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Joint Receipt and Certainty Equivalents of Gambles*

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Several rules have been proposed for calculating the utility of the joint receipt of two amounts of money in terms of their individual utilities and/or the utility of their sum. This paper explores how the rules can be generalized to gambles. A concept of a structure of gambles with a joint receipt operation $\oplus$ and a certainty equilvalence (CE) mapping is described (Definitions 1 and 2). Monotonicity of $\oplus$ relative to preference permits simple generalizations of the rules to gambles (Theorem 1), and general properties of the utility function are related to general properties of CE (Theorem 2). The latter establishes that several apparently plausible properties are inconsistent, which raises empirical questions. For the restrictive case of lotteries (random variables), it is shown that if the convolution operation is monotonic in preference, then the CE mapping is onto addition (Theorem 3). It is argued that gamblers, at least, must fail monotonicity of convolution. Finally, a homogeneous scaling property is introduced that leads to a class of joint receipt operations that have not yet received attention (Theorem 4).

1. INTRODUCTION

The joint receipt of two (or more) things is an everyday experience. The mail often includes two or more checks or a mix of checks and bills; gifts typically are opened in clusters; etc. Recently, Thaler and others (Linville & Fischer, 1991; Thaler, 1985; Thaler & Johnson, 1990) have explored empirically ways in which such joint receipts of sums of money are evaluated. And Luce (1991, 1992b; Luce & Fishburn, 1991) introduced the operation of joint receipt into axiomatizations of rank-and sign-dependent theories for the utility (abbreviated, RSDU) of gambles that generalize Kahneman and Tversky's (1979; Tversky & Kahneman, 1992) prospect theory to arbitrary finite gambles with uncertain as well as risky events.

In studying the joint receipt of pure sums of money, Thaler proposed what he called the "hedonic rule." He assumed the existence of a utility function U over money and, according to Fishburn and Luce's (1993) interpretation[1] of his rule, the utility of the joint receipt of two sums x and y, $x \oplus y$, is given by the larger of the utility of the sum and the sum of utilities, i.e.,

$$U(x \oplus y) = \max[U(x+y), U(x) + U(y)], \tag{1}$$

Fishburn and Luce (1995) explored the consequences of Eq. (1) on the assumption that U is concave for gains and either concave or convex for losses. The complexity of Eq. (1) arises when $xy < 0$.

This rule is, of course, quite special and others have been suggested, although none has been as thoroughly studied as Eq. (1). For example, D. von Winterfeldt proposed[2]

$$U(x \oplus y) = \alpha U(x+y) + (1-\alpha)[U(x) + U(y)], \tag{2}$$

where α may differ in the four quandrants of gains and losses. Again, this rule presupposes that the crucial terms are the utility of the sum and the sum of the utilities. To my knowledge this rule has not been studied.

Whenever $x + y$ arises, as in Eqs. (1) and (2), one can always write it as

$$x + y = U^{-1}[U(x)] + U^{-1}[U(y)], \tag{3}$$

and so all of the examples described can be thought of as special cases of the general proposition that there is a function F: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is strictly increasing in each variable such that

$$U(x \oplus y) = F[U(x), U(y)]. \tag{4}$$

Note that for rules such as Eqs. (1) and (2) it is not immediately obvious how to extend them to general gambles (uncertain alternatives) which involve an assignment of

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[1] Thaler's text is ambiguous as to what consistutes the hedonic rule: Eq. (1) itself or Thaler's description of the regions where he believed $U(x+y) > U(x) + U(y)$ to hold assuming Eq. (1) and that U is concave for gains and convex for losses. As Fishburn and Luce (1995) showed, the two are not exactly equivalent, even under the assumptions about the shape of U.

[2] Personal communication, June 29, 1993.

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consequences to a finite partition of an uncertain event. In the most general case, no probability distribution is assumed to be provided, only a description of the events giving rise to the consequences. In generalizing the hedonic rule, the only issue is what to substitute for $x+y$. One possibility that comes to mind is to replace all occurences of gambles by their (monetary) certainty equivalents. If we let CE(g) denote the certainty equivalent of the gamble g, then, for example, the natural analog to Eq. (1) is

$$U(g \oplus h) = \max\{U[\mathrm{CE}(g) + \mathrm{CE}(h)], U(g) + U(h)\}. \quad (5)$$

A second possibility also arise in the special case of *lotteries*, i.e., gambles for which the consequences are money and the event partition has a probability distribution associated to it. In this case we may think of the lotteries as random variables with finitely many distinct values and of g and h as their distributions. Then, a natural analog of $+$ is the sum of the (independent) random variables, which has as its distribution the convolution of g and h, denoted $g*h$. So, for that case the second natural generalization of Thaler's rule is

$$U(g \oplus h) = \max[U(g*h), U(g) + U(h)]. \quad (6)$$

Theorem 3 demonstrates that if the operation $*$ is monotonic in the preference order $\succsim$ (see footnote 4 in Definition 2 below), then Eq. (6) reduces to Eq. (5).

One question to be addressed are the conditions under which Eqs. (5) and (6) are not only generalizations of Eq. (1), but which are consequences of it as well. We begin, however, in the more general setting of Eq. (4).

2. CERTAINTY-EQUIVALENT STRUCTURES

Since the major trick in relating Eq. (5) to Eq. (1) is to introduce certainty equivalents of gambles, we must provide a formal structure within which that is feasible. To that end, consider the following.

DEFINITION 1. Suppose G is a set, $\succsim$ is a binary relation (of weak preference) on G, and CE is a function from G into the real numbers, $\mathbb{R}$. We say that $\mathcal{G} = \langle G, \succsim, \mathrm{CE}\rangle$ is a *certainty-equivalent structure*[3] *provided that*:

(1) $\mathbb{R} \subseteq G$.

(2) For $x \in \mathbb{R}$, $\mathrm{CE}(x) = x$.

(3) For $g, h \in G$, $g \succsim h$ iff $\mathrm{CE}(g) \geqslant \mathrm{CE}(h)$.

At this point, the compass of the set G is limited only to its elements being something for which judgments of preference make sense. The intended interpretation for G is described in greater detail in motivating Definition 2 in the next section.

[3] This is very similar to the structures studied by Rotar (1994).

LEMMA 1. *Suppose that $\mathcal{G}$ is a certainty-equivalent structure, then for all $g, h \in G$,*

(i) $g \sim \mathrm{CE}(g)$

(ii) CE *is into* $\mathbb{R}$.

(iii) *over* $\mathbb{R}$, $\succsim \equiv \geqslant$.

(iv) $\succsim$ *is a weak order.*

Proof. (i) Using properties (2) and (3) of a certainty-equivalent structure

$$g \succsim \mathrm{CE}(g) \quad \text{iff} \quad \mathrm{CE}(g) \geqslant \mathrm{CE}[\mathrm{CE}(g)] = \mathrm{CE}(g),$$

which is possible iff $g \sim \mathrm{CE}(g)$.

(ii) It is an immediate consequence of property (2) that CE is onto $\mathbb{R}$.

(iii) Suppose $x, y \in \mathbb{R}$, then by properties (1), (2), and (3),

$$x \succsim y \quad \text{iff} \quad x = \mathrm{CE}(x) \geqslant \mathrm{CE}(y) = y.$$

(iv) That $\succsim$ is a weak order is an immediate consequence of properties (1) and (3) and the fact that $\geqslant$ is a total order. ∎

It should be pointed out that empirically realizing the concept of a certainty equivalent so that property (3) holds in somewhat problematic. There is ample evidence by now that judged certainty equivalents do not, in general, agree with choice-determined ones and that mixing choices with judged certainty equivalents leads to apparent violations of both transitivity and consequence monotonicity of gambles (Bostic, Herrnstein, & Luce, 1990; Mellers, Weiss, & Birnbaum, 1992; Tversky, Slovic, & Kahneman, 1990). In general, experience suggests that less difficulty arises, although at considerable experimental cost, when one uses choice-determined CEs involving some sort of interleaved, up–down procedure. A typical one, called PEST, is described in Cho, Luce, and von Winterfeldt (1994). A more efficient method is used in Tversky and Kahneman (1992), but it has not be calibrated against PEST or judged certainty equivalents.

3. MONOTONIC JOINT RECEIPT OPERATION

Although the following mathematical results do not depend upon the exact interpretation of G, certain properties to be assumed are motivated primarily by the following interpretation: G encompases all pure sums of money, all finite gambles that we may wish to include, and the entities that are inductively defined from these sums of money and gambles using a binary operation of joint receipt finitely many times. With that in mind, we introduce the fourth major property.

DEFINITION 2. Suppose $\mathscr{G} = \langle G, \succsim, \mathrm{CE} \rangle$ is a certainty-equivalent structure and $\oplus$ is a closed binary operation on G. Then $\langle \mathscr{G}, \oplus \rangle$ is a *joint-receipt, certainty-equivalent* (abbreviated, *JRCE*) *structure* if

(4) The restriction of $\oplus$ to $\mathbb{R}$ is weakly commutative and monotonic[4] in $\succsim$.

Note that although $\oplus$ is assumed to be weakly commutative over $\mathbb{R}$ it is not assumed to be weakly associative. Indeed, none of the above rules for relating it to utility permit it to be associative, except in special cases.

It is implicit that each gamble in G is based upon an independent realization of the chance experiment underlying it. In particular, $g \oplus h$ means that two different experiments are run, one for g and the other for h. After the proof of Theorem 1 a detailed example illustrates why dependence among the gambles is problematic.

The first theoretical question considered is the conditions under which the operation $\oplus$ is monotonic over the entire set of gambles. This allows us to deduce Eq. (4) over G from its holding over $\mathbb{R}$. In particular, that permits one to derive Eq. (5) from Eq. (1). It should be pointed out that the (a priori) normative reasons for supposing $\oplus$ is monotonic are just as compelling as those for assuming the transitivity of $\succsim$ or consequence monotonicity of gambles. To my knowledge, its empirical validity has yet to be tested.

THEOREM 1. *Suppose that $\langle \mathscr{G}, \oplus \rangle$ is a JRCE structure. Then*

(i) *$\oplus$ is monotonic with respect to $\succsim$ iff*

$$\mathrm{CE}(g \oplus h) \sim \mathrm{CE}(g) \oplus \mathrm{CE}(h). \tag{7}$$

(ii) *Suppose $\oplus$ is monotonic. Then $x \oplus y \sim (\succ, \prec)\, x + y$, for all $x, y \in \mathbb{R}$, iff*

$$\mathrm{CE}(g \oplus h) = (>, <)\ \mathrm{CE}(g) + \mathrm{CE}(h) \quad \text{for all} \quad g, h \in G. \tag{8}$$

(iii) *Suppose that $U\colon \mathbb{R} \to \mathbb{R}$ is defined and order preserving over $\mathbb{R}$, and Eq. (4) holds for $\langle \mathbb{R}, \geqslant, \oplus \rangle$. Define U on all of G by*

$$U(g) := U[\mathrm{CE}(g)]. \tag{9}$$

Then $\oplus$ is monotonic iff Eq. (4), with U of Eq. (9), holds over $\mathscr{G}$ with strictly increasing F.

[4] As usual weakly commutative means $x \oplus y \sim y \oplus x$, where $\sim$ is the indifference relation of $\succsim$, and monotonic means $x \succsim y$ iff $x \oplus z \succsim y \oplus z$ for all $x, y, z \in \mathbb{R}$.

Proof. (i) By Lemma 1(i) and the assumed monotonicity of $\oplus$,

$$\mathrm{CE}(g \oplus h) \sim g \oplus h \sim \mathrm{CE}(g) \oplus \mathrm{CE}(h),$$

and by Lemma 1(iv) Eq. (7) follows.

Conversely, suppose that Eq. (7) holds and monotonicity is violated, i.e., for some $g, g', h \in G$, $g' \succ g$, and $g' \oplus h \precsim g \oplus h$. By property (3), $\mathrm{CE}(g') > \mathrm{CE}(g)$ and by Eq. (7),

$$\begin{aligned}\mathrm{CE}(g') \oplus \mathrm{CE}(h) &= \mathrm{CE}(g' \oplus h) \leqslant \mathrm{CE}(g \oplus h)\\ &= \mathrm{CE}(g) \oplus \mathrm{CE}(h).\end{aligned}$$

By property (4), $\oplus$ is monotonic over $\mathbb{R}$, so $\mathrm{CE}(g') \precsim \mathrm{CE}(g)$ which, by Lemma 1(iii), implies that $\mathrm{CE}(g') \leqslant \mathrm{CE}(g)$, a contradiction.

(ii) Equation (8) is an immediate consequence of Eq. (7) and Lemma 1(iii).

(iii) Using Eqs. (4), (7), and (9) we have

$$\begin{aligned}U(g \oplus h) &= U[\mathrm{CE}(g \oplus h)]\\ &= U[\mathrm{CE}(g) \oplus \mathrm{CE}(h)]\\ &= F[U[\mathrm{CE}(g)], U[\mathrm{CE}(h)]]\\ &= F[U(g), U(h)],\end{aligned}$$

which is Eq. (4) for gambles. The converse is trivial. ∎

Assuming Thaler's hedonic rule for money, Eq. (1), then by Theorem 1(iii) the monotonicity of $\oplus$ yields its generalization Equation (5) for gambles.

P. C. Fishburn has pointed out[5] that the monotonicity of $\oplus$ is almost surely inconsistent with some gambles being based on a common underlying experiment. To demonstrate the difficulty, consider an event E that is subjectivity as likely to occur as not; i.e., for all $x, y \in \mathbb{R}$, $(x, E, y) \sim (y, E, x)$. For some particular x and y, let $g = (x, E, y)$ and $h = (y, E, x)$, and so $g \sim h$. Assuming the gambling operation is idempotent, i.e., for all $x \in \mathbb{R}$, $x \sim (x, E, x)$, $\oplus$ is commutative and monotonic, and the same experiment determines E in both g and h, we prove that

$$x \oplus y \sim \mathrm{CE}(g) \oplus \mathrm{CE}(g). \tag{10}$$

By idempotence, commutativity, and the commonness of E,

$$\begin{aligned}x \oplus y &\sim (x \oplus y, E, x \oplus y) \sim (x \oplus y, E, y \oplus x)\\ &\equiv (x, E, y) \oplus (y, E, x) \equiv g \oplus h.\end{aligned}$$

[5] Personal communication, August 20, 1993.

It should be noted that this argument presumes that no distinction is made between the equivalent formulations, $(x\oplus y, E, y\oplus x)$ and $(x, E, y)\oplus(y, E, x)$, of the same fact, namely, that $x\oplus y$ is received if E occurs and $y\oplus x$ if E fails to occur. This intermediate result, together with Lemma 1(i) and the monotonicity of $\oplus$ yields

$$x\oplus y\sim g\oplus h\sim g\oplus g\sim \mathrm{CE}(g)\oplus\mathrm{CE}(g),$$

from which Lemma 1(iv) implies Eq. (10).

Now, consider $x=\$1000$ and $y=\$0$, then for me,

$$\mathrm{CE}(g)<\$450, \qquad \$1000\oplus\$0>\$450\oplus\$450,$$

but using Lemma 1(iii) and property (4),

$$\$1000\oplus\$0\succ\$450\oplus\$450\succ\mathrm{CE}(g)\oplus\mathrm{CE}(g),$$

contrary to Eq. (10).

4. AN EMPIRICAL RESULT AND SOME IMPLICATIONS ABOUT CES AND $\oplus$

Another rule relating utility U to joint receipt $\oplus$, one that does not involve anything concerning $x+y$ and which is stated for gambles rather than just certain amounts, was arrived at by Luce and Fishburn (1991). For gains and losses separately, it was derived from certain assumptions about how $\oplus$ distributes[6] over gambles in the context of binary RSDU. The extension to mixed gains and losses was purely an extrapolation subject to the requirement that $\oplus$ remain monotonic. The form they arrived at and axiomatized is

$$U(g\oplus h)=\begin{cases} A(+)\,U(g)+B(+)\,U(h)+C(+)\,U(g)\,U(h), & g\succ 0,\quad h\succ 0\\ A(+)\,U(g)+B(-)\,U(h), & g\succsim 0,\quad h\precsim 0\\ A(-)\,U(g)+B(+)\,U(h), & g\precsim 0,\quad h\succsim 0\\ A(-)\,U(g)+B(-)\,U(h)-C(-)\,U(g)\,U(h), & g\prec 0,\quad h\prec 0,\end{cases} \tag{11}$$

where $A(\cdot)>0$, $B(\cdot)>0$, and $C(\cdot)\geqslant 0$. Note that $\oplus$ is commutative if and only if $A(\cdot)=B(\cdot)$. The special case of pure additivity (i.e., $A(\cdot)=B(\cdot)=1$ and $C(\cdot)=0$) was invoked by Luce and Fishburn (1991) in arriving at a version of RSDU and by Luce (1992b) in developing a theory of certainty equivalents. (See Note Added in Proof at the end of the article.)

[6] A closely related property was called "segregation" in Cho *et al.* (1994).

Some consequences of this RSDU of CEs have been studied empirically in Cho *et al.* (1994). Among other things, their, albeit limited, data suggest that CE over $\oplus$ is *subadditive* for gains and *superadditive* for losses. It is less clear what holds for mixed gains and losses, although *additivity* is a possibility. Assuming this to be correct, then stated formally the three cases are:

$$\mathrm{CE}(g\oplus h)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}\mathrm{CE}(g)+\mathrm{CE}(h), \quad \begin{cases} g\succ 0,\ h\succ 0\\ g\succsim 0\succsim h\\ g\prec 0,\ h\prec 0.\end{cases} \tag{12}$$

Similarly, we can define U over $\oplus$ as being subadditive, additive, and superadditive on an interval $I\subseteq\mathbb{R}$, provided that for all $x, y\in I$,

$$U(x\oplus y)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix}U(x)+U(y). \tag{13}$$

Note that if U is *concave*[7] on I, then it is subadditive over $+$; and if it is *convex*, then it is superadditive over $+$.

THEOREM 2. *Suppose that $\langle\mathscr{G},\oplus\rangle$ is a monotonic JRCE structure. Suppose that for gains CE is subadditive over $\oplus$. Then, for gains,*

(i) *U is concave implies U is subadditive over $\oplus$;*

(ii) *U is superadditive or additive over $\oplus$ implies U is superadditive over $+$.*

Suppose that for losses CE is superadditive over $\oplus$. Then, for losses

(iii) *U convex implies U that superadditive over $\oplus$;*

(iv) *U is subadditive or additive over $\oplus$ implies U is subadditive over $+$.*

Proof. (i)

$$\begin{aligned} U(g\oplus h)&=U[\mathrm{CE}(g)\oplus\mathrm{CE}(h)] && \text{(by Eq. (7))}\\ &<U[\mathrm{CE}(g)+\mathrm{CE}(h)] && \text{(by Eq. (12))}\\ &\leqslant U[\mathrm{CE}(g)]+U[\mathrm{CE}(h)] && \text{(by } U \text{ is concave)}\\ &=U(g)+U(h) && \text{(by Lemma 1(i)).}\end{aligned}$$

(ii) Let $x=\mathrm{CE}(g)>0$ and $y=\mathrm{CE}(h)>0$, then

$$\begin{aligned} U(x+y)&=U[\mathrm{CE}(g)+\mathrm{CE}(h)] && \text{(definitions)}\\ &>U[\mathrm{CE}(g\oplus h)] && \text{(by Eq. (12))}\end{aligned}$$

[7] U is *concave* over I iff for all $x, y\in I$ and $0<\alpha<1$, $U[\alpha x+(1-\alpha)\,y]>\alpha U(x)+(1-\alpha)\,U(y)$; *convex* just means the reversed inequality.

$$= U[\mathrm{CE}(g) \oplus \mathrm{CE}(h)] \qquad \text{(by Eq. (7))}$$

$$\geqslant U[\mathrm{CE}(g)] + U[\mathrm{CE}(h)] \qquad \text{(by Eq. (13))}$$

$$= U(x) + U(y) \qquad \text{(definitions).}$$

Since by definition of JRCE, $\mathbb{R} \subseteq G$, this proves that for gains U is superadditive over $+$.

The proofs of (iii) and (iv) are analogous. ∎

Note that for gains, both Thaler's rule, Eq. (1), and the Luce–Fishburn rule, Eq. (11) with $A(\cdot) = B(\cdot) = 1$ imply that U is superadditive or additive over $\oplus$. Thus, if Cho *et al.*'s (1994) empirical finding is correct, Theorem 2(ii) means that U must be superadditive over $+$. Prevailing wisdom, including Thaler's (1985) empirical findings, suggest that, on the contrary, U apparently is concave for gains and so it is subadditive over $+$. The present argument thus implies, then, that at least one of the following statements must be in error for gains:

U is concave over $+$.

U is superadditive or additive over $\oplus$.

CE is subadditive over $\oplus$.

$\oplus$ is monotonic.

Probably the evidence is weakest for U being superadditive or additive over $\oplus$, and so both the hedonic rule, Eq. (1), and the Luce–Fishburn rule, Eq. (11), with $A(\cdot) = B(\cdot) = 1$ are suspect.

For losses, the situation is slightly more complex. Thaler's rule continues to imply that U is superadditive or additive over $\oplus$ for losses, and, by the data of Cho *et al.* (1994), CE for losses is also superadditive over $\oplus$. Because the directions of the inequalities do not work together, no conclusion follows. In contrast, the Luce–Fishburn rule, Eq. (11), with $A(\cdot) = B(\cdot) = 1$ says that U over $\oplus$ is subadditive for losses and so by Theorem 2(iv) U must be subadditive over $+$ for losses. Again, this is counter to the prevailing belief that the utility function is convex and so it is superadditive over $+$ for losses. Thus, Eq. (11) is most suspect. The implications of this are significant; either the binary RSDU model is wrong or the segregation assumption $(x, E, 0) \oplus y \sim (x \oplus y, E, 0 \oplus y)$, for gains and losses separately, is wrong because together these assumptions imply Eq. (11) for gains and losses.

The possibility must be entertained that perhaps we are dealing with different kinds of subjects, some of whom are concave–convex, others of whom are convex–concave, and still others satisfy the other two possibilities. Then depending on the mix of subjects we happen to have in an experiment, the data yield average evidence favoring one or the other. Such a possibility of individual difference is raised again in Section 6. Attempts probably should be made to partition the data into these four possiblities prior to analysis.

The von Winterfeldt suggestion, Eq. (2), fares much better. Its natural analog for gambles is

$$U(g \oplus h) = \alpha U[\mathrm{CE}(g) + \mathrm{CE}(h)] + (1-\alpha)[U(g) + U(h)]. \tag{14}$$

Observe that U is subadditive (additive) (superadditive) over $\oplus$ iff it is subadditive (additive) (superadditive) over $+$. Thus, the function is not in conflict with Theorem 2.

5. LOTTERIES AND THE MONOTONICITY OF CONVOLUTION

We now specialize to the situation in which the gambles in G are actually lotteries, i.e., distributions of random variables with suitable closure properties, and our aim is to show that Eq. (6) is closely related to the monotonicity of both $\oplus$ and the convolution operator $*$.

THEOREM 3. *Suppose that $\langle \mathscr{G}, \oplus \rangle$ is a JRCE structure of lotteries such that convolution $*$ is a closed operation*[8]. *Then*

(i) *$*$ is monotonic with respect to $\succsim$ iff*

$$\mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h). \tag{15}$$

Assume, further, that $U\colon G \to \mathbb{R}$ is order preserving, $*$ is monotonic, and Eq. (1) holds over $\mathbb{R}$. Then

(ii) *$\oplus$ is monotonic iff Eq. (6) holds.*

Proof. (i) Suppose that $*$ is monotonic. By Lemma 1(i), $g \sim \mathrm{CE}(g)$, so

$$\mathrm{CE}(g * h) \sim g * h \sim \mathrm{CE}(g) * \mathrm{CE}(h).$$

However, over $\mathbb{R}$ $*$ is $+$, so $\mathrm{CE}(g) * \mathrm{CE}(h) = \mathrm{CE}(g) + \mathrm{CE}(h)$ and so by Lemma 1(iii) Eq. (15) follows.

Now, suppose Eq. (15) holds and $g' \succsim g$, then

$$g' * h \sim \mathrm{CE}(g' * h) = \mathrm{CE}(g') + \mathrm{CE}(h) \geqslant \mathrm{CE}(g) + \mathrm{CE}(h) = \mathrm{CE}(g * h) \sim g * h$$

and conversely, thus proving that $*$ is monotonic.

(ii) Suppose that $\oplus$ is monotonic, then

$$U(g \oplus h) = U[\mathrm{CE}(g \oplus h)] \qquad \text{(Lemma 1(i))}$$

[8] As usual, closed means that if $g, h \in G$, then $g * h \in G$.

$= U[\mathrm{CE}(g) \oplus \mathrm{CE}(h)]$ (Eq. (7))

$= \max\{U[\mathrm{CE}(g)+\mathrm{CE}(h)],\ U[\mathrm{CE}(g)] + U[\mathrm{CE}(h)]\}$ (Eq. (1))

$= \max\{U[\mathrm{CE}(g*h)],\ U[\mathrm{CE}(g)] + U[\mathrm{CE}(h)]\}$ (Eq. (15))

$= \max[U(g*h), U(g) + U(h)],$ (monotonicity of $*$, Lemma 1(i))

which is Eq. (6).

Conversely, suppose Eq. (6) and $g' \succsim g$. Then using the monotonicity of $*$ as well as that of $+$,

$$\begin{aligned} U(g' \oplus h) &= \max[U(g' * h), U(g') + U(h)] \\ &\geqslant \max[U(g*h), U(g) + U(h)] \\ &= U(g \oplus h). \end{aligned}$$

Moreover, $\succ$ goes into $>$. Because U is order presevering, this proves that $\oplus$ is monotonic. ∎

Fishburn suggested[9] that it might be more natural to define convolution in terms of $\oplus$ than $+$. Let us speak of that as JR-convolution, which is defined formally as

$$g \odot h(z) = \sum_x \sum_y [g(x)\, h(y) : x \oplus y \sim z].$$

Assuming that $\odot$ and $\oplus$ are both monotonic, then using the fact that for $x, y \in \mathbb{R}$, $x \odot y \sim x \oplus y$, and Lemma 1(i),

$$\begin{aligned} g \odot h &\sim \mathrm{CE}(g) \odot \mathrm{CE}(g) \sim \mathrm{CE}(g) \\ &\oplus \mathrm{CE}(h) \sim \mathrm{CE}(g \oplus h) \sim g \oplus h, \end{aligned}$$

whence $g \odot h \sim g \oplus h$. So it is not a new operation in the monotonic case.

6. GAMBLERS ARE NOT MONOTONIC IN CONVOLUTION

In a JRCE of lotteries, suppose that $*$ is monotonic. Define $g(n) = g(n-1) * g$ and $g(1) = g$. If $g \succ 0$, then by the monotonicity of $*$ we see that for all n, $g(n) \succ 0$.

Let $E(g)$ denote the expected value of a lottery g. If for some decision maker, whom I refer to as a *gambler*, there is a g such that both

$$g \succ 0 \qquad \text{and} \qquad E(g) < 0, \tag{16}$$

then I venture to gues that there is some n for which $g(n) \prec 0$. The reason is that $E[g(n)] = nE(g) < 0$ grows increasingly negative with n, whereas its standard deviation grows only as $\sqrt{n}\,\mathrm{SD}(g)$. It is difficult to imagine that these convolved gambles will not sooner or later prove to be less preferred than 0. But this violates the monotonicity of $*$.

It should be noted that the concept of a gambler given in Eq. (16) depends in many ways on the social context–casino, experiment, insurance purchases. Any one who buys insurance is a gambler according to Eq. (16).

Note unrelated to these remarks is the fact that, by Theorem 3, CE is a homomorphic map of $\langle G, \succsim, *\rangle$ onto $\langle \mathbb{R}, \geqslant, +\rangle$ and so, by Hölder's theorem, the convolution structure is necessarily Archimedean. So, even worse than the example given, it means that for any h, however desirable, there must be an n such that $g(n) \succ h$. This is totally implausible.

Vladimir I. Rotar pointed out to me[10] that the line of argument here is closely related to results that Rotar and Sholomitsky (1994) have obtained concerning monotonicity of convolution.

The above remarks make clear that there is empirical interest in studying the monotonicity of $\oplus$ and $*$. For the latter, it seems essential to deal separately with people who are gamblers in the context of the experiment, in the sense of Eq. (16), from those who are not.

7. FORMS FOR $U(x \oplus y)$ DERIVED FROM A HOMOGENEITY CONDITION

In this section we consider a scaling property that seems fairly plausible and that can be used to get forms for $U(x \oplus y)$ that are still different from those of Thaler, Eq. (5), von Winterfeldt, Eq. (2), or Luce and Fishburn, Eq. (11). The general idea is to ask what happens when x and y are simultaneously rescaled by a factor k. One idea is that $x \oplus y$ is in some sense rescaled by a factor that depends on k. We capture this idea as follows.

DEFINITION 3. Suppose that $\langle \mathscr{G}, \oplus\rangle$ os a JRCE structure. $\langle \mathscr{G}, \oplus\rangle$ is said to be *CE-homogeneous* if and only if there is a strictly increasing function $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $x, y \in \mathbb{R}$ and all $k \in \mathbb{R}^+$,

$$kx \oplus ky \sim \phi(k)\, \mathrm{CE}(x \oplus y). \tag{17}$$

It is not difficult to see that if U is a power function, then both Thaler's rule, Eq. (1), and von Winterfeldt's, Eq. (2), both satisfy CE-homogeneity with $\phi(k) = k$. The Luce–Fishburn rule, Eq. (11) does not satisfy it even in the special case U is a power function unless $C(\cdot) = 0$.

[9] Personal communication, August 20, 1993.

[10] Personal communications, October 5 and 8, 1993.

We pursue the consequences of CE-homogeneity within a class of structures that have a distinguished singular point (Luce, 1992a), which in this interpretation can be thought of as the status quo, but that are quite regular on either side of the status quo. There are three lemmas leading up to Theorem 4, which describes the resulting structures, which include Thaler and von Winterfeldt's forms as special cases.

LEMMA 2. *Suppose that* $\langle \mathscr{G}, \oplus \rangle$ *is a JRCE structure that is CE-homogeneous and* $0 \oplus 0 \sim 0$. *Then, there exists positive constants* α, β, *and* ρ *such that for* $k \in \mathbb{R}_+$ *and* $x \in \mathbb{R}$,

$$\phi(k) = k^\rho, \qquad x \oplus 0 \sim \begin{cases} \alpha x^\rho, & x > 0, \\ -\beta(-x)^\rho, & x < 0. \end{cases} \tag{18}$$

Proof. Because $\oplus$ is monotonic on $\mathbb{R}$, there exists a strictly increasing function $\vartheta\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x \oplus 0 \sim \vartheta(x)$. By the hypothesis, $0 \oplus 0 \sim 0$ and so, by Lemma 1(iii), $\vartheta(0) = 0$. Set $y = 0$ in Eq. (17), then by condition 2 of Definition 1,

$$\begin{aligned} \vartheta(kx) &\sim kx \oplus 0 \sim \phi(k)\,\mathrm{CE}(x \oplus 0) \sim \phi(k)\,\mathrm{CE}[\vartheta(x)] \\ &= \phi(k)\,\vartheta(x). \end{aligned}$$

By Lemma 1(iii),

$$\vartheta(kx) = \phi(k)\,\vartheta(x).$$

Then, setting $k = 1$ yields $\phi(1) = 1$. Next, setting $x = 1$ shows that $\phi(k) = \vartheta(k)/\vartheta(1)$, thus yielding the functional equation over $\mathbb{R}_+$,

$$\vartheta(kx) = \vartheta(k)\,\vartheta(x)/\vartheta(1),$$

whose strictly increasing solutions for $x > 0$ are well known to be power functions of the form shown in Eq. (18), with $\alpha = \vartheta(1) > \vartheta(0) = 0$. For the negative domain, one sets $x = -1$ and proceeds in an analogous fashion. The exponents in the two domains are forced to be equal because of their common relation to ϕ. ∎

Given these special power function forms and following the convention that x^ρ has its usual meaning when $x > 0$ and $-(-x)^\rho$ when $x < 0$, then consider the defined operation $\otimes$,

$$x \otimes y := x^{1/\rho} \oplus y^{1/\rho}. \tag{19}$$

LEMMA 3. *Suppose that the hypotheses of Lemma 2 obtain, and define* $\otimes$ *by Eq.* (19). *Then for all* $k > 0$ *and all* $x \in \mathbb{R}$,

(i) $kx \otimes ky \sim k(x \otimes y)$;

(ii) $x \otimes 0 = \alpha x$, $x \geqslant 0$; *or* βx, $x < 0$.

(iii) $\otimes$ *is monotonic if and only if* $\oplus$ *is monotonic.*

Proof. (i) and (ii) are easily verified by using Eqs. (18) and (19). (iii) is an immediate consequence of the definition of $\otimes$ in terms of $\oplus$. ∎

In developing a theory of scale types, Narens (1981a; 1981b) first introduced the following concepts, and Luce (1992a) generalized them to structures with singular points (see below). The concepts and many of the results are summarized in Chapter 20 of Luce, Krantz, Suppes, and Tversky (1990).

A structure is *N-point unique* if it has the property that whenever an automorphism of the structure has N distinct fixed points,[11] then it is the identity map. A structure is *finitely unique* if it is N-point unique for some finite N.

A point of the structure is *singular* if it is a fixed point of all autormorphisms. Clearly, an N-point unique structure with a non-trivial automorphism has at most N singular points.

An automorphism is a *translation* if its only fixed points are the singular ones.

A structure is said to be (*translation*) *M-point homogeneous between singular points* if given any two ordered sequences of M distinct points lying in an interval between adjacent singular points, then there is a (translation) automorphism that takes the one sequence onto the other.

LEMMA 4. *Suppose the hypotheses of Lemma 2 obtain and that* $\otimes$ *is monotonic. Let* $\mathscr{H} = \langle G, \succsim, \mathrm{CE}, \otimes \rangle$, *where* $\otimes$ *is defined in Eq.* (19), *and suppose that* $\mathscr{H}/\sim$ *is finitely unique. Then* $\mathscr{H}/\sim$ *is* 1*-point unique,* 0 *is its unique singular point, and the structure is translation* 1*-point homogeneous on each side of* 0.

Proof. Because CE is onto and Eq. (7) follows from the monotonicity of $\oplus$, it is easy to see that $\mathscr{H}/\sim$ is isomorphic ($\approx$) to $\langle \mathbb{R}, \geqslant, \otimes \rangle$. For each $k > 0$, define the map $\alpha_k(x) = kx$. We show that α_k is an automorphism. Obviously it is order preserving and by Lemma 3(i),

$$\alpha_k(x \otimes y) = k\,\mathrm{CE}(x \otimes y) = kx \otimes ky = \alpha_k(x) \otimes \alpha_k(y).$$

As is easily verified, $\alpha_{1/k} = \alpha_k^{-1}$. So $\{\alpha_k : k \in \mathbb{R}_+\}$ is a group of translations under function composition.

Next we show that this is actually the entire group of automorphisms. Suppose that β is an automorphism. For some $x > 0$, select k so that $\alpha_k(x) = \beta(x)$. Define $x(n)$ inductively by $x(1) = x$ and $x(n) = x(n-1) \otimes x$. By the

[11] An *automorphism* of a structure is any isomorphism of the structure onto itself. For present purposes we do not distinguish automorphisms that differ only within equivalence classes of $\sim$. If α is an automorphism, x is a *fixed point* of α iff $\alpha(x) \sim x$.

monotonicity of $\otimes$ (Lemma 3(iii)), $x(n) \succ x(n-1)$, so these are countably many distinct points. But observe, by induction $\beta[x(n)] \sim a_k[x(n)]$, whence by the finite uniqueness of $\mathscr{H}/\sim$, $\beta \equiv \alpha_k$.

Clearly 0 is a fixed point of every automorphism and, except for $k=1$, the unique one. Thus, the automorphisms are the translations, and it is well known that if the translations from a group, they are 1-point unique (see, for example, Luce *et al.*, 1990, p. 118).

Finally, for any x and y with $xy>0$, define $k=y/x$ and we see that the structure is translation 1-point homogeneous on either side of 0. ∎

THEOREM 4. *Suppose that* $\langle \mathscr{G}, \oplus \rangle$ *is a monotonic JRCE structure with the following properties*:

(1) *$\mathscr{G}$ is CE-homogeneous.*

(2) $0 \oplus 0 \sim 0$.

(3) *$\mathscr{H}/\sim\, = \langle \mathbb{R}, \geqslant, \otimes \rangle$ is finitely unique ($\otimes$ is defined by Eq. (19)).*

(4) *$\mathscr{G}$ is solvable in the sense that for each $g \in G$, there exists $g' \in G$ such that*

$$g \oplus g' \sim 0.$$

Let α, β, and ρ be defined as in Lemma 2. Then, there exist a function $U\colon G \to \mathbb{R}$, characterized by three constants $\sigma > 0$, $U(1) > 0$, and $U(-1) < 0$, and two functions $f_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, such that

(i) *U is order preserving.*

(ii) $$U(x) = \begin{cases} U(1)\, x^{\sigma}, & x \geqslant 0 \\ U(-1)(-x)^{\sigma}, & x < 0. \end{cases} \tag{20}$$

(iii) *The f_1 are strictly increasing and the $f_1(x)/x$ are strictly decreasing.*

(iv) *Following the convention for the meaning of x^{ρ} when $x<0$,*

$$U(x \oplus y) = \begin{cases} U(y^{\sigma})\, f_1[U(x^{\rho})/U(y^{\rho})], & y > 0, \\ \alpha^{\sigma} U(x^{\rho}), \quad y = 0, & x \geqslant 0, \\ \beta^{\sigma} U(x^{\rho}), \quad y = 0, & x < 0, \\ U(y^{\rho})\, f_2[U(x^{\rho})/U(y^{\rho})], & y < 0. \end{cases} \tag{21}$$

Proof. By Lemmas 3 and 4, the structure $\mathscr{H}/\sim\, \approx \langle \mathbb{R}, \geqslant, \otimes \rangle$ meets the conditions of the corollary to Theorem 5 of Luce (1992a) and so there exists an order-preserving function U, positive constants C_i, and functions f_i, $i=1, 2$, such that

$$U(x \otimes y) = \begin{cases} U(y)\, f_1[U(x)/U(y)], & y > 0, \\ C_1\, U(x), \quad y = 0, & x \geqslant 0, \\ C_1\, U(x), \quad y = 0, & x < 0, \\ U(y)\, f_2[U(x)/U(y), & y < 0. \end{cases} \tag{22}$$

Using the definition of Eq. (19), $x \otimes y = x^{\rho} \otimes y^{\rho}$, in Eq. (22) thus yields Eq. (21) except for the determination of the constants C_i, which is done below.

The same corollary also says that the translations of $\mathscr{H}/\sim$ map into multiplication by positive constants, one for gains and a power of it for losses. Thus, there exists a function $\psi\colon \mathbb{R}_+ \times \{+, -\} \to \mathbb{R}_+$ such that for all $k>0$ and all x,

$$U(kx) = U[\alpha_k(x)] = \psi(k, \operatorname{sgn} x)\, U(x).$$

Thus, by the usual argument, U must be a power function with, say, exponent $\sigma(+)$ for $x>0$ and $\sigma(-)$ for $x<0$, thus yielding Eq. (20) but with different exponents for gains and losses. Substituting this form into Eq. (17) and using Eq. (18) forces $\sigma(+) = \sigma(-) = \sigma$.

To determine the constants C_i observe that from Eq. (22) and Lemma 3(ii) we have for $x>0$,

$$U(x \otimes 0) = U(\alpha x) = C_1\, U(x).$$

Substituting Eq. (20) for U yields $C_1 = \alpha^{\sigma}$. A simular argument for $x<0$ yields $C_2 = \beta^{\sigma}$. ∎

The formulation of Theorem 4 suffers from the defect that the property that $\mathscr{H}/\sim$ is finitely unique is stated in terms of the defined quantity $\otimes$ rather than the primitive of the system, $\oplus$. I do not know of a convenient way to formulate finite uniqueness of $\mathscr{H}/\sim$ in terms of $\mathscr{G}$.

Of course, no difficulty is encountered when $\phi(k)=k$ in Eq. (17), in which case $\rho=1$ in Eqs. (19) and (21) and so $\otimes = \oplus$ in Eq. (19). The special case $f_1(z) = f_2(z) = 1+z$ is the one that Luce and Fishburn (1991) actually use in arriving at the RSDU representation. Although they did not assume U to be a power function in that derivation, in subsequent applications of the theory that assumption has also been made.

The form for $\oplus$ given by Eq. (21) encompasses Thaler's hedonic rule and von Winterfeldt's averaging one, provided that $U(-1) = -U(1)$. The argument rests on the observation that for such a power function form for U, both

$$U(x+y) = U(y)\left(1 + \frac{U^{1/\rho}(x)}{U^{1/\rho}(y)}\right)^{\rho},$$

$$U(x) + U(y) = U(y)\left(1 + \frac{U(x)}{U(y)}\right)$$

hold, and so both rules can be put in the form of Eq. (21). Equation (21) does not encompass the Luce–Fishburn rule, Eq. (11).

I am unaware of any results of data that bear on the suitability of other realizations of Eq. (21) for describing joint receipt. Because it rests heavily on the accuracy of the assumption of CE-homogeneity, Eq. (17), the property clearly needs testing.

8. CONCLUSIONS

The major conclusion that I draw from the above results is that the concept of a JRCE structure of gambles exhibits a fair number of interesting properties, especially when $\oplus$ is monotonic in preferences. I have not attempted here to relate this structure seriously to the event–outcome structure normally studied in utility theories for gambles. For the additive utility rule and less so for the rule of Eq. (11) some results are known vis-à-vis rank- and sign-dependent utility (Luce, 1991, 1992b; Luce & Fishurn, 1991). But clearly, much more needs to be done in order to understand fully how each type of structure constraints the other.

At this point there seem to be many options, and I believe that we need serious experimental studies to guide our future developments. We must examine the various properties of $\oplus$, $*$, and CE that have arisen in Theorems 1–4 with an eye to narrowing down the possibilities.

Note Added in Proof. Luce and Fishburn (1991) argued, on the grounds of monotonicity of $\oplus$, that $C(\cdot)$ had to be $\geqslant 0$ in Eq. (11), and they assumed $A(\cdot) = B(\cdot) = 1$ and $C(\cdot) = 0$ in deriving the rank-dependent form for the utility of gains and losses separately. Subsequently Luce and Fishburn (Luce, R. D., & Fishburn, P. C. (submitted). A note on deriving rank-dependent utility using additive joint receipts) recognized that, with U suitably bounded, $C(\cdot) \leqslant 0$ also satisfies monotonicity, in which case U is subadditive for gains and superadditive for losses. This change alters significantly the conclusions drawn about Eq. (11) in Section 4. They also show that exactly the same rank-dependent model derives from $A(\cdot) = B(\cdot) = 1$ with no restriction on $C(\cdot)$ beyond being non-positive. This generalization is significant because it means that the plausible property that $x \oplus y = x + y$ does not force $U(x) = cx$, $c > 0$.

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