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Tests of Hypotheses about Certainty Equivalents and Joint Receipt of Gambles

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The present experiment was designed to test whether choice-induced certainty equivalents (CEs) and joint receipt (JR) of gambles exhibit certain properties such as the equality of JR and convolution, monotonicity of convolution, monotonicity of JR over gambles, additivity of JR over gambles and money, segregation of a common consequence, additive segregation when JR is replaced by +, and several other derivative properties. Subjects were partitioned into "gamblers" and "nongamblers" by their performance on screening gambles. Assuming that CE is order preserving, monotonicity of JR and additivity of JR over gambles were both rejected whereas additivity of JR over money, segregation, and additive segregation were all sustained for gamblers and nongamblers. For gamblers, convolution is not monotonic but is equivalent to JR and segregation and additive segregation are not equivalent. For nongamblers, convolution is monotonic but is not equivalent to JR and segregation and additive segregation are probably equivalent.

INTRODUCTION

The purpose of this paper is to study empirically the various properties relating joint receipts and certainty equivalents of gambles. To that end, we describe these two concepts and possible empirical questions along with their current role in theories of decision making.

Joint Receipt and Certainty Equivalents

Joint receipt (JR). Joint receipt is the operation of receiving two or more things at once—e.g., gifts at special occasions, bills and checks in the daily mail, shopping, etc. It can be viewed as a mathematical (nonnumerical) operation in much the same way that placing objects on a pan balance is viewed as an operation. Despite the ubiquitousness of joint receipt, this concept as a mathematical operation has played no role in the study of utility until recently. This is somewhat surprising considering that the operation of physical concatenation (putting things together) has played a significant role in constructing some important physical measures—e.g., mass, length, and charge—and that behavioral scientists have bemoaned the lack of such operations in their area. The problem with many—most—pairs of a behavioral attribute and a concatenation operation is that behavioral attributes often do not exhibit the following crucial feature of physical attribute–operation pairs: if a and b possess the attribute, then so does their concatenation, denoted $a \bigcirc b$. For example, whereas individual people have measurable intelligence, no meaning has been given to the intelligence of two people as one unit. However, in the case of subjective values, if one values both a and b, then surely one also values their joint receipt $a \oplus b$, where $\oplus$ denotes the binary operation of joint receipt. Thus, perhaps, this operation can be effectively used in measuring utility.

The study of decision making under risk and uncertainty concerns preference relations over a set of risky or uncertain alternatives. Usually these alternatives consist of an assignment of consequences to corresponding events, in which case they are called *gambles* or *prospects.* If probabilities are attached to the events or are given in lieu of the events and if the consequences are sums of money, they are sometimes called *lotteries.* In this paper, we use the generic word gambles for lotteries. The joint receipt of two gambles means playing the two gambles independently and receiving one of the several possible combinations of outcomes. We provide a specific example in the next section.

Some economists who have discussed this issue have observed a possible difficulty in adopting the operational aspect of joint receipt and have evolved a way of deemphasizing it. They have pointed out that some goods exhibit a decided complementarity: If one is

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stranded in the wilderness, bullets alone or a gun alone each have very little value, but bullets together with a gun of appropriate caliber might well be enormously valuable. Although such complementarity sometimes is clearly significant, we doubt that it plays any significant role when people are thinking of money and gambles with money consequence. Additionally, for the most part, economists have ignored the issue of complementarity and have treated joint receipt as a commodity vector (sometimes called a bundle) consisting of different goods. They have not made any attempt to study their interplay in terms of operations.

The literature on joint receipts is not large. One of the earliest appearances was in an empirical paper of Slovic and Lichtenstein (1968) followed by a paper of Payne and Braunstein (1971). The next major use of the concept was in a series of papers initiated by Thaler (1985; Fishburn & Luce, 1995; Linville & Fischer, 1991; Thaler & Johnson, 1990) who studied the utility of two jointly received gambles or sums of money of the same or mixed sign. Independently of that work, Luce (1991, 1992) and Luce and Fishburn (1991, 1995) developed a choice-based utility theory, called rank- and sign-dependent utility (RSDU) theory, based on the joint receipt operation of gambles and several underlying assumptions related to the operation. We will introduce these assumptions later.

Joint receipt and convolution. When studying sums of money and gambles with money consequences, one important possible and plausible meaning for the operation $\oplus$ of joint receipt involves numerical sums. Suppose x and y are sums of money, it is certainly reasonable to suppose pure additivity,

$$x \oplus y = x + y. \qquad (1)$$

For example, it says that if a day's mail brings a check for \$100 and a bill for \$75, this is the same as receiving a single check for \$25. The empirical validity of this equation is debatable. On the one hand, Tversky and Kahneman (1992), in criticizing Luce and Fishburn's (1991) axiomatization of rank- and sign-dependent utility, took Eq. (1) for granted. They observed that this coupled with another assumption of Luce and Fishburn, i.e., utility over joint receipt is additive, results in the unacceptable conclusion that utility for money is proportional to money.[1] On the other hand, Thaler (1985) studied Eq. (1) using "real life" scenarios such as winning \$25 and \$50 in two separate world series pools versus winning \$75 in a single one. He found that $x \oplus y > x + y$ seems to be true for gains, that $x \oplus y \sim x + y$ for losses, and sometimes $>$ and sometimes $\sim$ for mixed gains and losses. He proposed the following hypotheses as consistent with his observations: the utility function is concave for gains, convex for losses, and the following *hedonic rule* obtains:

$$U(x \oplus y) = \max\,[U(x) + U(y),\ U(x + y)].$$

Subsequent empirical work (Linville & Fisher, 1991; Thaler & Johnson, 1990) and theoretical work (Luce, 1995) cast considerable doubt on the accuracy of the hedonic rule. Thus, it is important to test the empirical validity of Eq. (1).

If Eq. (1) holds, then what are its implications for the joint receipt of gambles g and h, $g \oplus h$? First, consider the specific gambles

$$g = (x,p;y) \quad \text{and} \quad h = (u,q;v),$$

where, in g, x is the consequence with probability p and y with probability $1\text{-}p$, and h is similar. If the chance events underlying g and h are independent, we see that for independently realized g and h, the joint receipt of the two gambles is the complex gamble

$$g \oplus h = (x \oplus u, pq;\ x \oplus v, p(1\text{-}q);\ y \oplus u, (1\text{-}p)q;\ y \oplus v, (1\text{-}p)(1\text{-}q)).$$

In the present experiment, $g \oplus h$ was implemented by presenting two gambles which were connected by an "&" sign (see Fig. 1). The subjects were told that they would play each gamble, first one and then independently the other, and the experimenter specified all the possible outcome combinations and the corresponding probabilities. By applying Eq. (1), right side of $g \oplus h$ is reduced to what is called the *convolution* of g and h,

$$g * h = (x + u, pq;\ x + v, p(1\text{-}q);\ y + u, (1\text{-}p)q;\ y + v, (1\text{-}p)(1\text{-}q)),$$

where the pairs of possible outcomes of the two gambles are added and the corresponding probabilities are multiplied, resulting in a four outcome gamble. Treating g and h as distributions of independent random

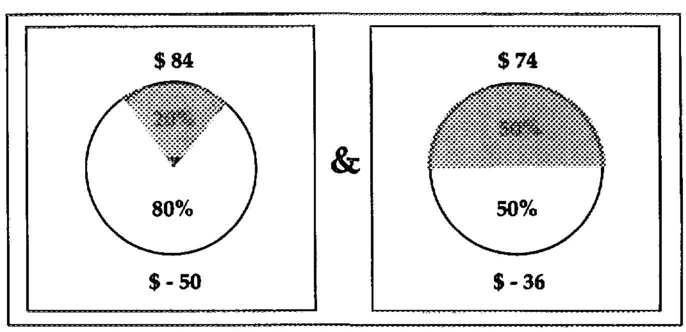

FIG. 1. A typical display of a joint receipt of two gambles.

[1] Recently, Luce and Fishburn (1995) have shown that, even if Eq. (1) holds, the same rank- and sign-dependent representation can be derived under the assumption that utility is nonadditive.

variables, $g * h$ is the distribution of their sum. Thus, a second empirical question is to test the hypothesis that $\oplus = *$, i.e., for all g and h,

$$g \oplus h = g * h \quad (2)$$

Of course, Eq. (1), which is a special case of Eq. (2), can hold without Eq. (2) holding. However, one might argue that Eq. (2) should be true for a "rational" person.

Certainty equivalents (CEs). In most theories of individual decision making under risk and uncertainty so far developed, preference is treated as a qualitative relation. However, collecting choice data is time-consuming: For n gambles, we need to compare $n(n-1)/2$ pairs. Additionally, because preferences for the same pair of alternatives seem to vary, we must collect multiple responses for each pair of alternatives and use the general tendency of the data to establish the direction of preference (Mosteller & Nogee, 1951; Davidson, Suppes, & Siegel, 1957). As a result, many studies have employed the indirect approach of establishing a certainty equivalent (CE) for each gamble, which requires only n estimates for n gambles, and then constructing the preference ordering indirectly from the CEs of the gambles. The *certainty equivalent* of a gamble is by definition the amount of money for which a person is indifferent when offered a choice between the money and the gamble. By this we mean that a larger sum of money would be chosen over the gamble, whereas the gamble would be chosen over a smaller sum.

One common way of estimating CE is to ask people to judge directly the CE of a gamble and the estimated CE is called judged CE. Another way is to estimate CE indirectly from the choice between a gamble and a monetary amount and the estimated CE is called the choice-induced CE (or simply choice CE). Most studies have employed judged CEs without establishing properties that need to be satisfied in order to study preference relations indirectly via certainty equivalents. A major reason is that most authors have assumed that CEs—however they are determined—are nothing but choice indifference points. This simple view was put in question by the discovery of the preference-reversal phenomenon: for some pairs of gambles g and h, many subjects choose g over h and yet when asked to assign CEs to the gambles, they assign a larger CE to h than g, i.e., $\mathrm{CE}(h) > \mathrm{CE}(g)$ (Lichtenstein & Slovic, 1971). Many subsequent studies have confirmed that discovery. Perhaps the most vivid and systematic demonstration is that of Mellers, Chang, Birnbaum, and Ordóñez (1992).

The preference-reversal phenomenon demonstrates that the preference order constructed indirectly from the judged CEs of gambles does not preserve the preference order exhibited when the gambles are compared directly. In other words, the judged CEs do not satisfy an order-preserving property.

To estimate the choice-derived CEs, Bostic, Herrnstein, and Luce (1990) adopted a traditional psychophysical up-down method called Parameter Estimation by Sequential Testing (PEST). The PEST procedure estimates CE from the repetitive presentation of monetary amount paired with a gamble. They found that for some gambles, there is a substantial difference between judged and choice-induced CEs and, by using the choice CEs, the preference-reversal phenomenon is reduced to the noise level of the estimates. Their results imply that choice-induced CEs satisfy the order-preserving property.

Another important property that CE needs to satisfy is consequence monotonicity: if a single money outcome x in gamble g is replaced by $x' > x$, yielding gamble g', then $\mathrm{CE}(g') > \mathrm{CE}(g)$. This property can be viewed as a special case of the order-preserving property where only one outcome is replaced by another. Mellers, Weiss, and Birnbaum (1992) have shown that when subjects give a judged CE for each gamble, then for gambles with a zero (or near to zero) outcome, monotonicity fails when this small outcome, which occurs with probability less than .10 or .15, is replaced by a somewhat larger outcome. von Winterfeldt, Chung, Luce, and Cho (1995) replicated this finding, although not as strongly, but they also did not reject consequence monotonicity when the PEST procedure is used to establish CEs.

Relating joint receipt and certainty equivalents. Luce (1991, 1992) and Luce and Fishburn (1991) developed a rank- and sign-dependent utility theory based on several assumptions about the joint receipt operation. In testing the empirical validity of these assumptions, Cho, Luce, and von Winterfeldt (1994) adopted an indirect method of estimating CEs. As RSDU theory is a choice-based theory, they estimated choice-derived CEs using a PEST procedure: Subjects were asked to choose between playing two given gambles separately and receiving a monetary amount. A choice CE for the JR of two gambles was derived by systematically changing the monetary amount. Two assumptions, called duplex decomposition and segregation,[2] were sustained using median tests. Slovic and Lichtenstein (1968) first studied the former assumption in the context of exploring the importance of variance of outcomes, and found that the assumption was sustained (see also Payne & Braunstein, 1971). Luce's

[2] Because we do not study duplex decomposition in this paper, we do not state it explicitly. The other assumption, segregation, is explicitly defined below.

(1992) theory of certainty equivalents also involved the same basic axioms of segregation and duplex decomposition. The third assumption underlying the RSDU theory was the additivity of certainty equivalents over joint receipt, i.e., $\text{CE}(g \oplus h) = \text{CE}(g) + \text{CE}(h)$. Cho *et al.* (1994) tested a special case of this assumption where one of the components was a sum of money, y, instead of a gamble, i.e., $\text{CE}[(x,p;0) \oplus y)] = \text{CE}(x,p;0) + y$. Although the additivity was not statistically rejected using median tests, there was a strong tendency for subadditivity for gains and superadditivity for losses, i.e., $\text{CE}[(x,p;0) \oplus y)] < \text{CE}(x,p;0) + y$, $x, y > 0$ and $\text{CE}[(x,p;0) \oplus y)] > \text{CE}(x,p;0) + y$, $x, y < 0$. These results were not compatible with those of Thaler and Johnson (1990) who reported support for the additivity for gains but not for losses. Since Thaler *et al.* studied the joint receipts of sums of money with "real scenarios," the two studies are not exactly comparable.

The empirical results of the test of additivity of CEs raise questions about how the CE of a joint receipt is related to the CEs of its several components. Specifically, if x and y represent sums of money, g and h denote gambles, and $\oplus$ the operation of joint receipt, then what are the necessary and sufficient assumptions in expressing $\text{CE}(x \oplus y)$, $\text{CE}(x \oplus g)$, and $\text{CE}(g \oplus h)$ in terms of x, y, $\text{CE}(g)$, and $\text{CE}(h)$? Motivated by this question, Luce (1995) studied the relation between joint receipt and certainty equivalents of gambles. The present paper tests a number of the assumptions from that paper, along with the tests of the questions discussed before. As was discussed earlier, in constructing the preference relations to test these assumptions, we estimate choice CEs.

Assumptions about CE and JR

We divide the several assumptions regarding CEs and JR into two groups: background ones that we believe do not need to be tested either because they are obvious (trivial) or because they have been shown to hold; and more substantive hypotheses that are less obviously true and therefore need to be tested. Here we introduce only the substantive ones, but the whole network of assumptions, which is called a joint receipt certainty equivalence (JRCE) structure, is described in the appendix.

Suppose G is a set of gambles with $g,h \in G$, including (as a special case) all possible sums of money which is modeled as the set $\Re$ of real numbers; $\oplus$ is a binary operation on G; G^c is the binary closure[3] of G under the operation $\oplus$; $\succsim$ is a binary preference relation on G; and CE is a function from G into the real numbers $\Re$. Then the assumptions can be stated as follows:

(H1) CE IS ORDER-PRESERVING OVER G^c. *For* $g,h,g',h' \in G$, $g \oplus h \succsim g' \oplus h'$ *if and only if* $\text{CE}(g \oplus h) \succsim \text{CE}(g' \oplus h')$. (3)

(The identification of the hypotheses, H1, H2, and H4, follows the notion of the appendix.) This assumption is essential when we use CEs to study preferences involving JR of gambles, just as the order-preserving property is basic when using CEs to order gambles. It is conceivable that this assumption may not hold. However, we did not attempt to test it directly in this study for two reasons. First, in a preference reversal study, Bostic, Herrnstein, and Luce (1990) found the order-preserving property of choice-based CEs over gambles, G, to be consistent with the data. Because this assumption seems quite justified over the set G of gambles, we extrapolated that result to the case of joint receipt of gambles. Second, testing the assumption is only possible if we have, in addition to estimating CEs, a method to determine $g \oplus h \succsim g' \oplus h'$ directly, which is not easy to do because the procedure is time-consuming and the data are noisy.

Equality of joint receipt and convolution. As was noted earlier, one of our key questions is whether Eq. (2) holds. Given that we are working with CEs, this has to be restated[4] as

$$\text{CE}(g \oplus h) = \text{CE}(g * h). \tag{2'}$$

We will call this equation *the equality of joint receipt and convolution.*

Additivity of JR over money. Eq. (1) is the special case of Eq. (2). Assuming $\text{CE}(x) = x$ and CE is order-preserving over sums of money, its testable form is

$$\text{CE}(x \oplus y) = x + y \tag{1'}$$

We call this assumption *the additivity of joint receipt for money* (or more briefly, *additivity of* JR_M).

Monotonicity of convolution. As discussed before, one possible way that a person might treat the joint receipt of lotteries is as their statistical convolution. When one thinks of the lotteries g and h as random variables, $g * h$ is the distribution of the sum of these two independent random variables. Thus, it is natural to consider whether the following monotonicity condition holds.

(H4) MONOTONICITY OF CONVOLUTION. *For all* $f,g,h \in G$,

$$g \succsim h \text{ if and only if } g * f \succsim h * f. \tag{4}$$

We must transform Eq. (4) into a testable form involving CEs. The most obvious version,

[3] The binary closure means that G^c contains G and all elements of the form $g \oplus h$, where $g,h \in G$.

[4] We will use the same number for a property and for its testable equivalence involving CEs, but with a prime on the latter.

$$\text{CE}(g) \geqslant \text{CE}(h) \text{ if and only if } \text{CE}(g * f) \geqslant \text{CE}(h * f)$$

is somewhat tricky to test because it may well hold for some combinations of f,g,h and not for others. It would be better if we could find a condition, preferably an equality rather than an inequality, that is equivalent to monotonicity of $*$. To that end, we have the following result from Luce (1995): *Suppose that* $\langle G^c, \succsim, \oplus\rangle$ *is a JRCE structure of lotteries such that convolution* $*$ *is a closed operation. Then,* $*$ *is monotonic with respect to* $\succsim$ *if and only if*

$$\text{CE}(g * h) = \text{CE}(g) + \text{CE}(h). \qquad (4')$$

In the present study, we call this property *the monotonicity of convolution.*

Monotonicity of JR over gambles. A closely related question is whether $\oplus$ is monotonic. If we should find that $\oplus = *$ and $*$ is monotonic, then we know that $\oplus$ is monotonic. However, if either of these is false, we cannot conclude anything about the monotonicity of $\oplus$. This property can be formulated as:

(H2) MONOTONICITY OF JR OVER GAMBLES. *For all* $f,g,h \in G$,

$$g \succsim h \text{ if and only if } g \oplus f \succsim h \oplus f. \qquad (5)$$

This is a very useful property in building theories involving JR. Indeed, we know of no theory based on a binary operation that does not make this assumption, explicitly or implicitly. Of course, the question is how to relate Eq. (5) to the CE of gambles. Luce (1995), using H1 and the background assumptions, proved the following theorem: *Suppose that* $\langle G^c, \succsim, \oplus\rangle$ *is a JRCE structure. Then, for all* $g,h \in G$,

(i) $\oplus$ *is monotonic with respect to* $\succsim$ *if and only if*

$$\text{CE}(g \oplus h) \sim \text{CE}(g) \oplus \text{CE}(h). \qquad (5')$$

(ii) *Suppose* $\oplus$ *is monotonic. Then,*
$x \oplus y \sim (>,<)\ x + y$, *for all* $x,y \in \Re$, *if and only if*

$$\text{CE}(g \oplus h) = (>,<)\ \text{CE}(g) + \text{CE}(h). \qquad (6)$$

For an empirical realization, we can rewrite $\text{CE}(g) \oplus \text{CE}(h)$ as $\text{CE}[\text{CE}(g) \oplus \text{CE}(h)]$ by using that CE is order-preserving and $\text{CE}(x) = x$. Thus, we can restate Eq. (5′) as: $\oplus$ being monotonic is equivalent to

$$\text{CE}(g \oplus h) = \text{CE}[\text{CE}(g) \oplus \text{CE}(h)]. \qquad (5'')$$

Put another way, monotonicity of JR is equivalent to being able to substitute for each gamble its CE. We call the test of Eq. (5″) *the monotonicity of JR over gambles,* (or more briefly, *monotonicity of* JR_G) as given in the subsection heading. We call the test of Eq. (6) *the additivity of joint receipt over gambles* (or more briefly, *additivity of* JR_G).

Although monotonicity of joint receipt was assumed in the several theories mentioned above and in an empirical test of other assumptions about joint receipt (Cho *et al.*, 1994), it has not itself been tested.

Additivity of JR_G *over CEs.* In obtaining data to test the preceding hypotheses, we will also be able to check one additional relation by examining $\text{CE}[\text{CE}(g) \oplus \text{CE}(h)] = \text{CE}(g) + \text{CE}(h)$, which is called *additivity of* JR_G *over CEs.* This indirect relation is in principle the same as the test of additivity of JR_M.

Segregation. We turn next to the idea of "adding" or "subtracting" a common amount to each consequence arising in a gamble. Although Tversky and Kahneman (1986) used the term segregation broadly to describe the editing of a gamble, they did not provide concrete rules. One specific form of an editing rule (see below) was assumed in deriving the rank-dependence for gains and losses separately in Luce and Fishburn's rank- and sign-dependent theory. There are four different terms to be considered for $x,y > 0$, or $x,y < 0$:

(a) $\text{CE}(x \oplus y,p;y)$

(b) $\text{CE}[(x,p;0) \oplus y]$

(c) $\text{CE}(x + y,p;y)$

(d) $\text{CE}(x,p;0) + y$.

The relation of (a) and (b) being equal, which we think of as the fundamental idea, is called *segregation.* The other two arise from (a) and (b) by replacing $\oplus$ by $+$. Cho *et al.* (1994) examined the equality of (b) and (c), and the median data did not reject the hypothesis that they are equal. They also tested the equality of (b) and (d) and rejected that equality. Rather, they found subadditivity [(b) < (d)] for gains and superadditivity [(b) > (d)] for losses. This implies that at least one of the following is wrong: CE is order-preserving over JR (H1), monotonicity of $\oplus$ (H2), or the additivity of JR_M. Moreover, that study leaves in doubt how (c) and (d) are related; we call their equality

$$\text{CE}(x + y,p;y) = \text{CE}(x,p;0) + y, \qquad (7)$$

additive segregation.

One goal of the present experiment is to explore the resulting subnetwork of six possible equalities. We collect sufficient data to estimate each of the four terms and so we can test segregation and additive segregation as well as the remaining four comparisons among the terms.

Gamblers vs Nongamblers

There is everyday evidence suggesting that either Eq. (2), i.e., $g \oplus h = g * h$, or monotonicity of convolution may not hold for some people: Millions of people daily purchase lottery tickets which, taking into account the purchase price, have a small expected value, usually about −$.50. Letting 0 denote the status quo, g the lottery, EV(g) its expected value after purchase, and $\succ$ preference, we have

$$g \succ 0 \quad \text{and} \quad \mathrm{EV}(g) < 0. \tag{8}$$

Let g' and g be two different tickets of the same type, then if convolution, $*$, is monotonic we have that $g' * g \succ 0 * 0 = 0$ and $\mathrm{EV}(g' * g) = \mathrm{EV}(g') + \mathrm{EV}(g) = 2\mathrm{EV}(g) < 0$. If we let g^n denote n independent convolutions of g, we see that monotonicity implies

$$g^n \succ 0 \quad \text{and} \quad \mathrm{EV}(g^n) = n\mathrm{EV}(g) < 0.$$

Of course, the standard deviation of n convolutions grows much more slowly than the mean, namely, as $\sqrt{n}$ versus n. Thus, for sufficiently large n, we expect a change in preference, $g^n \prec 0$, because the probability of an unusually large payoff relative to the mean decreases with increasing n. Certainly very few people buy more than a few lottery tickets at a time. So, there appears to be something inconsistent with the convolution analysis and, thus, it may be worth studying: either Eq. (2) is false, i.e., $\oplus \neq *$, or $*$ is not monotonic in preferences.

Moreover, this line of argument suggests that there might be at least two classes of people whose behavior may be different. A lot of people exhibit the gambling behavior of Eq. (8) on a regular basis, but a lot do not. Perhaps these two groups see $\oplus$ and $*$ differently, and so we probably should make an effort to partition our subjects according to their disposition to prefer lotteries with small negative expected values. Except for Schneider and Lopes (1986), who collected data separately for risk-prone and risk-averse people in the study of risk, most studies use aggregated data and do not attempt to distinguish among subjects in any way. Since we need to distinguish among subjects and, due to procedural difficulties, we are not able to get sufficient data to study them individually, we will be content simply to partition them prior to analyzing the data. To this end, we constructed 10 gambles with EV of losing $2 and classed the subjects by their CEs for these gambles. The specific procedure for a partition is described under the results and conclusions section.

Summary of the assumptions to be tested. Table 1 summarizes all of the tests to be made. We distinguish those hypotheses that are tested by indirect manipulation of gamble sets and those that are to be examined indirectly by recombining the preexisting conditions.

EXPERIMENT

Subjects

Forty undergraduate students at the University of California at Irvine were recruited for the experiment through a campus advertisement. Each subject participated in three different sessions and received $36.

Stimuli

The stimuli were presented on a computer screen. A typical display of a binary gamble is shown in Fig. 2. A balance beam was used to represent indifference between money on the right side and a gamble on the left side. The money amounts were always written inside a green box. The gamble was characterized by a pie chart, where the sizes of the colored pie segments were proportional to the probabilities of receiving the corresponding outcomes. An enlarged version of this gamble was also presented above it, where the actual probabilities were written inside the two pie segments and the corresponding outcomes were written outside the circle along the arc of each pie region. The subject's task was to choose either the money amount or the

TABLE 1
Conditions To Be Tested in the Experiment

Hypothesis	Condition
Direct tests	
Equality of JR and convolution	$\mathrm{CE}(g \oplus h) = \mathrm{CE}(g*h)$
Additivity of $\mathrm{JR_M}$	$\mathrm{CE}(x \oplus y) = x + y$
Monotonicity of convolution	$\mathrm{CE}(g*h) = \mathrm{CE}(g) + \mathrm{CE}(h)$
Monotonicity of $\mathrm{JR_G}$	$\mathrm{CE}(g \oplus h) = \mathrm{CE}[\mathrm{CE}(g) \oplus \mathrm{CE}(h)]$
Additivity of $\mathrm{JR_G}$	$\mathrm{CE}(g \oplus h) = \mathrm{CE}(g) + \mathrm{CE}(h)$
Segregation (a) = (b)	$\mathrm{CE}(x \oplus y,p;y) = \mathrm{CE}[(x,p;0) \oplus y]$
Additive segregation (c) = (d)	$\mathrm{CE}(x + y,p;y) = \mathrm{CE}(x,p;0) + y$
Indirect tests	
Additivity of $\mathrm{JR_G}$ over CEs	$\mathrm{CE}[\mathrm{CE}(g) \oplus \mathrm{CE}(h)] = \mathrm{CE}(g) + \mathrm{CE}(h)$
Variants on segregation	
(a) = (c)	$\mathrm{CE}(x \oplus y,p;y) = \mathrm{CE}(x + y,p;y)$
(a) = (d)	$\mathrm{CE}(x \oplus y,p;y) = \mathrm{CE}(x,p;0) + y$
(b) = (c)	$\mathrm{CE}[(x,p;0) \oplus y] = \mathrm{CE}(x + y,p;y)$
(b) = (d)	$\mathrm{CE}[(x,p;0) \oplus y] = \mathrm{CE}(x,p;0) + y$

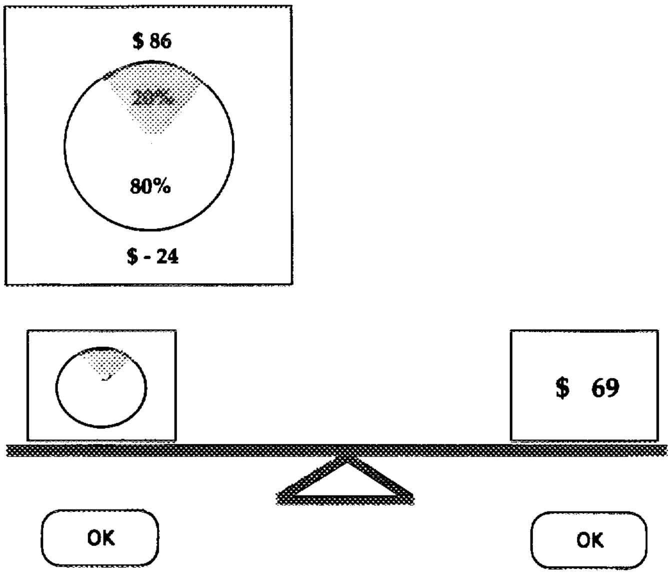

FIG. 2. A typical display of a binary gamble.

gamble, and to click the corresponding OK button on the bottom to show his/her preference.

Since some hypotheses have common conditions with others, some stimuli were used in several conditions. The hypotheses can be partitioned into two groups in relation to the types of stimuli: One group—the equality of JR and convolution, the monotonicity of JR_G, the monotonicity of convolution, and the additivity of JR_G—involves the manipulation of two gambles. The other group—the additivity of JR_M, segregation, and additive segregation—involves the manipulation of two sums of money, gains and losses separately.

TABLE 2

Sets of Test Stimuli and Screening Stimuli with Their EVs

Stimulus type	Gamble	EV
g_1	(84,.2;–50)	–23.2
g_2	(96,.5;–44)	26.0
g_3	(104,.9;–38)	89.8
h_1	(63,.2;–40)	–19.4
h_2	(74,.5;–36)	19.0
h_3	(90,.9;–32)	77.8
$g_1 * h_1$	(147,.04;44,.16;13,.16;–90,.64)	–42.6
$g_1 * h_2$	(158,.1;48,.1;24,.4;–86,.4)	–4.2
$g_2 * h_3$	(186,.45;64,.05;46,.45;–76,.05)	103.8
$g_3 * h_1$	(167,.18;64,.72;25,.02;–78,.08)	70.4
$g_3 * h_3$	(194,.81;72,.09;52,.09;–70,.01)	167.6
Segregation	(39 ⊕ 80,.2;80)	87.8
	(45 ⊕ 60,.5;60)	82.5
	(87 ⊕ 50,.9;50)	128.3
Screening gambles	(99,.1;–13)	-1.8
	(86,.2;–24)	–2.0
	(68,.3;–32)	–2.0
	(72,.4;–51)	–1.8
	(44,.5;–48)	–2.0
	(86,.5;–90)	–2.0
	(54,.6;–86)	–2.0
	(23,.7;–60)	–1.9
	(17,.8;–78)	–2.0
	(8,9;–92)	–2.0

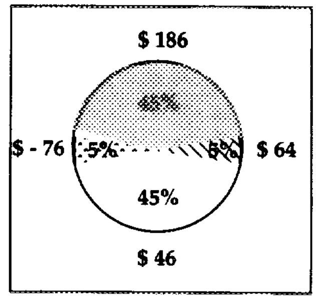

FIG. 3. A typical display of a convolved gamble.

The gambles were constructed as follows. There were two types of binary gambles of mixed outcomes: $g_i = (x_i,p_i;y_i)$ and $h_i = (u_i,q_i;v_i)$, where $i = 1,2,3$, $x_i,u_i > 0$ and $y_i,v_i < 0$, with the values shown in Table 2. A total of five pairs—(g_1,h_1), (g_1,h_2), (g_2,h_3), (g_3,h_1), and (g_3,h_3)—were selected to avoid similar outcomes in their convolution. These five pairs were always used in constructing the gamble pairs for joint receipt ($g \oplus h$), convolution ($g * h$), and additivity over gambles [CE(g) + CE(h)]. Because g and h were mixed gambles, the tests of assumptions involving these gambles were limited to the mixed case, but not to the gains or losses.

Equality of JR and convolution. To estimate CE($g \oplus h$), pairs of binary gambles were presented in the format of the joint receipt shown in Fig. 1, where the "&" represents playing the two gambles independently and sequentially. To estimate CE($g * h$), the gamble of $g * h$ was constructed by adding the pairs of possible outcomes and multiplying the corresponding probabilities of the two gambles. Specifically, let $g = (x,p;y,1\text{-}p)$ and $h = (u,q;v,1\text{-}q)$. Then, as noted earlier,

$$g * h = (x + u,pq;x + v,p(1\text{-}q);y + u,\\ (1\text{-}p)q;y + v,(1\text{-}p)(1\text{-}q)).$$

The colors red, green, blue, and yellow were used to represent the probabilities. A typical display[5] is shown in Fig. 3. There were five pairs of $g \oplus h$ and five of $g * h$.

Monotonicity of convolution. Because the condition of estimating CE($g * h$) was the same as above, no new

[5] Throughout the experiment, the stimulus display always consists of a gamble, its enlarged version, a sum of money, and a balance beam as in Fig. 2. Because only the display of a gamble was changed according to the conditions, we present the enlarged gamble only, omitting the rest of the display.

stimuli were needed for this condition. To estimate CE(g) + CE(h), the CEs for the binary gambles g and h were first estimated and simply added. Thus, 6 binary gambles were constructed for this test.

Monotonicity of JR_G. Because the condition of estimating CE($g \oplus h$) was the same as above, no new stimuli were needed for this condition. To estimate CE[CE(g) $\oplus$ CE(h)], we first estimated the CEs of the g and h gambles from each subject and later presented the two CEs of each pair in a format of joint receipt. Each CE was represented by a pie chart with 100% chance of receiving the outcome, either CE(g) or CE(h). Therefore, the entire pie chart was filled by one color, blue. As the 6 binary gambles were already estimated, the newly constructed gambles were 5 pairs of joint receipts, CE(g) $\oplus$ CE(h).

Additivity of JR_G. Because CE($g \oplus h$), CE(g), and CE(h) were estimated, no further estimates were necessary to test Eq. (1).

The other set of hypotheses involves a manipulation of two amounts of money, x and y. The outcomes were either gains only, $x,y > 0$, or losses only, $x,y < 0$. For gains, the sum x = \$39 was paired with y = \$80 and p = .2, x = \$45 with y = \$60 and p = .5, and x = \$87 with y = \$50 and p = .9. These stimuli are shown in Table 2. For losses, only the sign of the outcome was reversed.

Additivity of JR_M. To estimate CE($x \oplus y$), we used the same display as was used for CE[CE(g) $\oplus$ CE(h)]. Therefore, x and y were represented by two whole pie charts, each with 100% probability. The condition $x + y$ is simply a calculation, not a measurement. Thus, there were six gambles, three for gains and three for losses.

Segregation. To estimate CE($x \oplus y,p;y$), a binary gamble was presented as above except that the two outcomes, x and y, were written side by side along the arc of the corresponding probability. The typical display is shown in Fig. 4. To estimate CE[($x,p;0$) $\oplus$ y], a binary gamble ($x,p;0$) and an amount of money, y, were presented together in a format of joint receipt. The typical display is shown in Fig. 5. Thus, there were 12 gambles, 6 of type ($x \oplus y,p;y$) and 6 of type [($x,p;0$) $\oplus$ y].

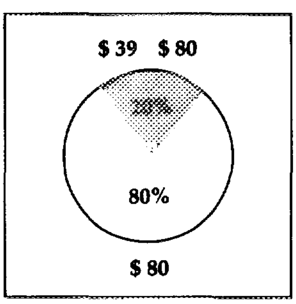

FIG. 4. A typical display of a binary gamble used in segregation test.

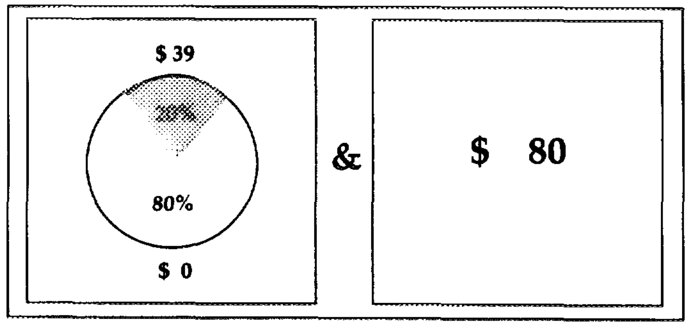

FIG. 5. A typical display of a joint receipt of a gamble and an amount of money.

Additive segregation. To estimate CE($x + y,p;y$), a binary gamble of type ($z,p;y$), where $z = x + y$, was presented. For CE($x,p;0$) + y, we first estimated CE($x,p;0$) and simply added y to it. The display of ($x,p;0$) was the one of a binary gamble. Again, there were 12 gambles, 6 of type ($z,p;y$) and 6 of type ($x,p;0$).

Classification of gamblers/nongamblers. To uncover a subject's attitude about risk, 10 additional mixed gambles with an EV of approximately −\$2 were constructed. We call these *screening gambles.* They are also shown in Table 2.

In total, there were 10 + 6 + 5 + 6 + 12 + 12 + 10 = 61 test gambles. In addition, four sets of filler gambles were constructed by multiplying the outcomes of test gambles by ⅓, ½, 2, or 3. The role of filler gambles is explained in the procedure section.

Procedure

Sessions. The experiment consisted of three sessions on separate, but not necessarily consecutive, days. All 40 subjects participated in all three sessions. In the first session, there were 19 test gambles—3 type g and type h gambles, 5 type $g * h$, 5 type $g \oplus h$, and 3 screening gambles—along with 4 × 19 = 76 filler gambles. In the second session, there were 20 test gambles—5 type CE(g) $\oplus$ CE(h), 6 type ($x \oplus y,p;y$), 6 type ($x,p;0$) $\oplus$ y, and 3 screening gambles—and 4 × 20 = 80 filler gambles. In the last session, there were 22 test gambles—6 type ($x + y,p;y$), 6 type ($x,p;0$), 6 type ($x \oplus y$), and the remaining 4 screening gambles—and 4 × 22 = 88 filler gambles.

Estimating CEs. As was remarked earlier, it is an issue how best to estimate the CEs. Although judged CEs, which are obtained by asking subjects to state directly the amount of money that they see as indifferent to the gamble, require little time to collect, we know that they are not good estimates of choice CEs upon which the theory is based (Bostic, Herrnstein, & Luce, 1990; Mellers, Weiss, & Birnbaum, 1992; Tversky, Sattath, & Slovic, 1988). In this study, we have elected to estimate the choice CEs. To that end, we adopted the PEST procedure, which Bostic *et al.* (1990) and Cho *et al.* (1994) used in the study of decision making.

This estimation procedure was implemented as follows: On each trial, the subject had to choose between a gamble (or a pair of jointly received gambles) and an amount of money. The computer monitored the internal consistency of the subject's response in the following sense. If either the money amount exceeded the largest possible outcome of the gamble and the subject selected the gamble or if the money amount was less than the smallest possible outcome of the gamble and the subject selected the money, then a warning message appeared asking the subject to reconsider his/her choice. The same stimulus was repeatedly presented until the subject provided a reasonable choice.

In the first round, the whole set of test gambles, not including any filler gambles, was randomized and presented individually. The initial money amount for each gamble was chosen according to a uniform distribution over the range of the gamble, i.e., over the interval from the smallest to the largest outcome. The initial step size—an increase or decrease in the money amount—was set at ⅕ of the range of the outcomes. After each trial, the computer modified the money amount by either adding or subtracting the initial step size according to whether the subject chose the gamble or the money, and stored the modified money amount until that stimulus next appeared in the second round.

In the second round, the whole set of test gambles was rerandomized and presented with their corresponding modified amounts of money. If, in this round, the choice was opposite to the previous one, then the step size was reduced by half. If the choice was the same as the previous round, then the step size was not changed. In succeeding rounds, the above rule was followed. However, when the choice was the same on several successive rounds, the step size was doubled as follows: (1) before the first reversal, the step size was doubled after three choices in the same direction, and (2) after the first reversal, it was doubled after two choices in the same direction. After that, the step size was repeatedly doubled until a reversal occurred.

Each test gamble appeared in each round until the step size became less than 1/50 of the range of outcomes in that gamble, called *the stopping step size.* At this point the gamble was removed, and its choice CE was estimated to be the average of the lowest accepted and the highest rejected money amounts among the last series of three responses.

When a test gamble reached the PEST criterion for removal, it was replaced by a filler gamble of the same type, chosen from the first set of filler gambles. If a replaced filler gamble was removed while there were still test gambles in the set, it was replaced by a filler gamble of the same type, chosen from the second set of filler gambles, and so on. Four sets of filler gambles sufficed. The purpose of the filler gambles was to help mask the removal of test gambles and to keep the average reappearance interval of each test gamble constant throughout the experiment.

The experiment was terminated when all of the test gambles were replaced by filler gambles. The procedure was the same for all sessions of the experiment. Each session took from 1 to 2 hours, totaling as much as six hours.

TABLE 3
Several Classes of Gamblers and Nongamblers

Gambler			Nongambler		
Type	Criterion	*N*	Type	Criterion	*N*
Strict	≥8	10	Lenient	≤7	30
Moderate	≥7	14	Moderate	≤6	26
Lenient	≥6	20	Strict	≤5	20

RESULTS AND CONCLUSIONS

Types of Analyses

Classification of gamblers/nongamblers. As was explained in the Introduction, we felt that it was important to analyze the data separately according to a subject's behavior on the 10 screening gambles. Because, for most subjects, the CEs of the 10 gambles were not uniformly larger (or smaller) than the status quo, 0, we used the number of times that each subject showed $CE(g) > 0$ as a measure of his/her attitude toward gambling.[6] Table 3 gives the nomenclature that we have used and the resulting sample sizes. Note that the criterion means the number of positive CEs out of the 10 CEs for the screening gambles. Observe that the class of lenient types includes all of the moderate type subjects and more, and the class of moderate types includes all of the strict ones and more. Thus, for example, the 14 subjects classed as moderate gamblers consist of the 10 subjects listed as strict gamblers plus an additional 4. We present the data just for the moderate gamblers and nongamblers. For the other two classifications, we simply state the results in the text, along with the *z*-scores, when necessary.[7]

[6] We also classified the gamblers and non-gamblers according to the relation between CEs and EV, rather than the status quo. The pattern of the data was basically the same.

[7] The data tables for the other classifications are available upon request.

We also asked subjects some questions about their gambling behavior such as visits to Las Vegas, purchasing lottery tickets, and whether they have engaged in any form of gambling behavior such as playing poker with money. All of these had no significant correlations with our behavioral criterion. Thus, it may be inappropriate to use the labels "gamblers" and "nongamblers" for our classification, but whatever the label, this distinction partitions the subjects in a useful manner.

Data analysis. Since there was a fair amount of difference between individuals, it is more desirable to compare the two CE conditions for each individual rather than to calculate overall means or medians of CEs for the conditions. We call the former "individual" data analysis. In doing individual data analysis, we compare the two values of CE for each subject and each gamble pair and combine the frequencies of <, =, > across all gambles under each hypothesis. We assume that, under the hypothesis of an equality of two CEs, the distribution of differences should be symmetric when an equality holds and biased positively or negatively when an inequality holds. Thus, we adopted the nonparametric sign test to test the equality of the two frequencies of < and > categories.

Several questions arise regarding the individual data analysis and the non-parametric sign test. Is it legitimate to combine the frequencies across the gamble pairs? An objection may be that each subject provides data for more than one pair, and thus the observations for the different pairs may not be independent. It would be more appropriate to apply the sign test for each gamble pair. However, additional difficulties may arise using this approach: Because the gamble pairs under each hypothesis may not show a consistent pattern, we need to establish a criterion for when to reject the hypothesis in light of mixed evidence. Moreover, for the tests of all hypotheses with the exception of some cases of segregation, each of the gamble pairs provided a similar direction of skewness, even though each case did not give rise to a statistically significant skewness. Combining frequencies across the gamble pairs will more readily reject the null hypothesis. In our case we are interested in accepting the null hypothesis, namely equality of the CEs in the two conditions, and combining the frequencies across gambles will result in lower probability of a false positive, and hence greater power of our statistical analysis.

How do we compare the two CEs and how do we treat the ties? One way is simply to compare the strict numerical values of the two CEs and to categorize them as <, =, >, and to delete the ties for the sign-test. A criticism of this method lies in the fact that the CEs are estimates, and it seems desirable to construct a defining interval for judging the equality of two CEs, rather than simply to compare the numerical values of the two estimated CEs. Our defining interval made use of the stopping rule of the PEST procedure. Recall that each test gamble was removed when the step size dropped below 1/50 of the range of the outcomes. This stopping rule means that we do not know the CEs to any greater accuracy than this number. So, we define the indifference interval to be one half of the sum of stopping step sizes of the two CEs. The resulting indifference intervals were \$1–\$2 for the tests of the segregation and the additive segregation, and \$3–\$5 for the remaining tests. Denote the CE of the gamble in the first and second position of the test condition by CE_1 and CE_2, respectively, and let D denote the corresponding indifference interval. We define $CE_1 <' CE_2$ if numerically $CE_1 - CE_2 < -D$; $CE_1 >' CE_2$ if $CE_1 - CE_2 > D$; and $CE_1 =' CE_2$ if $|CE_1 - CE_2| \leq D$. The number of subjects whose CE_1 <′, =′, >′ CE_2 were counted separately for each gamble pair, and combined across the gamble pairs. Although the sign test assumes that the frequencies for ties are small and the effect of deleting the ties are negligible, in the case of using the indifference interval, the frequencies in an equal category are large and it does not seem to be appropriate to delete the equal frequencies for the sign test. Thus, we decided to partition the equality into two parts: the equality defined using the interval and the strict numerical equality. In other words, we define $CE_1 =' CE_2$ if $-D < CE_1 - CE_2 < 0$ or $0 < CE_1 - CE_2 < D$; $CE_1 = CE_2$ if $CE_1 - CE_2 = 0$. We call the former "equality" and the latter "tie." For the sign test, we delete the tie frequencies and split the equality frequencies into the other two categories.

We analyzed the data in both ways, i.e., with and without the defined interval of "equality," and compared the results of both analyses. The results were virtually identical except for very specific cases of several hypotheses. Table 4 shows the z-scores for those four cases where the conclusions from the two analyses are not fully consistent. A major difference was found for the test of additivity of JR_M for the gambles of losses with nongamblers, but not for the gambles of gains or for the gamblers. When the two CEs were compared strictly, the distributions were asymmetric for the moderate and strict nongamblers ($z = 2.57$ and $z = 2.04$, $p < .05$) whereas with the interval the distributions were symmetric. These statistical results of asymmetry seem unrealistic considering the median CE difference of $CE(x \oplus y)$ and $x + y$ is \$1 for the gambles with EVs of \$165–\$199. Except for the test of additivity of JR_M, the other tests show different results for only one of the categorizations. For comparison, we provide the z-scores for all three categorizations. Notice that, except for the additivity of JR_M, we can draw

TABLE 4

z-Scores for the Hypotheses Which Show Different Results between the Analyses with or without Using the Interval

Category	z-Score	
	Without interval	With interval
Additivity of JR_M for losses		
Lenient nongamblers	1.76	−0.32
Moderate nongamblers	2.57*	−0.17
Strict nongamblers	2.04*	−0.37
Monotonicity of convolution		
Lenient nongamblers	−1.56	−1.15
Moderate nongamblers	−2.20*	−1.76
Strict nongamblers	−1.11	−1.21
Segregation 3 for losses		
Lenient nongamblers	−1.88	−1.88
Moderate nongamblers	−1.53	−2.0*
Strict nongamblers	−1.21	−1.62
Segregation 6 for gains		
Lenient nongamblers	−1.83	−0.97
Moderate nongamblers	−1.73	−0.92
Strict nongamblers	−1.99*	−1.32

* $p < .05$.

the same general conclusions from both analyses when all three categorizations are considered. Thus, we report only the results from the analysis using the interval, and draw conclusions based on them. In all data tables, we present (in parentheses) the frequencies for exact ties under the category of equality.

Although the statistical power of our analysis may have been increased by combining frequencies across gambles, difficulties in accepting the null hypotheses may still remain. In the above analysis, we accept the null hypotheses provided the data fail to reject it. The conclusions would be more convincing if we had additional evidence to support the equality of the two CEs. A possible approach is to use a confidence interval. Note that if the hypothesis $CE_1 = CE_2$ is true, then excluding the actual observed ties but splitting the equality frequencies, the observed inequalities $CE_1 <' CE_2$ and $CE_1 >' CE_2$ should occur, on average, equally often. Thus, from the observed proportion, one may compute the 95% confidence interval for the underlying probability, and see if $p = 0.5$ is included in that interval. These results are presented in tables along with the z-scores for the sign test. For most hypotheses, the conclusions from the confidence interval agreed with the conclusions drawn from the sign tests except for one case: For the test of (a) = (d) under segregation for the gambles of gains with gamblers, the confidence intervals did not include $p = 0.5$ even though the sign tests in general did not reject the null hypothesis. However, for the strict gamblers the z-score was significant ($p < .05$) and those for the other two classifications were nearly significant, which is almost consistent with the confidence interval results. In the following discussion, we only explicitly state the results of confidence interval if they were inconsistent with those of z-scores. We also only considered accepting the hypotheses of equality if both the sign test and confidence-interval test failed to reject these hypotheses.

The details of the inferences from the data to our hypotheses, although not especially complex, are relegated to the Appendix, where a set of background hypotheses are listed as B1–B7; the set of basic hypotheses, about which we are uncertain, are listed as H1–H4; and a summary of the data is listed as D1–D8. Although the hypothesis H1 that CE is order-preserving over joint receipt underlies many of the inferences, it is not testable using the present methods. Thus, the conclusions are stated assuming that H1 is true whenever joint receipt is involved. Clearly, H1 needs to be tested directly in the future.

It should be noted that we have only tested H1, H2, and H4 in the case of gambles with mixed gains and losses as outcomes (see Table 2). We have not tested pure gains and pure losses; this is under investigation.

Results and Conclusions about Convolution and JR

Additivity of JR_M. The frequency distributions and z-scores for the test of this assumption are shown in Table 5 separately for the gambles of gains and losses. For both gamblers and nongamblers, the frequency distributions were quite symmetric regardless of the sign. The frequency distributions for the strict and lenient gamblers and nongamblers were symmetric as well. The results imply that $CE(x \oplus y) = x + y$, which is the simplest form of the convolution. This result differs from that of Thaler (1985). Since the two experiments are not comparable in that Thaler provided a somewhat elaborate scenario and we did not, further studies are needed to understand more fully the source of difference.

Monotonicity of convolution. The frequency distributions and z-scores for the test of this assumption are shown in Table 6. For moderate gamblers, the frequency distributions were asymmetric, and the result for the strict gamblers was the same ($z = 2.57, p < .05$). However, the distributions for the lenient gamblers were symmetric ($z = 1.41$, $p > .05$). We conclude that $CE(g * h) \neq CE(g) + CE(h)$ for gamblers. Conversely, for moderate nongamblers, the distributions were symmetric. The results were the same for the lenient and strict nongamblers ($z = -1.15$, $p > .05$ and $z = -1.21$, $p > .05$, respectively). We conclude that the results imply the equality of two conditions for nongamblers.

TABLE 5
Frequency Distributions and z-Scores for the Test of Additivity of JR_M

Category	$CE_1 <' CE_2$	$CE_1 =' (=) CE_2$	$CE_1 >' CE_2$	z-Score	Confidence interval
Gains					
Gamblers	2	28 (11)	1	0.0	$0.35 < p < 0.68$
Nongamblers	0	52 (25)	1	−0.27	$0.36 < p < 0.62$
Losses					
Gamblers	0	15 (25)	2	−0.49	$0.22 < p < 0.64$
Nongamblers	1	31 (44)	2	−0.17	$0.31 < p < 0.63$

The equality $CE(g * h) = CE(g) + CE(h)$ for nongamblers implies that the convolution is monotonic for nongamblers (Proposition II(iii) in the Appendix). The inequality for gamblers implies the nonmonotonicity of $*$ and agrees with Luce's (1995) argument based on considering n-fold convolutions of the gamble. These differences certainly reinforce partitioning the subjects into gamblers and nongamblers.

Equality of JR and convolution. The frequency distributions and z-scores for the test of this assumption are also shown in Table 6. For moderate gamblers, the distributions were symmetric; those for the strict and lenient gamblers were also symmetric ($z = 0.14, p > .05$ and $z = 1.52, p > .05$, respectively). For nongamblers, the result was opposite: for all three categories, the frequency distributions are significantly asymmetric ($z = 3.23, p < .01$ for the lenient nongamblers; $z = 2.53, p < .05$ for the strict nongamblers). The results imply that JR is the same as convolution for gamblers, but not for nongamblers. This finding also reinforces the idea of partitioning the subjects as we did.

One natural issue to consider is the degree to which the two sets of data in testing the monotonicity of convolution and the equality of JR and convolution provide the same information about the CE relations. For that, we examined the covariance of the two CE conditions. The results are shown in Table 7, separately for the moderate gamblers and nongamblers. Unfortunately, these data do not indicate a strong correlation.

Results and Conclusions about Monotonicity and Additivity of JR_G

Monotonicity of JR_G. The frequency distributions and z-scores for the test of this assumption are shown in Table 8 for the moderate categories. For both gamblers and nongamblers, the frequency distributions were asymmetric. The results were the same for the strict and lenient categories. This implies that $CE(g \oplus h)$ differs from $CE[CE(g) \oplus CE(h)]$, contrary to what we had expected. Indeed, the evidence is that predominantly

$$CE(g \oplus h) < CE[CE(g) \oplus CE(h)]. \quad (9)$$

Because the hypotheses of CE being order-preserving over joint receipt of gambles (H1) and of the monotonicity of JR_G (H2) together predict that the two CEs should be equal, either one or both of the hypotheses must be wrong (Proposition II(i) in the Appendix). At the moment we are uncertain as to which is the culprit, although we suspect the monotonicity of joint receipt because the procedure used to determine CEs practically insures that the estimated CEs should be order-preserving.

Additivity of JR_G. The frequency distributions and z-scores for the test of this assumption are also shown in Table 8 for the moderate categories. For both gamblers and nongamblers, the frequency distributions

TABLE 6
Frequency Distributions and z-Scores for the Tests of Monotonicity of Convolution and Equality of JR and Convolution

Category	$CE_1 <' CE_2$	$CE_1 =' (=) CE_2$	$CE_1 >' CE_2$	z-Score	Confidence interval
Monotonicity of convolution					
Gamblers	40	13 (1)	16	2.89**	$0.56 < p < 0.78$
Nongamblers	40	28 (1)	61	−1.76	$0.34 < p < 0.50$
Equality of JR and convolution					
Gamblers	30	14 (1)	25	0.48	$0.42 < p < 0.65$
Nongamblers	69	25 (3)	33	3.19**	$0.56 < p < 0.72$

* $p < .05$.
** $p < .01$.

TABLE 7

Covariation of the Tests for Monotonicity of Convolution and the Equality of Joint Receipt and Convolution for the Moderate Gamblers and Nongamblers

		CE(g) + CE(h) vs CE($g * h$)		
		<′	=′	>′
		Moderate gamblers		
CE($g \oplus h$)	<′	21	1	3
vs	=′	5	6	4
CE($g * h$)	>′	14	7	9
		Moderate nongamblers		
CE($g \oplus h$)	<′	20	8	5
vs	=′	10	9	9
CE($g * h$)	>′	10	12	47

were asymmetric. The results were the same for the strict and lenient categories. This implies that CE($g \oplus h$) is not equal to CE(g) + CE(h). Again, contrary to what we had expected, the results showed that CE($g \oplus h$) < CE(g) + CE(h) for both gamblers and nongamblers. However, on the assumption that $x \oplus y \sim x + y$, this inequality follows from the Eq. (9) (Proposition 1(i) and (ii) in the appendix). Notice that we tested the hypothesis, $x \oplus y \sim x + y$, called the additivity of JR_M, and it was sustained. So the conclusion is reaffirmed that at least one of the first two hypotheses, i.e., H1 and H2, is in doubt.

Additivity of JR_G over CEs. The frequency distributions and *z*-scores for the test of this indirect assumption are also shown in Table 8. For the moderate level of both gamblers and nongamblers, the frequency distributions were not asymmetric. The results were the same for the strict and moderate levels. This implies the equality of the two conditions. These results were virtually the same as those for additivity of JR_M and extends the conclusion of additivity to the mixed outcomes.

Locus of the violations of monotonicity and additivity of JR_G. To locate the source of violation of monotonicity according to the signs of CE(g) and CE(h), for each pair of the gambles g and h, we categorized all the subjects into three groups based on their CE evaluations. If both CE(g) > 0 and CE(h) > 0, they were classed in the positive (evaluation) group, if both CE(g) < 0 and CE(h) < 0, they were classed in the negative (evaluation) group, and otherwise they were classed in the mixed (evaluation) group. For each group, we applied the individual data analysis. The frequency distributions, combined across the five gamble pairs, are shown in Table 9. The results of the sign test clearly showed that the monotonicity of JR_G was violated by the positive and mixed groups, and the additivity of JR_G was violated by the positive group and just missed being significantly violated for the mixed group. Neither assumption was violated by the negative group.

The results in Table 9 show that the violation of, presumably, monotonicity is localized to those pairs of gambles for which at least one gamble is perceived a positive. We are not certain what to make of that fact. Robin Keller (personal communication) has suggested the following argument. The joint receipt $g \oplus h$, being the receipt of gambles, is necessarily more risky than CE(g) $\oplus$ CE(h), which amounts just to a sum of money. As has been demonstrated in many studies, people tend to be risk averse for gains and risk seeking for losses. Thus Eq. (9) should hold for positive groups and CE($g \oplus h$) $\geqslant$ CE[CE(g) $\oplus$ CE(h)] = CE(g) $\oplus$ CE(h) for negative groups. Although the data in Table 8 are not significant for negative groups, they do go in the direction predicted by Keller's argument.

We also examined the covariation between the monotonicity of JR_G and the additivity of JR_G. As there was virtually no difference between moderate gamblers and nongamblers, the data from both gamblers and nongamblers were combined, and are shown in Table 10. The correlation is very high.

TABLE 8

Frequency Distributions and *z*-Scores for the Tests of Monotonicity and Additivity of JR_G

Category	CE_1 <′ CE_2	CE_1 =′ (=) CE_2	CE_1 >′ CE_2	*z*-Score	Confidence interval
Monotonicity of JR_G					
Gamblers	40	10 (2)	18	2.55*	$0.54 < p < 0.76$
Nongamblers	78	19 (0)	33	3.95**	$0.59 < p < 0.75$
Additivity of JR_G					
Gamblers	38	18 (1)	13	2.89**	$0.56 < p < 0.78$
Nongamblers	72	24 (3)	31	3.55**	$0.58 < p < 0.74$
Additivity of JR_G over CEs					
Gamblers	3	50 (15)	2	0.0	$0.38 < p < 0.64$
Nongamblers	6	91 (27)	6	0.0	$0.41 < p < 0.60$

* $p < .05$.
** $p < .01$.

TABLE 9
Frequency Distribution and z-Scores as a Function of the Signs of CE(g) and CE(h)

Signs of CE(g) and CE(h)	$CE_1 <' CE_2$	$CE_1 <' (=) CE_2$	$CE_1 >' CE_2$	z-Score	Confidence interval
Monotonicity of JR					
Positive	68	18 (1)	13	5.43**	$0.69 < p < 0.85$
Negative	12	4 (1)	19	−1.01	$0.26 < p < 0.56$
Mixed	38	7 (0)	19	2.38*	$0.53 < p < 0.76$
Additivity of JR_G					
Positive	68	22 (2)	8	5.96**	$0.68 < p < 0.85$
Negative	10	7 (1)	18	−1.35	$0.22 < p < 0.52$
Mixed	32	13 (1)	18	1.76	$0.49 < p < 0.72$

* $p < .05$.
** $p < .01$.

Future Work on Order-Preserving and Monotonicity

Because the order-preserving property of CE over JR and the monotonicity of joint receipt are both basic to the theories so far developed, it is disconcerting to find that at least one of them is violated. Of the two, probably the former is the more crucial to testing several properties of JR by estimating CEs and using the results to evaluate choice theories; whereas, the latter seems more crucial in developing theories of choice. Nonetheless, at least one must be wrong. Which one will have to be decided by further empirical investigations. Note that due to the nature of stimulus construction, we tested the monotonicity hypothesis using only mixed gambles, but not using gambles of gains or losses. Further empirical tests are needed for the remaining two cases as well. To verify the order-preserving property of CE over JR, one needs to test directly the hypothesis that for $g,h,g',h' \in G$,

$$g \oplus h \succsim g' \oplus h' \text{ iff } CE(g \oplus h) \geqslant CE(g' \oplus h').$$

As for the monotonicity of JR_G, it is not clear whether monotonicity of JR_G would be violated in some global fashion—for $f,g,h \in G$, $g \succsim h$ and $g \oplus f \succsim h \oplus f$—or whether the violation occurs because one evaluates the $\oplus$ in $g \oplus h$ in a different fashion from the $\oplus$ in $CE(g) \oplus CE(h)$.

TABLE 10
Covariation of the Tests for Monotonicity of Joint Receipt and Additivity of Joint Receipt over Gambles

		CE(g ⊕ h) vs CE[CE(g) ⊕ CE(h)]		
		<′	=′	>′
CE(g ⊕ h)	<′	109	1	0
vs	=′	9	30	7
CE(g) + CE(h)	>′	0	0	44

Note. Moderate gamblers and nongamblers are combined.

Results and Conclusions about Segregation

Since there were three pairs of gambles in the gambles of gains and losses each, the frequencies were combined across these three pairs.

Segregation and additive segregation. The frequency distributions and z-scores for the tests of these assumptions are shown in Table 11 for the moderate category. None of the distributions exhibited significant asymmetry. The results were the same for the other two categories. These results imply that, for both gamblers and non-gamblers, (a) = $CE(x \oplus y,p;y)$ is equal to (b) = $CE[(x,p;0) \oplus y]$ and (c) = $CE(x + y,p;y)$ is equal to (d) = $CE(x,p;0) + y$. Thus, both segregation and additive segregation assumptions are sustained.

As before, the covariance data are given in Table 12. It does not appear that there is a powerful common factor to these two versions of segregation.

Variants of segregation. The frequency distributions and z-scores for the tests of these assumptions are shown in Table 13 for the gambles of gains and losses separately. For the moderate category, the results showed that almost all the inequalities occurred for the gamble of gains, but not for the gambles of losses. The results for the strict and lenient categories were the same. For the gambles of gains, the gamblers generally showed more inequalities than the nongamblers and the directions of inequalities were opposite for the gamblers and the nongamblers. However, the results for the gambles of gains were slightly different for the different classifications of gamblers and nongamblers: We present the z-scores for the strict and lenient categories in Table 14. In general, the equality failed to hold for all the gamblers, but not for the nongamblers except for the test of (b) = (d). Specifically, for the tests of (a) = (c) and (b) = (c), the equality failed to hold for all classes of gamblers, but not for the nongamblers. For the test of (a) = (d), the equality failed to hold only for the strict gamblers ($z = 2.31$, $p < .05$), but

TABLE 11
Frequency Distributions and z-Scores for the Test of Segregation and Additive Segregation

Category	$CE_1 <' CE_2$	$CE_1 <' (=) CE_2$	$CE_1 >' CE_2$	z-Score	Confidence interval
		Segregation			
Gains					
Gamblers	13	9 (5)	15	−0.33	$0.31 < p < 0.62$
Nongamblers	33	12 (2)	31	0.11	$0.40 < p < 0.62$
Losses					
Gamblers	21	6 (4)	11	1.46	$0.47 < p < 0.77$
Nongamblers	27	9 (7)	35	−0.95	$0.33 < p < 0.55$
		Additive segregation			
Gains					
Gamblers	11	5 (9)	17	−1.04	$0.25 < p < 0.56$
Nongamblers	24	16 (3)	35	−1.15	$0.32 < p < 0.54$
Losses					
Gamblers	11	8 (9)	14	−0.35	$0.30 < p < 0.62$
Nongamblers	34	13 (5)	26	0.94	$0.45 < p < 0.67$

not for any other classifications. We conclude that (a) = (d) for the nongamblers. However, the results of z-scores and confidence interval for the gamblers imply a strong tendency for (a) ≠ (d). For the test of (b) = (d), the equality held only for the lenient gamblers and for the strict nongamblers: for the other cases, the equality failed to hold. Thus, we conclude (b) ≠ (d) for both gamblers and nongamblers of gains.

Given that segregation, (a) = (b), and that additive segregation, (c) = (d), both hold, what do we expect to obtain for the variants? The variant (a) = (c) follows simply by replacing $x \oplus y$ by $x + y$, which as we saw in Table 5 holds for both types of subjects for both gains and losses. Nevertheless, we find empirically that (a) < (c) holds for the gamblers with gains. It was sustained for the other cases. This is a surprising inconsistency for which we have no explanation.

The variant (b) = (d) is expected to hold if monotonicity, $CE(g \oplus y) = CE(g) + y$, is correct; otherwise it should fail. Because monotonicity is rejected for gains, with $CE(g \oplus y) < CE(g) + y$ (Table 8), we predict that (b) < (d) for gains and = otherwise. We observed that the former holds for gamblers with gains, but not with nongamblers, and the latter for both cases with losses.

TABLE 12
Covariation of the Tests for Segregation and Additive Segregation for the Moderate Gamblers and Nongamblers

		$CE(x \oplus y,p;y)$ vs $CE[(x,p;0) \oplus y]$		
		<′	=′	>′
		Moderate gamblers		
$CE(x + y,p;y)$	<′	11	5	6
vs	=′	12	14	5
$CE(x,p;0) + y$	>′	11	5	15
		Moderate nongamblers		
$CE(x + y,p;y)$	<′	23	8	27
vs	=	12	11	14
$CE(x,p;0) + y$	>′	25	11	25

Beyond this, if one is willing to assume that transitivity of these inferences holds—which, given the statistical nature of the conclusions, is a somewhat heroic assumption—additional inconsistencies appear. They are most easily seen by recasting Tables 11 and 13 into the diagrams of Fig. 6 that show acceptances (double sided arrows) and rejections (directional arrows) of the several null hypotheses for gains and losses and for the gamblers and nongamblers. One sees perfect consistency for the gamblers with losses, one inconsistency each for the nongamblers with both gains and losses, and two inconsistencies for the gamblers with gains. We are uncertain what to make of these results, but suspect there may be issues of reliability arising from somewhat small sample sizes.

Recall that Cho *et al.* (1994) tested for the equalities (b) = (c) and (b) = (d) and, using median data, concluded that the former held and the latter failed for both the gambles of gains and losses. We reanalyzed these data using the individual analysis and the results of the sign tests were unchanged. In our earlier paper, we did not partition the subjects into gamblers and nongamblers. When we combine all of our subjects in the present study, we found neither equality was rejected regardless of the sign of gambles. The results from the two studies are not fully consistent, which may arise from different numbers of subjects and different proportional combinations of gamblers and nongamblers.

Overall Summary of the Results

If we assume CE is order-preserving over gambles, we are led to conclude for gamblers that convolution is

TABLE 13

Frequency Distributions and z-Scores for the Test of Variants of Segregation

Category	$CE_1 <' CE_2$	$CE_1 <' (=) CE_2$	$CE_1 >' CE_2$	z-Score	Confidence interval
		(a) = (c)			
Gains					
Gamblers	23	10 (3)	6	2.56*	$0.56 < p < 0.83$
Nongamblers	31	11 (3)	33	−0.23	$0.37 < p < 0.59$
Losses					
Gamblers	19	7 (6)	10	1.50	$0.48 < p < 0.78$
Nongamblers	20	15 (6)	37	−2.0*	$0.27 < p < 0.49$
		(a) = (d)			
Gains					
Gamblers	22	5 (4)	11	1.78	$0.50 < p < 0.79$
Nongamblers	27	12 (3)	36	−0.92	$0.33 < p < 0.55$
Losses					
Gamblers	19	10 (2)	11	1.11	$0.45 < p < 0.74$
Nongamblers	32	6 (6)	34	−0.12	$0.37 < p < 0.60$
		(b) = (c)			
Gains					
Gamblers	9	5 (3)	25	−2.56*	$0.17 < p < 0.44$
Nongamblers	37	13 (2)	26	1.26	$0.47 < p < 0.68$
Losses					
Gamblers	18	7 (4)	13	0.81	$0.42 < p < 0.72$
Nongamblers	32	14 (4)	28	0.35	$0.41 < p < 0.64$
		(b) = (d)			
Gains					
Gamblers	23	6 (4)	9	2.11*	$0.53 < p < 0.81$
Nongamblers	22	12 (1)	43	−2.28*	$0.27 < p < 0.48$
Losses					
Gamblers	17	5 (4)	16	0.16	$0.37 < p < 0.68$
Nongamblers	33	10 (6)	29	0.35	$0.41 < p < 0.64$

* $p < .05$.
** $p < 0.01$.

TABLE 14

z-Scores for the Test of Variants of Segregation for the Gambles of Gains

Category	z-Score	Confidence interval	Category	z-Score	Confidence interval
	(a) = (c)			(a) = (d)	
Gamblers			Gamblers		
Strict	2.97**	$0.62 < p < 0.9$	Strict	2.31*	$0.55 < p < 0.87$
Lenient	2.38*	$0.54 < p < 0.78$	Lenient	1.74	$0.49 < p < 0.74$
Nongamblers			Nongamblers		
Lenient	−0.22	$0.35 < p < 0.6$	Lenient	−0.97	$0.29 < p < 0.53$
Strict	−0.26	$0.39 < p < 0.6$	Strict	−1.32	$0.34 < p < 0.55$
	(b) = (c)			(b) = (d)	
Gamblers			Gamblers		
Strict	−3.21**	$0.08 < p < 0.36$	Strict	3.08**	$0.63 < p < 0.92$
Lenient	−2.27*	$0.23 < p < 0.47$	Lenient	0.94	$0.44 < p < 0.69$
Nongamblers			Nongamblers		
Lenient	1.29	$0.48 < p < 0.72$	Lenient	−2.45*	$0.26 < p < 0.50$
Strict	1.56	$0.47 < p < 0.67$	Strict	−1.82	$0.27 < p < 0.47$

* $p < .05$.
** $p < .01$.

nonmonotonic and that convolution is identical to joint receipt. For nongamblers, exactly the opposite holds. In all other respects, gamblers and nongamblers are pretty much the same: both groups violate monotonicity of joint receipt of mixed gambles; over money ⊕ equals +; and segregation and additive segregation both hold. Several inconsistencies are found when the two terms of segregation are compared separately to the two terms of additive segregation.

DISCUSSION

A major difficulty with our data, as with most data on individual decision making under risk and uncertainty, is noise. There are at least two sources. One is the noisiness of the individual data. We attempted to reduce this difficulty by estimating the choice CEs and inferring a preference order by comparing the two CE conditions for each subject. However, these estimates remain somewhat noisy, and there is no obvious criterion for concluding the equality of two estimated CEs. Better statistical methods are needed for this task. Moreover, accepting our criterion as reasonable, the data lead us to question both whether CE is order-preserving over JR and whether JR is monotonic over (at least mixed) gambles. To test these hypotheses directly, we must evaluate the noisy choices directly, independent of CE estimates.

The other source of noise is substantial individual differences which impugn any analysis of aggregated data. We have clearly demonstrated appreciable differences between those we have called "gamblers" and "nongamblers." We very much doubt that this captures all of the individual differences and so it is rash to suppose that each of these groups is in fact homogeneous.

The approach of traditional psychophysics to such problems is simply to present the choice pairs repeatedly, develop psychometric curves, and then deal with each subject individually. There are at least two practical difficulties in implementing such an approach with gambles. One is the amount of time it takes. A second is that, unlike the psychophysical case where subjects work at the limits of their sensory systems, in the present context it is easy to imagine subjects developing strategies for the task. For example, after many repetitions of the same pair of gambles, even in the context where they are well separated by many other pairs, they may begin to remember their previous decisions.

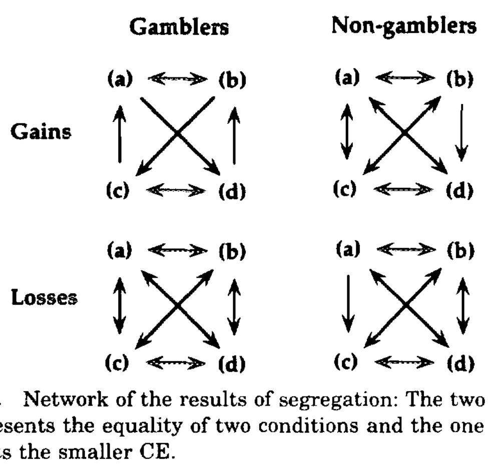

FIG. 6. Network of the results of segregation: The two sided arrow represents the equality of two conditions and the one sided arrow points the smaller CE.

Despite the possible problems, this direction must be pursued if we are to get direct tests of the two most fundamental hypotheses about CEs and if we are ever going to acquire enough data on subjects to evaluate their individual performance relevant to the several hypotheses being tested.

If we discover that CEs are order preserving, it would be desirable to develop a faster choice procedure than PEST to estimate them. von Winterfeldt *et al.* (1995) explored one method which did not work. Tversky and Kahneman (1992) used another that is somewhat faster than PEST. A comparison of the two procedures is currently under investigation.

If we become convinced that JR really is nonmonotonic for gambles, then a certain amount of new theoretical work is indicated. First, one needs to understand something about the possible numerical representations of nonmonotonic operations. Given that it is very plausible that the JR operation ⊕ is commutative ($g \oplus h \sim h \oplus g$) and associative [$f \oplus (g \oplus h) \sim (f \oplus g) \oplus h$ for gains and losses separately, then the class of representations is surely decidedly limited. Second, once these representations are understood, one needs to return to Luce and Fishburn's (1991, 1995) development of the rank- and sign-dependent utility representation for gambles, which rested on segregation and duplex decomposition, and work out the necessarily more complex representation arising from the nonmonotonic JR representation.

APPENDIX: INFERENCES FROM DATA AND HYPOTHESES

Luce (1995) explored the consequences of a number of hypotheses about certainty equivalences in structures with an operation of joint receipt. The purpose of this appendix is to appraise the impact of our data on these hypotheses. We partition the assumptions into two types: background ones that we do not believe need to be tested in the context of this type of experiment and hypotheses that, although highly plausible, are less secure. The background ones are further partitioned according to whether or not the operation $\oplus$ of joint receipt is involved. Next, we provide an idealized summary of the data, partitioned, where appropriate, by "gamblers" and "nongamblers." And finally we establish several propositions that follow from subsets of these properties. These conclusions were summarized in the text under Discussion and Conclusions.

Background Assumptions Involving CE and Preference

Let G denote a set of gambles including all pure consequences $\Re$, i.e., the real numbers. Suppose that over G, there is a weak preference relation $\succsim$ and a certainty equivalent mapping CE from G into $\Re$.

B1 $\Re \subseteq G$.

B2. For x in $\Re$, $\mathrm{CE}(x) = x$.

B3. CE is order preserving over G, i.e., $g \succsim h$ is equivalent to $\mathrm{CE}(g) \geqslant \mathrm{CE}(h)$.

B4. Consequence monotonicity, i.e., if g' is obtained from g by replacing one consequence x by x', where $x' \succsim x$, then $g' \succsim g$.

The following was established in Luce (1995):

LEMMA 1. *Given B1–B4, then*

(i) $g \sim CE(g)$.

(ii) *CE is onto* $\Re$.

(iii) *Over* $\Re$, $\succsim$ *is identical to* $\geqslant$.

(iv) $\succsim$ *is a weak ordering of G, i.e., connected and transitive.*

Background Hypotheses Involving Joint Receipt $\oplus$ as Well as CE

The structure $\langle G, \succsim, CE\rangle$ is assumed to be imbedded in its closure, $\langle G^c, \succsim, \mathrm{CE}, \oplus\rangle$, under the binary operation $\oplus$, which is interpreted as joint receipt. Both $\succsim$ and CE are extended to G^c. The members of G^c can be thought of as generalized gambles.

B5. $\succsim$ on G^c is a weak order.

B6. $g \oplus h \sim \mathrm{CE}(g \oplus h)$.

B7. $\oplus$ over $\Re$ is weakly commutative and monotonic increasing in each argument, i.e., $x \oplus y \sim y \oplus x$ and $x \geqslant y$ is equivalent to $x \oplus z \succsim y \oplus z$.

Note that part (i) of Lemma 1 and B6 are virtually forced to hold by the way in which CE is determined experimentally. Moreover, the study of Bostic, Herrnstein, and Luce (1990) supports B3 when CE is determined using a choice procedure. Consequence monotonicity, B4, was supported experimentally using PEST-determined CEs by von Winterfeldt, Chung, Luce, and Cho (1995). The extension of B1 to B5 does not seem very suspect, although it has not been studied explicitly. Finally, B7 for pure money consequences seems very plausible although it has not been directly tested. It is, of course, true if $x \oplus y \sim x + y$.

Substantive Hypotheses

H1. CE is order preserving over G^c.

H2. $\oplus$ over G^c is monotonic increasing in each argument.

H3. Consequence monotonicity for gambles holds with consequences in $\Re \oplus \Re$.

H4. For g and h in G, let $g * h$ denote their convolution. Then $*$ is monotonic.

Several immediate consequences of these hypotheses should be mentioned. First, H1 implies B5. B1–B3, B6–B7, and H1 imply that H2 is equivalent to:

$$\mathrm{CE}(g \oplus h) = \mathrm{CE}(g) \oplus \mathrm{CE}(h). \quad \text{(A1)}$$

[Luce, 1995, Theorem 1(i)]. Under the same assumptions, H4 is equivalent to

$$\mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h). \quad \text{(A2)}$$

[Luce, 1995, Theorem 3(i)].

Qualitative Summary of Data

The following are all statements that appear to be sustained by the data. Of course one can never be fully sure about establishing empirically the equality of the certainty equivalents of two gambles or jointly received gambles. In these cases, we must distinguish two types of subjects whom we have called "gamblers" and "nongamblers." For all real consequences x and y and gambles g and h in G:

D1. $\mathrm{CE}(g \oplus h) < \mathrm{CE}[\mathrm{CE}(g) \oplus \mathrm{CE}(h)]$ for gains (Tables 8, 9).

D2. $\mathrm{CE}(g \oplus h) < \mathrm{CE}(g) + \mathrm{CE}(h)$ for gains (Tables 8, 9).

D3. $\mathrm{CE}[\mathrm{CE}(g) \oplus \mathrm{CE}(h)] = \mathrm{CE}(g) + \mathrm{CE}(h)$ (Table 8).

D4. $\mathrm{CE}(x \oplus y) = x + y$ (Table 5).

D5. $\mathrm{CE}(x \oplus y,p;y) = \mathrm{CE}[x,p;0) \oplus y]$ (Table 11).

D6. $\mathrm{CE}(x + y,p;y) = \mathrm{CE}\ (x,p;0) + y$ (Table 11).

D7a. For nongamblers: $\mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h)$ (Table 6).

b. For gamblers: $\mathrm{CE}(g * h) < \mathrm{CE}(g) + \mathrm{CE}(h)$ (Table 6).

D8a. For nongamblers: $\mathrm{CE}(g * h) > \mathrm{CE}(g \oplus h)$ (Table 6).

b. For gamblers: $\mathrm{CE}(g * h) = \mathrm{CE}(g \oplus h)$ (Table 6).

Propositions

The first proposition concerns just relations among the data and it does not require any of the background assumptions. The second and third accept the background assumptions and concern relations among the data and hypotheses.

PROPOSITION I. (i) *Given D3, D1 is equivalent to D2.*
(ii) *D4 implies D3.*
(iii) *D7 and D8 imply D2.*
D2 and D7a imply D8a.
D2 and D8b imply D7b.

Proof: (i) and (ii) are both obvious.
(iii) For a nongambler, assume D7a and D8a, then

$$\mathrm{CE}(g \oplus h) < \mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h),$$

which is D2. Assume D2 and D7a, then

$$\mathrm{CE}(g \oplus h) < \mathrm{CE}(g) + \mathrm{CE}(h) = \mathrm{CE}(g * h),$$

which is D8a.
For a gambler, assume D7b and D8b, then

$$\mathrm{CE}(g \oplus h) < \mathrm{CE}(g * h) = \mathrm{CE}(g) + \mathrm{CE}(h),$$

which is D2. Assume D2 and D8b, then

$$\mathrm{CE}(g * h) = \mathrm{CE}(g \oplus h) < \mathrm{CE}(g) + \mathrm{CE}(h),$$

which is D7b.

PROPOSITION II. *Suppose that B1–B7 hold.*
(i) *D1 implies not both H1 and H2.*
(ii) *Suppose H1, H3, and D4. Then* (*a*) = (*c*), *i.e.,*

$$\mathrm{CE}(x \oplus y,p;y) = \mathrm{CE}(x + y,p;y). \qquad \text{(A3)}$$

Suppose H1, H2, and D4, then (*b*) = (*d*), *i.e.,*

$$\mathrm{CE}[(x,p;0) \oplus y] = \mathrm{CE}(x,p;0) + y. \qquad \text{(A4)}$$

(iii) *D7a is equivalent to H4 for nongamblers.*
(iv) *H1 and D8 imply*

$$g \oplus h \quad g * h \qquad \begin{matrix}\text{for nongamblers}\\ \text{for gamblers}\end{matrix} \qquad \text{(A5)}$$

Proof: (i) Suppose D1, H1, and H2 all hold, then

$$\begin{aligned} g \oplus h &< \mathrm{CE}(g) \oplus \mathrm{CE}(h) && \text{(D1 and H1)}\\ &\sim \mathrm{CE}(g \oplus h) && \text{[H2 and Equation (A1)]}\\ &\sim g \oplus h && \text{(B6)} \end{aligned}$$

which contradicts B5.
(ii) Assuming H1, H3, and D4,

$$\begin{aligned} \mathrm{CE}(x + y,p;y) &= \mathrm{CE}[\mathrm{CE}(x \oplus y),p;y] && \text{(B6, H3)}\\ &= \mathrm{CE}(x,p;0) + y, && \text{(D4, B4)} \end{aligned}$$

which is Equation (A3).
Assuming H1, H2, and D4,

$$\begin{aligned} \mathrm{CE}(x,p;0) \oplus y] &= \mathrm{CE}\{\mathrm{CE}[x,p;0) \oplus y]\} && \text{(B2, H2)}\\ &= \mathrm{CE}(x,p;0) + y, && \text{(D4, B2)} \end{aligned}$$

which is Equation (A4).
The other conclusions are immediate.
(iii) Equation (A2).
(iv) Immediate.

PROPOSITION III. *Suppose B1–B3, B5–B7, D3, and D7 hold. If Eq. (A5), then Eq. (A1) is false. Assuming H1, then CE is not monotonic over ⊕ (H2).*

Proof: We distinguish between nongamblers and gamblers by putting the inequalities for the latter in parentheses when they differ:

$$\begin{aligned} g \oplus h &< (\sim)\, g*h && \text{[Equation (A1)]}\\ &\sim \mathrm{CE}(g*h) && \text{[Lemma 1(i)]}\\ &= (<)\, \mathrm{CE}(g) + \mathrm{CE}(h) && \text{[D7a (D7b)]}\\ &= \mathrm{CE}[\mathrm{CE}(g) \oplus \mathrm{CE}(h)] && \text{(D3)}\\ &\sim \mathrm{CE}(g) \oplus \mathrm{CE}(h). && \text{(B6)} \end{aligned}$$

Thus, Equation (A1), and so H2, is violated.

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