in this section and the next are due to Marschak and his collaborators. The reader interested in several notions that we shall not cover here should consult Becker, DeGroot, and Marschak (1963a), Block and Marschak (1960), and Marschak (1960). Some of the proofs that we give are taken directly from these papers; the remainder are somewhat different because we have chosen to organize the results somewhat differently. We make no attempt to indicate which is which.

the subject attending to different aspects of the choice situation. Were it possible to specify what it is that causes him to attend to one aspect rather than another aspect, a deterministic model might very well be appropriate. As long as these causes are unknown, we may very well have to be satisfied with probability models.

In contrast to algebraic utility theory, which largely has not been the product of psychologists, probabilistic theories have had a long development in psychology and only recently have begun to be welded together with the algebraic utility ideas. The Fechner-Thurstone development of probability models for psychophysics and psychoacoustics corresponds to what in the present context are called strong and random utility models. The recent history of developments in probability models is rather tangled, but much of it can be recovered from our references to specific ideas and results.

The probability models all assume that when X is presented on trial n, there exists a probability $p_{X,n}(x)$ that $x \in X$ is chosen. The models in this section include the added assumption that these probabilities are independent of the trial on which x is presented, so the subscript n is suppressed. The assumption that these are probabilities and that a choice is made at each opportunity is summarized by

$$\begin{gathered} p_X(x) \geq 0, \qquad \text{for all } x \in X, \\ \sum_{x \in X} p_X(x) = 1. \end{gathered} \tag{17}$$

In practice, these probabilities are estimated by presenting X repeatedly to the subject, often interleaved among presentations of a number of other subsets of A; and the proportion of times x is chosen when X is presented is taken as the estimate of $p_X(x)$.

The theoretical problem is to discover what mathematical constraints the subject imposes on the probabilities beyond those automatically implied by probability theory. Each theory makes such a proposal, and the empirical problem, discussed in Sec. 8, is to decide how accurate the proposals actually are. A number of the theories do not attempt to describe the probability structure for all subsets of A but just for the two-element sets. In these binary preference theories it is convenient to write $p(x, y)$ for $p_{\{x,y\}}(x)$.

5.2 Constant Utility Models

Although much data suggest that algebraic utility models are inadequate, the intuitive idea of representing the strength of preference in terms of a numerical utility function—in terms of a subjective scale—is much too appealing to be abruptly dropped. Indeed, no one has yet dropped it at all; every theory that we examine includes such a notion, although in at least one case it can be dispensed with easily if one chooses.

Basically, at present, there are only two approaches to how response probabilities and utility scales relate. In the approach characterized by the constant utility models discussed here, the utility function is a fixed numerical function over the outcomes, and the response probabilities are some function of the scale values of the relevant outcomes. The choices are assumed to be governed directly by these response probabilities. In the other approach, defined by the random utility models of Sec. 5.3, the utility function is assumed to be randomly determined on each presentation, but once selected the subject's decision is unequivocally determined by the relevant utilities just as in the algebraic models. Probabilistic behavior arises from the randomness of the utility function, not from the decision rule. We first present the constant utility models.

Perhaps the weakest probabilistic generalization of the fundamental decision rule of the algebraic models is the one that simply says that a tendency to choose x over y exists if and only if the utility of x is greater than the utility of y. This we formalize as:

Definition 16. *A* weak (binary) utility model *is a set of binary preference probabilities for which there exists a real-valued function w over A such that*

$$p(x, y) \geq \tfrac{1}{2} \quad \textit{if and only if} \quad w(x) \geq w(y), \qquad x, y \in A. \tag{18}$$

If no other requirements are imposed, then it is not difficult to see that w must be an ordinal scale, that is, w is unique up to strictly monotonic increasing transformations.

Every algebraic theory of utility provides us with a weak binary utility model, and conversely, by means of the following definition: if $x, y \in A$, then

$$x \geq y \quad \text{if and only if} \quad p(x, y) \geq \tfrac{1}{2}.$$

A set of axioms on $\geq$ that are sufficient to establish the existence of a real-valued, order-preserving utility function w can be immediately restated into an equivalent set of axioms about the binary probabilities using Def. 16. If the axioms on $\geq$ yield a scale stronger than ordinal, then clearly the corresponding weak utility model has a scale of exactly the same strength. Thus, for example, if $A = A_1 \times A_2$ and if $\geq$ satisfies the axioms of Luce and Tukey (1964) (see Sec. 2.3), then the equivalent axioms for binary probabilities yield a representation theorem of the form: there exist real-valued functions u_1 on A_1 and u_2 on A_2 such that

$$p[(a, x), (b, y)] \geq \tfrac{1}{2} \quad \text{if and only if} \quad u_1(a) + u_2(x) \geq u_1(b) + u_2(y),$$

and u_1 and u_2 are interval scales with a common unit.

Next in level of strength is the Fechnerian model, most familiar from classical psychophysics (see Chapter 4, Sec. 2).

Definition 17. *A* strong (or Fechnerian) (binary) utility model *is a set of binary preference probabilities for which there exist a real-valued function u over A and a cumulative distribution function ϕ such that*

(i) $\phi(0) = \frac{1}{2}$, *and*
(ii) *for all $x, y \in A$ for which $p(x, y) \neq 0$ or* 1,

$$p(x, y) = \phi[u(x) - u(y)]. \tag{19}$$

It is clear that by an appropriate change in ϕ, u can be transformed by positive linear transformations without affecting the representation, and it can be shown (Block & Marschak, 1960, p. 104) that these are the only possible transformations—that u is an interval scale—when A is a continuum and ϕ is strictly increasing.

Theorem 29. *Any strong (binary) utility model in which the preference probabilities are different from* 0 *or* 1 *is also a weak utility model, but not conversely.*

PROOF. Suppose that $p(x, y) \geq \frac{1}{2}$, then the monotonicity of ϕ and $\phi(0) = \frac{1}{2}$ imply, by (ii) of Def. 17, that $u(x) - u(y) \geq 0$. So u is the weak utility function.

The failure of the converse is easily shown by examples.

Although the idea that strength of preference, as measured by the binary choice probabilities, should be a monotonic function of a scale of strength of preference, of utility, is an appealing one, examples have been suggested that make it dubious as a general hypothesis. The following was offered by L. J. Savage[6] as a criticism of the next model to come, but it applies equally well to the strong binary model.

Suppose that a boy must select between having a pony x and a bicycle y and that he wavers indecisively between them. According to the strong utility model, $u(x)$ must be approximately equal to $u(y)$. The bicycle dealer, trying to tip the scale in his favor, suddenly brings forth another bicycle z which, although basically the same as y, is better in minor ways, such as having a speedometer. One can well believe that the boy will still be indifferent, or nearly so, between x and z. The minor differences between the two bicycles do not help much to resolve his indecision between a pony and a bicycle—any bicycle within reason. According to the model, we are forced to conclude that the utility difference between the two bicycles, $u(y) - u(z)$, is small when evaluated in terms of preferences relative to x.

[6] In correspondence with Luce.

Nevertheless, were the boy forced by his impatient father to confine his choice to the two bicycles, he might very well exhibit a strong preference for one over the other, forcing us to conclude that the utility difference $u(y) - u(z)$ is not small, contrary to the earlier conclusion. If this can happen—and the introspections of several people suggest that it can—then the strong binary utility model is too strong to describe these preferences.

We return to this problem after first stating an even stronger model.

Definition 18. *A* strict binary utility model *is a set of binary preference probabilities for which there exists a positive real-valued function v over A such that for all $x, y \in A$ for which $p(x, y) \neq 0$ or 1,*

$$p(x, y) = \frac{v(x)}{v(x) + v(y)}. \tag{20}$$

The strict utility scale is determined up to multiplication by a positive constant, that is, it is a ratio scale.

The strict binary utility model has been investigated by a number of authors: Abelson & Bradley (1954); Becker, DeGroot, & Marschak (1963a); Block & Marschak (1960); Bradley (1954a,b, 1955); Bradley & Terry (1952); Ford (1957); Gulliksen (1953); Luce (1959); Marschak (1960); Thurstone (1930); and Törnqvist (1954).

Theorem 30. *Any strict binary utility model is also a strong utility model, but not conversely.*

PROOF. Let v be the strict utility function. Define $u = \log v + b$, and let ϕ be the logistic distribution function, $\phi(\zeta) = 1/(1 + e^{-\zeta})$, then

$$\begin{aligned} p(x, y) &= \frac{1}{1 + v(y)/v(x)} \\ &= \frac{1}{1 + \exp\{-[u(x) - u(y)]\}} \\ &= \phi[u(x) - u(y)]. \end{aligned}$$

The failure of the converse is easily shown by examples.

These last two models have two rather peculiar features. The first, which is also shared by the weak utility model, is their restriction to binary choices. Although many experiments involve only binary presentations, larger presentation sets could easily be used, and one would expect an adequate theory to treat all possible presentation sets. Second, these theories are stated only for probabilities different from 0 and 1; nothing is asserted about how the 0 and 1 probabilities relate to the others or to each other, which is the reason we were able to omit the restriction to probabilities different from 0 or 1 in the hypothesis of Theorem 30, but not in that of Theorem 29. Some authors have suggested that the non-0 or 1 models are suitable idealizations to reality, on a par with the continuous idealizations to discrete phenomena in physics, and so there really is no problem. Others, for example, Block and Marschak (1960, p. 122), have cited simple economic examples where small changes in one component of a commodity bundle change a choice probability from about $\frac{1}{2}$ to 1 in a highly discontinuous fashion. Many feel that these apparent discontinuities are a deep problem in the understanding of preferences.

A generalization of the strict binary model has been suggested that attempts to overcome both these problems (Luce, 1959). In the original presentation, the 0-1 difficulty was dealt with in what, to some critics, seemed a distinctly artificial manner; however, as was shown in Chapter 4, Sec. 3.2, this apparently artificial formulation is implied by the following simpler assumption.

Definition 19. *A set of preference probabilities defined for all the subsets of a finite set A satisfy the* choice axiom *provided that for all x, Y, and X such that $x \in Y \subseteq X \subseteq A$,*

$$p_Y(x) = p_X(x \mid Y),$$

whenever the conditional probability exists.

What the choice axiom says is that a choice from Y is just that, independent of what else may have been available. That is to say, even though X may have been presented, if we look only at those occasions when the choices were made from Y, then the probability of choosing x from Y, $p_X(x \mid Y)$, is exactly the same as the probability of choosing x from Y, $p_Y(x)$, when only Y was presented in the first place.

Theorem 31. *If the choice axiom holds and A is finite, then there exists a ratio scale v on A such that for any $p_Y(x) \neq 0, 1$,*

$$p_Y(x) = \frac{v(x)}{\sum_{y \in Y} v(y)}. \tag{21}$$

PROOF. Equation 24 of Sec. 3.2 of Chapter 4 (p. 221).

We see that for non-0 or 1 probabilities, the choice axiom generalizes the strict binary utility model; indeed, the property asserted in Theorem 31 has been called the *strict utility model* by Block and Marschak (1960).

Of course, the pony-bicycle example discussed is applicable to the strict utility models because they are special cases of the strong ones. Nevertheless, it is instructive to see how it violates the choice axiom. [Substantially, this same example was discussed by Debreu (1960b) in his review of Luce (1959).] Let x again denote the pony and let y and z be two nearly identical bicycles with compensating features so that the boy is indifferent between each of the pairs; hence $v(x) = v(y) = v(z)$. According to Theorem 31, the probability of $\frac{1}{2}$ of choosing the pony x when only one bicycle is offered is dropped to $\frac{1}{3}$ when they are both offered. Savage and Debreu feel that this conclusion is incorrect—that the two bicycles would be treated as a single alternative, later to be analyzed into a final choice if the pony is rejected.

The argument seems convincing, and it is closely related to the observation (Luce, 1959, pp. 132–133) that we cannot expect the choice axiom to hold for over-all decisions that are divided in some manner into two or more intermediate decisions. It appears that such criticisms, although usually directed toward specific models, are really much more sweeping objections to all our current preference theories. They suggest that we cannot hope to be completely successful in dealing with preferences until we include some mathematical structure over the set of outcomes that, for example, permits us to characterize those outcomes that are simply substitutable for one another and those that are special cases of others. Such functional and logical relations among the outcomes seem to have a sharp control over the preference probabilities, and they cannot long be ignored.

5.3 Random Utility Models

The random utility models are more similar in spirit to the algebraic models than are the constant utility ones. They are the same to this extent: the subject is assumed to choose the outcome that has the largest utility value at the time of choice. They differ in that the utilities are no longer assumed to stay put. Each time a choice is to be made, a utility function is selected according to some postulated probability mechanism, and the outcome is determined algebraically by it. The most familiar psychological mechanism of this type is Thurstone's model in which the utilities are determined by multivariate normal distribution functions, but more complex mechanisms are possible.

In the next definition we use the concept of a (finite-dimensional) random vector. Let A be the finite set of alternatives and let $\mathbf{U}$ be a function defined on A such that, for each x in A, $\mathbf{U}(x)$ is a random variable. Then we call $\mathbf{U}$ a *random vector on A*. In general, no assumption is made about dependency relations between the random variables. Thus without further assumptions the following equality cannot be simplified:

$$Pr[\mathbf{U}(x) \geq \mathbf{U}(y),\, y \in Y] = \int_{-\infty}^{\infty} Pr[\mathbf{U}(x) = t,\, \mathbf{U}(y) \leq t,\, y \in Y]\, dt.$$

If the random variables are independent, then the right-hand side simplifies to

$$\int_{-\infty}^{\infty} Pr[\mathrm{U}(x) = t] \prod_{y \in Y - \{x\}} Pr[\mathrm{U}(y) \leq t]\, dt.$$

Definition 20. *A* random utility model *is a set of preference probabilities defined for all subsets of a finite A for which there is a random vector* $\mathbf{U}$ *on A such that for $x \in Y \subseteq A$,*

$$p_Y(x) = Pr[\mathrm{U}(x) \geq \mathrm{U}(y), y \in Y]. \tag{22}$$

If the definition is only asserted for the binary preference probabilities, that is,

$$p(x, y) = Pr[\mathrm{U}(x) \geq \mathrm{U}(y)],$$

then the model is called a binary *random utility model. If the random vector* $\mathbf{U}$ *consists of components that are independent random variables, then we say the model is an* independent *random utility model.*

The primary theoretical results so far obtained about random utility models establish relations with the constant utility models, which we now present, and with certain observable properties, which we discuss in Sec. 5.6.

Theorem 32. *For choice probabilities different from* 0 *and* 1, *any strict utility model is an independent random utility model, but not conversely.*

PROOF.[7] Suppose that v is the scale of the strict utility model. Define $\mathbf{U}$ to be the random vector whose components are independent random variables with probability densities

$$Pr[\mathrm{U}(x) = t] = \begin{cases} v(x)e^{v(x)t}, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0. \end{cases}$$

Consider

$$\begin{aligned} Pr[\mathrm{U}(x) \geq \mathrm{U}(y), y \in Y] &= \int_{-\infty}^{\infty} Pr[\mathrm{U}(x) = t] \prod_{y \in Y - \{x\}} Pr[\mathrm{U}(y) \leq t]\, dt \\ &= \int_{-\infty}^{0} v(x)e^{v(x)t} \prod_{y \in Y - \{x\}} \left[\int_{-\infty}^{t} v(y)e^{v(y)\tau}\, d\tau\right] dt \\ &= \int_{-\infty}^{0} v(x)e^{v(x)t} \prod_{y \in Y - \{x\}} e^{v(y)t}\, dt \\ &= \int_{-\infty}^{0} v(x) \exp\left(\sum_{y \in Y} v(y)t\right) dt \\ &= \frac{v(x)}{\sum_{y \in Y} v(y)}. \end{aligned}$$

[7] This proof is due to Eric Holman and A. A. J. Marley. It has the advantages over Block's and Marschak's (1960, p. 110) proof that it does not use any ranking notions, that it is constructive, and that it is a good deal shorter.

Note that any monotonic transformation f of $\mathbf{U}$ in the preceding example again provides a random utility interpretation of a given strict utility model provided that we replace t by $f^{-1}(t)$ in the right side of the defining equation. It is conjectured that these are the only reasonably well-behaved examples, but no proof has yet been devised. In any event, just about any random utility model one cares to write down is not a strict utility model, so the failure of the converse of the theorem is easy to establish.

Although certain other relations exist among the random, weak, and strong utility models that are easy to work out directly, we shall not do so because they can be deduced from later results using only the transitivity of implication. It is sufficient to note the results here.

There are (binary) random utility models that are not weak, and therefore not strong, utility models.

There are strong, and therefore weak, utility models that are not random utility models.

The question also arises how the strong and the random models are related when both are binary. No complete answer seems to be known; however, the following is of interest.

Theorem 33. *Any strong utility model for which the distribution function ϕ is the difference distribution of two independent and identically distributed random variables is a binary random utility model.*

PROOF. Let u be the strong utility function and define the random variable $\mathbf{U}(x) = u(x) + \boldsymbol{\epsilon}(x)$, where for $x \neq y$, $\boldsymbol{\epsilon}(x)$ and $\boldsymbol{\epsilon}(y)$ are independent and identically distributed and ϕ is the distribution function of $\boldsymbol{\epsilon}(x) - \boldsymbol{\epsilon}(y)$. Then

$$\begin{aligned} Pr[\mathbf{U}(x) \geq \mathbf{U}(y)] &= Pr[u(x) + \boldsymbol{\epsilon}(x) \geq u(y) + \boldsymbol{\epsilon}(y)] \\ &= Pr[\boldsymbol{\epsilon}(y) - \boldsymbol{\epsilon}(x) \leq u(x) - u(y)] \\ &= \phi[u(x) - u(y)] \\ &= p(x, y). \end{aligned}$$

5.4 Observable Properties

None of the models just described has ever been tested experimentally in any complete fashion. Indeed, save for the choice axiom, they are all stated in terms of nonobservable utility functions, and so it is impossible to test them completely until we know conditions that are necessary and sufficient to characterize them in terms of the preference probabilities themselves, for only these can be estimated from data. No results of this generality are now known, and so experimenters have been content to examine various observable properties that are necessary consequences of at least some of the models.

In this subsection we list several of the more important properties, although not all that Marschak and his colleagues have investigated theoretically; in Sec. 5.5 we cite some of the interrelations among them; and in Sec. 5.6, the relations between them and the several models are described.

Consider three outcomes x, y, and z and the corresponding binary probabilities $p(x, y)$, $p(y, z)$, and $p(x, z)$. If we think of these as represented in the plane by arrows of lengths $p(x, y)$, $p(y, z)$, and $p(x, z)$, then it is not unreasonable that we should be able to construct a triangle from them. That is to say, the length corresponding to the x-z pair should not exceed the sum of the lengths corresponding to the x-y and y-z pairs. If so, then the following property must be met by the data.

Definition 21. *A set of binary preference probabilities satisfies the* triangle condition *if for every* $x, y, z \in A$,

$$p(x, y) + p(y, z) \geq p(x, z). \tag{23}$$

Marschak (1960, p. 317) attributes the triangle condition to Guilbaud (1953). It is in many ways the weakest property that has been suggested.

The next group of ideas are all attempts to extend to the probabilistic situation the notion that preferences are transitive. The term "stochastic" used in the definition is inappropriate (see the discussion in Sec. 1.3), but it is so ingrained in the literature that we make no attempt to change it here.

Definition 22. *Whenever* $\min [p(x, y), p(y, z)] \geq \frac{1}{2}$, *the binary preference probabilities are said to satisfy*

(i) weak (stochastic) transitivity *provided that*

$$p(x, z) \geq \tfrac{1}{2};$$

(ii) moderate (stochastic) transitivity *provided that*

$$p(x, z) \geq \min [p(x, y), p(y, z)]; \tag{24}$$

(iii) strong (stochastic) transitivity *provided that*

$$p(x, z) \geq \max [p(x, y), p(y, z)]. \tag{25}$$

Marschak (1960, p. 319) attributes the definitions of weak and strong transitivity to Valavanis-Vail (1957) and that of moderate (sometimes called *mild*) transitivity to Georgescu-Roegen (1958) and Chipman (1958).

Menger (1951) proposed a multiplicative condition on triples of binary probabilities which Morrison (1963) modified slightly by requiring the same hypothesis as in the stochastic transitivity conditions. We adopt Morrison's form of the definition.

Definition 23. *A set of binary preference probabilities satisfy the* multiplicative condition *if for every* $x, y, z \in A$ *such that* $\min[p(x, y), p(y, z)] \geq \frac{1}{2}$, *then* $p(x, y)p(y, z) \leq p(x, z)$.

We turn next to a binary property, due to Davidson and Marschak (1959), which involves four elements and which has already been discussed for algebraic models in Sec. 2.4. It is suggested by the strong utility model. Suppose that that model holds, then by the strict monotonicity of ϕ, $p(w, x) \geq p(y, z)$ is equivalent to $u(w) - u(x) \geq u(y) - u(z)$, which in turn is equivalent to $u(w) - u(y) \geq u(x) - u(z)$ and that is equivalent to $p(w, y) \geq p(x, z)$. Thus we are led to consider the following property.

Definition 24. *A set of binary preference probabilities satisfy the* quadruple condition *provided that* $p(w, x) \geq p(y, z)$ *implies* $p(w, y) \geq p(x, z)$.

Our final binary property was pointed out by Luce (1959). It says, in essence, that the probability of an intransitive cycle of independent choices $x > y$, $y > z$, $z > x$ is exactly the same as the probability of $x > z$, $z > y$, $y > x$.

Definition 25. *A set of binary preference probabilities not equal to* 0 *or* 1 *satisfy the* product rule[8] *if for every set of three distinct elements*[9] $x_1, x_2, x_3 \in A$,

$$p(x_1, x_2)p(x_2, x_3)p(x_3, x_1) = p(x_1, x_3)p(x_3, x_2)p(x_2, x_1). \tag{26}$$

[8] This term appears to be due to Estes (1960, p. 272). Suppes and Zinnes (1963, p. 49) introduced the term "multiplication condition," which, however, is too similar to the "multiplicative condition" of Def. 23.

[9] One might expect Def. 25 to have been stated for n elements rather than three; however, the following result, which was pointed out to us by R. M. Rose, shows that this is unnecessary.

If a set of binary preference probabilities satisfies the product rule, then for any set of $n \geq 3$ *distinct elements* $x_1, x_2 \ldots, x_n \in A$.

$$p(x_1, x_2)p(x_2, x_3) \ldots p(x_n, x_1) = p(x_1, x_n)p(x_n, x_{n-1}) \ldots p(x_2, x_1). \tag{27}$$

PROOF. By hypothesis it is true for $n = 3$. We proceed inductively, assuming Eq. 27 for n and showing it for $n + 1$. Let $x_1, x_2, \ldots x_n, x_{n+1}$ be distinct elements of A. By the induction hypothesis, Eq. 27 can be written

$$\frac{p(x_1, x_2)p(x_2, x_3) \ldots p(x_{n-1}, x_n)}{p(x_n, x_{n-1}) \ldots p(x_3, x_2)p(x_2, x_1)} = \frac{p(x_1, x_n)}{p(x_n, x_1)}$$

Substituting,

$$\frac{p(x_1, x_2)p(x_2, x_3) \ldots p(x_{n-1}, x_n)p(x_n, x_{n+1})p(x_{n+1}, x_1)}{p(x_1, x_{n+1})p(x_{n+1}, x_n)p(x_n, x_{n-1}) \ldots p(x_3, x_2)p(x_2, x_1)} = \frac{p(x_1, x_n)p(x_n, x_{n+1})p(x_{n+1}, x_1)}{p(x_1, x_{n+1})p(x_{n+1}, x_n)p(x_n, x_1)}$$

$$= 1,$$

by the product rule.

Aside from the choice axiom (Def. 19), which is stated in terms of observable probabilities, the only simple observable condition that has been suggested for nonbinary choice probabilities and investigated in detail is the weak and extremely reasonable condition that adding a new alternative to a choice set never increases the probabilities of choosing an old alternative.[10] This is, perhaps, the weakest form of the notion of the independence of added alternatives, whereas the strongest probabilistic form is the choice axiom. We formalize this condition.

Definition 26. *A set of preference probabilities defined for the subsets of A is* regular *if for all x, X and Y such that $x \in X \subseteq Y \subseteq A$,*

$$p_X(x) \geq p_Y(x).$$

5.5 Relations among the Observable Properties

In this subsection we establish a number of implications and counterexamples among the properties just defined, and in Sec. 5.6 we do the same thing for the relations between the models of Secs. 5.2 and 5.3 and these observable properties. A summary of these results, plus those of Secs. 5.2 and 5.3, are given in Fig. 5 for probabilities different from 0 and 1. From these results plus the fact that a binary property or model never implies a

[10] A closely related notion has been proposed which, however, we do not list in the text since little has been published on it. Suppose $q_Y(x)$ denotes the probability that x is discarded from Y as not being the most preferred outcome—but not necessarily the least preferred outcome. The proposed relation between these probabilities and the choice probabilities is, for all $x \in Y \subseteq A$,

$$p_Y(x) = \sum_{z \in Y - \{x\}} q_Y(z) p_{Y-\{z\}}(x). \tag{*}$$

This equation arises if we suppose that the choice from Y is arrived at by first discarding an alternative z as not the choice and then a choice is made from the reduced set $Y - \{z\}$. If this scheme is repeated, the ultimate choice is a binary one. The only published result (Luce, 1960) is that if the choice probabilities satisfy the strict utility model, then

$$q_Y(x) = \frac{1 - p_Y(x)}{|Y| - 1},$$

where $|Y|$ denotes the number of elements in Y. It is easy to show by examples that Eq. (*) does not imply either the triangle condition or weak stochastic transitivity, which according to Fig. 5 are the two weakest binary properties. Since this chapter was written, A. A. J. Marley has proved that if the choice probabilities satisfy the regularity condition (Def. 26), then probabilities q_Y exist such that Eq. (*) holds. The proof is too long to include in a footnote.

nonbinary one, it is possible, using only the transitivity of implication, to establish the relations holding between any pair of notions involved in Fig. 5. For example, consider regularity and the product rule. Since regularity is defined for nonbinary probabilities and the product rule only for binary ones, the latter cannot imply the former. Suppose that regularity implies the product rule, then we have the chain

$$\begin{aligned}\text{random utility} &\xrightarrow{41} \text{regularity} \longrightarrow \text{product rule} \xrightarrow{40} \text{quadruple}\\ &\xrightarrow{39} \text{strong transitivity} \xrightarrow{37} \text{moderate transitivity}\\ &\xrightarrow{38} \text{multiplicative,}\end{aligned}$$

contrary to Theorem 42. As a second example, consider the strong utility model and the multiplicative condition. The chain

$$\begin{aligned}\text{strong utility} &\xrightarrow{47} \text{quadruple} \xrightarrow{39} \text{strong transitivity}\\ &\xrightarrow{37} \text{moderate transitivity} \xrightarrow{38} \text{multiplicative}\end{aligned}$$

shows that strong utility implies the multiplicative condition. The converse implication is false, for if it were true then we would have the chain

$$\text{multiplicative} \longrightarrow \text{strong utility} \xrightarrow{29} \text{weak utility} \xrightarrow{45} \text{weak transitivity,}$$

contrary to Theorem 36.

The remainder of this section is devoted to stating and proving the theorems that establish these various relationships.

Theorem 34. *Regularity implies the triangle condition, but not conversely.*[11]

PROOF. Let $Y = \{x, y, z\}$. Assuming regularity, we know that $p(x, y) \geq p_Y(x)$, $p(y, z) \geq p_Y(y)$, and $p(z, x) > p_Y(z)$. Adding and using Eq. 17 (p. 332),

$$p(x, y) + p(y, z) + p(z, x) \geq p_Y(x) + p_Y(y) + p_Y(z) = 1,$$

which is equivalent to the triangle condition.

Theorem 35. *The multiplicative condition implies the triangle condition, but not conversely.*

PROOF. Suppose that the multiplicative condition holds and that the triangle condition is false, that is, for some $x, y, z \in A$, $p(x, y) + p(y, z) < p(x, z) \leq 1$. With no loss of generality, we may suppose $p(y, z) < \frac{1}{2}$;

[11] In this and other similar theorems the converse does not hold because a binary property does not imply a nonbinary one. This usually will not be noted explicitly.

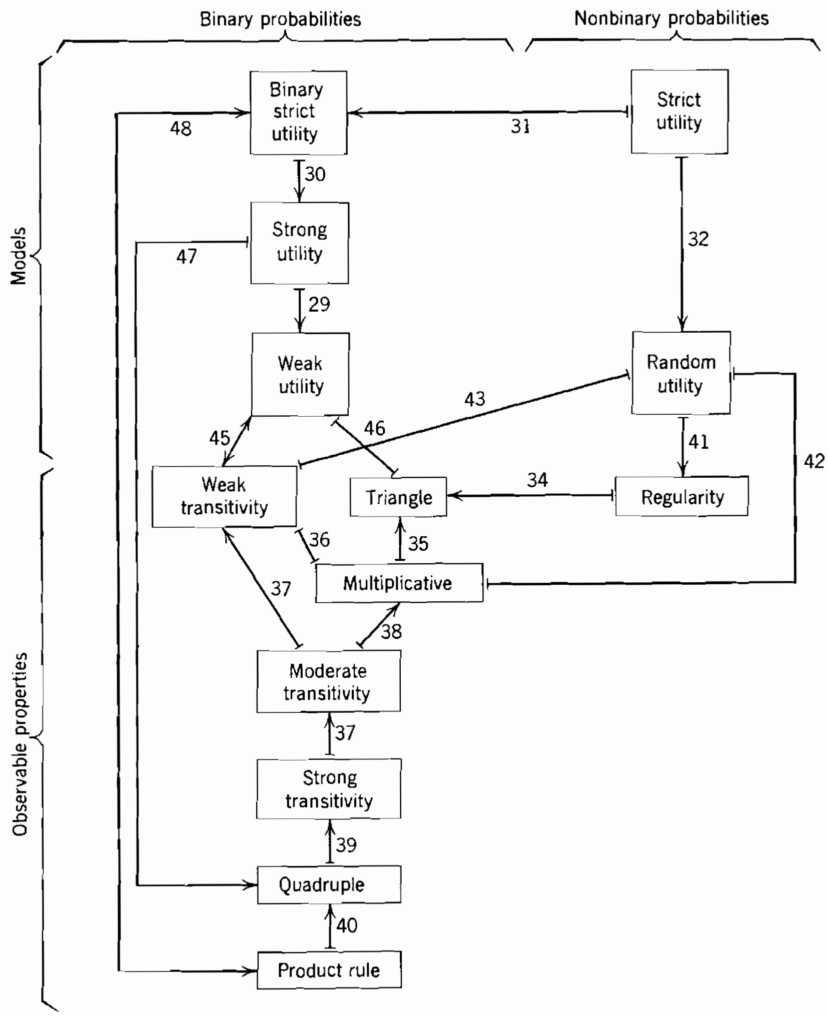


Fig. 5. A summary of the implications (→) and failures of implication (⊣) among models and observable properties on the assumption that the probabilities are different from 0 and 1 and that the set of all alternatives is finite. The number beside a line indicates the relevant theorem. The relation between any two concepts can be deduced using only the transitivity of implication from these relations plus the fact that a binary model or property never implies a nonbinary one (see text).

hence $p(z, y) > \frac{1}{2}$. There are, therefore, two possible (nonvacuous) cases of the multiplicative condition to consider.

(i) $p(x, z) \geq \frac{1}{2}$, in which case our hypothesis implies

$$\begin{aligned} p(x, z)p(z, y) &> [p(x, y) + p(y, z)]p(z, y) \\ &= p(x, y)[1 - p(y, z)] + p(y, z)p(z, y) \\ &= p(x, y) + p(y, z)[p(z, y) - p(x, y)] \\ &\geq p(x, y), \end{aligned}$$

because, by hypothesis, $p(x, y) < 1 - p(y, z) = p(z, y)$. Since this contradicts the multiplicative condition, we turn to the other possibility.

(ii) $p(y, x) \geq \frac{1}{2}$, in which case we have

$$\begin{aligned} p(z, y)p(y, x) &= [1 - p(y, z)]p(y, x) \\ &> [1 + p(x, y) - p(x, z)]p(y, x) \\ &= [p(z, x) + p(x, y)][1 - p(x, y)] \\ &= p(z, x) + p(x, y)[1 - p(z, x) - p(x, y)] \\ &= p(z, x) + p(x, y)[p(x, z) - p(x, y)] \\ &\geq p(z, x), \end{aligned}$$

because by hypothesis $p(x, z) - p(x, y) > p(y, z) > 0$. Since this contradicts the multiplicative condition, we have shown that the triangle condition holds.

The converse does not hold, for example, when

$$p(x, y) = p(y, z) = p(z, x) = \tfrac{2}{3}.$$

Theorem 36. *Neither the multiplicative condition nor weak stochastic transitivity implies the other.*

PROOF. The binary probabilities $p(x, y) = p(y, z) = 0.60$ and $p(x, z) = 0.36$ satisfy the multiplicative condition but not weak transitivity.

The binary probabilities $p(x, y) = p(y, z) = 0.9$ and $p(x, z) = 0.6$ satisfy weak transitivity but not the multiplicative condition.

Theorem 37. *Strong transitivity is strictly stronger than moderate transitivity, which in turn is strictly stronger than weak transitivity.*

PROOF. The implications follow immediately from the definitions. The failure of the opposite implications is easily shown by simple examples.

Theorem 38. *Moderate stochastic transitivity implies the multiplicative condition, but not conversely.*

PROOF. If $\min [p(x, y), p(y, z)] \geq \frac{1}{2}$, then moderate transitivity implies

$$\begin{aligned} p(x, z) &\geq \min [p(x, y), p(y, z)] \\ &\geq p(x, y)p(y, z). \end{aligned}$$

The converse is false, for if it were true, then by Theorem 37 the multiplicative condition would imply weak transitivity, contrary to Theorem 36.

Theorem 39. *The quadruple condition implies strong stochastic transitivity, but not conversely.*

PROOF. Suppose that $p(x, y) \geq \frac{1}{2}$, then since $p(z, z) = \frac{1}{2}$ for any $z \in A$, the quadruple condition implies $p(x, z) \geq p(y, z)$. A similar argument using $p(y, z) \geq \frac{1}{2} = p(x, x)$ establishes strong transitivity.

The converse does not hold because $p(x, y) = p(x, z) = \frac{1}{3}$ and $p(y, z) = \frac{2}{3}$ satisfy strong transitivity but not the quadruple condition:

$$p(x, y) = \tfrac{1}{3} = p(x, z), \text{ but } p(x, x) = \tfrac{1}{2} < p(y, z) = \tfrac{2}{3}.$$

Theorem 40. *For binary probabilities different from* 0 *and* 1, *the product rule implies the quadruple condition, but not conversely.*

PROOF. Suppose that $p(w, x) \geq p(y, z)$, then rewriting the product rule on $\{w, x, y, z\}$,

$$\begin{aligned} \frac{p(z, x)}{p(x, z)} \frac{p(w, y)}{p(y, w)} &= \frac{p(w, x)p(z, y)}{p(x, w)p(y, z)} \\ &\geq 1, \end{aligned}$$

from which it follows that $p(w, y) \geq p(x, z)$.

Simple counterexamples show that the converse is false.

5.6 Relations between the Models and the Observable Properties

Following the plan shown in Fig. 5, we next establish the basic connections between the models and the observable properties. We begin with the weaker models and work up to the stronger ones.

Theorem 41. *Any random utility model is regular, but not conversely.*

PROOF. Let $x \in X \subseteq Y$, then by Def. 20,

$$\begin{aligned} p_X(x) &= Pr[\mathbf{U}(x) \geq \mathbf{U}(y), y \in X \subseteq Y] \\ &\geq Pr[\mathbf{U}(x) \geq \mathbf{U}(y), y \in Y] \\ &= p_Y(x). \end{aligned}$$

The proof that the converse is false is given as a corollary to Theorem 49 of Sec. 6.1.

Theorem 42. *Neither being a random utility model nor satisfying the multiplicative condition implies the other.*

PROOF. Consider $A = \{x, y, z\}$, let ϵ be a number such that $0 < \epsilon < 1$, and let $\mathbf{U} = \langle \mathbf{U}(x), \mathbf{U}(y), \mathbf{U}(z)\rangle$ be the random vector with the following distribution

$$Pr(\mathbf{U} = \mathbf{u}) = \begin{cases} \epsilon & \text{if } \mathbf{u} = \langle 3, 2, 1\rangle \\ \dfrac{1-\epsilon}{2} & \text{if } \mathbf{u} = \langle 1, 3, 2\rangle \text{ or } \langle 2, 1, 3\rangle \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\begin{aligned} p(x, y) &= Pr[\mathbf{U}(x) \geq \mathbf{U}(y)] \\ &= Pr[\mathbf{U} = \langle 3, 2, 1\rangle] + Pr[\mathbf{U} = \langle 2, 1, 3\rangle] \\ &= \epsilon + \frac{1-\epsilon}{2} \\ &= \frac{1+\epsilon}{2} \\ &> \frac{1}{2} \end{aligned}$$

In like manner,

$$\begin{aligned} p(y, z) &= \frac{1+\epsilon}{2} \\ p(x, z) &= \epsilon. \end{aligned}$$

Thus

$$\begin{aligned} p(x, y)p(y, z) - p(x, z) &= \left(\frac{1+\epsilon}{2}\right)^2 - \epsilon \\ &= \left(\frac{1-\epsilon}{2}\right)^2 \\ &> 0, \end{aligned}$$

and the multiplicative condition is violated.

The implication the other way fails because the multiplicative condition restricts only the binary probabilities.

Theorem 43. *Neither being a random utility model nor satisfying weak stochastic transitivity implies the other.*

PROOF. Consider $A = \{x, y, z\}$, let ϵ be a number such that $0 < \epsilon < \frac{1}{6}$, and let $\mathbf{U} = \langle \mathbf{U}(x), \mathbf{U}(y), \mathbf{U}(z)\rangle$ be a random vector with the following distribution:

$$Pr(\mathbf{U} = \mathbf{u}) = \begin{cases} \epsilon + \frac{1}{6} & \text{if} \quad \mathbf{u} = \langle 3, 2, 1\rangle, \langle 1, 3, 2\rangle, \text{ or } \langle 2, 1, 3\rangle \\ -\epsilon + \frac{1}{6} & \text{if} \quad \mathbf{u} = \langle 1, 2, 3\rangle, \langle 2, 3, 1\rangle, \text{ or } \langle 3, 1, 2\rangle \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\begin{aligned}
p(x, y) &= Pr[\mathbf{U}(x) \geq \mathbf{U}(y)] \\
&= Pr(\mathbf{u} = \langle 3, 2, 1\rangle) + Pr(\mathbf{u} = \langle 2, 1, 3\rangle) + Pr(\mathbf{u} = \langle 3, 1, 2\rangle) \\
&= \epsilon + \tfrac{1}{6} + \epsilon + \tfrac{1}{6} - \epsilon + \tfrac{1}{6} \\
&= \epsilon + \tfrac{1}{2}
\end{aligned}$$

Similarly,

$$p(y, z) = \epsilon + \tfrac{1}{2}$$
$$p(x, z) = -\epsilon + \tfrac{1}{2}.$$

Clearly, weak transitivity is not met.

Although it is not indicated on Fig. 5, perhaps it is worth noting that by imposing some fairly strong regularity conditions on the random utility model it is possible to show that strong stochastic transitivity must hold and that under slightly stronger conditions we can obtain an expression for the weak utility function, which by Theorems 37 and 45 must exist. It will be convenient to use the notation

$$F(t, x, y) = Pr[\mathbf{U}(x) \leq t] - Pr[\mathbf{U}(y) \leq t]$$
$$f(t, x, y) = \frac{dF(t, x, y)}{dt} = Pr[\mathbf{U}(x) = t] - Pr[\mathbf{U}(y) = t].$$

Theorem 44. *Suppose that a set of binary preference probabilities is a binary independent random utility model.*

(i) *If for $x, y \in A$, $F(t, x, y)$ is either nonnegative for all t or nonpositive for all t, then strong stochastic transitivity is satisfied;*
(ii) *if, in addition,* $\lim_{t\to\infty} tF(t, x, y) = \lim_{t\to-\infty} tF(t, x, y) = 0$, *then*

$$w(x) = E[\mathbf{U}(x)] = \int_{-\infty}^{\infty} tPr[\mathbf{U}(x) = t]\, dt$$

is a weak utility function.

PROOF. We first note that

$$\begin{aligned}
p(x, z) - p(x, y) &= p(y, x) - p(z, x) \\
&= \int_{-\infty}^{\infty} Pr[\mathbf{U}(x) = t]F(t, z, y)\, dt.
\end{aligned}$$

Because $F(t, z, y)$ has the same sign for all t and $Pr[\mathbf{U}(x) = t] \geq 0$, it follows that $p(x, z) - p(x, y)$ has the same sign as $F(t, x, y)$.

(i) Suppose $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$. Thus

$$0 \leq p(x, y) - \tfrac{1}{2} = p(x, y) - p(x, x)$$
$$0 \leq p(y, z) - \tfrac{1}{2} = p(y, z) - p(y, y),$$

and so $F(t, y, x) \geq 0$ and $F(t, z, y) \geq 0$. Therefore $p(x, z) - p(x, y) \geq 0$ and $p(x, z) - p(y, z) = p(z, y) - p(z, x) \geq 0$, which proves strong stochastic transitivity.

(ii) Consider

$$\begin{aligned} w(x) - w(y) &= E[\mathbf{U}(x) - \mathbf{U}(y)] \\ &= \int_{-\infty}^{\infty} tf(t, x, y)\, dt \\ &= tF(t, x, y)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(t, x, y)\, dt \\ &\gtrless 0 \quad \text{if and only if} \quad -F(t, x, y) = F(t, y, x) \gtrless 0, \end{aligned}$$

where we have integrated by parts and used the hypothesis of part (ii). But

$$p(x, y) - \tfrac{1}{2} = p(x, y) - p(x, x) \gtrless 0 \quad \text{if and only if} \quad F(t, y, x) \gtrless 0,$$

which proves that

$$p(x, y) \geq \tfrac{1}{2} \quad \text{if and only if} \quad w(x) \geq w(y).$$

It should be noted that Thurstone's Case V model (see Vol. I, p. 55), which assumes independent normal distributions with equal variances, satisfies all the conditions of this theorem.

Theorem 45. *A weak utility model satisfies weak stochastic transitivity. The converse is not true in general, but it is when A is finite.*

PROOF. If $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$, then in a weak utility model there is a function w such that $w(x) \geq w(y) \geq w(z)$, and therefore $p(x, y) \geq \frac{1}{2}$.

To show that the converse is false, consider points in the plane, $x = (x_1, y_1)$, and for $\alpha, \beta > \frac{1}{2}$, $\alpha \neq \beta$, define

$$p(x, y) = \begin{cases} \alpha & \text{if } x_1 > y_1 \\ \beta & \text{if } x_1 = y_1 \quad \text{and} \quad x_2 > y_2 \\ \frac{1}{2} & \text{if } x_1 = y_1 \quad \text{and} \quad x_2 = y_2. \end{cases}$$

We first show that these probabilities satisfy moderate stochastic transitivity and, therefore by Theorem 37, weak transitivity. Assuming $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$, there are nine cases to consider; we look at only three because the rest are similar.

1. $p(x, y) = \alpha = p(y, z)$, then $x_1 > y_1 > z_1$, so $p(x, z) = \alpha$.

2. $p(x, y) = \alpha$, $p(y, z) = \beta$, then $x_1 > y_1 = z_1$, so $p(x, z) = \alpha$.

3. $p(x, y) = \beta$, $p(y, z) = \frac{1}{2}$, then $x_1 = y_1$, $x_2 > y_2$, $y_1 = z_1$, and $y_2 = z_2$, so $x_1 = z_1$ and $x_2 > z_2$; hence $p(x, z) = \beta$.

Now, the relation $\geq$ defined by $x \geq y$ if and only if $p(x, y) \geq \frac{1}{2}$ holds if and only if $x_1 > y_1$ or $x_1 = y_1$ and $x_2 \geq y_2$. Thus, if these probabilities were to satisfy the weak utility model, there would have to be an order-preserving function of this relation, contrary to what we showed in Sec. 2.1, but when A is finite, it is immediately evident that an order-preserving function exists.

Theorem 46. *Neither being a weak utility model nor satisfying the triangle condition implies the other.*

PROOF. For $A = \{x, y, z\}$ and $p(x, y) = 0.1$, $p(y, z) = 0.6$, and $p(x, z) = 0.8$, the weak utility model holds with $w(x) = 1$, $w(y) = 2$, and $w(z) = 0$, but the triangle condition does not because $p(x, y) + p(y, z) = 0.7 < 0.8 = p(x, z)$.

If the converse held, then we would have the chain

$$\text{multiplicative} \xrightarrow{35} \text{triangle} \longrightarrow \text{weak utility} \xrightarrow{45} \text{weak transitivity},$$

contrary to Theorem 36.

Theorem 47. *A strong utility model satisfies the quadruple condition, but not conversely.*

PROOF. A proof of the first part was given in the discussion leading to the quadruple condition.

Debreu (1958) showed that the quadruple condition does not imply the strong utility model in general, but that it does when the following condition of *stochastic continuity* is met: for every $x, y, z, \in A$ and every real α such that $p(x, y) < \alpha < p(x, z)$, there exists a $w \in A$ such that $p(x, w) = \alpha$.

Theorem 48. *A strict utility model satisfies the product rule, but not conversely except in the binary case when they are equivalent.*

PROOF. To show that being a (binary) strict utility model implies satisfaction of the product rule, substitute Eq. 20 (p. 335) into both sides of Eq. 26 (p. 341) and cancel the denominators. The converse, which is not true in general, is true in the binary case: For $a \in A$, define $v(x) = p(x, a)/p(a, x)$, then from $p(x, y)p(y, a)p(a, x) = p(x, a)p(a, y)p(y, x)$, we see that

$$\begin{aligned} p(x, y) &= \frac{p(x, a)/p(a, x)}{p(x, a)/p(a, x) + p(y, a)/p(a, y)} \\ &= \frac{v(x)}{v(x) + v(y)}. \end{aligned}$$

6. GENERAL PROBABILISTIC RANKING THEORIES

In the algebraic studies of preference, no distinction is currently drawn between theories of choice and of ranking. In the simplest models, knowing which outcome is best is equivalent to knowing the utilities of all the outcomes of the presented set, and because utilities are numerical this is equivalent to a ranking of the presentation set. However, once we entertain the possibility that decisions may be governed by a probability process, the connections between choices and ranking are no longer at all obvious; it is a theoretical and experimental problem of some magnitude to try to find out what they are.

Very little theoretical work has yet been done, and to our knowledge there are no relevant experiments. The theoretical work is due to Block and Marschak (1960), Luce (1959, p. 68–74), and Marschak (1960).

Two general modes of attack on the theoretical problem have been tried. One may assume either that the choice probabilities can be expressed as some relatively simple and "natural" function of the ranking probabilities or that the ranking probabilities can be expressed simply in terms of the choice probabilities. Our next two subsections present the only serious ideas that have been put forward.

6.1 A Function Relating Choice to Ranking Probabilities

We begin with a simple example. Suppose that $X = \{x, y, z\}$ is presented many times and that the subject is required to rank order the three outcomes. This gives us estimates of the ranking probabilities $p(xyz)$, $p(yzx)$, etc., where xyz denotes the ranking in which x is first, y second, and z third, etc. If we only knew these probabilities and we were asked what is the probability that the subject will choose x over y, the most obvious thing to do is to calculate the total probability of all the rankings in which x is rated better than y, that is, to assume

$$p(x, y) = p(xyz) + p(xzy) + p(zxy). \tag{28}$$

Indeed, Coombs (1958) invoked just this assumption (applied to sets of four elements) to estimate binary choice probabilities (see Sec. 8.2).

As has been pointed out, this is not a necessary connection, and models are easily devised in which it does not hold. For example, suppose that the rankings of three alternatives are generated in the following way from the choice probabilities. A pair of outcomes is chosen at random, that is, with probability $\frac{1}{3}$, and they are ordered according to the relevant binary choice probability. One of these two outcomes is then chosen at random, that is, with probability $\frac{1}{2}$, and it is compared with the third outcome according to the relevant binary choice probability. This may or may not produce a ranking of the three outcomes; if not, the two not previously compared are now ordered, which produces a ranking of all three. It is not difficult to show that

$$p(x, y) = \frac{p(xyz) - p(zxy)}{p(xyz) - p(zxy) + p(yxz) - p(zyx)},$$

which is obviously different from Eq. 28.

Nevertheless, the idea embodied in Eq. 28 seems worth exploration. To do so, some notation is needed.

Let A denote the finite set of all n outcomes, and let R denote the set of $n!$ rankings of A. Thus $\rho \in R$ is a particular ranking of A. Denote by ρ_i the element of A that is ranked in the ith position by ρ, and denote by $\rho(x)$ the rank order position of $x \in A$ under the ranking ρ. The set of rankings for which $x \in A$ is placed above all the elements in $Y \subseteq A$ is denoted by $R(x, Y)$, that is,

$$R(x, Y) = \left\{\rho \mid \rho \in R \quad \text{and} \quad \rho(x) > \rho(y), \quad \text{all} \quad y \in Y - \{x\}\right\}.$$

Block and Marschak (1960, p. 107) proved the following interesting result.

Theorem 49. *A set of preference probabilities p_Y, $Y \subseteq A$ is a random utility model (Def. 20, p. 338) if and only if there exists a probability distribution p over the set R of rankings of A such that*

$$p_Y(x) = \sum_{\rho \in R(x, Y)} p(\rho). \tag{29}$$

PROOF. Suppose, first, that the preference probabilities satisfy a random utility model. Define

$$p(\rho) = Pr[\mathbf{U}(\rho_1) > \mathbf{U}(\rho_2) > \ldots > \mathbf{U}(\rho_n)],$$

then

$$\begin{aligned} p_Y(x) &= Pr[\mathbf{U}(x) > \mathbf{U}(y), y \in Y - \{x\}] \\ &= \sum_{\rho \in R(x, Y)} Pr[\mathbf{U}(\rho_1) > \mathbf{U}(\rho_2) > \ldots > \mathbf{U}(\rho_n)] \\ &= \sum_{\rho \in R(x, Y)} p(\rho). \end{aligned}$$

Conversely, because we can place A into one-to-one correspondence with the set I_n of the first n integers, we may just as well assume $A = I_n$ and that R is the set of permutations of I_n ordered by magnitude. Now, for any vector $u = (u_1, u_2, \ldots, u_n)$ with real components u_i, we define the random vector $\mathbf{U}$ by

$$Pr[\mathbf{U} = u] = \begin{cases} p(u) & \text{if } u \in R \\ 0 & \text{if } u \notin R. \end{cases}$$

It is easy to see that

$$\begin{aligned} p_Y(x) &= \sum_{u \in R(x,y)} p(u) \\ &= \sum_{u \in R(x,Y)} Pr[\mathbf{U} = u] \\ &= Pr[\mathbf{U}(x) > \mathbf{U}(y),\ y \in Y - \{x\}]. \end{aligned}$$

Corollary. *A model that satisfies the regularity condition need not be a random utility model.*

PROOF. For $A = \{1, 2, 3, 4\}$, it is easy to see from Eq. 29 that a necessary condition for the random utility model to hold is that

$$\begin{aligned} p_{\{2,3\}}(2) + p_A(2) - p_{\{1,2,3\}}(2) - p_{\{2,3,4\}}(2) \\ = p(1423) + p(4123) \\ \geq 0. \end{aligned}$$

Let $0 < \epsilon < \frac{1}{6}$ and let

$$\begin{aligned} p_A(2) &= \epsilon \\ p_A(i) &= (1 - \epsilon)/3, \qquad i = 1, 3, 4 \\ p_S(i) &= \begin{cases} \frac{1}{3} \text{ if } S \text{ has 3 elements} \\ \frac{1}{2} \text{ if } S \text{ has 2 elements} \end{cases} \end{aligned}$$

It is clear that these choice probabilities satisfy regularity, but that

$$\begin{aligned} p_{\{2,3\}}(2) + p_A(2) - p_{\{1,2,3\}}(2) - p_{\{2,3,4\}}(2) &= \tfrac{1}{2} + \epsilon - \tfrac{1}{3} - \tfrac{1}{3} \\ &< -\tfrac{1}{6} + \tfrac{1}{6} \\ &= 0, \end{aligned}$$

and so the random utility model is not satisfied.

6.2 A Function Relating Ranking to Choice Probabilities

Now we look at the same problem of how choice and ranking probabilities are related from the other point of view, namely, how the ranking probabilities can be expressed as a function of the choice probabilities. And again there is a natural model. It seems most reasonable that a person might rank order a set of outcomes by first selecting what strikes him as the best outcome and assigning it rank one; this outcome is then discarded, and he selects the best outcome from the remaining set and assigns it rank two; and so on, until the set is exhausted. Once stated, this seems like the only sensible mechanism, but there are alternatives. For example, he might simply reverse the procedure; first select the least satisfactory outcome and rank it last, then the least satisfactory from the remaining set and rank it next to last, etc. The relation between these two procedures is considered shortly. Or he might proceed as in the example of Sec. 6.1 by choosing a pair of elements at random, order them, and then compare one of these with another outcome chosen at random, etc. Note that this last scheme is prone to give a partial order or an intransitive relation when there are more than three elements, but it is not obvious that this is descriptively wrong. To generate a consistent ranking of numerous alternatives, for example, of student applications, is not considered particularly easy by most people who have ever had to do it.

Recall that if ρ is a rank ordering of A, then ρ_i denotes the element of A that the has ith rank. We may formulate the first model proposed previously as asserting that for all $\rho \in R$,

$$p(\rho) = p_A(\rho_1)p_{A-\{\rho_1\}}(\rho_2) \dots p(\rho_{n-1}, \rho_n). \tag{30}$$

Nothing much seems to be known about Eq. 30 by itself, but when coupled with Eq. 29 of Sec. 6.1, Block and Marschak (1960, p. 109) have proved the following strong result, the first part of which generalizes a result in Luce (1959, p. 72).

Theorem 50. *If a set of preference probabilities p_Y, $Y \subseteq A$, is a strict utility model (Theorem 31, p. 336), then there exist ranking probabilities p such that p and p_Y satisfy both Eqs. 29 and 30; conversely, if there exist preference and ranking probabilities that satisfy both Eqs. 29 and 30, then the set of preference probabilities is a strict utility model.*

PROOF. Suppose that the preference probabilities form a strict utility model. If we define the ranking probabilities by Eq. 30, then it is sufficient to show that Eq. 29 holds. Observe, first, that the set $R(x, Y)$ is the union of the following disjoint sets: $R(x, A)$; the sets of rankings where $z_1 \in A - Y$ is ranked first and x second; the sets where $z_1 \in A - Y$ is ranked first, $z_2 \in A - Y - \{z_1\}$ is ranked second, and x third, etc.

$$\begin{aligned}\sum_{\rho \in R(x,Y)} p(\rho) = p_A(x) &+ \sum_{z_1 \in A-Y} p_A(z_1)p_{A-\{z_1\}}(x) \\ &+ \sum_{z_1 \in A-Y} \sum_{z_2 \in A-Y-\{z_1\}} p_A(z_1)p_{A-\{z_1\}}(z_2)p_{A-\{z_1,z_2\}}(x) \\ &+ \dots \\ &+ \sum_{z_1 \in A-Y} \dots p_A(z_1) \dots p_{Y \cup \{z_k\}}(z_k)p_Y(x).\end{aligned}$$

For any Z such that $Y \subseteq A - Z$, *the strict utility model implies*

$$
\begin{aligned}
p_{A-Z}(x) &= p_Y(x) p_{A-Z}(Y) \\
&= p_Y(x)[1 - p_{A-Z}(A - Z - Y)].
\end{aligned}
$$

Substituting these expressions in the previous equation,

$$
\begin{aligned}
\sum_{\rho \in R(x,Y)} p(\rho) &= p_Y(x)\Big\{1 - p_A(A - Y) + \sum_{z_1 \in A-Y} p_A(z_1) \\
&\quad \times [1 - p_{A-\{z_1\}}(A - Y - \{z_1\})] \\
&\quad + \sum_{z_1 \in A-Y} \sum_{z_2 \in A-Y-\{z_1\}} [p_A(z_1) p_{A-\{z_1\}}(z_2)] \\
&\quad \times [1 - p_{A-\{z_1,z_2\}}(A - Y - \{z_1, z_2\}] \\
&\quad + \dots \\
&\quad + \sum_{z_1 \in A-Y} \dots p_A(z_1) \dots p_{Y \cup \{z_k\}}(z_k)\Big\} \\
&= p_Y(x)\Big[1 - p_A(A - Y) + p_A(A - Y) \\
&\quad - \sum_{z_1 \in A-Y} p_A(z_1) p_{A-\{z_1\}}(A - Y - \{z_1\}) \\
&\quad + \sum_{z_1 \in A-Y} p_A(z_1) p_{A-\{z_1\}}(A - Y - \{z_1\}) - \dots\Big] \\
&= p_Y(x).
\end{aligned}
$$

Conversely, if Eqs. 29 and 30 hold, then we have

$$
\begin{aligned}
p_Y(x) &= p_A(x) + \sum_{z_1 \in A-Y} p_A(z_1) p_{A-\{z_1\}}(x) \\
&\quad + \sum_{z_1 \in A-Y} \sum_{z_2 \in A-Y-\{z_1\}} p_A(z_1) p_{A-\{z_1\}}(z_2) p_{A-\{z_1,z_2\}}(x) \\
&\quad + \dots + \sum_{z_1 \in A-Y} \dots p_A(z_1) \dots p_{Y \cup \{z_k\}}(z_k) p_Y(x),
\end{aligned}
$$

where $p_Z(w) = 0$ when $w \notin Z$. Clearly, for $Y = A$, the strict utility model holds. To prove it generally, we perform a decreasing induction on the size of Y. By the induction hypothesis,

$$
p_{A-\{z_1\}}(x) = \frac{p_A(x)}{p_A(A - \{z_1\})}, \qquad p_{A-\{z_1,z_2\}}(x) = \frac{p_A(x)}{p_A(A - \{z_1, z_2\})}, \quad \text{etc.}
$$

Substituting these into the preceding equation yields

$$\begin{aligned} p_Y(x) &= p_A(x)\Bigg[1 + \sum_{z_1 \in A-Y} \frac{p_A(z_1)}{p_A(A - \{z_1\})} \\ &\qquad + \sum_{z_1 \in A-Y} \sum_{z_2 \in A-Y-\{z_1\}} \frac{p_A(z_1)p_{A-\{z_1\}}(z_2)}{p_A(A - \{z_1, z_2\})} + \dots\Bigg] \\ &\qquad + \Bigg[\sum_{z_1 \in A-Y} \dots p_A(z_1) \dots p_{Y \cup \{z_k\}}(z_k)\Bigg] p_Y(x) \\ &= p_A(x)F(Y) + p_Y(x)G(Y), \end{aligned}$$

where $F(Y)$ and $G(Y)$ are independent of x. Rewriting,

$$p_Y(x) = p_A(x)\left[\frac{F(Y)}{1 - G(Y)}\right],$$

and so

$$\frac{p_Y(x)}{p_Y(x')} = \frac{p_A(x)}{p_A(x')},$$

which yields the strict utility model if we set $v(x) = p_A(x)$.

6.3 An "Impossibility" Theorem

Our final result about rankings concerns the relation of the ranking model embodied in Eq. 30 and the corresponding model based upon choices of least preferred outcomes. We let p_Y denote the usual preference probabilities and p_Y^* the choice probabilities for the least preferred element of Y. Exactly what empirical interpretation should be given to "choosing the least preferred element of Y" is not clear. If in estimating p_Y we simply ask the subject to select from Y the element he most prefers, then it seems reasonable to estimate p_Y^* by asking which element he least prefers. However, if we estimate p_Y by giving him the element that he selects, then what is a comparable payoff procedure to estimate p_Y^*? One possibility is to reward him with an element chosen at random from $Y - \{x\}$ when he says he least prefers x. In any event, let us suppose that p_Y^* exists.

Now suppose that a single random utility model underlies both of these judgments. The probability that he gives the ranking ρ when he is asked to order the outcomes from the most preferred to the least preferred is, as before,

$$p(\rho) = Pr[\mathbf{U}(\rho_1) > \mathbf{U}(\rho_2) > \dots > \mathbf{U}(\rho_n)].$$

The probability of the ranking ρ when he is asked to order the elements from the least preferred to the most preferred is

$$p^*(\rho) = Pr[\mathbf{U}(\rho_1) < \mathbf{U}(\rho_2) < \ldots < \mathbf{U}(\rho_n)].$$

Observe that if we let ρ^* denote the reverse of ranking ρ, that is,

$$\rho_1^* = \rho_n, \ldots, \rho_i^* = \rho_{n+1-i}, \ldots, \rho_n^* = \rho_1, \text{ then}$$

$$p^*(\rho^*) = p(\rho). \tag{31}$$

A similar argument based upon Eq. 29 yields

$$p(x, y) = p^*(y, x), \qquad \text{for} \quad x, y \in A. \tag{32}$$

Finally, let us suppose that Eq. 30 holds both for the starred and the unstarred probabilities, that is,

$$p(\rho) = p_A(\rho_1)p_{A-\{\rho_1\}}(\rho_2) \ldots p_{\{\rho_{n-1},\rho_n\}}(\rho_{n-1}) \tag{33}$$

$$p^*(\rho) = p_A^*(\rho_1)p_{A-\{\rho_1\}}^*(\rho_2) \ldots p_{\{\rho_{n-1},\rho_n\}}^*(\rho_{n-1}). \tag{34}$$

Theorem 51. *Suppose that the preference and ranking probabilities for the most preferred outcome, p_Y and p, and for the least preferred outcome, p_Y^* and p^*, satisfy the same random utility model (and hence satisfy Eqs.* 31 *and* 32). *If the two ranking probabilities are given by Eqs.* 33 *and* 34, *that is, if both p_Y and p_Y^* satisfy strict utility models, then for all $x \in Y \subseteq A$,*

$$p_Y(x) = p_Y^*(x) = \frac{1}{|Y|},$$

where $|Y|$ denotes the number of elements in Y.

PROOF. Let the two strict utility scales be v and v^*, respectively, where with no loss of generality we may assume

$$\sum_{x \in A} v(x) = \sum_{x \in A} v^*(x) = 1.$$

For any $x, y \in A$, let ρ be that ranking of A for which $\rho_1 = x$, $\rho_2 = y$, and let σ be that ranking of A for which $\sigma_1 = y$, $\sigma_2 = x$, and $\sigma_i = \rho_i$, $i = 3, \ldots, n$. By Eq. 33,

$$\frac{p(\rho)}{p(\sigma)} = \frac{p_A(x)p_{A-\{x\}}(y)p_{A-\{x,y\}}(\rho_3) \ldots}{p_A(y)p_{A-\{y\}}(x)p_{A-\{x,y\}}(\rho_3) \ldots}$$

$$= \frac{\dfrac{v(x)}{\sum_{z \in A} v(z)} \dfrac{v(y)}{\sum_{z \in A-\{x\}} v(z)}}{\dfrac{v(y)}{\sum_{z \in A} v(z)} \dfrac{v(x)}{\sum_{z \in A-\{y\}} v(z)}}$$

$$= \frac{1 - v(y)}{1 - v(x)}.$$

By Eqs. 34 and 32,

$$
\begin{aligned}
\frac{p^*(\rho^*)}{p^*(\sigma^*)} &= \frac{p_A^*(\rho_1^*)p_{A-\{\rho_1{}^*\}}^*(\rho_2^*)\cdots p_{\{\rho_{(n-1)}{}^*,\rho_n{}^*\}}^*(\rho_{n-1}^*)}{p_A^*(\sigma_1^*)p_{A-\{\sigma_1{}^*\}}^*(\sigma_2^*)\cdots p_{\{\sigma_{(n-1)}{}^*,\sigma_n{}^*\}}^*(\sigma_{n-1}^*)} \\
&= \frac{p_A^*(\rho_n)p_{A-\{\rho_n\}}^*(\rho_{n-1})\cdots p^*(y, x)}{p_A^*(\sigma_n)p_{A-\{\sigma_n\}}^*(\sigma_{n-1})\cdots p^*(x, y)} \\
&= \frac{p^*(y, x)}{p^*(x, y)} \\
&= \frac{p(x, y)}{p(y, x)} \\
&= \frac{v(x)}{v(y)}
\end{aligned}
$$

By Eq. 31, $p^*(\rho^*) = p(\rho)$ and $p^*(\sigma^*) = p(\sigma)$, so

$$\frac{1 - v(y)}{1 - v(x)} = \frac{v(x)}{v(y)}.$$

Since x and y were arbitrary, $v(x) =$ constant independent of x. Since $p^*(x, y) = p(y, x)$, it also follows that $v^*(x) =$ constant independent of x. Thus

$$p_Y(x) - p_Y^*(x) = 1/|Y|.$$

This result is due to Block and Marschak (1960, p. 111); it generalizes to any n a result proved for $n = 3$ in Luce (1959, p. 69).

Clearly, Theorem 51 is an impossibility theorem. Its conclusion is unacceptable on empirical grounds; hence its hypotheses cannot all be correct assumptions in a theory of behavior. Block and Marschak, who did not make explicit in their formal statement of the theorem that they assumed a common random utility model and, in particular, that Eqs. 31 and 32 hold, interpreted Theorem 51 as powerful evidence against the strict utility model (or, what is the same, against Eqs. 33 and 34). There are, in our view, at least two alternative possibilities. First, one can question the assumption (embodied in Eqs. 31 and 32) that a common random utility model underlies the choice of the most preferred and of the least preferred outcomes and of the ranking from best to worst and from worst to best. As has been previously pointed out (Luce, 1959), the assumption $p(\rho) = p^*(\rho^*)$ is not obviously correct as a description of behavior. Second, one can question whether the whole problem makes any sense at all, that is, whether choices of least preferred outcomes have any operational meaning. Thus, although the conclusion of Theorem 51 is empirically unacceptable, it is not evident to us which of its hypotheses should be rejected.

7. PROBABILISTIC CHOICE THEORIES FOR UNCERTAIN OUTCOMES

By an experiment that has uncertain outcomes, we mean one in which knowledge of the stimulus presentation and of the response is insufficient to determine the actual "elementary" or "certain" outcome received by the subject. Rather, the stimulus and response determine a function ω from trials to the set A of certain outcomes; this was called a conditional outcome schedule in Sec. 1.4 of Chapter 2, Vol. I. Throughout this section we shall assume that the conditional outcome schedules are all simple random ones in the sense that for $x \in A$

$$\pi(x \mid \omega) = Pr[\omega(n) = x]$$

is independent of the trial number n.

Another way of characterizing uncertain outcomes is to say that the presentation plus the response determines a set $\{E(x)\}$ of mutually exclusive and exhaustive chance events that are in one-to-one correspondence with the set of certain outcomes A. A particular outcome x is delivered to the subject if and only if the event $E(x)$ occurs. The connection between these two ways of speaking is essentially that between random variables and sample spaces.

For most purposes of this section it is convenient to use the event language and to subscript symbols so that E_i is the event corresponding to outcome x_i, π_i is the probability of event E_i occurring, and if f is a function defined over A, $f_i = f(x_i)$.

In summary, then, an *uncertain outcome* is simply a probability distribution π over the set A of certain outcomes. Some authors speak of uncertain outcomes as wagers or gambles.

7.1 Expected Utility Models

The first idea that comes to mind to deal with uncertain outcomes is to combine the major algebraic concept of utility theory, namely, the expected utility hypothesis, with one or another of the general probability models described in Secs. 5.2 and 5.3.

Let $\pi^j = (\pi_1{}^j, \pi_2{}^j, \ldots, \pi_n{}^j)$, $j = 1, 2, \ldots, m$, denote m uncertain outcomes defined over a set $A = \{x_i\}$ of n certain outcomes. Of course, some of the $\pi_i{}^j$ may equal 0, meaning that that outcome x_i cannot be received by the subject when π^j is selected. Let $\mathscr{A} = \{\pi^j\}$. The defining properties of the several models are the following.

WEAK EXPECTED UTILITY. *There is a real-valued function w over A such that*

$$p(\pi^j, \pi^k) \geq \tfrac{1}{2} \quad \textit{if and only if} \quad \sum_{i=1}^{n} \pi_i{}^j w_i \geq \sum_{i=1}^{n} \pi_i{}^k w_i.$$

STRONG EXPECTED UTILITY. *There is a real-valued function u over A and a distribution function ϕ such that*

$$p(\pi^j, \pi^k) = \phi\left(\sum_{i=1}^{n} \pi_i{}^j u_i - \sum_{i=1}^{n} \pi_i{}^k u_i\right)$$

provided that $p(\pi^j, \pi^k) \neq 0$ or 1.

STRICT EXPECTED UTILITY. *There is a positive real-valued function v over A such that*

$$p_{\mathscr{A}}(\pi^j) = \frac{\sum_{i=1}^{n} \pi_i{}^j v_i}{\sum_{k=1}^{m} \sum_{i=1}^{n} \pi_i{}^k v_i}$$

provided that $p_{\mathscr{A}}(\pi^j) \neq 0$ *or* 1.

In their presentation of these models, Becker, DeGroot, and Marschak (1963*a*) state a weaker form of the strict and strong expected utility model that involves two functions, u and v, and assumes that the relevant scale values are

$$v\left(\sum_{i=1}^{n} \pi_i{}^j u_i\right).$$

This, however, does not seem to us to be the natural generalization of these utility models, so we have formulated them in terms of just one function.

Still another possibility was suggested by Suppes (1961). He considered a learning situation in which a subject must choose from a set $\mathscr{A}$ of gambles all of the same form: if $\zeta \in \mathscr{A}$, then with probability $\pi(\zeta)$ the subject is reinforced (that is, given a desired outcome) and with probability $1 - \pi(\zeta)$ nothing happens. Using a one-element stimulus sampling model (see Chapter 10, Vol. II), he arrived at a Markov chain to describe the transition from one trial to the next. The asymptotic choice probability turns out to be of the form

$$p_{\mathscr{A}}(\zeta) = \frac{v(\zeta)}{\sum_{\eta \in \mathscr{A}} v(\eta)},$$

where

$$v(\zeta) = \frac{1}{\epsilon(\zeta)[1 - \pi(\zeta)]}$$

and $\epsilon(\zeta)$ is the probability that the stimulus element is conditioned to another response when ζ is not reinforced. We describe this model (for two responses) somewhat more carefully in Sec. 7.3; here we merely point out its similarity in form to the strict utility model and the considerable difference in the form of $v(x)$ from that assumed in the strict expected-utility model.

RANDOM EXPECTED UTILITY. *There is a random vector* $\mathbf{U}$ *whose components are in one-to-one correspondence with the elements of A such that*

$$p_{\mathscr{A}}(\pi^j) = Pr\left(\sum_{i=1}^{n} \pi_i^{\,j}\mathbf{U}_i \geq \sum_{i=1}^{n} \pi_i^{\,k}\mathbf{U}_i, \qquad k = 1, 2, \ldots, m\right).$$

The first three models can readily be rephrased as subjective expected-utility models merely by replacing the objective probabilities $\pi_i^{\,j}$ with subjective ones. The last one can be extended in at least two different ways: by postulating a constant subjective probability function to replace $\pi_i^{\,j}$ or by treating subjective probability as a random vector, just as utility is treated in this model.

At the moment, however, all such generalizations are really quite idle because next to nothing is known about any of these models, let alone their generalizations. An exception is the following special, but testable, result about the strict and random expected utility models (Becker, DeGroot, & Marschak, 1963a).

Theorem 52. *Let* $\mathscr{A} = \langle \pi^j \rangle$ *consist of* $m + 1$ *uncertain outcomes for which the* $(m + 1)$st *is the average of the other m in the sense that*

$$\pi_i^{m+1} = \frac{1}{m}\sum_{j=1}^{m} \pi_i^{\,j}.$$

If the strict expected utility model is satisfied, then

$$p_{\mathscr{A}}(\pi^{m+1}) = \frac{1}{m+1}\,.$$

If the random expected utility model is satisfied, then

$$p_{\mathscr{A}}(\pi^{m+1}) = 0.$$

PROOF. For the strict expected utility model, we have

$$\begin{aligned} p_{\mathscr{A}}(\pi^{m+1}) &= \frac{\sum_{i=1}^{n} \pi_i^{m+1} v_i}{\sum_{j=1}^{m}\sum_{i=1}^{n} \pi_i^{\,j} v_i + \sum_{i=1}^{n} \pi_i^{m+1} v_i} \\ &= \frac{\sum_{i=1}^{n} \pi_i^{m+1} v_i}{m\sum_{i=1}^{n} \pi_i^{m+1} v_i + \sum_{i=1}^{n} \pi_i^{m+1} v_i} \\ &= \frac{1}{m+1}\,. \end{aligned}$$

For the random expected utility model, it is easy to see that equalities among the random variables occur with probability 0, so

$$\begin{aligned} p_{\mathscr{A}}(\pi^{m+1}) &= Pr\left(\sum_{i=1}^{n} \pi_i^{m+1} \mathbf{U}_i > \sum_{i=1}^{n} \pi_i^{k} \mathbf{U}_i, \qquad k = 1, 2, \ldots, m\right) \\ &= Pr\left[\frac{1}{m} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \pi_i^{j} \mathbf{U}_i\right) > \sum_{i=1}^{n} \pi_i^{k} \mathbf{U}_i, \qquad k = 1, 2, \ldots, m\right] \\ &= 0, \end{aligned}$$

because the average of several quantities can never exceed each of them.

Corollary. *A strict expected utility model need not be a random expected utility model.*

At first glance, this corollary may seem surprising in the light of Theorem 32 of Sec. 5.3, but it should be clear that they are not inconsistent.

It is easy to see that a proof paralleling that of Theorem 29 of Sec. 5.2 establishes that a strong expected utility model is a weak expected utility model. As stated, a strict expected utility model need not be a strong one, although this is true for the weaker definitions given by Becker, De Groot, and Marschak.

Nothing much else seems to be known about these models, except for one experiment based on Theorem 52 (see Sec. 8.4).

7.2 Decomposition Assumption

In this subsection we confine our attention to uncertain outcomes of the simplest possible form, namely, those with nonzero probability on only two certain outcomes. It is convenient to introduce a slightly different notation from what we have been using. By xEy we denote the uncertain outcome from which $x \in A$ is received by the subject if event E occurs and $y \in A$ is received if E does not occur. If we let $\mathscr{E}$ denote the set (actually, Boolean algebra) of events used to generate these binary uncertain outcomes, then $\mathscr{A} = A \times \mathscr{E} \times A$ is the set of possible uncertain outcomes. Certain outcomes are included in $\mathscr{A}$ by taking E to be the certain event.

As we have been assuming all along, preference probabilities that the subject chooses the (uncertain or certain) outcome ζ when $\mathscr{X}$ is presented, $p_{\mathscr{X}}(\zeta)$, $\zeta \in \mathscr{X} \subseteq \mathscr{A}$, are assumed to exist. In addition, we postulate that the subject is able to make judgments about which of several events is most likely to occur—he must be able to do this, at least crudely, if he is to make any sense of selecting among uncertain outcomes. Thus, when $E \in \mathscr{D} \subseteq \mathscr{E}$, we suppose that a probability $q_{\mathscr{D}}(E)$ exists that the subject selects E as the event in $\mathscr{D}$ that is most likely to occur. We speak of these as *judgment probabilities.*

Suppose that xEy and xDy are presented and the subject must select one. It is not unreasonable that he should try to decompose his choice into two simpler ones by asking: Which outcome, x or y, would I rather receive and, independent of this, which event, E or D, do I believe is more likely to occur? It is clear that he should prefer xEy to xDy in just two of the four possible cases (ignoring indifferences): when x is preferred to y and E is judged more likely than D, and when y is preferred to x and D is judged more likely than E. If so and if the two decisions—preference of outcomes and judgments of events—are statistically independent, then it is clear that the following condition must hold (Luce, 1958, 1959).

Decomposition Assumption. *For all $x, y \in A$ and $E, D \in \mathscr{E}$,*

$$p(xEy, xDy) = p(x, y)q(E, D) + p(y, x)q(D, E). \tag{35}$$

The following trivial result shows that if the decomposition assumption holds, it is possible to estimate the binary judgment probabilities from the binary preference probabilities.

Theorem 53. *If the decomposition assumption holds and if $x, y \in A$ are such that $p(x, y) \neq \frac{1}{2}$, then*

$$q(E, D) = \frac{p(xEy, xDy)p(x, y) - p(xDy, xEy)p(y, x)}{p(x, y)^2 - p(y, x)^2}.$$

The primary criticism of the decomposition assumption is that judgments of relative likelihood may not be independent of preference decisions. Certainly, this can be arranged by logically interlocking the outcomes with the events. L. J. Savage[12] suggested the following example. Let x and y be two greyhound dogs, E the event that x wins over y, and $D = \bar{E}$ the complementary event that y wins over x in a single race between them. Thus you will receive one of the dogs, but which one depends upon which gamble you choose and which dog wins the race. If you propose to race the dog received in the future, it seems clear that in the absence of other information about them you would prefer the dog that wins the race. Thus you would choose xEy with probability 1 over $x\bar{E}y$, even though because of ignorance about the dogs $p(x, y) = q(E, \bar{E}) = \frac{1}{2}$. Clearly, this violates the decomposition assumption.

Usually in the laboratory, and often outside of it, events and outcomes are logically unrelated, and so examples of this kind are not particularly relevant except to show that independence is not a uniformly valid assumption. Nevertheless, considerable experimental evidence shows that the outcomes of an uncertain choice may affect judgments of likelihood; however, one must be very careful about concluding that these studies disprove the decomposition assumption. Consider, for example, the work

[12] In correspondence with Luce.

of Irwin and his students (Irwin, 1953; Marks, 1951; and unpublished data). In the simplest, and most relevant, version of these experiments, the subject predicts whether or not he will draw a marked card from a small deck of cards that has a known number of marked and unmarked ones. The payoffs are as shown:

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\underline{\text{Event}}} \\ & & \text{Predicted} & \text{Not Predicted} \\ \text{Event} & E & x & y \\ \text{Occurs} & \bar{E} & y & x \end{array}$$

In our terms, the subject must choose between xEy and $x\bar{E}y$. F. W. Irwin and G. Snodgrass (unpublished) have shown that if, in addition to this payoff matrix, you reward (or punish) the subject by giving (or taking away) a fixed sum of money whenever E occurs, independent of what was predicted, then his frequency of predicting the event appears to be a monotonic increasing function of this "irrelevant" extra payment. Although this finding is quite reasonably described as proving that an interaction exists between preferences and judgments of event likelihood, it does not really bear upon the decomposition assumption because the choice is between $(x + a)Ey$, and $x\bar{E}(y + a)$, where a is the amount paid when E occurs.

The main theoretical results that are known about the decomposition assumption concern its conjunction with other probabilistic assumptions. For example, we can easily prove the following result.

Theorem 54. *If the decomposition assumption holds and there exist* $x, y \in A$ *such that* $p(x, y) = 1$, *then any property satisfied by the binary preference probabilities* p *is also satisfied by the binary judgment probabilities* q.

PROOF. Because $p(x, y) = 1$, the decomposition assumption implies that for any $E, D \in \mathscr{E}$, $q(E, D) = p(xEy, xDy)$, and so any proposition about the binary q probabilities can be immediately replaced by an equivalent statement about the p probabilities.

In some cases, the transfer of a binary property from the p's to the q's can be proved without assuming the existence of x and y such that $p(x, y) = 1$. For example, we can prove the following result.

Theorem 55. *Suppose that the decomposition assumption holds and that there exist* $x, y \in A$ *such that* $p(x, y) \neq \frac{1}{2}$. *If the binary preference probabilities satisfy strong stochastic transitivity, then the binary judgment probabilities do also.*

PROOF. Let $E, D \in \mathscr{E}$ be given. It is not difficult to show that strong stochastic transitivity implies that for all $\zeta \in \mathscr{A}$, $p(xEy, \zeta) - p(xDy, \zeta)$ has the same sign. For any $F \in \mathscr{E}$, set $\zeta = xFy$, apply the decomposition assumption, and collect terms:

$$p(xEy, xFy) - p(xDy, xFy) = [p(x, y) - p(y, x)][q(E, F) - q(D, F)].$$

Because the left side has the same sign for all F and because $p(x, y) - p(y, x) \neq 0$, it follows that $q(E, F) - q(D, F)$ has the same sign for all F, thus proving strong stochastic transitivity for the judgment probabilities.

The other results in Luce (1958) are too intricate to be described here. Suffice it to say that they are concerned with conditions which, when added to the decomposition axiom, permit us to show that the preference probabilities satisfy a strong utility model. Here we prove a closely related, but simpler, theorem that shows how the distribution functions of the strong utility model are limited when both the decomposition assumption and the expected utility hypothesis are satisfied.

Theorem 56. *Suppose that there exist a (utility) function u from* $\mathscr{A} = A \times \mathscr{E} \times A$ *into a bounded real interval, which we take to be* $[0, 1]$ *with no loss of generality, a (subjective probability) function s from* $\mathscr{E}$ *into* $[0, 1]$, *and continuous distribution functions P and Q such that*

1. *if* $\bar{E}$ *is the complement of E, where* $E, \bar{E} \in \mathscr{E}$, *then* $s(\bar{E}) = 1 - s(E)$;
2. *for all* $x, y \in A$, $E \in \mathscr{E}$, $u(xEy) = u(x)\, s(E) + u(y)[1 - s(E)]$;
3. *for all* $\zeta, \eta \in \mathscr{A}$, $p(\zeta, \eta) = P[u(\zeta) - u(\eta)]$;
4. *for all* $E, D \in \mathscr{E}$, $q(E, D) = Q[s(E) - s(D)]$; *and*
5. *the decomposition assumption is satisfied.*

If A and $\mathscr{E}$ *are dense in the sense that for every* $\alpha \in [-1, 1]$, *there exist* $x, y \in A$ *and* $E, D \in \mathscr{E}$ *such that* $u(x) - u(y) = \alpha = s(E) - s(D)$, *and if* $P(1) \neq \frac{1}{2}$, *then there exist constants* $\epsilon > 0$ *and* $k > 0$ *such that*

$$P(\alpha) = \begin{cases} \dfrac{1 + k\alpha^{\epsilon}}{2}, & \text{if} \quad 0 \leq \alpha \leq 1, \\[2ex] \dfrac{1 - k\,|\alpha|^{\epsilon}}{2}, & \text{if} \ -1 \leq \alpha < 0 \end{cases}$$

$$Q(\alpha) = \begin{cases} \dfrac{1 + \alpha^{\epsilon}}{2}, & \text{if} \quad 0 \leq \alpha \leq 1, \\[2ex] \dfrac{1 - |\alpha|^{\epsilon}}{2}, & \text{if} \ -1 \leq \alpha < 0. \end{cases}$$

PROOF. Let $\alpha, \beta \in [-1, 1]$, and choose $x, y \in A$ and $E, D \in \mathscr{E}$ such that $\alpha = u(x) - u(y)$ and $\beta = s(E) - s(D)$. From the expected utility hypothesis, we see that

$$\alpha\beta = [u(x) - u(y)][s(E) - s(D)] = u(xEy) - u(xDy).$$

Thus, by the decomposition and strong utility assumptions,

$$\begin{aligned} P(\alpha\beta) &= P[u(xEy) - u(xDy)] \\ &= p(xEy, xDy) \\ &= p(x, y)q(E, D) + p(y, x)q(D, E) \\ &= P(\alpha)Q(\beta) + [1 - P(\alpha)][1 - Q(\beta)]. \end{aligned} \tag{36}$$

Setting $\alpha = 1$ in Eq. 36,

$$P(\beta) = Q(\beta)[2P(1) - 1] + 1 - P(1).$$

Because $P(1) \neq \frac{1}{2}$,

$$Q(\beta) = \frac{P(\beta) + P(1) - 1}{k},$$

where $k = 2P(1) - 1$. Substituting this in Eq. 36 and simplifying yields

$$P(\alpha\beta) = \frac{2P(\alpha)P(\beta) - P(\alpha) - P(\beta) + P(1)}{k}.$$

Define $f(\alpha) = [2P(\alpha) - 1]/k$. Substituting this and simplifying yields

$$f(\alpha\beta) = f(\alpha)f(\beta). \tag{37}$$

Note that $f(1) = 1$.

Because P is continuous and monotonic increasing, so is f. It is well known that when $\alpha \geq 0$ and $f(1) = 1$ the only continuous monotonic solutions of Eq. 37 are $f(\alpha) = \alpha^{\epsilon}$, where $\epsilon > 0$. Because $P(-\alpha) = 1 - P(\alpha)$, if follows that $f(-\alpha) = -f(\alpha)$, and so for $\alpha < 0$, the solution is $f(\alpha) = -f(-\alpha) = -|\alpha|^{\epsilon}$. Substituting these back into the expressions for P and Q, we obtain the forms stated earlier. Because P is monotonic increasing and $P(0) = \frac{1}{2}$, $k = 2P(1) - 1 > 0$.

Theorem 56 shows that the combination of the strong utility model with the expected utility hypothesis and the decomposition assumption is really very restrictive—the forms of the distribution functions are determined. It is clear that unless we introduce some *ad hoc* assumptions about 0 and 1 probabilities, this model does not apply to situations where the certain outcomes are simply ordered on a single dimension, for example, when they are sums of money or, more generally, amounts of something.

Completely different, but also very restrictive, conclusions arise when we combine the decomposition assumption with the choice axiom (Def. 19, p. 336), even when we do not postulate the expected utility hypothesis.

Theorem 57. *Suppose that the preference probabilities p satisfy the choice axiom, that the judgment probabilities q also satisfy the choice axiom, and that the binary p's and q's satisfy the decomposition assumption. For* $E, D \in \mathscr{E}$, *define* $E \sim D$ *if and only if* $q(E, D) = \frac{1}{2}$. *Then, either, for all* $x, y \in A$, $p(x, y) = 0, \frac{1}{2}$, *or* 1, *or* $\sim$ *is an equivalence relation having at most three equivalence classes.*

The proof of this result is too long to include here; see Luce (1959, pp. 80–82).

Although experimental evidence is difficult to cite, everyone is confident from his experience and introspections that judgment probabilities permit more than three equivalence classes of events, and so we must conclude that when the assumptions of Theorem 57 are met, the preference probabilities over A actually form a simple algebraic structure of the type discussed in Secs. 2 and 3. As was pointed out earlier, we expect this to happen with money outcomes. It is less clear that preferences are algebraic when the outcomes are different in kind, for example, a pony versus a bicycle. If empirical evidence is produced to show that preferences are truly probabilistic among some certain outcomes, then the assumptions of Theorem 57 are too strong to describe behavior in response to those outcomes. No matter what its descriptive status may ultimately be, Theorem 57 is of interest as an "existence proof" of a behavioral theory that leads to strong, discontinuous restrictions on some of the preference probabilities without these restrictions being particularly apparent in the initial assumptions.

Very little else is known about the consequences of the hypotheses of Theorem 57 except for one special result of some empirical interest that is appropriately described in the next section.

7.3 Several Theories Applied to a Special Case

Both learning and utility theories attempt to describe behavior when the outcomes are uncertain. In learning theory, the problem is approached by asking what remains invariant when the response probabilities change from trial to trial in a fixed choice situation. In utility theory, it is approached by assuming asymptotic behavior and asking what remains invariant in several related choice situations. Of course, learning models have the empirical advantage that they explicitly describe the sequential structure to be expected in the data. If we confine our attention to the asymptotic predictions of learning theories, then it is reasonable to expect that we can find experiments where both classes of theories predict the behavior. Comparisons of this sort may help considerably in choosing among the various theories.

The special experiment that we shall consider involves two uncertain outcomes, both consisting of the same probability distribution over a pair of certain outcomes. It is most easily represented in tabular form:

$$\begin{array}{lcc} & & \text{Choice} \\ & & \begin{array}{cc} 1 & 2 \end{array} \\ \text{Event} & \pi & \\ & & \left[\begin{array}{cc} x_{11} & x_{12} \\ x_{21} & x_{22} \end{array}\right]. \\ \text{Probability} & 1-\pi & \end{array}$$

Thus, if the subject chooses the uncertain alternative 2, he receives outcome x_{12} with probability π and outcome x_{22} with probability $1 - \pi$.

We confine our attention to one prediction, namely, the plot of $p = p(1, 2)$ versus π when the outcomes x_{ij} are held fixed. Experimentally, we develop such a plot by carrying out a series of runs at different values of π and estimating p from what appear to be asymptotic data. We shall examine what ten different theories, six learning and four utility, say about the nature of this plot.

A ONE-ELEMENT STIMULUS-SAMPLING MODEL. Suppes (1961) suggested the following stimulus-sampling model for this experiment under the added restriction that $x_{12} = x_{21} =$ no experimenter delivered outcome to the subject. We suppose that there is a single stimulus element that is conditioned at the beginning of any trial to one of the two responses. The conditioning of the element dictates which response is made. If the subject is reinforced, that is, if either x_{11} or x_{22} is the outcome, then it is assumed that the conditioning is not changed; if, however, he is not reinforced, it is assumed that the conditioning changes to the other response with a fixed probability ϵ_i, where i denotes the response that was made. The one-step transition trees for the conditioning of the element are shown in Fig. 6. These trees may be summarized in terms of the one-step transition matrix for the Markov chain:

$$\begin{array}{ccc} & & \text{Trial } n+1 \\ & & \begin{array}{cc} \quad 1 \quad\quad\quad\quad & \quad\quad 2 \end{array} \\ \text{Trial } n & \begin{array}{c} 1 \\ 2 \end{array} & \left[\begin{array}{cc} \pi + (1-\pi)(1-\epsilon_1) & (1-\pi)\epsilon_1 \\ \pi\epsilon_2 & 1 - \pi + \pi(1-\epsilon_2) \end{array}\right]. \end{array}$$

If we let p_n denote the probability of choosing response 1 on trial n, then the transition equation is easily seen to be

$$p_{n+1} = [\pi + (1-\pi)(1-\epsilon_1)]p_n + \pi\epsilon_2(1-p_n).$$

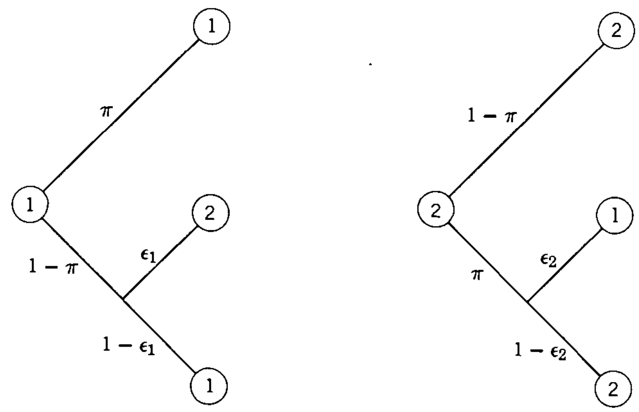


Fig. 6. One-step transition trees of the one-element stimulus-sampling model.

At asymptote, $E(p_{n+1}) = E(p_n) = p_\infty$. So taking expectations and the limit as $n \to \infty$, we find

$$p_\infty = \frac{\pi}{\pi + (1 - \pi)\epsilon_1/\epsilon_2}. \tag{38}$$

It is clear that if $\epsilon_1 = \epsilon_2$, which may hold when $x_{11} = x_{22}$, then $p_\infty = \pi$. This is the well-publicized prediction of probability matching. More generally, we have $p_\infty \gtreqless \pi$ if and only if $\epsilon_2 \gtreqless \epsilon_1$. Thus the model predicts that the response probability either overshoots π for all values of π, matches it, or undershoots it for all values of π. (Several variants of this model are also considered in Suppes and Atkinson, 1960, Chapter 11.)

OBSERVING-RESPONSE MODEL. A theoretical approach that also originates in the tradition of stimulus-sampling theory but that adds a new concept is the one that postulates an implicit response by the organism prior to the observed response. This implicit response has been referred to as an observing, orienting, or approach-avoidance response in the literature. The basic ideas are presented in Audley (1960), Bower (1959), and Estes (1960). A simplified version of the Estes model is formulated and tested in Suppes and Atkinson (1960, Chapter 11) and Atkinson (1961); their version is essentially defined by the following five assumptions, further simplified here to the case of two responses.

1. On every trial each response has an approach value of 1 or 0.
2. At the start of each trial one of the two responses is randomly observed.
3. If the approach value of the observed response is 1, that response is made. If the approach value is 0, the other response is observed.

conditions, namely,

1. Weak ordering. *R is a* weak ordering of $A_1 \times A_2$.

2. Cancellation. *For all $a, b, f \in A_1$ and $p, x, y \in A_2$, if $(a, p)R(f, y)$ and $(f, x)R(b, p)$, then $(a, x)R(b, y)$.*

The second axiom can be cast in an apparently simpler form by defining a new relation D on $A_1 \times A_2$ which corresponds to a comparison of differences instead of sums of components (see Sec. 2.4), namely,

$$(a, x)D(b, y) \quad \textit{if and only if} \quad (a, y)R(b, x).$$

Then the cancellation axiom is simply the assertion that D is transitive.

The third axiom is not open in that it postulates that there are adequate elements in A_1 and A_2 so that certain equations can always be solved.

3. Solution of equations. *For any $a, b \in A_1$ and $x, y \in A_2$, there exist $f \in A_1$ and $p \in A_2$ such that*

$$(a, x)I(f, y) \quad \text{and} \quad (a, x)I(b, p).$$

From these three assumptions it is not difficult to show that the condition of independence, Eq. 3, holds, and so the following induced orderings R_1 on A_1 and R_2 on A_2 are well defined:

$$aR_1b \quad \text{if and only if,} \quad \text{for} \quad x \in A_2, \qquad (a, x)R(b, x),$$

$$xR_2y \quad \text{if and only if,} \quad \text{for} \quad a \in A_1, \qquad (a, x)R(a, y).$$

With these definitions, we may introduce the main constructive device used by Luce and Tukey in their proof and in terms of which they introduced their final axiom. A nontrivial *dual standard sequence* (dss) is any set of elements of the form $\{(a_i, x_i) \mid i$ any positive or negative integer or $0\}$ for which

(i) if $i \neq j$, then not $a_i I_1 a_j$ and not $x_i I_2 x_j$;
(ii) for all i, $(a_i\, x_i)I(a_{i-1}, x_{i-1})$;
(iii) for all i, $(a_{i+1}, x_i)I(a_i, x_{i+1})$.

The final axiom is

4. Archimedean. *For any nontrivial dss $\{(a_i, x_i)\}$ and for any $(b, y) \in A_1 \times A_2$, there exist integers m and n such that*

$$(a_n, x_n)R(b, y)R(a_m, x_m).$$

With these four axioms, the following result can be proved.

Theorem 5. *If $\mathscr{A} = \langle A_1 \times A_2, R\rangle$ satisfies Axioms* 1 *to* 4, *then there are real-valued functions u on $A_1 \times A_2$, u_1 on A_1, and u_2 on A_2 such that for all $a, b \in A_1$ and $x, y \in A_2$,*

(i) $(a, x)R(b, y)$ *if and only if* $u(a, x) \geq u(b, y)$;
(ii) aR_1b *if and only if* $u_1(a) \geq u_1(b)$;
(iii) xR_2y *if and only if* $u_2(x) \geq u_2(y)$;
(iv) $u(a, x) = u_1(a) + u_2(x)$.

Moreover, any other functions u', u_1', and u_2' satisfying (i) to (iv) *are related to u, u_1, and u_2 by linear transformations of the following form:*

$$\begin{aligned} u_1' &= \alpha u_1 + \beta, \\ u_2' &= \alpha u_2 + \gamma, \\ u' &= \alpha u + \beta + \gamma, \end{aligned}$$

where $\alpha > 0$.

The utility representation of a dss is particularly simple in that successive elements in the dss are equally spaced in utility, that is, if $\{(a_i, x_i)\}$ is a nontrivial dss, then for all integers n

$$\begin{aligned} u(a_n, x_n) - u(a_0, x_0) &= n[u(a_1, x_1) - u(a_0, x_0)], \\ u_1(a_n) - u_1(a_0) &= n[u_1(a_1) - u_1(a_0)], \\ u_2(x_n) - u_2(x_0) &= n[u_2(x_1) - u_2(x_0)]. \end{aligned}$$

Much the same representation was arrived at by Debreu (1960a) except that some of his assumptions are topological (see Sec. 2.2) rather than entirely algebraic. Specifically, in the two-component case he assumed that:

(1) *R is a weak ordering of $A_1 \times A_2$.*

(2) *A_1 and A_2 are both connected and separable topological spaces.* (The notion of a separable space was given in Def. 5; a topological space is *connected* if it cannot be partitioned into two nonempty, disjoint, open subsets.)

(3) *All sets of the form*

$$\{(a, x) \in A_1 \times A_2 \mid (a, x)P(b, y)\}$$

and

$$\{(a, x) \in A_1 \times A_2 \mid (b, y)P(a, x)\}$$

are open for all $(b, y) \in A_1 \times A_2$.

(4) *R is an independent relation in the sense of Eq.* 3; *and*

(5) *The equivalence relations I_1 on A_1 and I_2 on A_2 defined in terms of I with, respectively, the second and the first component fixed,* (by 4, I_1 and I_2 are unique, independent of the choice of the fixed element) *have at least two equivalence classes.*

where

$$K = \pi(b_{11} - b_{12}) + (1 - \pi)(b_{21} - b_{22})$$

and

$$p_\infty = \frac{b_{12}\pi + b_{22}(1 - \pi)}{K}. \tag{42}$$

It has been shown that if $K \neq 0$ and if the limit of the expected value of v_n is not zero or infinity, then the asymptotic expected mean is given by Eq. 42 (Luce, 1959, p. 116). The condition $K \neq 0$ is seen to be violated when the events are experimenter-controlled, that is, when $b_{11} = b_{12}$ and $b_{21} = b_{22}$. The general situation is rather complicated. For example, if $\pi b_{12} + (1 - \pi)b_{22} < 0$ and $\pi b_{11} + (1 - \pi)b_{21} > 0$, then $p_\infty = 1$.

There is some formal resemblance between the asymptotic mean of the beta model, Eq. 42, and of the experimenter-controlled linear model, Eq. 41 (and so of the one-element stimulus-sampling model, Eq. 38). The difference is that the beta model includes a $1 - \pi$ term in the numerator, whereas the others do not. Of course, it is not really fair to compare these two predictions, because the linear model has been worked out only for experimenter-controlled events and so it is less general.

A THRESHOLD MODEL. In Secs. 2.3 and 5.2 of Chapter 3, Vol. I, a threshold model was described for experiments with two presentations and two responses. The main idea is that as a result of the presentation the subject enters into one of two "detection" states, the probability depending on the stimulus presented. The response made is assumed to be governed by the state that the subject is in and by a response bias that he controls.

In the present experiment the presentations are the same on all trials, that is, there are no discriminative stimuli to indicate to the subject what is going to happen. Nevertheless, it is conceivable that some subjects behave as if they think that there are cues in the situation that can be used to tell which event is going to occur. If so, their behavior might be described by the threshold model with the added restriction that the two conditional probabilities of going into the detect state are, in fact, identical. If we let q denote this common probability, then a lower limb bias yields,

$$p_n = t_n q,$$

and an upper limb bias yields

$$p_n = q + u_n(1 - q).$$

Presumably, the biases change as a result of experience, and so they can be described by a learning process. Assuming the linear process of Sec. 5.2 of Chapter 3, we find that

$$t_\infty = u_\infty = \frac{\pi}{\pi + (1 - \pi)\theta_2/\theta_1},$$

where θ_1 and θ_2 are learning rate parameters. If we assume that there is a value π_0 such that the subject uses a lower limb bias for $\pi < \pi_0$ and an upper limb bias otherwise, we have for the asymptotic mean probability

$$p_\infty = \begin{cases} \dfrac{\pi q}{\pi + (1 - \pi)\theta_2/\theta_1}, & \text{if } \pi < \pi_0 \\ q + \dfrac{\pi(1 - q)}{\pi + (1 - \pi)\theta_2/\theta_1}, & \text{if } \pi > \pi_0. \end{cases} \tag{43}$$

Note that when the learning rates are equal, $\theta_1 = \theta_2$, then both parts are linear in π; the lower limb starts at (0, 0) and has slope q, the upper limb ends at (1, 1) and has slope $1 - q$, and a discontinuity separates them at $\pi = \pi_0$. Of course, it is likely that π_0 must be treated as a random variable even for a single subject, and certainly it must be for a population of subjects. Thus in practice we cannot expect to see a discontinuous function, but rather an average of several. The difference is sketched in Fig. 7.

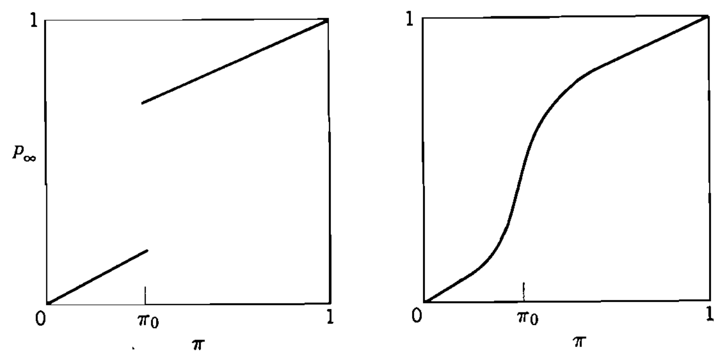


Fig. 7. Threshold model prediction of p_∞ versus π with π_0 fixed and random.

STRICT EXPECTED UTILITY MODEL. The general strict expected utility model was stated in Sec. 7.1; we simply specialize it to the present experiment. Let $v_{ij} = v(x_{ij})$; then

$$p_\infty = \frac{v_{11}\pi + v_{21}(1 - \pi)}{(v_{11} + v_{12})\pi + (v_{21} + v_{22})(1 - \pi)}. \tag{44}$$

Superficially, this prediction seems to be the same as that of the beta model, Eq. 42; however, there is an important difference. All the v's in this model are positive, whereas some of the b's in Eq. 42 may be negative. Indeed, if we assume that x_{11} is preferred to x_{12} and x_{22} to x_{21}, then it is reasonable to suppose that $\beta_{11}, \beta_{12} > 1 > \beta_{21}, \beta_{22}$. Thus, in particular, $b_{22} = \log \beta_{22} < 0$. Yet if we try to equate the coefficients of π and

$1 - \pi$ between Eqs. 42 and 44, we find that $v_{21} = b_{22} < 0$, which is impossible in the strict utility model.

Note that if we demand $p_\infty = 0$ when $\pi = 0$ and $p_\infty = 1$ when $\pi = 1$, then $v_{12} = v_{21} = 0$ and Eq. 44 becomes

$$p_\infty = \frac{\pi}{\pi + (1 - \pi)v_{22}/v_{11}},$$

which is exactly the same form as Eq. 41 of the experimenter-controlled linear model. If v_{12} and $v_{21} \neq 0$, it is easy to see that p_∞ is defined for all π, $0 \leq \pi \leq 1$, and that for $\pi = 0, p_\infty > 0$ and for $\pi = 1, p_\infty < 1$.

DECOMPOSITION AND CHOICE MODEL. Luce (1959, pp. 86–88) showed that if the assumptions of Theorem 57 of the previous Section are satisfied (namely, the decomposition assumption and the choice axiom for both the p's and q's), if for $x, y \in A, p(x, y) = 0, \frac{1}{2}$, or 1, and if in particular

$$p(x_{11}, x_{21}) = p(x_{12}, x_{22}) = 1,$$

then p must be a monotonic increasing step function of π. We do not reproduce the proof here. It should be noted that no results are known about the location or breadth of the steps; the theorem merely says that the function must be a step function.

UTILITY OF VARIABILITY MODEL. Siegel (1959, 1961) suggested an asymptotic utility model that, in addition to considering the utilities of the outcomes, incorporates a term to represent the value to the subject of sheer variation in his responses. Siegel was led to consider this modification of the usual algebraic expected utility models because of their clear experimental inadequacy. He first worked it out for the two-response case, such as the probability prediction experiments in which the subject attempts to predict which of two mutually exclusive events will occur, and later Siegel and McMichael (1960) generalized it to n responses.

Siegel supposed that utility arises from three sources: u_{11} is contributed to the total whenever the event that has probability π of occurring is correctly predicted, u_{22} is contributed whenever the complementary event is correctly predicted, and u is a contribution due entirely to variability in the subject's responses, where it is assumed that variability is measured by $p(1 - p)$, p being the probability of response 1. Thus the expected utility is

$$E(u) = u_{11}\pi p + u_{22}(1 - \pi)(1 - p) + up(1 - p).$$

To find the probability p_∞ that maximizes expected utility, we simply calculate the derivative of $E(u)$ with respect to p and set it equal to 0. This yields

$$p_\infty = \frac{\pi(a_1 + a_2) + (1 - a_2)}{2}, \tag{45}$$

where

$$a_1 = \frac{u_{11}}{u} \quad \text{and} \quad a_2 = \frac{u_{22}}{u}.$$

It is not difficult to see that Eq. 45 is a special case of Eq. 42 that arose from the beta model. Specifically, if we choose the b's so that $b_{11} - b_{12} = b_{21} - b_{22}$, then Eq. 42 reduces to

$$p_\infty = \frac{\pi(b_{12} - b_{22}) + b_{22}}{b_{21} - b_{22}},$$

which is the same as Eq. 45 with

$$a_1 = \frac{2b_{12}}{b_{21} - b_{22}} - 1 \quad \text{and} \quad a_2 = 1 - \frac{2b_{22}}{b_{21} - b_{22}}.$$

Therefore any data supporting Siegel's asymptotic formula can equally well be interpreted as supporting the beta model formula.

RELATIVE EXPECTED LOSS MINIMIZATION (RELM) RULE. In discussing his data (see Sec. 8.3), Edwards (1956) introduced the following somewhat *ad hoc* proposal for the relation between p_∞ and π. He began with Savage's idea (see Sec. 3.4) that decision rules for uncertain situations should be based not upon the payoff matrix, but upon the matrix of the differences between what the outcome could have been had the subject known which event would occur and what was actually received. This is called the *loss matrix*. (This is the same as Savage's notion of a regret matrix, Sec. 3.4.) Assuming numerical payoffs, then the loss matrix in our simple 2 by 2 situation is

$$\begin{array}{cc} & \begin{array}{cc} \quad 1 & \qquad 2 \end{array} \\ \begin{array}{c} \pi \\ 1-\pi \end{array} & \begin{bmatrix} 0 & x_{11} - x_{12} \\ x_{22} - x_{21} & 0 \end{bmatrix} \end{array}$$

The expected loss for each response is therefore

$$EL_1 = \pi 0 + (1 - \pi)(x_{22} - x_{21})$$
$$EL_2 = \pi(x_{11} - x_{12}) + (1 - \pi)0.$$

The relative expected loss is

$$\frac{EL_2 - EL_1}{(EL_1 + EL_2)/2} = 2\,\frac{\pi(x_{11} - x_{12}) - (1 - \pi)(x_{22} - x_{21})}{\pi(x_{11} - x_{12}) + (1 - \pi)(x_{22} - x_{21})}.$$

Edwards' RELM rule is that the choice probability is a linear function of the relative expected loss, that is, there is a constant K such that

$$p = \tfrac{1}{2} + K\left(\frac{EL_2 - EL_1}{EL_1 + EL_2}\right).$$

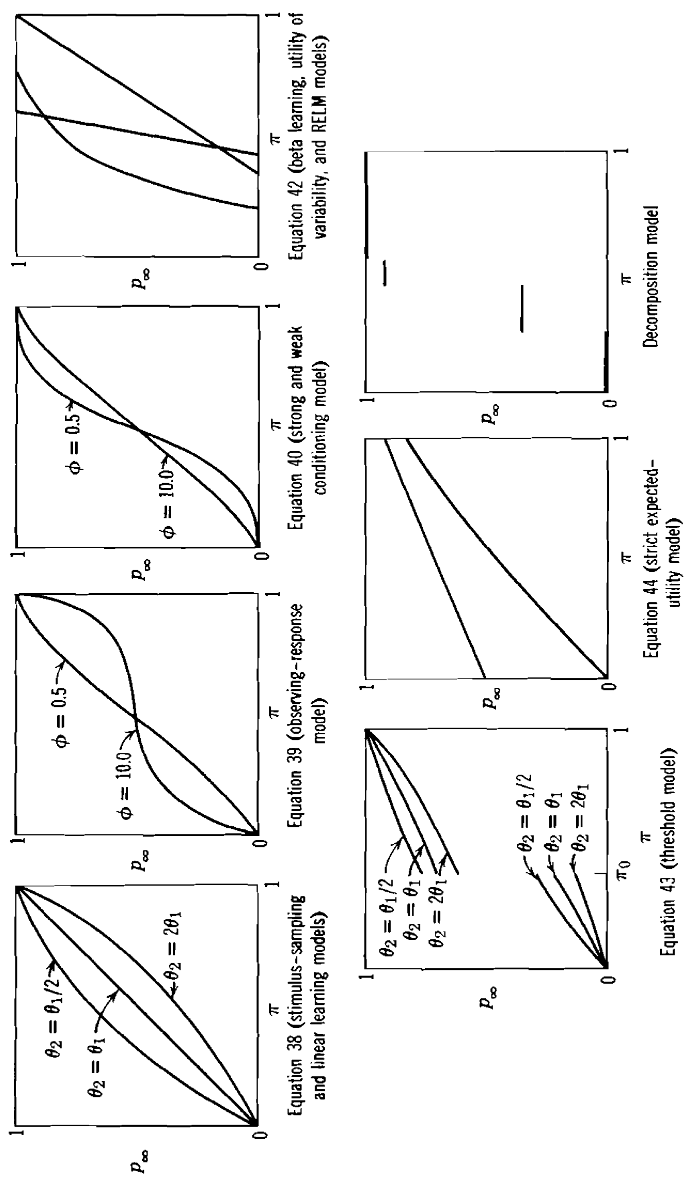


Fig. 8. Typical predictions of p_∞ versus π from ten different models.

It is not difficult to see that this is a special case of the beta model prediction. If the b's are chosen so that the following four equations hold, then the predictions are identical:

$$3(b_{22} + b_{12}) = b_{11} + b_{21}$$
$$b_{11} - b_{12} = x_{11} - x_{12}$$
$$b_{21} - b_{22} = x_{22} - x_{21}$$
$$\frac{b_{12}}{b_{11} - b_{12}} = K + \tfrac{1}{2}.$$

SUMMARY. Possibly the simplest way to see what these ten models predict is to plot specific examples of each equation. This we have done in Fig. 8. There are only seven plots because one pair and one triple of models do not lead to different predictions. The final seven are really quite different, and so we can hope to decide experimentally among them. The existing data are described in Sec. 8.3.

8. EXPERIMENTAL TESTS OF PROBABILISTIC MODELS

8.1 Some Experimental Issues

The experiments intended to test probabilistic theories of preference are neither numerous, extensive, nor particularly conclusive. It is not certain that we know yet just which experiments to perform or how best to implement them, and it is quite certain that we do not know how to analyze them. It may be well to amplify somewhat these summary observations before examining specific studies.

Consider, first, the design of an experiment that is intended to test a probabilistic theory. Because these theories presume asymptotic behavior, learning theorists argue that the same set of outcomes should be presented for as many trials as are needed to demonstrate empirically that the behavior is "asymptotic"—judging by Edwards' (1961a) data, something of the order of 500 trials are needed to stabilize human responses under his experimental conditions. When comparing a subject's response probabilities in different but related choice situations, equally long runs must be carried out for each different presentation set.

Aside from impracticality when one wishes to estimate response probabilities for very many different presentation sets, this approach has been criticized because it permits serious memory effects to intrude, and these are not incorporated into the theories. These critics argue that many presentation sets should be interleaved in some more or less random fashion, only later to be regrouped when estimating the response probabilities. Some have gone so far as to say that each presentation set should occur only once, but this makes it most difficult to test adequately any probabilistic theory: single observations do not lead to refined estimates of nonzero and one probabilities.

The primary objection to interleaving a number of different presentations is that there may exist appreciable, but not easily observed, sequential effects among the responses: if the general view embodied in the various mathematical theories of learning is approximately correct, we can hardly hope for asymptotic behavior when we continually alter the choice situation. We know of no demonstration of this a priori fear nor of any answer to it except the fond counterhope that people actually are better able to uncouple their responses to different presentations than current theories of learning lead us to believe.

Another difference of opinion exists concerning how much a subject should know about the chance events that are used to generate the uncertain outcomes. In experiments arising from the learning tradition, especially the probability prediction experiments, the subject is told little or nothing about the mechanism used to generate the conditional outcome schedules; indeed, his task is to discover the probability structure over the outcomes. It is in this context that hundreds of trials are needed before the behavior seems to stabilize. If, however, one is interested only in asymptotic behavior rather than in the transients of learning, it hardly seems efficient to force the subject to discover by his own crude, empirical means much of what can be conveyed accurately to him in words. So, other experimenters have gone to the other extreme of telling subjects everything they can about the generation of the outcome schedule except what will actually occur trial by trial. In some experiments the mechanism for generating the events, for example, dice, is part of the visible apparatus and the subject actually sees each chance event run off; in addition, he is sometimes given complete information about the probabilities of the several relevant events.

In general, experimenters either have coupled long experimental runs of a single presentation set with no a priori information about the conditional outcome schedule or they have coupled interleaved presentation sets with maximal a priori information about the generation of the schedule. The other two combinations do not seem to have been much used (an exception is Suppes and Atkinson, 1960), even though there are good reasons for trying a single presentation set with complete information: asymptotic behavior might occur quickly and the problem of interaction between different presentation sets would not exist.

In addition to these two broad issues of experimental design, numerous minor differences exist in the way studies are run, with as yet little or no agreement about how they might best be done. Thus, existing preference experiments are far less comparable to one another than are, say, many experiments in psychophysics or in parts of animal learning.

However we may decide to run such an experiment, once it is completed we face nasty statistical problems that have hardly begun to be formulated, let alone solved. Suppose that we are willing to assume that there is no variability other than binomial in our estimates of the response probabilities and that we wish to test whether the hypothesis of strong stochastic transitivity is met, that is, whenever $p(x, y) \geq \frac{1}{2}$ and $p(y, z) \geq \frac{1}{2}$, then $p(x, z) \geq \max [p(x, y), p(y, z)]$. It is clear that when we simply substitute estimates $\hat{p}$ for p in this or any other similar condition, we are going to make three kinds of errors: it may be that $\hat{p}(x, y) \geq \frac{1}{2}$ when, in fact, $p(x, y) < \frac{1}{2}$, causing us to check the condition when its hypotheses are not really satisfied; it may be that $\hat{p}(x, z) < \hat{p}(x, y)$ when, in fact, $p(x, z) \geq p(x, y)$, causing us to reject the condition when it is true; and it may be that $\hat{p}(x, z) \geq \hat{p}(x, y)$ when, in fact, $p(x, z) < p(x, y)$, causing us to accept it when it is false. Thus, we have the usual problem of trying to avoid both types of errors. Nothing much seems to be known about testing this hypothesis except under rather implausible assumptions (see Sec. 8.2); and no work has been done in those apparently simpler cases when a theory prescribes an explicit dependence of $p(x, z)$ upon $p(x, y)$ and $p(y, z)$, such as the product rule

$$p(x, z) = \frac{p(x, y)p(y, z)}{p(x, y)p(y, z) + p(z, y)p(y, x)} \tag{46}$$

which derives from the strict utility model.

Lacking satisfactory statistical methods, authors often simply report, for example, the percentage of violations of strong transitivity and, on some intuitive basis, they conclude whether the failures are sufficiently numerous to reject the hypothesis. In other situations, tables or plots of data are reported and the reader is left pretty much on his own to reach a conclusion. Because the results are almost never clear cut, one is left with a distinct feeling of inconclusiveness.

Let us turn now from the broad issues of experimentation in this area to the studies that have actually been performed. We have organized the experiments according to the theoretical proposition under test. In Sec. 8.2 we report those studies bearing on the stochastic transitivity properties, which are the only binary properties that experimenters have so far explored. In Sec. 8.3 we present data designed to get at the plot of p_∞ versus π in a two-response situation. Finally, in Sec. 8.4 are reported three studies that are concerned with probabilistic expected utility models. As far as we are aware, except for two minor results that are mentioned in footnotes, these are the only theoretical ideas that have been studied empirically.

8.2 Stochastic Transitivity

Of the various ideas about probabilistic choices, the most thoroughly examined are the three stochastic transitivity properties. The experimental results are none too consistent, except that weak transitivity seems to have stood up whenever it was examined; in several papers nothing was said about it, but we suspect that this means that it was satisfied. Edwards (1961b, p. 483) makes this comment.

> No experiment yet reported has created conditions deliberately designed to be unfavorable to transitivity, strong or weak, and ended up accepting even weak stochastic transitivity.[13] In short, as a basis for psychological theorizing, algebraic transitivity is dead, and stochastic transitivity, strong or weak, has yet to be exposed to the adverse climate of hostile experiments. It seems likely that conditions can be designed in which subjects choose intransitively most of the time (unpublished research so indicates); it is even possible that the direction of the intransitive cycles can be controlled by experimental manipulation.

We group the following experiments into three categories according to the number of observations that were used to estimate the response probabilities.

ONE OBSERVATION. Davidson and Marschak (1959) performed the first study in which each choice was made just once. A typical presentation of the experiment was

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} ZOJ \\ ZEJ \end{array} & \begin{bmatrix} -5¢ & +36¢ \\ -21¢ & -38¢ \end{bmatrix}, \end{array}$$

where each nonsense syllable refers to the labels on three faces of a die. Three dice with different pairs of syllables were used to generate the chance events. A total of 319 different choices were presented to each subject, of which 15 served as pretraining and to test whether or not the events had subjective probability of $\frac{1}{2}$ (a point that does not concern us here). The payoffs were actually carried out in 107 randomly selected cases.

[13] The wording of this sentence suggests that experiments have been performed that provide grounds for rejecting weak transitivity; we are not aware of which they are and Edwards does not give any references.

Each of the 17 student subjects, 11 men and 6 women, were run individually in three sessions of 35 to 55 minutes each that were spaced from one to five days apart.

There were 76 triples of presentations for which any of the transitivity conditions could be checked. (In addition, transitivity of "intervals" was examined, but we will not go into that.) Consider any triple of presentations (x, y), (y, z), and (z, x) and one response to each. Of the eight possible observations, six satisfy strict algebraic transitivity and two do not. The problem is to use the observations from a number of triples to infer whether or not it is reasonable to suppose that the underlying probability vectors $\langle p(x, y), p(y, z), p(z, x)\rangle$ satisfy one or another of the stochastic transitivity conditions. It is clear that any such vector must lie in the unit cube U, that those vectors satisfying weak transitivity lie in some subset W of U, and that those satisfying strong transitivity lie in another subset S, where, of course, $S \subset W \subset U$. We consider three hypotheses:

H_0: the distribution of all vectors is uniform over U,
H_w: the distribution of all vectors is uniform over W,
H_s: the distribution of all vectors is uniform over S.

Davidson and Marschak calculated the probabilities of an intransitive observation on the assumption that each hypothesis is true; the numbers are the following:

Hypothesis	Probability of an Intransitive Observation
H_0	0.2500
H_w	0.1875
H_s	0.1375

The decision rule that they investigated places the error probability of the hypothesis under test, H_w or H_s, equal to the error probability of the alternative hypothesis H_0. Thus, for example, H_w is accepted if the number r of intransitive cycles $\leq c$, where c is the number such that

$$Pr(r < c \mid H_0 \quad \text{is true}) = Pr(r \geq c \mid H_w \quad \text{is true}),$$

and this common probability is the significance level. With a sample size of 76, one accepts H_w if $r \leq 17$ and the significance level is 0.24, and one accepts H_s if $r \leq 15$ and the significance level is 0.10.

A second way to evaluate the results is to look at the likelihood ratio

$$L_w(r, n) = \frac{P_w(r, n)}{P_0(r, n)},$$

where $P_w(r, n)$ is the probability that, when H_w is true, exactly r intransitive observations will occur if n observations are made, and $P_0(r, n)$ is the same thing when H_0 is true. A similar quantity can be calculated for hypothesis H_s.

The data and the two likelihood ratios for the 17 subjects are shown in Table 4. It is seen that H_w is acceptable according to the decision criterion for all subjects and that H_s is rejected for two, J and N.

Table 4 Number of Intransitive Observations in 76 Triples and the Likelihood Ratios for H_w and H_s against H_0 as Reported by Davidson and Marschak (1959)

Subject	Number of Intransitive Observations	Likelihood Ratios: H_w against H_0	Likelihood Ratios: H_s against H_0
A	4	100	2,100
B	10	11	26
C	11	7.6	12
D	11	7.6	12
E	1	300	20,000
F	9	16	54
G	5	69	1,000
H	4	100	2,100
I	4	100	2,100
J	16	1.2	0.3
K	8	23	110
L	2	210	9,300
M	5	69	1,000
N	16	1.2	0.3
O	7	33	240
P	7	33	240
Q	14	2.5	1.3

It is noteworthy that the number of intransitive observations dropped over the three sessions: the group percentages were 13.9, 10.9, and 6.8, with an over-all average of 10.4%.

Davidson and Marschak concluded that the experiment provides no evidence for rejecting weak transitivity and that in only two cases is strong transitivity in serious doubt. The difficulty with this conclusion, as they recognized, is that the experimental design was such that these tests are not very powerful. As Morrison (1963) has pointed out, if each pair of n stimuli is presented, the maximum proportion of intransitive triples is

$(n + 1)/4(n - 2)$, which for large n approaches $\frac{1}{4}$. Thus, unless the subject is trying very hard to be intransitive—is *diabolic* in Morrison's terms—we have little hope of rejecting either null hypothesis, H_w or H_s, whether or not weak and strong stochastic transitivity are false. Moreover, the formulation of the hypotheses in terms of uniform distributions over the relevant regions of the unit cube is most questionable; however, without a far more complete theory of behavior, it is not clear what distribution we should assume.

Table 5 Scale Values Obtained by Applying Thurstone and Linear Programming Models to the Pair Comparisons of 282 Men and Women Expressing Preference for Ten Categories of Newspaper Content (Sanders, 1961, p. 27)

	Thurstone Model		Linear Programming Model	
Content Category	Scale Value	Interval	Scale Value	Interval
Government News	1000		1000	
		242		276
Medical and Health News	758		724	
		43		69
Feature Columns	715		655	
		22		69
Editorial Comment	693		586	
		121		69
Economic News	572		517	
		20		69
Science News	552		448	
		241		138
Crime and Accidents	311		310	
		1		69
Sports	310		241	
		135		103
Personal and Social News	175		138	
		175		138
Comics	0		0	

For further discussion of the statistical problems when only one observation is made per presentation, see Block and Marshak (1960) and Morrison (1963).

A large-scale study of newspaper content preference was made by Sanders (1961) who utilized the Thurstone pair comparison model and the linear programming model of Davidson, Suppes, and Siegel (1957). A sample of 282 men and women were asked to state their preferences for ten categories of newspaper content presented in pair comparisons. The results of applying the two models to the data are shown in Table 5.

Table 6 Number of Circular Triads Observed in 45 Pair Comparison Choices by 282 Men and Women (Sanders, 1961, p. 23)

Number of Circular Triads	Number of Subjects	Number of Circular Triads	Number of Subjects
0	58	11	9
1	35	12	6
2	30	13	8
3	24	14	4
4	20	15	2
5	16	16	2
6	17	17	1
7	14	18	1
8	13	20	1
9	12	21	1
10	7	30	1

To avoid a possible confusion we remark that the linear programming model was applied to the proportions of group preferences, and not to the individual choices, as described in Sec. 4.2. With the scale values transformed to a standard [0, 1000] range it is clear from the table that the two models yield very similar preference scales.

Although Sanders did not analyze his data for violations of weak or strong stochastic transitivity, he did count the number of circular triads observed in the 45 pair comparisons made by each subject. These results are shown in Table 6. If choices were made on a completely random basis, the expected number of circular triads is 30. As Table 6 shows, almost all the 282 subjects produced a far smaller number of circular triads. In fact, the majority produced not more than three such triads.

A FEW OBSERVATIONS. Two studies fall into this small sample range. Papandreou et al. (1957) prepared alternatives of the following type.

A triple $c = (c_1, c_2, c_3)$ consists of imaginary alternatives

$$c_1 = 3 \text{ tickets to } x \text{ and } 1 \text{ to } y,$$
$$c_2 = 2 \text{ tickets to } x \text{ and } 2 \text{ to } y,$$
$$c_3 = 1 \text{ ticket to } x \text{ and } 3 \text{ to } y,$$

where each ticket was valued at \$3.60 and x and y were chosen from ten activities, five of which were "esthetic" such as a play, opera, etc., and five of which were "athletic" such as a baseball game, a tennis match, etc. These outcomes were described in some detail to the subjects. Each of the three binary choices from each triple was offered six times, so that a total of $6 \times 3 \times 45 = 810$ choices were made. The various orders of things were randomized in the usual ways. A total of 18 student subjects participated, their responses being obtained over a period of $2\frac{1}{2}$ weeks.

After these data were collected, the experiment was criticized on the grounds that if a person were to prefer strongly one kind of activity to the other, then he would only consider the preferred one and so the numbers of tickets of that activity would be controlling and would automatically ensure transitivity. This quite reasonable objection led the authors to replicate the study with five new subjects using triples in which each c_i was constructed from two basic commodities, but no commodity appeared in more than one c_i of the triple. Thus each triple involved six basic commodities. Twenty-one triples of this sort were constructed, to which 11 irrelevant ones were added forming a set of 32 that was used. Each subject responded eight times to each pair during a period of from 1 to 2 days.

A statistical model, which we do not reproduce, was proposed in their paper; from it strong stochastic transitivity follows. The hypothesis tested was whether or not the observations were consistent with the model, and therefore with strong transitivity. The test used was the likelihood ratio λ of the null hypothesis that the model is true against the alternative hypothesis that the model is not true, with critical regions defined in terms of the parameter space of the model. The frequency distributions of λ are given in Table 7. The authors point out that the statistical properties of this λ-test are not known.

> However, the whole problem of statistical methodology has not been pushed further because the evidence provided by the data is so overwhelming by any criterion. [p. 7]

The second small sample study is by Chipman (1960b). The events in this experiment were generated by drawings from match boxes that contained a total of 100 heads and stems of broken matches. In one class of events, the only ones we consider here, the subjects knew the composition of stems and heads; in the other, they knew only the composition of a rigged sample of 10 that they saw. Three pairs of payoffs were associated with the events: (25¢, −25¢), (25¢, 0), and (0, −25¢). Because a pair of payoffs could be assigned to heads and stems in either order, the same choice was generated in two ways. With three pairs of payoffs, this means that there were six uncertain outcomes associated with each pair of events. Assuming that the decomposition assumption (Sec. 7.2) is correct, as Chipman did, this yields six independent observations from which to estimate the probabilities.

Table 7 The Frequency Distributions of Likelihood Ratios Reported by Papandreou et al. (1957)

Study	$\lambda = 1$	$\lambda < 1$	Unclassifiable
1	704	101	5
2	89	13	2

A total of ten male students participated in the study.

There are only two triples of uncertain outcomes in this study for which stochastic transitivity can be tested.[14] For one triple, all subjects satisfied weak and moderate transitivity and four appeared not to satisfy strong transitivity. For the other triple, weak and so moderate and strong transitivity were violated by one subject and strong by five others. Chipman did not express an opinion about how these results should be interpreted, but he cautioned the reader about the smallness of the sample size. To get some notion of what we might expect with small samples, we carried out Monte Carlo runs for three sets of three probabilities satisfying strong stochastic transitivity (in fact, satisfying Eq. 46, p. 379 that derives from the strict utility model); the percentages of violations of the several transitivity properties are shown in Table 8.[15]

A MODERATE NUMBER OF OBSERVATIONS. Again, two studies fall into this category. In the first, Coombs (1958, 1959) obtained preference reports on shades of gray from four subjects, two men and two women.

[14] Chipman also looked at several other things, among them a single direct test of the equation

$$p_X(x) = p_Y(x)p_X(Y)$$

that arises from the strict utility model and from the choice axiom when the several probabilities are different from 0 and 1. The test seems inappropriate, however, because one of the probabilities was estimated to be 0 for all ten subjects, in which case the equation is not necessarily expected to hold. It does not.

[15] We wish to thank Richard Willens for carrying out these calculations.

The stimuli were 12 gray chips, ranging from almost white to almost black. The presentation sets consisted of four chips, and the subjects were instructed to rank them according to their preferences—preferences for what purpose was not specified. Each of the 495 different presentation sets was presented twice. Each pair of stimuli occurred in 45 of the sets, and the number of times that one was ranked higher than the other in the 90 opportunities for comparison was accepted as an estimate of the binary choice probability. As we pointed out in Sec. 6.1, this involves a

Table 8 Percentage of Failures of Weak, Moderate, and Strong Stochastic Transitivity Found in Estimates of Probabilities from Monte Carlo Runs Subject to True Probabilities that Satisfy Strong Stochastic Transitivity.

For each of the three sets of true probabilities that satisfy the product rule (Eq. 46, p. 379) estimated probabilities were calculated for sample sizes (n) of 5, 9, and 19. The percentage of failures of the several transitivity conditions were calculated from, respectively, 200, 100, and 50 Monte Carlo runs. The "circular" cases, for which there is an ordering x, y, z such that $\hat{p}(x, y)$, $\hat{p}(y, z)$, and $\hat{p}(z, x)$ are all $> \frac{1}{2}$, can be counted either as a single or triple failure of all three conditions; both tabulations are given.

				Per Cent Failures When Circular Cases Are Counted Once			Per Cent Failures When Circular Cases Are Counted Three Times		
Probabilities	n	N	Per Cent Circular Triples	Weak	Moderate	Strong	Weak	Moderate	Strong
$p(x, y) = 0.60$	5	200	22.5	22.5	32.0	51.4	46.5	53.1	66.6
$p(y, z) = 0.60$	9	100	17.0	17.0	25.0	57.0	38.0	44.1	67.8
$p(x, z) = 0.69$	19	50	6.0	6.0	26.0	58.0	16.1	33.9	62.5
$p(x, y) = 0.60$	5	200	0.0	0.0	2.0	17.0			
$p(y, z) = 0.90$	9	100	0.0	0.0	2.0	26.0			
$p(x, z) = 0.93$	19	50	0.0	0.0	0.0	20.0			
$p(x, y) = 0.80$	5	200	0.5	0.5	3.0	16.0	1.5	4.0	16.8
$p(y, z) = 0.80$	9	100	0.0	0.0	1.0	14.0			
$p(x, z) = 0.94$	19	50	0.0	0.0	2.0	4.0			

strong assumption that need not be true. Aside from this untested assumption and from the question whether it is really meaningful to ask in the abstract for preferences of shades of gray, this study offers the most convincing data about stochastic transitivity in the literature.

The first question is whether weak transitivity is satisfied or, put another way, whether we can order the stimuli for each subject in such a way that the probability that a stimulus with higher rank was preferred to one with lower rank is always at least $\frac{1}{2}$. Surprisingly, this was possible without exception for all four subjects. Thus, since there is no reason to question weak transitivity, we turn to strong transitivity.

The motivation for the remainder of the analysis depends on a particular random utility model suggested by Coombs (1958) as a probabilistic generalization of his unfolding technique (see end of Sec. 2.4). For a

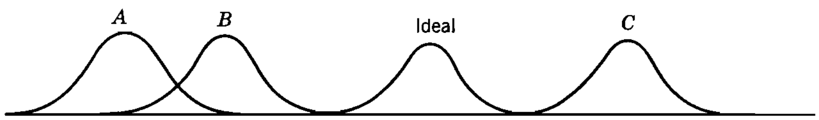


Fig. 9. Distributions of stimuli and an ideal point in Coombs' random utility model.

formal statement of the model, see Coombs, Greenberg, and Zinnes (1961). In words, the main idea is that random utilities can be assigned not only to the stimuli but also to the subject, representing his ideal outcome, and that preferences are determined by the absolute values of the deviations from the subject's ideal. Specifically, if **U** is the random vector over stimuli and **I** the random variable associated with the subject, then $-|\mathbf{U}(x) - \mathbf{I}|$ is the random utility assigned to stimulus x in the usual random utility model (Def. 20, p. 338) and so

$$p_Y(x) = Pr[|\mathbf{U}(x) - \mathbf{I}| \leq |\mathbf{U}(y) - \mathbf{I}|, y \in Y].$$

Let us suppose, for the sake of argument, that each of the several random variables is independently distributed according to some unimodal distribution, just as is usually assumed in the Thurstonian models. Now, consider the preferences between pairs of stimuli that are generated by the model. If both stimuli have distributions that lie on the same side of the distribution of the ideal point, as A and B do in Fig. 9, then the particular value of the ideal is irrelevant in the comparison. Such a pair Coombs calls *unilateral.* If they lie on opposite sides, as A and C do, then the particular value of the ideal random variable makes a great deal of difference in the response made by the subject. Such a pair he calls *bilateral.*

Next, consider triples of stimuli, as we do when testing strong stochastic transitivity. If we think of the continuum as being folded 180° about the mean ideal point, as in Fig. 10, then we can distinguish three possibilities. All three of the stimuli may lie on one side of the ideal, in which case they are said to form a *unilateral triple*. Examples are A, B, and C and D, E, and F in Fig. 10. Or two may lie on one side and one on the other side. If the isolated stimulus falls between the other two on the folded scale, as

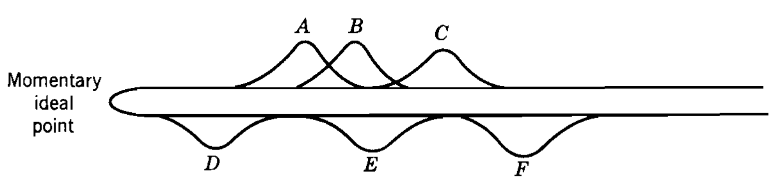


Fig. 10. Random utility distributions when the scale is folded about the momentary ideal point.

E falls between A and C, then the triple is called *bilateral split;* otherwise, when the isolated one either is nearer the ideal than the other two (D, B, C) or is further than the other two from the ideal (A, B, F), then the triple is called *bilateral adjacent*.

Note that the regions of overlap among the three distributions of a unilateral triple are unaffected by the fact that the ideal is a random variable. Thus, making the Case V Thurstone or weaker assumptions (see Theorem 44, p. 348), the general prediction of strong stochastic transitivity holds for unilateral triples. For bilateral triples, variability in the ideal point in effect increases the variability of the distributions when bilateral pairs are compared, but leaves it unchanged when unilateral ones are compared. An increase in the variability of the distributions of a given pair simply drives the corresponding choice probability towards $\frac{1}{2}$. Thus variability in the ideal point enhances the chance of finding strong transitivity satisfied by bilateral split triples and reduces the chance for bilateral adjacent ones. So, according to this model, we expect strong transitivity to be satisfied most often by bilateral split ones.

Table 9 Number of Violations of Strong Stochastic Transitivity Reported by Coombs (1958)

	Type of Triple					
	Bilateral Split		Unilateral		Bilateral Adjacent	
Subject	Cases	Violations	Cases	Violations	Cases	Violations
1	34	0	20	1	66	18
2	24	0	56	2	40	24
3	38	0	20	2	62	29
4	38	0	20	10	62	48
Totals	134	0	116	15	230	119

To test this hypothesis, we must classify each triple into one of the three types. Coombs did this by assuming that the underlying ordering of the stimuli was by brightness, and he then asked whether a mean ideal point could be chosen for each subject so that the folded scale predicted the observed preference ordering. It turned out that this was possible. Once this was done, then each triple was classified and violations of strong stochastic transitivity were counted. The results are shown in Table 9. The confirmation of his prediction is most impressive, and one cannot help but take the model seriously in spite of the questionable assumption underlying the estimates of binary choice probabilities from ranking data.

Perhaps the most important unresolved problems about the Coombsian model are a characterization of those preference situations for which the concept of an ideal is appropriate (it seems to be for this experiment, but it is much more questionable for money outcomes) and derivations of more of its mathematical properties. An interesting theoretical discussion of Coombs' experiment from a somewhat different viewpoint is to be found in Restle (1961, Chapter 4).

The second study having a moderate number of observations per presentation set was reported by Griswold and Luce (1962). Five subjects made choices among 74 different binary uncertain alternatives that were each presented, depending on the subject, from 32 to 50 times in random order. In 34 of the presentations the outcomes were sums of money ranging from 1 cent to 50 cents, and in the remainder they were packages of cigarettes of different brands. The chance events were generated by a simple pinball machine, and the events were run off after each choice were made. No payoffs were made during the course of a session, but at the end of each session one of the money and one of the cigarette pairs were chosen at random and the subject was paid off according to the outcome that had previously been determined.

With the cigarette outcomes, it was found that strong stochastic transitivity appeared to be violated in 18 out of 67 possible cases. Arguments based upon apparent shifts in preference among the cigarette brands were adduced to suggest that 27% violations may not be as unfavorable as it seems. For money outcomes, only cases of pure transitivity—where $p(x, y)$ and $p(y, z)$ are both estimated to be one—are reported; there was only one violation of transitivity in 56 cases, which seems impressive. The difference between the money and cigarette brand outcomes again hints that preferences among well-ordered outcomes may not satisfy the same model as those among more complex outcomes.[16]

8.3 Plots of p_∞ versus π

A fairly large number of two-response studies have been concerned with the dependence of the asymptotic choice probability p_∞ on the outcome probability π. The vast majority of these experiments were performed as tests of learning theory ideas, and as a result they are not as well suited to our purposes as they might first seem. We know of only two experimental studies of this plot that were prompted by preference ideas.

PROBABILITY PREDICTION. The learning studies that we examine have come to be known as probability prediction experiments. They vary

[16] The other main hypothesis that was tested in this study was the decomposition assumption (Sec. 7.2), and the data were interpreted as giving it modest support.

considerably in their details, and a number of them were devoted to issues of interest to learning theorists which, however, are not of immediate concern here. It does not seem appropriate to try to describe each study in detail, especially since most of these details seem to have only second-order effects on the plot of p_∞ versus π, so we simply outline the main features that are common to all designs.

The subject is repeatedly required to predict which of two events, for example, which of two lights, will occur on a trial. These events are programmed according to a simple random schedule[17] with probabilities π and $1 - \pi$; these probabilities are not known to the subjects at the beginning of the experiment. In most cases, one and only one of the two events occurred on a given trial, but in some the two conditional outcome schedules were independent. In these cases, the experimenter had the option of informing the subject what would have happened if he had made the other response. Anywhere from 60 to 1000 trials have been run, with the more recent studies tending toward longer runs. It is generally conceded today that the behavior is not likely to be asymptotic before two or three hundred trials, and possibly not even then.

The data reported in the literature are always averages for groups of subjects, the smallest group size being four and the largest well over 100. Considerable evidence (not much of it published) indicates that these group results can be quite misleading. The distribution of estimated p_∞ over subjects often seems not to be binomial, and sometimes there is little doubt but that it is bimodal, with the valley of the distribution coinciding roughly with the group mean probability.

In 1956, Edwards presented a table of all the asymptotic results that had been reported up to 1955. He gave "... an estimate of the asymptote (subjective, but as unbiased as I can make it) ..." If one compares his estimates with the original data, it appears that when the data curves had not leveled off to their asymptotic values Edwards extrapolated reasonable, smooth curves to determine his estimates of the asymptotes. As this is so highly subjective, we prefer simply to report the observed values averaged over the last few blocks of trials and to note whether or not, in our opinion, these are actually asymptotic values. In all nonasymptotic cases, the curves are increasing for $\pi > \frac{1}{2}$ and decreasing for $\pi < \frac{1}{2}$. Our estimates are given in Tables 10 and 11. The studies have been separated according to whether no explicit payoff matrix was used (Table 10) or whether it was part of the design (Table 11). The payoffs were cents except in two cases where they were points. The reader may work out for

[17] Schedules other than simple random have been used, but we confine our attention to the simple random ones.

Table 10 "Asymptotic" Group Mean Probabilities from Two-Response Probability Prediction Experiments with Simple Random Schedules and No Payoff Matrices

(The number of trials shown is the total run for each condition.)

Experimenter	Size of Group	No. of Trials	π	Est. p_∞	Comments
Grant, Hake, & Hornseth (1951)	37	60	0.00	0.00	
			0.25	0.24*	
			0.50	0.55	
			0.75	0.78*	
			1.00	1.00	
Jarvik (1951)	29	87	0.60	0.57	
	21		0.67	0.67	
	28		0.75	0.76	
Hake & Hyman (1953)	10	240	0.50	0.50	
			0.75	0.77	
Burke, Estes, & Hellyer (1954)	72	120	0.90	0.87	
Estes & Straughan (1954)	16	240	0.30	0.28*	
		120	0.50	0.48	
		120	0.85	0.87*	
Neimark (1956)	20	100	0.66	0.62	When outcome of other choice is unknown, 0.60 and 0.99
		66	1.00	1.00	
Gardner (1957)	24	450	0.60	0.62	
			0.70	0.72	
Engler (1958)	20	120	0.25	0.29*	
			0.75	0.71*	
Cotton & Rechtschaffen (1958)	24	450	0.60	0.64	
			0.70	0.74	
Neimark & Shuford (1959)	36 × 3 runs	100	0.67	0.63*	

Table 10 (*continued*)

Experimenter	Size of Group	No. of Trials	π	Est. p_∞	Comments
Rubinstein (1959)	44	126	0.67	0.65	The three conditions
	41		0.67	0.69*	differed in the
	37		0.67	0.78	representation of the event
Anderson & Whalen (1960)	18	300	0.50	0.52	
			0.65	0.67	
			0.80	0.82	
Suppes & Atkinson (1960)	30	240	0.60	0.59	
Edwards (1961a)	10	1000	0.30	0.11	
			0.40	0.31	
			0.50	0.40	
			0.60	0.69	
			0.70	0.83	
Myers et al. (1963)	20	400	0.60	0.62	
			0.70	0.75	
			0.80	0.87	
Friedman et al. (1964)	80	288	0.80	0.81	The third of three experimental sessions

* Definitely appears not to be asymptotic.

himself the general comparisons between these numerous experimental results and the models discussed in Sec. 7.3.

With the exception of Edwards' (1961a) 1000-trial study, the experiments without payoff matrices appear to confirm the probability matching hypothesis, at least for group averages. When, however, payoffs are included, it is clear from Table 11 that probability matching is not confirmed. This can also be seen in Fig. 11, where we have plotted the data from those studies in which two or more values of π were used and in which the payoff matrix was symmetric in the sense that $x_{11} = x_{22}$ and $x_{12} = x_{21}$. As Edwards pointed out in his 1956 paper, it seems as if the curve is approximately a straight line that crosses the 45° line of the matching prediction somewhere in the neighborhood of $\pi = \frac{1}{2}$ and

Table 11 "Asymptotic" Group Mean Probabilities from Two-Response Probability Prediction Experiments with Simple Random Schedules and Payoff Matrices

Except where noted, the payoffs are in cents. (The number of trials shown is the total run for each condition.)

Experimenter	Size of Group	No. of Trials	Payoff Matrix	π	Est. p_∞	Comments
Goodnow (1955)	10	120	$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.50	0.52	
	14			0.70	0.82*	
	14			0.90	0.96	
Edwards (1956)	24	150	$\begin{pmatrix} 10, & -5 \\ -5, & 10 \end{pmatrix}$	0.30	0.18*	
				0.50	0.48	
				0.60	0.62	
				0.80	0.96*	
Edwards (1956)	6	150	$\begin{pmatrix} 4, & -2 \\ -2, & 4 \end{pmatrix}$	0.50	0.59	When outcome of other choice is unknown, 0.59, 0.86, 0.94.
				0.70	0.85	
				0.80	0.98	
Edwards (1956)	6	150	$\begin{pmatrix} 4, & -2 \\ -2, & 12 \end{pmatrix}$	0.50	0.30	When outcome of other choice is unknown, 0.42, 0.63, 0.84, 0.93*.
				0.70	0.46*	
				0.80	0.80*	
				0.90	0.95*	
Galanter & Smith (1958)	24	100	$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$	0.75	0.72*	Chips
	30	200	–	0.75	0.78	Payoffs were known step-functions of the number of correct responses.
				0.75	0.90	
Nicks (1959)	72	380	$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$	0.50	0.58	Points
	144			0.67	0.71*	
	72			0.75	0.79*	
Siegel & Goldstein (1959)	4	300	–	0.75	0.75	
			$\begin{pmatrix} 5, & 0 \\ 0, & 5 \end{pmatrix}$	0.75	0.86	
			$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.75	0.95	
Suppes & Atkinson (1960)	24	60	$\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$	0.60	0.63*	Six pairs of alternatives were interleaved for a total of 360 trials.
				0.80	0.73*	
	30	240	$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.60	0.64	
	30	240	$\begin{pmatrix} 10, & -10 \\ -10, & 10 \end{pmatrix}$	0.60	0.69	
Siegel & Abelson (in Siegel, 1961)	20	300	$\begin{pmatrix} 5, & -5 \\ -5, & 5 \end{pmatrix}$	0.65	0.75	
				0.75	0.93	

Table 11 *(continued)*

Experimenter	Size of Group	No. of Trials	Payoff Matrix	π	Est. p_∞	Comments
Myers et al. (1963)	20	400	$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.60	0.65	
			$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.70	0.87	
			$\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}$	0.80	0.93	
			$\begin{pmatrix} 10, & -10 \\ -10, & 10 \end{pmatrix}$	0.60	0.71	
			$\begin{pmatrix} 10, & -10 \\ -10, & 10 \end{pmatrix}$	0.70	0.87	
			$\begin{pmatrix} 10, & -10 \\ -10, & 10 \end{pmatrix}$	0.80	0.95	

* Definitely appears not to be asymptotic.

intersects the 0 and 1 lines somewhat before π reaches 0 and 1. Whether it really is straight cannot be decided from the meager parametric data now available. It is also clear that both the exact slope of the line and its crossing point are functions of the payoff matrix (see Edwards, 1956; Galanter & Smith, 1958; and Siegel & Goldstein 1959).

One of the Suppes and Atkinson (1960) experiments bears a few words. It was explicitly designed to test the one-element stimulus-sampling model

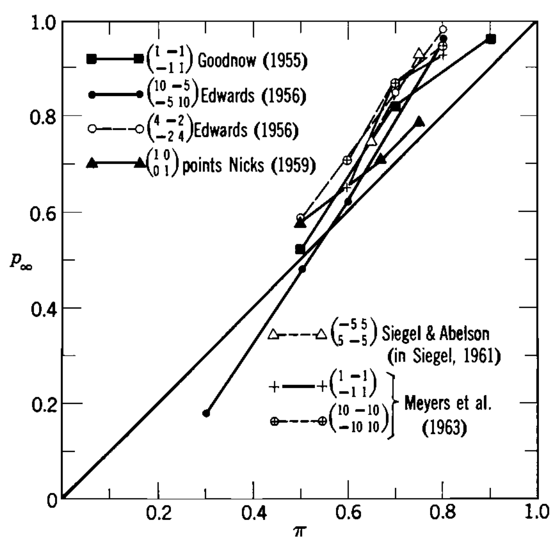


Fig. 11. "Asymptotic" values of p_∞ versus π from seven probability prediction experiments in which symmetric payoff matrices were used. (See Table 11 for more details.)

described in Sec. 7.3. There were four levers, each having a fixed probability of a one cent reward (the probabilities were 0.2, 0.4, 0.6, and 0.8). On each trial, the subject was told which two to select from. Each of the six pairs was presented sixty times, and in the analysis these sets of trials were dealt with separately as if they were independent of the others among which they had been interleaved. Thus, in contrast to the other experiments of this section, this one involved the interleaving of several presentation sets. These data, including some of their sequential features, were analyzed in considerable detail by Suppes and Atkinson in terms of the stimulus-sampling model, and they concluded that the correspondence between theory and data is not good. As far as the asymptotic results are concerned, the mean learning curves suggest that the subjects were not stabilized at the end of 60 trials. On the other hand, the fit of the observing-response model described in Sec. 7.3 to these data is much better.

A PREFERENCE EXPERIMENT. Luce and Shipley (1962) attempted to test the prediction of the decomposition assumption and the choice axiom that, under certain conditions on the payoff matrix, the plot of p_∞ versus π is a monotone increasing step function. They employed a design in which several different presentations were interleaved over trials and in which the subjects had complete information about the mechanism generating the chance events.

On each trial, the subject was presented with a card of the form:

$$\begin{array}{ccc} & & \text{Option} \\ & & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \text{Event} & \begin{array}{c} E \\ \tilde{E} \end{array} & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}. \end{array}$$

After he selected one of the options, the chance event E was run off to determine which outcome he was to receive. The outcomes were points, and the total accumulated during the experiment determined the subject's share of a \$200 bonus. Two different sets of three payoff matrices were used. Within each set, the matrices differed only by a linear transformation. For one set, the entries satisfied the inequalities needed to prove that p_∞ versus π is a step function; for the other set, these inequalities were not satisfied.

The chance events were generated by tumbling five dice in a wire cage, as in the Mosteller and Nogee (1951) experiment. The possible outcomes were ranked into hands, much as in poker, and each subject was given an ordered list of the 252 hands and the probabilities of beating each. Each outcome matrix was coupled with 15 different events that spanned a 0.2 probability range, which was selected on the basis of some preliminary runs. Each of the 90 event-matrix combinations occurred once during successive blocks of 90 trials for a total of 50 presentations each during the experiment. Within each 90-trial block, the order was randomized.

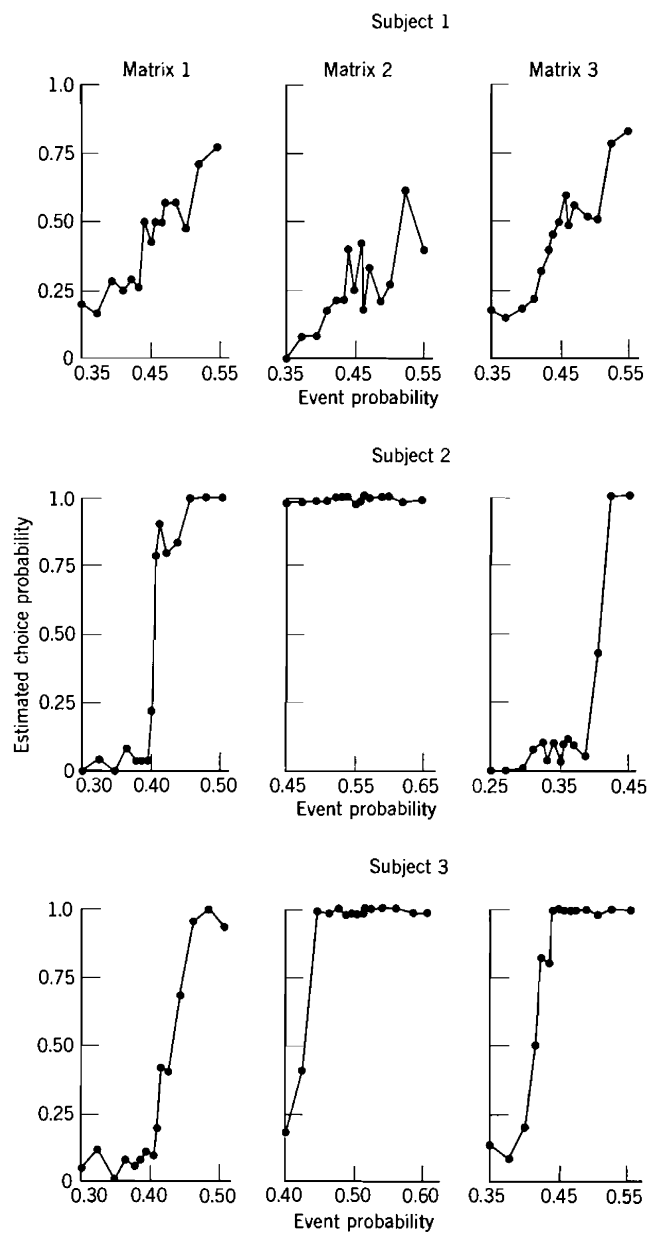
The five student subjects were run as a group by a single experimenter, but they received independent presentations.

For two of the subjects, the plots of $\hat{p}_\infty$ versus π are simple discontinuous functions from 0 to 1. The discontinuity for one subject was at $\pi = 0.4$, the rational breakpoint according to a maximization of expected money criterion, and for the other it was at 0.5. The data for the other three subjects are shown in Fig. 12. Although it is not easy to say just what mathematical function might underly these observations, when we compare them with the seven types of plots that arose in Sec. 7.3 (see Fig. 8, p. 376), they seem most consistent with the step functions of the decomposition-choice model. Luce and Shipley used Monte Carlo techniques to decide whether the observed "plateaus" might have resulted from binomial variability and a continuous ogive, and they concluded that this was most unlikely.

It is evident that these results and those from the probability prediction experiments are not particularly compatible. Among the differences that might contribute to the inconsistency are long runs of the same presentation versus interleaved presentations, averages over groups of subjects versus individual plots, and a possible lack of asymptotic stability in the preference experiment.

8.4 Probabilistic Expected Utility Models

RANDOM VERSUS STRICT EXPECTED UTILITY. It will be recalled that Becker, DeGroot, and Marschak (1963a) showed that if a choice is made from a set consisting of m uncertain alternatives plus the average of these m, then the average alternative is selected with probability 0 when the choice probabilities satisfy the random expected utility model and with probability $1/(m + 1)$ when they satisfy the strict expected utility model (see Theorem 52, p. 361). In a later paper (1963b) these same authors performed an experiment to test this differential prediction. Each choice set consisted of three uncertain alternatives, $a\frac{1}{2}d$, $c\frac{1}{2}b$, and the average of these two (that is, each outcome had probability $\frac{1}{4}$ of occurring), where the pure outcomes were sums of money ordered $a < b < c < d$. A total of 25 such sets were presented to each of 62 subjects, half of whom received no money payments and the other half were paid off in pennies. Since there were no obvious differences in behavior, the results were combined.


Subject 1
Matrix 1
Matrix 2
Matrix 3
1.0
0.75
0.50
0.25
0
0.35 0.45 0.55
0.35 0.45 0.55
0.35 0.45 0.55
Event probability
Subject 2
Estimated choice probability
1.0
0.75
0.50
0.25
0
0.30 0.40 0.50
0.45 0.55 0.65
0.25 0.35 0.45
Event probability
Subject 3
1.0
0.75
0.50
0.25
0
0.30 0.40 0.50
0.40 0.50 0.60
0.35 0.45 0.55
Event probability

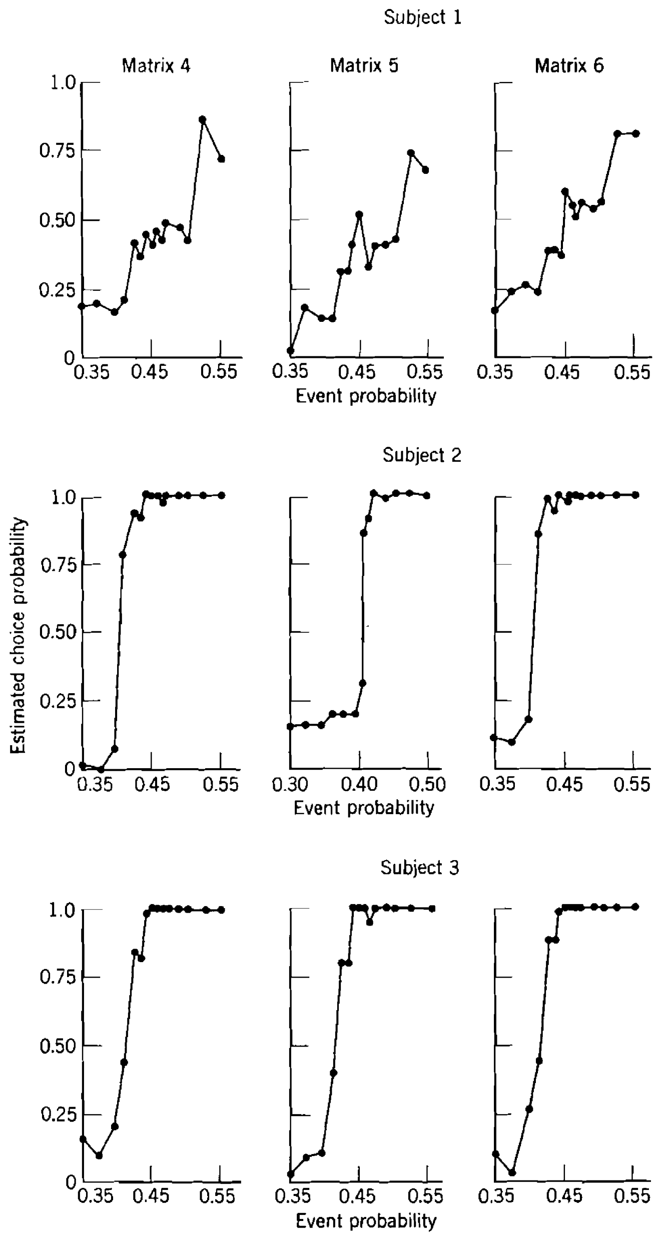


Fig. 12. "Asymptotic" values of p_∞ versus π for three subjects each with six different payoff matrices in a gambling experiment. Adapted by permission from Luce & Shipley (1962, p. 45).

There is little doubt that the random expected utility model is wrong: 60 of the 62 subjects chose the average alternative at least once. Moreover, the data also reject the strict expected utility model, at least for some subjects. That model predicts a binomial distribution with $p = \frac{1}{3}$. Table 12 compares the observed frequency distribution of the 62 subjects with the predicted one (using the normal approximation to the binomial distribution). It is evident that the discrepancy is larger than can be expected by chance. Becker et al. argue that one must conclude that at least 18% of the population fail to satisfy this model.

Table 12 Numbers of Subjects Selecting the Average Alternative (see text)

Number of Choices in 25 Trials of Average Alternative	Number of Subjects Observed	Predicted
≤ 4	18	2.04
5	5	2.88
6	2	5.06
7	11	7.75
8	7	9.80
9	2	10.38
10	8	9.26
11	2	6.86
12	1	4.33
≥ 13	6	3.68

REGULARITY. Debreu and Savage (see Sec. 5.2) have both suggested that $p(x, y) = p_{\{x,y,y'\}}(x)$ whenever y' is in some sense very similar to y and in particular when y' and y denote the same thing. This, it will be recalled, is not what we would expect if the strict utility model, among others, were true. Becker, DeGroot, and Marschak (1963c) reported an experimental test of this hypothesis. As in the previous experiment, let $a < b < c < d$ be money outcomes and let $x = \langle a, a, d, d\rangle$, $y = \langle c, c, b, b\rangle$, and $y' = \langle b, b, c, c\rangle$ be the uncertain outcomes in which each component has probability $\frac{1}{4}$ of occurring. Note that y and y' are similar in the sense that the pure outcomes occur with exactly the same probabilities, although they are associated with different events. Four choice sets were studied:

$$(x, y), \qquad (x, y'), \qquad (x, y, y), \qquad \text{and} \quad (x, y, y').$$

Focusing on the choice of x, there are four interesting comparisons:

1. (x, y) and (x, y, y),
2. (x, y) and (x, y, y'),
3. (x, y') and (x, y, y'),
4. (x, y') and (x, y, y).

Each of 62 subjects was presented with 25 choice sets of each of the four types; these were generated by different selections for a, b, c, and d. Let $n_2(i,j)$ denote the number of times in the jth comparison that subject i chose x from the two-element set and not from the three-element set, and let $n_3(i,j)$ denote the number of times that he chose x from the three-element set and not from the two-element one. If the Debreu-Savage null hypothesis is correct, the random variable $n_2(i,j)/[n_2(i,j) + n_3(i,j)]$ is binomially distributed with $p = \frac{1}{2}$. From an examination of each (i,j) pair separately, Becker et al. concluded that "the observed data provide no reason to doubt that Debreu's comments are valid for most of the population." However, it should be kept in mind that the sample sizes are very small (none exceeded 19, and over half were less than 10), and so the tests are not very powerful.

To increase the power somewhat, we can sum either over i (subjects) or over j (comparison sets). When we do the latter and look at the statistic t, in this case $= [n_2(i) - n_3(i)]/\sqrt{n_2(i) + n_3(i)}$, where $n_k(i) = \sum_{j=1}^{4} n_k(i,j)$, we find that t exceeds 2 for 11 subjects, is between -2 and 2 for 34, and is less than -2 for 7. The remaining 10 subjects continue to have sample sizes less than 10. Thus regularity seems to be violated by 7 of 52 subjects, and the Debreu-Savage hypothesis cannot be rejected for 34.

When we sum over subjects, the resulting t values are -0.75, $+1.78$, $+2.55$, and $+0.54$, respectively, for the four conditions. This suggests that comparisons 1 and 4 may be somewhat different from 2 and 3. Note that the identical alternative appeared twice in the three-element sets of comparisons 1 and 4; evidently there is some tendency to distinguish identity of the presentation from mere similarity, as in 2 and 3.

STRONG EXPECTED UTILITY. By extending the linear programming methods of Davidson, Suppes, and Siegel (1957), Dolbear (1963) applied to data a strong or Fechnerian expected utility model in the sense of Sec. 7.1. In order to develop a computationally feasible maximum-likelihood estimation procedure to determine the utility function, he assumed the following specific form of the cumulative distribution ϕ:

$$\phi[u(x) - u(y)] = \begin{cases} \frac{1}{2} \exp\,[u(x) - u(y)] & \text{if} \quad u(x) \le u(y), \\ 1 - \frac{1}{2} \exp\,[u(y) - u(x)] & \text{if} \quad u(x) \ge u(y). \end{cases}$$

This choice for ϕ was also motivated by its ability to fit fairly well Mosteller and Nogee's (1951) curves of the estimated choice probabilities plotted as a function of estimated expected utility differences.

We do not attempt to summarize Dolbear's method of using a linear programming method to approximate the maximum-likelihood function of the utilities; readers are referred to his dissertation for details.

He applied the strong expected utility model to data from an experiment in which ten subjects were run, five of whom were graduate students and five of whom were at an Air Force language school at Yale. Each subject participated in three sessions. During each session the subject was presented with 100 pairs of options, each of which had two outcomes, and he was required to choose one from each pair. One of these pairs was selected at random from each session and they were run off at the end of the experiment. The smaller outcome probability was either $\frac{1}{10}$, $\frac{1}{3}$, or $\frac{1}{2}$. The outcomes ranged from losing \$1.50 to winning \$9.75, which is a considerably wider range than has been used in most experiments with monetary outcomes.

For 250 choices made in three sessions Dolbear compared the number of correct predictions made by his strong expected utility model with the number made by the actuarial model and by the minimax decision rule. The remaining 50 choices were used to estimate the utility function. Averaging over subjects, the strong utility model correctly predicted 79.3% of the choices, the actuarial model 68.1%, and the minimax model 57.2%. Using a one-tail test of the null hypothesis that the strong utility and actuarial models have the same predictive ability, he found that the former was better at the 0.01 significance level for four subjects, at the 0.05 level for a fifth subject, at the 0.06 level for a sixth, and at 0.17 level for a seventh. Using the same test, the strong utility model was better than the minimax model at the 0.01 level for 8 of the 10 subjects.

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